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Parabolic Gradient Estimates and Harnack Inequalities for a Nonlinear Equation Under The Ricci Flow

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Abstract

When the Riemannian metric evolves under the Ricci flow, we investigate parabolic gradient estimates (Li–Yau's type and J. Li's type) for positive solutions to the nonlinear parabolic equation $(\Delta - \partial_t)u = (p+1)\frac{|\nabla u|^2}{u} + qu$ on the underlying manifold. Based on these gradient estimates, we derive associated Harnack inequalities, respectively.

Keywords Parabolic gradient estimate · Nonlinear parabolic equation · Ricci flow

Mathematics Subject Classification 35B45 · 35K55 · 53C44

1 Introduction

In 1980s, Li and Yau (1986) derived a gradient estimate, which was known as the Li–Yau estimate, for the heat equation on a complete Riemannian manifold. Moreover, they deduced Harnack inequalities. The Harnack inequality also applied to the Ricci flow by Hamilton (1993) and played an important role in solving the Poincaré conjecture Cao and Zhu (2006); Perelman (Perelman). After the fundamental work of Li and Yau (1986), the investigation of Li–Yau estimates for general parabolic partial differential equations of second order has drawn much attentions. To name a few, Yang (2008) proved the Li–Yau estimate for the equation on a Riemannian manifold of $\partial_t u = \Delta u + au \ln u + bu$ with $a, b \in \mathbb{R}$, which was introduced by Ma (2016). Moreover, Wu (2010) derived a Li–Yau estimate for the positive solutions to the equation of $\partial_t u = \Delta u - \langle \nabla \phi, \nabla u \rangle - au \ln u - qu$ on complete Riemannian manifolds. Moreover, Li (1991) proved different type parabolic gradient estimates and Harnack inequalities for positive solutions to a heat-type equation of $(\Delta - \partial_t)u + hu^{\alpha} = 0$ on manifolds.

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When the metric evolves along the Ricci flow, Bailesteanu et al. (2010) obtained a series of gradient estimates and Harnack inequalities for positive solutions to the heat equation. Li and Zhu (2016) showed Li–Yau estimates and associated Harnack inequalities for the parabolic partial differential equation of $\partial_t u = \Delta_t u + hu^p$ under the Ricci flow. They also Li and Zhu (2018) derived Li–Yau's gradient estimates for $(\Delta_t - \partial_t)u = qu + au(\ln u)^\alpha$ coupling with the Ricci flow.

Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow

$$\partial_t g(t) = -2Ric_{g(t)}. \tag{1.1}$$

Motivating by the works mentioned above, we consider a positive function $u = u(x,t)$ defined on the compact set $Q_{\rho,T} := \overline{B(\bar{x},\rho)} \times [0,T]$ solving the nonlinear parabolic equation of

$$(\Delta - \partial_t)u = (p+1)\frac{|\nabla u|^2}{u} + qu \tag{1.2}$$

with a nonzero function $p \in C^2(M^n)$ and a time-dependent function $q \in C^{2,1}\big(M^n \times [0,T]\big)$. Here Δ stands for the Laplacian of $g(x,t)$.

Throughout the paper, we define parabolic cylinders

$$Q_{\rho,t} := \overline{B(\bar{x},\rho)} \times [0,t] \subset M \times [0,T],$$

which is compact for any $0 < \rho < +\infty$.

The first result of this paper gives a Li–Yau's type gradient estimate for positive solutions of the nonlinear parabolic Eq. (1.2) in the case of $p > 0$ under the Ricci flow (1.1).

Theorem 1.1 *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M^n with $-Kg(t) \le Ric_{g(t)} \le Kg(t)$ on $Q_{\rho,T}$ for some constant $K > 0$ and $u(x,t)$ be a smooth positive solution to the nonlinear parabolic Eq. (1.2) with $p > 0$ on $Q_{\rho,T}$. Then for any $\beta > 1$, $0 < \epsilon < 1$ and $0 < a < \frac{1}{\beta}$, there exist positive constants $C_{1/2}$ and $\bar{C}$ so that*

$$\begin{aligned}
&p|\nabla f|^2 + \beta\partial_t f + \beta q \\
&\quad\le \frac{n\beta}{2a(1-\epsilon)\sigma_1}\left[A + \frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2} + \frac{\sigma_3^2}{(1-a\beta)(\beta-1)\sigma_1^2}\right.\\
&\qquad\left.+\frac{\theta_1}{2} + \frac{\beta K}{2(\beta-1)} + \frac{\sqrt{n}\sigma_4}{2(\beta-1)\sigma_1}\right] + \left(\frac{n\beta}{2a(1-\epsilon)\sigma_1}\right)^{\frac{1}{2}}\\
&\qquad\cdot\left[\frac{\beta^2 nK^2}{(1-a\beta)\sigma_1} + (\beta-1)\sigma_2\theta_1 + \sqrt{n}\beta\theta_2\right]^{\frac{1}{2}}
\end{aligned} \tag{1.3}$$

on $Q_{\frac{\rho}{2},T}$.

Here $0 < \sigma_1 \leq p \leq \sigma_2$, $|\nabla p| \leq \sigma_3$, $|Hess\ p| \leq \sigma_4$ *and* $|\nabla q| \leq \theta_1$, $|Hess\ q| \leq \theta_2$ *on* $Q_{\rho,T}$ *for some positive constants* σ_1, σ_2, σ_3, σ_4 *and* θ_1, θ_2. *Moreover,*

$$A := \frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K}+\frac{2}{\rho}\right)+\frac{C_{1/2}+2C_{1/2}^2}{\rho^2}+\frac{(\beta+1)\bar{C}+1}{t}+(\beta+1)C_{1/2}K.$$

It is not hard to derive the associated Harnack inequality.

Corollary 1.2 *Let* $(M^n, g(t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow* (1.1) *on an* n*-dimensional manifold* M^n *with* $-Kg(t) \leq Ric_{g(t)} \leq Kg(t)$ *on* $Q_{\rho,T}$ *for some constant* $K > 0$ *and* $u(x,t)$ *be a smooth positive solution to the nonlinear parabolic Eq.* (1.2) *with* $p > 0$ *on* $Q_{\rho,T}$. *Then for any* $\beta > 1$, $0 < \epsilon < 1$ *and* $0 < a < \frac{1}{\beta}$, *there exist positive constants* $C_{1/2}$ *and* $\bar{C}$ *so that*

$$\begin{aligned}\frac{u(y_2,s_2)}{u(y_1,s_1)} \leq \left(\frac{s_2}{s_1}\right)^{\frac{n(\beta\bar{C}+\beta+1)}{2a(1-\epsilon)\sigma_1}} \cdot \exp\Bigg\{&(s_2-s_1)\Bigg\{\theta_0+\frac{n}{2a(1-\epsilon)\sigma_1}\\ &\times\left[\frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K}+\frac{2}{\rho}\right)\right.\\ &+\frac{C_{1/2}+2C_{1/2}^2}{\rho^2}+(\beta+1)C_{1/2}K+\frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2}\\ &+\frac{\sigma_3^2}{(1-a\beta)(\beta-1)\sigma_1^2}+\frac{\theta_1}{2}\\ &\left.+\frac{\beta K}{2(\beta-1)}+\frac{\sqrt{n}\sigma_4}{2(\beta-1)\sigma_1}\right]+\sqrt{\frac{n\beta}{2a(1-\epsilon)\sigma_1}}\left[\frac{nK^2}{(1-a\beta)\sigma_1}+\frac{\beta-1}{\beta}\sigma_2\theta_1\right.\\ &\left.+\frac{\sqrt{n}\theta_2}{\beta}\right]^{\frac{1}{2}}\Bigg\}\Bigg\}. \qquad (1.4)\end{aligned}$$

for any (y_1, s_1), $(y_2, s_2) \in Q_{\frac{\rho}{2},T}$ *with* $0 < s_1 < s_2$.

Here, $0 < \sigma_1 \leq p \leq \sigma_2$, $|\nabla p| \leq \sigma_3$, $|Hess\ p| \leq \sigma_4$ *and* $|q| \leq \theta_0$, $|\nabla q| \leq \theta_1$, $|Hess\ q| \leq \theta_2$ *on* $Q_{\rho,T}$ *for some positive constants* σ_1, σ_2, σ_3, σ_4 *and* θ_0, θ_1, θ_2.

Inspired by the works of Li (1991), Li and Zhu (2016), we present a parabolic gradient estimate for positive solutions of the nonlinear parabolic Eq. (1.2) in the case of $p < 0$ under the Ricci flow (1.1).

