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Chains Homotopy in the Complement of a Knot in the Sphere S^3

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Abstract

If γ is a knot in $S^3 \cong \mathbb{H}^2_{\mathbb{C}} \subset \mathbb{P}^2_{\mathbb{C}}$, then the set $\Lambda(\gamma) \subset \mathbb{P}^2_{\mathbb{C}}$ is defined as the union of all the complex lines tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at points in the image of γ. The following result is obtained: the number of components of $\Omega(\gamma) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda(\gamma)$ is greater or equal to the number of distinct integers in the set $\{\ell(\gamma, C) : C$ is a positively oriented chain disjoint to $\gamma\}$, where $\ell(\gamma, C)$ denotes the linking number between γ and C.

Keywords Complex hyperbolic plane · Complex projective plane · Chain · Knot

Mathematics Subject Classification 51M10 · 57M25

1 Introduction

The results discussed in this paper arise as a natural problem in the theory of complex Kleinian groups: finding the number of components of the domain of discontinuity of a discrete subgroup $G \subset \mathrm{PU}(2, 1)$ acting on $\mathbb{P}^2_{\mathbb{C}}$. The domain of discontinuity, denoted

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$\Omega(G)$, is the complement in $\mathbb{P}^2_{\mathbb{C}}$ of the Kulkarni limit set denoted $\Lambda(G)$. This limit set $\Lambda(G)$ is obtained as the union of all the complex lines tangent to $\partial\mathbb{H}^2_{\mathbb{C}} \cong S^3$, the boundary 3-sphere of the complex hyperbolic space, at points in the set $L(G) \subset \partial\mathbb{H}^2_{\mathbb{C}}$, where $L(G)$ is the set of accumulation points of the orbit of some (any) point $p \in \overline{\mathbb{H}^2_{\mathbb{C}}}$ (see Navarrete 2006). In some cases, $L(G)$ happens to be a simple closed curve.

In this paper we study the related problem of finding the number of components of an open set, $\Omega(\gamma)$, associated to a knot $\gamma : S^1 \to S^3 \cong \partial\mathbb{H}^2_{\mathbb{C}}$ in the following way: Define $\Lambda(\gamma) := \bigcup_{p\in\gamma(S^1)} \ell_p$ where ℓ_p is the only complex line tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at the point p. The set $\Omega(\gamma)$ is the complement of $\Lambda(\gamma)$ in $\mathbb{P}^2_{\mathbb{C}}$.

In this paper we will be particularly interested on those curves obtained as the intersection of a complex projective line in $\mathbb{P}^2_{\mathbb{C}}$ and the boundary at infinity of the complex hyperbolic space, $\partial\mathbb{H}^2_{\mathbb{C}}$. These curves are called *chains* and there is a natural bijection between the complement in $\mathbb{P}^2_{\mathbb{C}}$ of the set $\overline{\mathbb{H}^2_{\mathbb{C}}} := \mathbb{H}^2_{\mathbb{C}} \cup \partial\mathbb{H}^2_{\mathbb{C}}$ and the set of non degenerate chains (chains which are not one single point). This bijection is given in the following way: $p \in \mathbb{P}^2_{\mathbb{C}} \setminus (\overline{\mathbb{H}^2_{\mathbb{C}}})$ corresponds to its *polar chain* C_p (see Sect. 3), and we obtain the following results:

Theorem 1.1 *Let us assume that p_0, p_1 are two points in $\Omega(\gamma) \cap (\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$ and C_{p_0}, C_{p_1} are the corresponding non degenerate polar chains. The points p_0 and p_1 can be joined by a continuous path $\nu : I \to \Omega(\gamma) \cap (\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$ if and only if there exists a continuous map*

$$H : I \times S^1 \to \partial\mathbb{H}^2_{\mathbb{C}}$$

such that $H(s, \cdot) : S^1 \to \partial\mathbb{H}^2_{\mathbb{C}}$ is a positive parametrization of the polar chain to $\nu(s)$, denoted $C_{\nu(s)}$, and $C_{\nu(s)} \cap \gamma(I) = \varnothing$