Theorem 1.3 *Let* $(M^n, g(t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow* (1.1) *on an* n*-dimensional manifold* M^n *with* $-Kg(t) \leq Ric_{g(t)} \leq Kg(t)$ *on* $Q_{\rho,T}$ *for some constant* $K > 0$ *and* $u(x,t)$ *be a smooth positive solution to the nonlinear parabolic Eq.* (1.2) *with* $p < 0$ *on* $Q_{\rho,T}$. *Then for positive constants* b, k_1 *and* k_2 *with* $k_1 b \geq 1$ *and* $\frac{2nb^2}{\min_{Q_{\rho,T}}(-p)}k_2 < 1$, *there exist positive constants* $C_{1/2}$ *and* $\bar{C}$ *so that*

$$
\begin{aligned}
&\frac{|\nabla u|^2}{u^2} - k_1 q - \frac{k_1 \partial_t u}{u} \\
&\leq \frac{nk_1^2}{1-k_2} B + \frac{k_1}{\sqrt{1-k_2}} \left(\sqrt{n}\theta_1 + n^{\frac{3}{4}} \sqrt{\theta_2 k_1} \right) \\
&\quad + \frac{\sqrt{n} k_1}{b(1-k_2)} \left[\frac{1}{n} \left(1 + \frac{\sigma_5}{b} - \frac{2}{k_1 b} \right)^2 - \frac{1}{k_2} \right]^{-\frac{1}{2}} \\
&\quad \cdot \left[1 + \frac{n\sigma_3^2}{b^2(1-k_2)} \left(1 + \frac{\sigma_5}{b} - \frac{1}{k_1 b} \right)^{-2} + \frac{nk_1^2 \sigma_6^2 C_{1/2}^2}{(1-k_2)\rho^2} \right. \\
&\quad \left. + 2(b+\sigma_6) k_1 K + 2k_1 b K \left(1 + \frac{\sigma_5}{b} - \frac{1}{k_1 b} \right) \right]
\end{aligned} \tag{1.5}
$$

on $Q_{\frac{\rho}{2},T}$, *where*

$$
B := \frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K} + \frac{2}{\rho}\right) + \frac{C_{1/2} + 2C_{1/2}^2}{\rho^2} + \frac{\bar{C}}{t} + \left(C_{1/2} + 2\right) K.
$$

Here $-\sigma_6 \leq p \leq -\sigma_5 < 0$, $|\nabla p| \leq \sigma_3$, $|Hess\ p| \leq \sigma_4$ *and* $|\nabla q| \leq \theta_1$, $|Hess\ q| \leq \theta_2$ *on* $Q_{\rho,T}$ *for some positive constants* σ_3, σ_4, σ_5, σ_6 *and* θ_1, θ_2.

The Harnack inequality associated to Theorem 1.3 is

Corollary 1.4 *Let* $(M^n, g(t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow* (1.1) *on an* n*-dimensional manifold* M^n *with* $-Kg(t) \leq Ric_{g(t)} \leq Kg(t)$ *on* $Q_{\rho,T}$ *for some constant* $K > 0$ *and* $u(x,t)$ *be a smooth positive solution to the nonlinear parabolic Eq.* (1.2) *with* $p < 0$ *on* $Q_{\rho,T}$. *Then for positive constants* b, k_1 *and* k_2 *with* $k_1 b \geq 1$ *and* $\frac{2nb^2}{\min_{Q_{\rho,T}}(-p)} k_2 < 1$, *there exist positive constants* $C_{1/2}$ *and* $\bar{C}$ *so that*

$$
\begin{aligned}
\frac{u(y_1,s_1)}{u(y_2,s_2)} &\leq \left(\frac{s_2}{s_1}\right)^{\frac{nk_1^2\bar{C}}{1-k_2}} \cdot \exp\left\{ (s_2 - s_1) \left\{ \theta_0 + \frac{nk_1^2}{1-k_2} E \right.\right. \\
&\quad + \frac{k_1}{\sqrt{1-k_2}} (\sqrt{n}\theta_1 + n^{\frac{3}{4}} \sqrt{\theta_2 k_1}) \\
&\quad + \frac{\sqrt{n} k_1}{b(1-k_2)} \left[\frac{1}{n} \left(1 + \frac{\sigma_5}{b} - \frac{2}{k_1 b} \right)^2 - \frac{1}{k_2} \right]^{-\frac{1}{2}} \\
&\quad \cdot \left[1 + + \frac{n\sigma_3^2}{b^2(1-k_2)} \left(1 + \frac{\sigma_5}{b} - \frac{1}{k_1 b} \right)^{-2} \right. \\
&\quad \left.\left.\left. + \frac{nk_1^2 \sigma_6^2 C_{1/2}^2}{(1-k_2)\rho^2} + 2(b+\sigma_6) k_1 K + 2k_1 b K \left(1 + \frac{\sigma_5}{b} - \frac{1}{k_1 b} \right) \right] \right\} \right\}.
\end{aligned} \tag{1.6}
$$

for any $(y_1, s_1), (y_2, s_2) \in Q_{\frac{\rho}{2},T}$ *with* $0 < s_1 < s_2$*, where*

$$E := \frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K} + \frac{2}{\rho}\right) + \frac{C_{1/2} + 2C_{1/2}^2}{\rho^2} + (C_{1/2} + 2)K.$$

This paper is arranged as follows. In Sect. 2, we introduce the geometric background of (1.2), moreover, we present a fundamental analytical result that we shall need. In Sect. 3, we prove Theorem 1.1 and Corollary 1.2. We finish the proof of Theorem 1.3 in Sect. 4.

2 Preliminaries

In fact, our consideration of the nonlinear parabolic differential Eq. (1.2) is motivated by the understanding of gradient generalized m-quasi-Einstein metrics. First of all, we recall the definition of a gradient generalized m-quasi-Einstein metric (see e.g. Barros and Gomes 2013; Besse 1987; Case et al. 2011).

A gradient generalized m-quasi-Einstein metric on a complete Riemannian manifold (M^n, g) is a choice of a smooth potential function $f : M^n \to \mathbb{R}$ as well as a smooth function $\lambda : M^n \to \mathbb{R}$ such that

$$Ric + Hess f - \frac{1}{m} df \otimes df = \lambda g, \tag{2.1}$$

where Ric denotes the Ricci tensor of g, while $0 < m < +\infty$ is a constant, $Hess$ and $\otimes$ stand for the Hessian and the tensorial product, respectively.

Taking the trace of both sides of (2.1), we have

$$R + \Delta f - \frac{|\nabla f|^2}{m} = n\lambda. \tag{2.2}$$

Then letting $u = e^f$, we find that

$$R + \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} - \frac{|\nabla u|^2}{mu^2} = n\lambda.$$

Therefore, we obtained a elliptic partial differential equation on Riemannian manifold (M^n, g) that

$$\Delta u = \frac{m+1}{m}\frac{|\nabla u|^2}{u} + (n\lambda - R)u. \tag{2.3}$$

When the Riemannian metric g evolves along the Ricci flow (1.1), we get a parabolic partial differential equation that

$$\partial_t u = \Delta u - \frac{m+1}{m}\frac{|\nabla u|^2}{u} - (n\lambda - R_{g(t)})u. \tag{2.4}$$

It is clear that (2.4) is a special case of (1.2).

Next, we present a smooth cut-off function satisfying a basic analytical result stated in the following lemma. (see also Li and Yau 1986; Bailesteanu et al. 2010; Li and Zhu 2018). Note that $r(x,t) := d_{g(t)}(x,\bar{x})$ is the distance function from some point $\bar{x} \in M^n$ with respect to the metric $g(t)$.

Lemma 2.1 *Given $\tau \in (0,T]$, there exists a smooth function $\overline{\Psi} : [0,\infty) \times [0,T] \to \mathbb{R}$ satisfying the following requirements:*

1. *The support of $\overline{\Psi}$ is a subset of $[0,\rho] \times [0,T]$, and $0 \le \overline{\Psi} \le 1$ in $[0,\rho] \times [0,T]$.*
2. *The equalities $\overline{\Psi} = 1$ and $\frac{\partial \overline{\Psi}}{\partial r}(r,t) = 0$ hold in $[0,\frac{\rho}{2}] \times [\tau,T]$ and $[0,\frac{\rho}{2}] \times [0,T]$, respectively.*
3. *The estimate $|\frac{\partial \overline{\Psi}}{\partial t}| \le \frac{\bar{C}\overline{\Psi}^{\frac{1}{2}}}{\tau}$ is satisfied on $[0,\infty) \times [0,T]$ for some $\bar{C} > 0$, and $\overline{\Psi}(r,0) = 0$ for all $r \in [0,\infty)$.*
4. *The inequalities $-\frac{C_\alpha \overline{\Psi}^\alpha}{\rho} \le \frac{\partial \overline{\Psi}}{\partial r} \le 0$ and $|\frac{\partial^2 \overline{\Psi}}{\partial r^2}| \le \frac{C_\alpha \overline{\Psi}^\alpha}{\rho^2}$ hold on $[0,\infty) \times [0,T]$ for every $\alpha \in (0,1)$ with some constant C_α dependent on α.*

Throughout this paper, we choose the cut-off function $\bar{\Psi}$ constructed in Lemma 2.1 for any fixed $\tau \in (0,T]$. Moreover, we define $\Psi : M^n \times [0,T] \to \mathbb{R}$ by

$$\Psi(x,t) = \overline{\Psi}(r(x,t),t).$$

In the rest of this section, we deal with $(\Delta - \partial_t)\Psi - 2\frac{|\nabla\Psi|^2}{\Psi}$ that will be used in the proof of main theorems.

To estimate $\Delta\Psi$, we divide the arguments into two cases:

- Case 1: $r(x,t) < \frac{\rho}{2}$. In this case, it follows from Lemma 2.1 that $\Psi(x,t) \equiv 1$ around (x,t). Therefore, $\Delta\Psi(x,t) = 0$.
- Case 2: $r(x,t) \ge \frac{\rho}{2}$. Since $Ric \ge -(n-1)K$ in $B(\bar{x},\rho)$ for any fixed $t \in [0,T]$, the Laplace comparison theorem (see e.g. Li 1993) implies

$$\Delta r \le (n-1)\sqrt{K}\coth(\sqrt{K}r) \le (n-1)\left(\sqrt{K} + \frac{1}{r}\right). \tag{2.5}$$

It follows that

$$\begin{aligned}\Delta\Psi(x,t) &= \frac{\partial\overline{\Psi}}{\partial r}\Delta r + \frac{\partial^2\overline{\Psi}}{\partial r^2}|\nabla r|^2 \\ &\ge -\frac{C_{1/2}\Psi^{1/2}}{\rho}(n-1)\left(\sqrt{K} + \frac{2}{\rho}\right) - \frac{C_{1/2}\Psi^{1/2}}{\rho^2}.\end{aligned}$$

Therefore, we obtain

$$\Delta\Psi(x,t) \ge -\frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K} + \frac{2}{\rho}\right) - \frac{C_{1/2}}{\rho^2}. \tag{2.6}$$

Next, we estimate $\partial_t \Psi$. $\forall\, x \in B(\bar{x}, \rho)$, let $\gamma : [0, a] \to M^n$ be a minimal geodesic connecting x and $\bar{x}$ at time $t \in [0, T]$. Then we have

$$\begin{aligned} \partial_t r(x,t) &= \partial_t \int_0^a |\dot{\gamma}(s)| ds \\ &= -\int_0^a Ric(\dot{\gamma}(s), \dot{\gamma}(s)) ds \\ &\le K r(x,t) \\ &\le K\rho. \end{aligned}$$

Combining with Lemma 2.1, we know that

$$\begin{aligned} |\partial_t \Psi| &\le |\partial_t \bar{\Psi}| + |\nabla \bar{\Psi}| \cdot |\partial_t r| \\ &\le \frac{\bar{C}\Psi^{\frac{1}{2}}}{\tau} + K C_{1/2} \Psi^{\frac{1}{2}} \\ &\le \frac{\bar{C}}{\tau} + K C_{1/2}. \end{aligned} \tag{2.7}$$

By Lemma 2.1, we obtain

$$\frac{|\nabla \Psi|^2}{\Psi} \le \frac{C_{1/2}^2}{\rho^2}. \tag{2.8}$$

It follows from (2.6), (2.7) and (2.8), we have

$$\begin{aligned} (\Delta - \partial_t)\Psi - 2\frac{|\nabla \Psi|^2}{\Psi} \ge &-\frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K} + \frac{2}{\rho}\right) - \frac{C_{1/2}}{\rho^2} \\ &-\frac{\bar{C}}{\tau} - C_{1/2}K - \frac{2C_{1/2}^2}{\rho^2}. \end{aligned} \tag{2.9}$$

For simplicity, we define

$$\tilde{A} := \frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K} + \frac{2}{\rho}\right) + \frac{C_{1/2} + 2C_{1/2}^2}{\rho^2} + \frac{\bar{C}}{\tau} + C_{1/2}K.$$

3 The Case of $p > 0$

In this section, we finish the proof of Theorem 1.1 and Corollary 1.2, which presents a Li–Yau estimate for positive solutions of (1.2) coupling with the Ricci flow (1.1). We proceed with following evolution inequality.

Lemma 3.1 *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M^n and $u(x,t)$ be a smooth positive solution to the nonlinear parabolic Eq. (1.2) with $p > 0$ on $Q_{\rho,T}$. If $f := \ln u$, $H := t\big(p|\nabla f|^2 + \beta\partial_t f + \beta q\big)$ for any $\beta > 0$ and $0 < a < \frac{1}{\beta}$, then we have*

$$
\begin{aligned}
&\Delta H-(1-\beta)\partial_t H\\
&\quad\geq \frac{2a\beta pt}{n}\left(p|\nabla f|^2+\partial_t f+q\right)^2-\frac{t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta|Ric|)^2\\
&\qquad+2p\langle\nabla H,\nabla f\rangle+2(1-\beta)pt\langle\nabla q,\nabla f\rangle+2\beta ptRic(\nabla f,\nabla f)\\
&\qquad-\sqrt{n}|\nabla f|^2t|Hess\ p|-\sqrt{n}\beta t|Hess\ q|-\frac{H}{t} \qquad (3.1)
\end{aligned}
$$

on $Q_{\rho,T}$.

Proof Since $u=e^f$, we have $\partial_t u=e^f\partial_t f$, $\nabla u=e^f\nabla f$ and $\Delta u=e^f(\Delta f+|\nabla f|^2)$. By (1.2), we have

$$e^f(\Delta f+|\nabla f|^2-\partial_t f)=(p+1)e^f|\nabla f|^2+qe^f,$$

i.e.,

$$\Delta f=p|\nabla f|^2+\partial_t f+q. \qquad (3.2)$$

Moreover, we have

$$\Delta f=\frac{H}{t}+(1-\beta)(\partial_t f+q). \qquad (3.3)$$

Note that

$$\nabla_i H=t\big(2p\nabla_i\nabla_j f\nabla_j f+\nabla_i p|\nabla f|^2+\beta\nabla_i\partial_t f+\beta\nabla_i q\big).$$

Moreover, we can derive that

$$
\begin{aligned}
&\Delta H\\
&=2pt|Hessf|^2+2pt\Delta\nabla_j f\nabla_j f+4Hessf(\nabla p,\nabla f)+\Delta p|\nabla f|^2+\beta\Delta\partial_t f+\beta\Delta q\\
&\geq\frac{2a\beta pt(\Delta f)^2}{n}+2(1-a\beta)pt|Hessf|^2+2pt\langle\nabla\Delta f,\nabla f\rangle+2ptRic(\nabla f,\nabla f)\\
&\quad+4Hessf(\nabla p,\nabla f)+\Delta p|\nabla f|^2+\beta\partial_t\Delta f-2\beta tR_{ij}\nabla_i\nabla_j f+\beta\Delta q\\
&=\frac{2a\beta pt}{n}(p|\nabla f|^2+\partial_t f+q)^2+2(1-a\beta)pt|Hess\ f|^2+4tHessf(\nabla p,\nabla f)\\
&\quad-2\beta tR_{ij}\nabla_i\nabla_j f+2pt\langle\nabla\left[\frac{H}{t}+(1-\beta)(\partial_t f+q)\right],\nabla f\rangle+2ptRic(\nabla f,\nabla f)\\
&\quad+\Delta p|\nabla f|^2t+\beta\Delta qt+\beta t\partial_t\left[\frac{H}{t}+(1-\beta)(\partial_t f+q)\right]. \qquad (3.4)
\end{aligned}
$$

By direct computations, we have

$$
\begin{aligned}
&2(1-a\beta)pt|Hessf|^2+4tHessf(\nabla p,\nabla f)-2\beta tR_{ij}\nabla_i\nabla_j f\\
&\quad\geq 2t[(1-a\beta)p|Hessf|^2-|Hess\ f|(2|\nabla p|\cdot|\nabla f|+\beta|Ric|)]\\
&\quad=2t\left[\sqrt{(1-a\beta)p}|Hessf|-\frac{2|\nabla p|\cdot|\nabla f|+\beta|Ric|}{2\sqrt{(1-a\beta)p}}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta|Ric|)^2\\
&\geq -\frac{t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta|Ric|)^2. \qquad (3.5)
\end{aligned}
$$

Applying (3.5) to (3.4), we obtain

$$
\begin{aligned}
\Delta H \geq\ & t\frac{2a\beta pt}{n}(p|\nabla f|^2+\partial_t f+q)^2-\frac{t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta|Ric|)^2\\
&+2p\langle\nabla H,\nabla f\rangle+2(1-\beta)pt\langle\nabla(\partial_t f+q),\nabla f\rangle+2ptRic(\nabla f,\nabla f)\\
&+\Delta p|\nabla f|^2 t+\beta\Delta qt-\beta\frac{H}{t}+\beta\partial_t H+\beta(1-\beta)t\partial_t(\partial_t f+q). \qquad (3.6)
\end{aligned}
$$

The definition of H implies that

$$
\begin{aligned}
\partial_t H &= \frac{H}{t}+t\partial_t(p|\nabla f|^2+\beta\partial_t f+\beta q)\\
&= \frac{H}{t}+2ptRic(\nabla f,\nabla f)+2pt\langle\nabla\partial_t f,\nabla f\rangle+\beta t\partial_t(\partial_t f+q), \qquad (3.7)
\end{aligned}
$$

where we used (1.1).

Combining (3.6) with (3.7), we have

$$
\begin{aligned}
&\Delta H-(1-\beta)\partial_t H\\
&\geq t\frac{2a\beta pt}{n}(p|\nabla f|^2+\partial_t f+q)^2-\frac{t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta|Ric|)^2\\
&\quad+2p\langle\nabla H,\nabla f\rangle+2(1-\beta)pt\langle\nabla(\partial_t f+q),\nabla f\rangle+2ptRic(\nabla f,\nabla f)\\
&\quad+\Delta p|\nabla f|^2 t+\beta\Delta qt-\beta\frac{H}{t}+\beta(1-\beta)t\partial_t(\partial_t f+q)-(1-\beta)\frac{H}{t}\\
&\quad-2(1-\beta)ptRic(\nabla f,\nabla f)-2(1-\beta)pt\langle\nabla\partial_t f,\nabla f\rangle-\beta(1-\beta)t\partial_t(\partial_t f+q)\\
&\geq \frac{2a\beta pt}{n}\big(p|\nabla f|^2+\partial_t f+q\big)^2-\frac{t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta|Ric|)^2\\
&\quad+2p\langle\nabla H,\nabla f\rangle+2(1-\beta)pt\langle\nabla q,\nabla f\rangle+2\beta ptRic(\nabla f,\nabla f)\\
&\quad-\sqrt{n}|\nabla f|^2 t|Hess\ p|-\sqrt{n}\beta t|Hess\ q|-\frac{H}{t}.
\end{aligned}
$$

□

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 From Lemma 3.1, we can get

$$
\begin{aligned}
&\Delta(\Psi H)-(1-\beta)\partial_t(\Psi H)\\
&\quad= H(\Delta-(1-\beta)\partial_t)\Psi+2\langle\nabla\Psi,\nabla H\rangle+\Psi(\Delta-(1-\beta)\partial_t)H\\
&\quad\geq H(\Delta-(1-\beta)\partial_t)\Psi+\frac{2}{\Psi}\langle\nabla\Psi,\nabla(\Psi H)\rangle-\frac{2|\nabla\Psi|^2}{\Psi}H\\
&\qquad+\Psi\Bigg[\frac{2a\beta pt}{n}\big(p|\nabla f|^2+\partial_t f+q\big)^2-\frac{t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta|Ric|)^2
\end{aligned}
$$

$$
\begin{aligned}
&+2p\langle\nabla H,\nabla f\rangle+2(1-\beta)pt\langle\nabla q,\nabla f\rangle-\sqrt{n}\beta t|Hess\ q|-\frac{H}{t}\\
&+2\beta ptRic(\nabla f,\nabla f)-\sqrt{n}|\nabla f|^2t|Hess\ p|\Big]\\
\geq\ & H(\Delta-(1-\beta)\partial_t)\Psi+\frac{2}{\Psi}\langle\nabla\Psi,\nabla(\Psi H)\rangle-\frac{2|\nabla\Psi|^2}{\Psi}H+2p\langle\nabla(\Psi H),\nabla f\rangle\\
&-2pH\langle\nabla\Psi,\nabla f\rangle+\Psi\Big[\frac{2a\beta pt}{n}\big(p|\nabla f|^2+\partial_t f+q\big)^2\\
&-\frac{t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta\sqrt{n}K)^2-2(\beta-1)pt|\nabla q||\nabla f|\\
&-\sqrt{n}\beta t|Hess\ q|-\frac{H}{t}-2\beta ptK|\nabla f|^2-\sqrt{n}|\nabla f|^2t|Hess\ p|\Big].
\end{aligned}
\tag{3.8}
$$

For fixed $\tau\in(0,T]$, let (x_1,t_1) be a maximum point for ΨH in $Q_{\rho,\tau}:=\overline{B(\bar{x},\rho)}\times[0,\tau]\subset Q_{\rho,T}$. It follows from (3.8) that at such point.

$$
\begin{aligned}
0\geq\ & H(\Delta-(1-\beta)\partial_t)\Psi-\frac{2|\nabla\Psi|^2}{\Psi}H-2pH\langle\nabla\Psi,\nabla f\rangle\\
&+\frac{2a\beta p\Psi t}{n}\big(p|\nabla f|^2+\partial_t f+q\big)^2\\
&-\frac{\Psi t}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta\sqrt{n}K)^2-2(\beta-1)p\Psi t|\nabla q||\nabla f|\\
&-\sqrt{n}\beta\Psi t|Hess\ q|-\frac{\Psi H}{t}-2\beta\Psi ptK|\nabla f|^2-\sqrt{n}|\nabla f|^2t|Hess\ p|.
\end{aligned}
\tag{3.9}
$$

Multiplying both sides of (3.9) by Ψt and define $I=\Psi H$, we will obtain

$$
\begin{aligned}
0\geq\ & \left[(\Delta-\partial_t)\Psi-2\frac{|\nabla\Psi|^2}{\Psi}+\beta\partial_t\Psi\right]It-2Ipt\langle\nabla\Psi,\nabla f\rangle\\
&+\frac{2a\beta p\Psi^2t^2}{n}\big(p|\nabla f|^2+\partial_t f+q\big)^2\\
&-\frac{\Psi^2t^2}{2(1-a\beta)p}(2|\nabla p|\cdot|\nabla f|+\beta\sqrt{n}K)^2-2(\beta-1)p\Psi^2t^2|\nabla q||\nabla f|\\
&-\sqrt{n}\beta\Psi^2t^2|Hess\ q|\\
&-\Psi I-2\beta p\Psi^2t^2K|\nabla f|^2-\sqrt{n}|\nabla f|^2\Psi t^2|Hess\ p|
\end{aligned}
\tag{3.10}
$$

at (x_1,t_1).

In the following, we deal with each term of the right-hand side of (3.10).

From Lemma 2.1, we get

$$
\langle\nabla\Psi,\nabla f\rangle\leq\frac{|\nabla\Psi|}{\Psi^{1/2}}\Psi^{1/2}|\nabla f|\leq\frac{C_{1/2}}{\rho}\Psi^{\frac{1}{2}}|\nabla f|.
\tag{3.11}
$$

Plugging (2.7), (2.9) and (3.11) into (3.10), we have

$$
\begin{aligned}
0 \geq & -\left(\tilde{A}+\frac{\beta \bar{C}}{\tau}+\beta C_{1/2} K\right) I t-\frac{2 p C_{1/2} I t}{\rho} \Psi^{\frac{1}{2}}|\nabla f| \\
& +\frac{2 a \beta p \Psi^2 t^2}{n}\left(p|\nabla f|^2+\partial_t f+q\right)^2 \\
& -\frac{2|\nabla p|^2|\nabla f|^2 \Psi^2 t^2}{(1-a \beta) p}-\frac{\beta^2 n K^2 \Psi^2 t^2}{(1-a \beta) p}-2(\beta-1) p \Psi^2 t^2|\nabla q||\nabla f| \\
& -\sqrt{n} \beta \Psi^2 t^2|Hess\ q| \\
& -\Psi I-2 \beta p \Psi^2 t^2 K|\nabla f|^2-\sqrt{n}|\nabla f|^2 \Psi t^2|Hess\ p| \qquad (3.12)
\end{aligned}
$$

at (x_1, t_1).

As in Chen and Chen (2009), Li and Zhu (2018), Yang (2008), we set $\omega = \frac{|\nabla f|^2(x_1,t_1)}{H(x_1,t_1)} \geq 0$. Therefore, $|\nabla f| = (\omega H)^{\frac{1}{2}}$ and

$$
p|\nabla f|^2+\partial_t f+q=p \omega H+\frac{1}{\beta}\left(\frac{H}{t}-p \omega H\right)=H\left(p \omega+\frac{1-p \omega t}{\beta t}\right).
$$

We can simplify (3.12) into the following inequality

$$
\begin{aligned}
0 \geq & -\left(\tilde{A}+\frac{\beta \bar{C}}{\tau}+\beta C_{1/2} K\right) I t-\frac{2 p C_{1/2} t}{\rho} \omega^{\frac{1}{2}} I^{\frac{3}{2}}+\frac{2 a p I^2}{n}[(\beta-1) p \omega t+1]^2 \\
& -\frac{2|\nabla p|^2 \omega I \Psi t^2}{(1-a \beta) p}-\frac{\beta^2 n K^2 \Psi^2 t^2}{(1-a \beta) p}-2(\beta-1) p \Psi^{\frac{3}{2}} t^2|\nabla q|(\omega I)^{\frac{1}{2}} \\
& -\sqrt{n} \beta \Psi^2 t^2|Hess\ q|-\Psi I-2 \beta p \Psi t^2 K \omega I-\sqrt{n} \omega I t^2|Hess\ p| \qquad (3.13)
\end{aligned}
$$

at (x_1, t_1).

By Cauchy's inequality, we get

$$
2(\omega I)^{\frac{1}{2}} \leq 1+\omega I, \qquad (3.14)
$$

and

$$
\frac{2 p C_{1/2} t}{\rho} \omega^{\frac{1}{2}} I^{\frac{3}{2}} \leq \frac{2 a \epsilon p I^2}{n \beta}[(\beta-1) p \omega t+1]^2+\frac{p C_{1/2}^2 \beta n \omega I t^2}{2 a \epsilon \rho^2[(\beta-1) p \omega t+1]^2} \qquad (3.15)
$$

for any $\epsilon \in (0,1)$.

Applying (3.14) and (3.15) to (3.13), we obtain

$$
\begin{aligned}
0 \geq & -\left(\tilde{A}+\frac{\beta \bar{C}}{\tau}+\beta C_{1/2} K\right) I t+\frac{2 a(1-\epsilon) p \beta I^2}{n \beta}[(\beta-1) p \omega t+1]^2 \\
& -\frac{p C_{1/2}^2 \beta n \omega I t^2}{2 a \epsilon \rho^2[(\beta-1) p \omega t+1]^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2|\nabla p|^2\omega I\Psi t^2}{(1-a\beta)p}-\frac{\beta^2 nK^2\Psi^2 t^2}{(1-a\beta)p}-(\beta-1)p\Psi^{\frac{3}{2}}t^2|\nabla q|(1+\omega I)\\
&-\sqrt{n}\beta\Psi^2 t^2|Hess\ q|-\Psi I-2\beta p\Psi t^2 K\omega I-\sqrt{n}\omega I t^2|Hess\ p| \qquad (3.16)
\end{aligned}
$$

at (x_1, t_1).

Note that $0 \le \Psi \le 1$ and $p > 0$. It follows from (3.16) that

$$
\begin{aligned}
&\frac{2a(1-\epsilon)pI^2}{n\beta}\Big[(\beta-1)p\omega t+1\Big]^2\\
&\le I\Big[(\tilde{A}+\frac{\beta\bar{C}}{\tau}+\beta C_{1/2}K)t+\frac{pC_{1/2}^2\beta n\omega t^2}{2a\epsilon\rho^2[(\beta-1)p\omega t+1]^2}+\frac{2|\nabla p|^2\omega t^2}{(1-a\beta)p}\\
&\quad+(\beta-1)pt^2|\nabla q|\omega+1+\beta pKt^2\omega+\sqrt{n}\omega t^2|Hess\ p|\Big]\\
&\quad+\frac{\beta^2 nK^2t^2}{(1-a\beta)p}+(\beta-1)p|\nabla q|t^2+\sqrt{n}\beta|Hess\ q|t^2 \qquad (3.17)
\end{aligned}
$$

at (x_1, t_1).

Since $\beta > 1$, we have $\big[(\beta-1)p\omega t+1\big]^2 \ge 1$ and

$$
\frac{p\omega t}{\big[(\beta-1)p\omega t-1\big]^2} \le \frac{1}{2(\beta-1)}.
$$

Therefore, (3.17) reduces to

$$
\begin{aligned}
I^2 \le\ & \frac{n\beta}{2a(1-\epsilon)p}\Bigg[\left(\tilde{A}+\frac{\beta\bar{C}}{\tau}+\beta C_{1/2}K\right)t+\frac{C_{1/2}^2\beta nt}{4a(\beta-1)\epsilon\rho^2}+\frac{|\nabla p|^2 t}{(1-a\beta)(\beta-1)p^2}\\
&+\frac{|\nabla q|t}{2}+1+\frac{\beta Kt}{2(\beta-1)}+\frac{\sqrt{n}|Hess\ p|t}{2(\beta-1)p}\Bigg]I\\
&+\frac{n\beta}{2a(1-\epsilon)p}\left[\frac{\beta^2 nK^2t^2}{(1-a\beta)p}+(\beta-1)p|\nabla q|t^2+\sqrt{n}\beta|Hess\ q|t^2\right] \qquad (3.18)
\end{aligned}
$$

at (x_1, t_1).

Recall an elementary fact: if $x^2 \le a_1x + a_2$ for some $a_1, a_2, x \ge 0$, then

$$
x \le \frac{a_1}{2}+\sqrt{a_2+\left(\frac{a_2}{2}\right)^2} \le \frac{a_1}{2}+\sqrt{a_2}+\left(\frac{a_2}{2}\right)^2 = a_1+\sqrt{a_2}.
$$

Then we get an upper bound for $I(x_1, t_1)$ that

$$
\begin{aligned}
I(x_1,t_1) \le\ & \frac{n\beta\tau}{2a(1-\epsilon)p}\Bigg[\tilde{A}+\frac{\beta\bar{C}}{\tau}+\beta C_{1/2}K+\frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2}\\
&+\frac{|\nabla p|^2(x_1)}{(1-a\beta)(\beta-1)p^2(x_1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\nabla q|(x_1,t_1)}{2}+\frac{1}{\tau}+\frac{\beta K}{2(\beta-1)}+\frac{\sqrt{n}|Hess\ p|(x_1)}{2(\beta-1)p}\Bigg] \\
&+\sqrt{\frac{n\beta}{2a(1-\epsilon)p(x_1)}}\Bigg[\frac{\beta^2 nK^2}{(1-a\beta)p(x_1)}+(\beta-1)p(x_1)|\nabla q|(x_1,t_1) \\
&+\sqrt{n}\beta|Hess\ q|(x_1,t_1)\Bigg]^{\frac{1}{2}}\tau \\
\le\ & \frac{n\beta\tau}{2a(1-\epsilon)\sigma_1}\Bigg[\tilde{A}+\frac{\beta\bar{C}}{\tau}+\beta C_{1/2}K+\frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2} \\
&+\frac{\sigma_3^2}{(1-a\beta)(\beta-1)\sigma_1^2}+\frac{\theta_1}{2}+\frac{1}{\tau}+\frac{\beta K}{2(\beta-1)} \\
&+\frac{\sqrt{n}\sigma_4}{2(\beta-1)\sigma_1}\Bigg]+\sqrt{\frac{n\beta}{2a(1-\epsilon)\sigma_1}}\Bigg[\frac{\beta^2 nK^2}{(1-a\beta)\sigma_1} \\
&+(\beta-1)\sigma_2\theta_1+\sqrt{n}\beta\theta_2\Bigg]^{\frac{1}{2}}\tau. \qquad (3.19)
\end{aligned}
$$

By the construction of Ψ, we have

$$
\sup_{Q_{\frac{\rho}{2},\tau}} H \le \sup_{Q_{\rho,\tau}}(\Psi H) = I(x_1,t_1)
$$

for all $t\in[0,\tau]$ with $\tau\le T$ is arbitrary. Therefore, we conclude that

$$
\begin{aligned}
p\frac{|\nabla u|^2}{u^2}+\beta\frac{\partial_t u}{u}+\beta q &= p|\nabla f|^2+\beta\partial_t f+\beta q \\
&\le \frac{n\beta}{2a(1-\epsilon)\sigma_1}\Bigg[A+\frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2}+\frac{\sigma_3^2}{(1-a\beta)(\beta-1)\sigma_1^2} \\
&\quad+\frac{\theta_1}{2}+\frac{\beta K}{2(\beta-1)}+\frac{\sqrt{n}\sigma_4}{2(\beta-1)\sigma_1}\Bigg]+\left(\frac{n\beta}{2a(1-\epsilon)\sigma_1}\right)^{\frac{1}{2}}\cdot \\
&\quad\Bigg[\frac{\beta^2 nK^2}{(1-a\beta)\sigma_1}+(\beta-1)\sigma_2\theta_1+\sqrt{n}\beta\theta_2\Bigg]^{\frac{1}{2}}
\end{aligned}
$$

on $Q_{\frac{\rho}{2},T}$, where

$$
A := \frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K}+\frac{2}{\rho}\right)+\frac{C_{1/2}+2C_{1/2}^2}{\rho^2}+\frac{(\beta+1)\bar{C}+1}{t}+(\beta+1)C_{1/2}K.
$$

□

We apply the Theorem 1.1 above to prove Corollary 1.2 by integrating along a space-time path joining any two points in M^n.

Proof of Corollary 1.2 It follows from Theorem 1.1 that

$$\begin{aligned}\frac{\partial_t u}{u} \leq \theta_0 + \frac{n}{2a(1-\epsilon)\sigma_1}\Bigg[&\frac{C_{1/2}}{\rho}(n-1)(\sqrt{K}+\frac{2}{\rho}) + \frac{C_{1/2}+2C_{1/2}^2}{\rho^2} \\ &+\frac{(\beta+1)\bar{C}+1}{t} + (\beta+1)C_{1/2}K + \frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2} + \frac{\sigma_3^2}{(1-a\beta)(\beta-1)\sigma_1^2} \\ &+\frac{\theta_1}{2} + \frac{\beta K}{\beta-1}\Bigg] \\ &+\left[\frac{n}{2a(1-\epsilon)\sigma_1\beta}\left((\beta-1)\sigma_2\theta_1 + \frac{\beta^2 n}{(1-a\beta)\sigma_1}K^2 + \beta\sqrt{n}\theta_2\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

For any $(y_1, s_1), (y_2, s_2) \in Q_{\frac{\rho}{2},T}$ with $0 < s_1 < s_2$, take the geodesic path $\gamma(t)$ from y_1 to y_2 at time s_1 parametrized proportional to arc length with parameter t starting at y_1 at time s_1 and ending at y_2 at time s_2. Now consider the path $(\gamma(t), t)$ in space-time and integrate (3.20) along γ, we get

$$\begin{aligned}\ln\frac{u(y_2,s_2)}{u(y_1,s_1)} \leq \frac{n(\beta\bar{C}+\beta+1)}{2a(1-\epsilon)\sigma_1}\ln\frac{s_2}{s_1} + (s_2-s_1)\Bigg\{&\theta_0 + \frac{\sigma_3^2}{(1-a\beta)(\beta-1)\sigma_1^2} \\ &+\frac{n}{2a(1-\epsilon)\sigma_1}\Bigg[\frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K}+\frac{2}{\rho}\right) + \frac{C_{1/2}+2C_{1/2}^2}{\rho^2} \\ &+\frac{\bar{C}+1}{t} + (\beta+1)C_{1/2}K + \frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2} + \frac{\theta_1}{2} + \frac{\beta K}{\beta-1}\Bigg] \\ &+\left[\frac{n}{2a(1-\epsilon)\sigma_1\beta}\left((\beta-1)\sigma_2\theta_1 + \frac{\beta^2 n}{(1-a\beta)\sigma_1}K^2 + \beta\sqrt{n}\theta_2\right)\right]^{\frac{1}{2}}\Bigg\}.\end{aligned}$$

for any given $(y_1, s_1), (y_2, s_2) \in Q_{\frac{\rho}{2},T}$ with $0 < s_1 < s_2$.

Now exponentiating and rearranging gives the desired result. □

As corollary of Theorem 1.1, we have a Li–Yau's type gradient estimate and associated Harnack inequality for (2.4), which has a closer relationship with the gradient generalized m-quasi-Einstein metric, coupling with the Ricci flow (1.1).

Corollary 3.2 *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.1) on an n-dimensional manifold M^n with $-Kg(t) \leq Ric_{g(t)} \leq Kg(t)$ on $Q_{\rho,T}$ for some constant $K > 0$ and $u(x,t)$ be a smooth positive solution to the nonlinear parabolic Eq. (2.4) on $Q_{\rho,T}$. Assume $\theta_1 = \sup_{Q_{\rho,T}} |\nabla(n\lambda - R_{g(t)})|$ and $\theta_2 = \sup_{Q_{\rho,T}} |Hess\ (n\lambda - R_{g(t)})|$, then for any $\beta > 1$, $0 < \epsilon < 1$ and $0 < a < \frac{1}{\beta}$, there exist positive constants $C_{1/2}$ and $\bar{C}$ so that*

$$\frac{|\nabla u|^2}{mu^2} + \beta\frac{\partial_t u}{u} + \beta q \leq \frac{mn\beta}{2a(1-\epsilon)}\Bigg[\frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K}+\frac{2}{\rho}\right) + \frac{C_{1/2}+2C_{1/2}^2}{\rho^2}$$

$$+\frac{\beta\bar{C}+\beta+1}{t}+(\beta+1)C_{1/2}K+\frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2}+\frac{\theta_1}{2}+\frac{\beta K}{\beta-1}\Big]$$
$$+\Big[\frac{mn\beta}{2a(1-\epsilon)}\Big(\frac{(\beta-1)\theta_1}{m}+\frac{\beta^2 nm}{2(1-a\beta)}K^2+\beta\sqrt{n}\theta_2\Big)\Big]^{\frac{1}{2}} \tag{3.21}$$

on $Q_{\frac{\rho}{2},T}$.

Corollary 3.3 *Let* $(M^n, g(t))_{t\in[0,T]}$ *be a complete solution to the Ricci flow* (1.1) *on an* n*-dimensional manifold* M^n *with* $-Kg(t) \le Ric_{g(t)} \le Kg(t)$ *on* $Q_{\rho,T}$ *for some constant* $K > 0$ *and* $u(x,t)$ *be a smooth positive solution to the nonlinear parabolic Eq.* (2.4) *on* $Q_{\rho,T}$. *Assume* $\theta_1 = \sup_{Q_{\rho,T}} |\nabla(n\lambda - R_{g(t)})|$ *and* $\theta_2 = \sup_{Q_{\rho,T}} |Hess\ (n\lambda - R_{g(t)})|$, *then for any* $\beta > 1$, $0 < \epsilon < 1$ *and* $0 < a < \frac{1}{\beta}$, *there exist positive constants* $C_{1/2}$ *and* $\bar{C}$ *so that*

$$\frac{u(y_2,s_2)}{u(y_1,s_1)} \le \Big(\frac{s_2}{s_1}\Big)^{\frac{mn(\beta\bar{C}+\beta+1)}{2a(1-\epsilon)}}\cdot\exp\Big\{(s_2-s_1)\Big\{\theta_0$$
$$+\frac{mn}{2a(1-\epsilon)}\Big[\frac{C_{1/2}}{\rho}(n-1)\Big(\sqrt{K}+\frac{2}{\rho}\Big)+\frac{C_{1/2}+2C_{1/2}^2}{\rho^2}$$
$$+(\beta+1)C_{1/2}K+\frac{C_{1/2}^2\beta n}{4a(\beta-1)\epsilon\rho^2}+\frac{\theta_1}{2}+\frac{\beta K}{\beta-1}\Big]$$
$$+\Big[\frac{mn}{2a(1-\epsilon)\beta}\Big(\frac{(\beta-1)\theta_1}{m}+\frac{\beta^2 nm}{2(1-a\beta)}K^2+\beta\sqrt{n}\theta_2\Big)\Big]^{\frac{1}{2}}\Big\}\Big\} \tag{3.22}$$

for any (y_1,s_1), $(y_2,s_2) \in Q_{\frac{\rho}{2},T}$ *with* $0 < s_1 < s_2$.

4 The Case of $p < 0$

In this section, we give a proof of Theorem 1.3, which presents a Li's type gradient estimate for positive solutions of (1.2) in the case of $p < 0$ under the Ricci flow (1.1).

As in Li (1991), Li and Zhu (2016), we define a new function

$$v = u^{-b},$$

where b is a small enough positive constant to be determined later. Then we have

$$\begin{aligned}(\Delta-\partial_t)v &= -bu^{-b-1}(\Delta-\partial_t)u + b(b+1)u^{-b-2}|\nabla u|^2\\ &= -b(p+1)u^{-b-2}|\nabla u|^2 - bqu^{-b} + b(b+1)u^{-b-2}|\nabla u|^2\\ &= \frac{b-p}{b}\frac{|\nabla v|^2}{v} - bqv,\end{aligned} \tag{4.1}$$

where we used (1.2) in the second equality.

Let k_1 be a positive constant such that $k_1 b \geq 1$. Similar to Li (1991), Li and Zhu (2016), we consider the evolution of the following three functions.

$$G_0 := |\nabla \ln v|^2 - k_1 b^2 q,$$
$$G_1 := \partial_t \ln v,$$

and

$$G := G_0 + k_1 b G_1.$$

Proposition 4.1 *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.1) and u be a smooth positive solution to the nonlinear parabolic Eq. (1.2) with $p < 0$ on $Q_{\rho,T}$. Then we have*

$$\begin{aligned}
&(\Delta - \partial_t)G_0 \\
&= \frac{2|Hessv|^2}{v^2} - \frac{2}{b}|\nabla \ln v|^2\langle \nabla p, \nabla \ln v\rangle - 4\frac{Hessv(\nabla \ln v, \nabla \ln v)}{v} + 2|\nabla \ln v|^4 \\
&\quad -2b(1 + k_1 p)\langle \nabla q, \nabla \ln v\rangle - \frac{2p}{b}\langle \nabla G_0, \nabla \ln v\rangle - k_1 b^2(\Delta - \partial_t)q
\end{aligned} \tag{4.2}$$

on $Q_{\rho,T}$.

Proof Note that

$$\nabla_i G_0 = \frac{2\nabla_i\nabla_j v\nabla_j v}{v^2} - \frac{2|\nabla v|^2\nabla_i v}{v^3} - k_1 b^2 \nabla_i q,$$

we have

$$\begin{aligned}
\Delta G_0 &= \frac{2|Hessv|^2}{v^2} + \frac{2\Delta\nabla_j v\nabla_j v}{v^2} - \frac{8Hessv(\nabla v, \nabla v)}{v^3} \\
&\quad -\frac{2|\nabla v|^2\Delta v}{v^3} + \frac{6|\nabla v|^4}{v^4} - k_1 b^2\Delta q \\
&= \frac{2|Hessv|^2}{v^2} + \frac{2\langle \nabla\Delta v, \nabla v\rangle}{v^2} + \frac{2Ric(\nabla v, \nabla v)}{v^2} - \frac{8Hessv(\nabla v, \nabla v)}{v^3} \\
&\quad -\frac{2|\nabla v|^2\Delta v}{v^3} + \frac{6|\nabla v|^4}{v^4} - k_1 b^2\Delta q,
\end{aligned} \tag{4.3}$$

where we used Bochner formula.

By (1.1), we obtain

$$\partial_t G_0 = \frac{2Ric(\nabla v, \nabla v)}{v^2} + \frac{2\langle \nabla\partial_t v, \nabla v\rangle}{v^2} - \frac{2|\nabla v|^2\partial_t v}{v^3} - k_1 b^2\partial_t q. \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$
\begin{aligned}
(\Delta - \partial_t)G_0 &= \frac{2|Hessv|^2}{v^2} + \frac{2\langle \nabla(\Delta - \partial_t)v, \nabla v\rangle}{v^2} - \frac{8Hessv(\nabla v, \nabla v)}{v^3} \\
&\quad - \frac{2|\nabla v|^2(\Delta - \partial_t)v}{v^3} + \frac{6|\nabla v|^4}{v^4} - k_1 b^2(\Delta - \partial_t)q \\
&= \frac{2|Hessv|^2}{v^2} - \frac{2|\nabla v|^2\langle \nabla p, \nabla v\rangle}{bv^3} + \frac{4(b-p)Hessv(\nabla v, \nabla v)}{bv^3} \\
&\quad - \frac{2(b-p)|\nabla v|^4}{bv^4} - \frac{2b\langle \nabla q, \nabla v\rangle}{v} - \frac{2bq|\nabla v|^2}{v^2} \\
&\quad - \frac{8Hessv(\nabla v, \nabla v)}{v^3} \\
&\quad - \frac{2|\nabla v|^2}{v^3}\left(\frac{(b-p)|\nabla v|^2}{bv} - bqv\right) + \frac{6|\nabla v|^4}{v^4} - k_1 b^2(\Delta - \partial_t)q \\
&= \frac{2|Hessv|^2}{v^2} - \frac{2}{b}|\nabla \ln v|^2\langle \nabla p, \nabla \ln v\rangle \\
&\quad - 4\left(1 + \frac{p}{b}\right)\frac{Hessv(\nabla \ln v, \nabla \ln v)}{v} \\
&\quad + 2\left(1 + \frac{2p}{b}\right)|\nabla \ln v|^4 - 2b\langle \nabla q, \nabla \ln v\rangle - k_1 b^2(\Delta - \partial_t)q \\
&= \frac{2|Hessv|^2}{v^2} - \frac{2}{b}|\nabla \ln v|^2\langle \nabla p, \nabla \ln v\rangle \\
&\quad - 4\frac{Hessv(\nabla \ln v, \nabla \ln v)}{v} + 2|\nabla \ln v|^4 \\
&\quad - 2b(1 + k_1 p)\langle \nabla q, \nabla \ln v\rangle - \frac{2p}{b}\langle \nabla G_0, \nabla \ln v\rangle \\
&\quad - k_1 b^2(\Delta - \partial_t)q,
\end{aligned} \tag{4.5}
$$

on $Q_{\rho,T}$, where we used (4.1) in the second equality. □

Similarly, we calculate $(\Delta - \partial_t)G_1$.

Proposition 4.2 *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.1) and u be a smooth positive solution to the nonlinear parabolic Eq. (1.2) with $p < 0$ on $Q_{\rho,T}$. Then we have*

$$
(\Delta - \partial_t)G_1 = -\frac{2(b-p)}{b}Ric(\nabla \ln v, \nabla \ln v) - \frac{2p}{b}\langle \nabla G_1, \nabla \ln v\rangle - b\partial_t q - \frac{2\langle Ric, \nabla^2 v\rangle}{v} \tag{4.6}
$$

on $Q_{\rho,T}$.

Proof By (1.1), we obtain

$$
\Delta G_1 = \frac{\Delta \partial_t v}{v} - \frac{2\langle \nabla \partial_t v, \nabla v\rangle}{v^2} - \frac{\Delta v \partial_t v}{v^2} + \frac{2|\nabla v|^2 \partial_t v}{v^3}
$$

$$= \frac{\partial_t \Delta v}{v} - \frac{2\langle Ric, \nabla^2 v\rangle}{v} - \frac{2\langle \nabla \partial_t v, \nabla v\rangle}{v^2} - \frac{\Delta v \partial_t v}{v^2} + \frac{2|\nabla v|^2 \partial_t v}{v^3}. \quad (4.7)$$

Furthermore, we have

$$\begin{aligned}
(\Delta - \partial_t) G_1 &= \frac{\partial_t (\Delta - \partial_t) v}{v} - \frac{2\langle Ric, \nabla^2 v\rangle}{v} - \frac{2\langle \nabla \partial_t v, \nabla v\rangle}{v^2} \\
&\quad - \frac{(\Delta - \partial_t) v \partial_t v}{v^2} + \frac{2|\nabla v|^2 \partial_t v}{v^3} \\
&= \frac{\partial_t \left(\frac{(b-p)|\nabla v|^2}{bv} - bqv \right)}{v} - \frac{2\langle Ric, \nabla^2 v\rangle}{v} - \frac{2\langle \nabla \partial_t v, \nabla v\rangle}{v^2} \\
&\quad - \frac{\left(\frac{(b-p)|\nabla v|^2}{bv} - bqv \right) \partial_t v}{v^2} + \frac{2|\nabla v|^2 \partial_t v}{v^3} \\
&= \frac{2(b-p) Ric(\nabla v, \nabla v)}{bv^2} + \frac{2(b-p)\langle \nabla \partial_t v, \nabla v\rangle}{bv^2} \\
&\quad - \frac{(b-p)|\nabla v|^2 \partial_t v}{bv^3} - b\partial_t q - bq \partial_t \ln v + \frac{2\langle Ric, \nabla^2 v\rangle}{v} \\
&\quad - \frac{2\langle \nabla \partial_t v, \nabla v\rangle}{v^2} - \frac{(b-p)|\nabla v|^2 \partial_t v}{bv^3} + bq \partial_t \ln v + \frac{2|\nabla v|^2 \partial_t v}{v^3} \\
&= -\frac{2(b-p)}{b} Ric(\nabla \ln v, \nabla \ln v) - \frac{2p}{b} \langle \nabla G_1, \nabla \ln v\rangle \\
&\quad - b\partial_t q - \frac{2\langle Ric, \nabla^2 v\rangle}{v} \qquad (4.8)
\end{aligned}$$

on $Q_{\rho,T}$, where we used (4.1) in the second equality. □

Then we derive the evolution inequality for G.

Lemma 4.3 *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete solution to the Ricci flow (1.1) and u be a smooth positive solution to the nonlinear parabolic Eq. (1.2) with $p < 0$ on $Q_{\rho,T}$. Then for some $0 < k_2 < 1$, we have*

$$\begin{aligned}
(\Delta - \partial_t) G &\geq \frac{2(1-k_2) G^2}{n k_1^2 b^2} + \frac{4(1-k_2)}{n k_1 b} \left(1 - \frac{p}{b} - \frac{1}{k_1 b} \right) |\nabla \ln v|^2 G \\
&\quad + 2(1-k_2) \left[\frac{1}{n} \left(1 - \frac{p}{b} - \frac{1}{k_1 b} \right)^2 - \frac{1}{k_2} \right] |\nabla \ln v|^4 - \frac{2}{b} |\nabla \ln v|^3 |\nabla p| \\
&\quad - 2b(1 + k_1 p)|\nabla q||\nabla \ln v| - \frac{2p}{b} \langle \nabla G, \nabla \ln v\rangle - \sqrt{n} k_1 b^2 |Hess q| \\
&\quad - 2(b-p) k_1 Ric(\nabla \ln v, \nabla \ln v) - \frac{2k_1 b \langle Ric, \nabla^2 v\rangle}{v}. \qquad (4.9)
\end{aligned}$$

on $Q_{\rho,T}$.

Proof Recall that $G = G_0 + k_1 b G_1$. It follows from Propositions 4.1 and 4.2 that

$$\begin{aligned}
&(\Delta - \partial_t)G \\
&= \frac{2|Hessv|^2}{v^2} - \frac{2}{b}|\nabla \ln v|^2 \langle \nabla p, \nabla \ln v\rangle - 4\frac{Hessv(\nabla \ln v, \nabla \ln v)}{v} + 2|\nabla \ln v|^4 \\
&\quad -2b(1+k_1 p)\langle \nabla q, \nabla \ln v\rangle - \frac{2p}{b}\langle \nabla G_0, \nabla \ln v\rangle - k_1 b^2 (\Delta - \partial_t) q \\
&\quad -2(b-p)k_1 Ric(\nabla \ln v, \nabla \ln v) - 2k_1 p \langle \nabla G_1, \nabla \ln v\rangle - k_1 b^2 \partial_t q \\
&\quad -\frac{2k_1 b\langle Ric, \nabla^2 v\rangle}{v} \\
&\geq 2(1-k_2)\frac{|Hessv|^2}{v^2} + 2\left(1 - \frac{1}{k_2}\right)|\nabla \ln v|^4 - \frac{2}{b}|\nabla \ln v|^3 |\nabla p| \\
&\quad -2b(1+k_1 p)|\nabla q||\nabla \ln v| - \frac{2p}{b}\langle \nabla G, \nabla \ln v\rangle - \sqrt{n} k_1 b^2 |Hessq| \\
&\quad -2(b-p)k_1 Ric(\nabla \ln v, \nabla \ln v) - \frac{2k_1 b\langle Ric, \nabla^2 v\rangle}{v},
\end{aligned} \tag{4.10}$$

for any $0 < k_2 < 1$, where we used Cauchy's inequality.

It follows from (4.1) that

$$\begin{aligned}
\frac{\Delta v}{v} &= \partial_t \ln v + \frac{b-q}{b}|\nabla \ln v|^2 - bq \\
&= \frac{G}{k_1 b} + \left(1 - \frac{p}{b} - \frac{1}{k_1 b}\right)|\nabla \ln v|^2
\end{aligned} \tag{4.11}$$

Furthermore, using Cauchy's inequality again and then applying (4.11), we have

$$\begin{aligned}
&(1-k_2)\frac{|Hessv|^2}{v^2} + \left(1 - \frac{1}{k_2}\right)|\nabla \ln v|^4 \\
&\geq \frac{1-k_2}{n}\left(\frac{\Delta v}{v}\right)^2 - \frac{1-k_2}{k_2}|\nabla \ln v|^4 \\
&= \frac{1-k_2}{n}\left[\frac{G}{k_1 b} + \left(1 - \frac{p}{b} - \frac{1}{k_1 b}\right)|\nabla \ln v|^2\right]^2 - \frac{1-k_2}{k_2}|\nabla \ln v|^4 \\
&= \frac{(1-k_2)G^2}{n k_1^2 b^2} + \frac{2b(1-k_2)}{n k_1 b}\left(1 - \frac{p}{b} - \frac{1}{k_1 b}\right)|\nabla \ln v|^2 G \\
&\quad +(1-k_2)\left[\frac{1}{n}\left(1 - \frac{p}{b} - \frac{1}{k_1 b}\right)^2 - \frac{1}{k_2}\right]|\nabla \ln v|^4.
\end{aligned} \tag{4.12}$$

(4.9) follows immediately by plugging (4.12) into (4.10). □

In the following, we finish the proof of Theorem 1.3.

Proof of Theorem 1.3 We only consider the case of $G \geq 0$.

By Lemma 4.3, we have

$$
\begin{aligned}
\Psi(\Delta-\partial_t)(\Psi G) \geq\ & \frac{2(1-k_2)(\Psi G)^2}{nk_1^2b^2}+\frac{4(1-k_2)}{nk_1b}\left(1-\frac{p}{b}-\frac{1}{k_1b}\right)|\nabla \ln v|^2\Psi^2 G \\
& +2(1-k_2)\left[\frac{1}{n}\left(1-\frac{p}{b}-\frac{1}{k_1b}\right)^2-\frac{1}{k_2}\right]\Psi^2|\nabla \ln v|^4-\frac{2}{b}\Psi^2|\nabla \ln v|^3|\nabla p| \\
& -2b(1+k_1p)\Psi^2|\nabla q||\nabla \ln v|-\frac{2p}{b}\Psi\langle\nabla(\Psi G),\nabla \ln v\rangle+\frac{2p}{b}\Psi G\langle\nabla\Psi,\nabla \ln v\rangle \\
& -\sqrt{n}k_1b^2\Psi^2|Hessq|-2(b-p)k_1\Psi^2 Ric(\nabla \ln v,\nabla \ln v) \\
& -\frac{2k_1b\Psi^2\langle Ric,\nabla^2 v\rangle}{v} \\
& +2\langle\nabla\Psi,\nabla(\Psi G)\rangle+\Psi G\left[(\Delta-\partial_t)\Psi-2\frac{|\nabla\Psi|^2}{\Psi}\right] \\
\geq\ & \frac{2(1-k_2)(\Psi G)^2}{nk_1^2b^2}+\frac{4(1-k_2)}{nk_1b}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)|\nabla \ln v|^2\Psi^2 G \\
& +2(1-k_2)\left[\frac{1}{n}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)^2-\frac{1}{k_2}\right]\Psi^2|\nabla \ln v|^4-\frac{2\sigma_3}{b}\Psi^2|\nabla \ln v|^3 \\
& -2b\theta_1\Psi|\nabla \ln v|-\frac{2p}{b}\Psi\langle\nabla(\Psi G),\nabla \ln v\rangle-\frac{2\sigma_6}{b}\Psi G|\nabla\Psi||\nabla \ln v| \\
& -\sqrt{n}k_1b^2\theta_2-2(b+\sigma_6)k_1K\Psi|\nabla \ln v|^2-2k_1bK\Psi^2|\frac{\Delta v}{v}| \\
& +2\langle\nabla\Psi,\nabla(\Psi G)\rangle+\Psi G\left[(\Delta-\partial_t)\Psi-2\frac{|\nabla\Psi|^2}{\Psi}\right].
\end{aligned}
\tag{4.13}
$$

For fixed $\tau\in(0,T]$, let (x_2,t_2) be a maximum point for ΨG in $Q_{\rho,\tau}:=\overline{B(\bar{x},\rho)}\times[0,\tau]\subset Q_{\rho,T}$. It follows from (4.13), (4.11) and (2.9) that at such point:

$$
\begin{aligned}
0\geq\ & \frac{2(1-k_2)(\Psi G)^2}{nk_1^2b^2}+\frac{4(1-k_2)}{nk_1b}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)|\nabla \ln v|^2\Psi^2 G \\
& +2(1-k_2)\left[\frac{1}{n}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)^2-\frac{1}{k_2}\right]\Psi^2|\nabla \ln v|^4-\frac{2\sigma_3}{b}\Psi^2|\nabla \ln v|^3 \\
& -2b\theta_1\Psi|\nabla \ln v|-\frac{2\sigma_6}{b}\Psi G|\nabla\Psi||\nabla \ln v|-\sqrt{n}k_1b^2\theta_2-2(b+\sigma_6)k_1K\Psi|\nabla \ln v|^2 \\
& -2K\Psi G-2k_1bK\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)\Psi|\nabla \ln v|^2-\tilde{A}\Psi G.
\end{aligned}
\tag{4.14}
$$

Since $k_1b\geq 1$ and $\frac{2nb^2}{\sigma_5}k_2<1$, it is easy to verify that

$$
\frac{1}{n}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)^2-\frac{2}{k_2}>0,
$$

and

$$\frac{1-k_2}{nk_1 b}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right) > 0, \tag{4.15}$$

By Cauchy's inequality and the fact of $0 \le \Psi \le 1$, we have

$$\begin{aligned} -\frac{2\sigma_3}{b}\Psi^2|\nabla \ln v|^3 \ge & -\frac{1-k_2}{n}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right)^2\Psi^2|\nabla \ln v|^4 \\ & -\frac{n\sigma_3^2}{b^2(1-k_2)}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right)^{-2}\Psi|\nabla \ln v|^2, \end{aligned} \tag{4.16}$$

and

$$-2b\theta_1\Psi|\nabla \ln v| \ge -\Psi|\nabla \ln v|^2 - b^2\theta_1^2. \tag{4.17}$$

Using Cauchy's inequality and Lemma 2.1, we obtain

$$\begin{aligned} & -\frac{2\sigma_6}{b}\Psi G|\nabla\Psi||\nabla \ln v| \\ & \ge -\frac{(1-k_2)(\Psi G)^2}{nk_1^2 b^2} - \frac{nk_1^2\sigma_6^2}{1-k_2}\frac{|\nabla\Psi|^2}{\Psi}\Psi|\nabla \ln v|^2 \\ & \ge -\frac{(1-k_2)(\Psi G)^2}{nk_1^2 b^2} - \frac{nk_1^2\sigma_6^2 C_{1/2}^2}{(1-k_2)\rho^2}\Psi|\nabla \ln v|^2. \end{aligned} \tag{4.18}$$

Plugging (4.15) to (4.18) into (4.14), we get

$$\begin{aligned} 0 \ge & \frac{(1-k_2)(\Psi G)^2}{nk_1^2 b^2} - (2K+\tilde{A})\Psi G - b^2\theta_1^2 - \sqrt{n}k_1 b^2\theta_2 \\ & +(1-k_2)\left[\frac{1}{n}\left(1+\frac{\sigma_5}{b}-\frac{2}{k_1 b}\right)^2 - \frac{1}{k_2}\right]\Psi^2|\nabla \ln v|^4 \\ & -\left[\frac{n\sigma_3^2}{b^2(1-k_2)}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right)^{-2} + 1 + \frac{nk_1^2\sigma_6^2 C_{1/2}^2}{(1-k_2)\rho^2}\right. \\ & \left. +2(b+\sigma_6)k_1 K + 2k_1 bK\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right)\right]\Psi|\nabla \ln v|^2 \\ \ge & \frac{(1-k_2)(\Psi G)^2}{nk_1^2 b^2} - (2K+\tilde{A})\Psi G - b^2\theta_1^2 - \sqrt{n}k_1 b^2\theta_2 \\ & -\frac{1}{1-k_2}\left[\frac{1}{n}\left(1+\frac{\sigma_5}{b}-\frac{2}{k_1 b}\right)^2 - \frac{1}{k_2}\right]^{-1} \\ & \cdot\left[1+\frac{n\sigma_3^2}{b^2(1-k_2)}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right)^{-2} + \frac{nk_1^2\sigma_6^2 C_{1/2}^2}{(1-k_2)\rho^2}\right. \\ & \left. +2(b+\sigma_6)k_1 K + 2k_1 bK\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right)\right]^2, \end{aligned} \tag{4.19}$$

at (x_2, t_2), i.e.,

$$(\Psi G)^2(x_2,t_2) \leq \frac{nk_1^2b^2}{1-k_2}(2K+\tilde{A})\Psi G + \frac{nk_1^2b^4\theta_1^2}{1-k_2} + \frac{n^{\frac{3}{2}}k_1^3b^4\theta_2}{1-k_2} + \frac{nk_1^2b^2}{(1-k_2)^2}\left[\frac{1}{n}\left(1+\frac{\sigma_5}{b}-\frac{2}{k_1b}\right)^2-\frac{1}{k_2}\right]^{-1} \cdot\left[1+\frac{n\sigma_3^2}{b^2(1-k_2)}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)^{-2}+\frac{nk_1^2\sigma_6^2C_{1/2}^2}{(1-k_2)\rho^2} + 2(b+\sigma_6)k_1K + 2k_1bK\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)\right]^2. \quad (4.20)$$

Recall an elementary fact: if $x^2 \leq a_1x + a_2$ for some $a_1, a_2, x \geq 0$, then

$$x \leq \frac{a_1}{2} + \sqrt{a_2 + \left(\frac{a_2}{2}\right)^2} \leq \frac{a_1}{2} + \sqrt{a_2} + \left(\frac{a_2}{2}\right)^2 = a_1 + \sqrt{a_2}.$$

Then we get an upper bound for $(\Psi G)(x_2, t_3)$ that

$$(\Psi G)(x_2,t_2) \leq \frac{nk_1^2b^2}{1-k_2}(2K+\tilde{A}) + \frac{k_1b^2}{\sqrt{1-k_2}}(\sqrt{n}\theta_1 + n^{\frac{3}{4}}\sqrt{\theta_2k_1}) + \frac{\sqrt{n}k_1b}{1-k_2}\left[\frac{1}{n}\left(1+\frac{\sigma_5}{b}-\frac{2}{k_1b}\right)^2-\frac{1}{k_2}\right]^{-\frac{1}{2}} \cdot\left[1+\frac{n\sigma_3^2}{b^2(1-k_2)}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)^{-2}+\frac{nk_1^2\sigma_6^2C_{1/2}^2}{(1-k_2)\rho^2} + 2(b+\sigma_6)k_1K + 2k_1bK\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1b}\right)\right]. \quad (4.21)$$

By the construction of Ψ, we have

$$\sup_{Q_{\frac{\rho}{2},\tau}} G \leq \sup_{Q_{\rho,\tau}}(\Psi G) = (\Psi G)(x_2,t_2)$$

for all $t \in [0, \tau]$ with $\tau \leq T$ is arbitrary. Therefore, we conclude that

$$\begin{aligned} G &= |\nabla \ln v|^2 - k_1b^2q + k_1b\partial_t \ln v \\ &= \frac{b^2|\nabla u|^2}{u^2} - k_1b^2q - \frac{k_1b^2\partial_t u}{u} \\ &\leq \frac{nk_1^2b^2}{1-k_2}B + \frac{k_1b^2}{\sqrt{1-k_2}}(\sqrt{n}\theta_1 + n^{\frac{3}{4}}\sqrt{\theta_2k_1}) \\ &\quad + \frac{\sqrt{n}k_1b}{1-k_2}\left[\frac{1}{n}\left(1+\frac{\sigma_5}{b}-\frac{2}{k_1b}\right)^2-\frac{1}{k_2}\right]^{-\frac{1}{2}} \end{aligned}$$

$$\cdot\left[1+\frac{n\sigma_3^2}{b^2(1-k_2)}\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right)^{-2}+\frac{nk_1^2\sigma_6^2C_{1/2}^2}{(1-k_2)\rho^2}\right.$$
$$\left.+2(b+\sigma_6)k_1K+2k_1bK\left(1+\frac{\sigma_5}{b}-\frac{1}{k_1 b}\right)\right] \tag{4.22}$$

on $Q_{\frac{\rho}{2},T}$, where

$$B:=\frac{C_{1/2}}{\rho}(n-1)\left(\sqrt{K}+\frac{2}{\rho}\right)+\frac{C_{1/2}+2C_{1/2}^2}{\rho^2}+\frac{\bar{C}}{t}+(C_{1/2}+2)K.$$

□

Remark 4.4 As we proved Theorem 1.1, it is not hard to derive Corollary 1.4 by similar arguments as in the proof of Corollary 1.2.

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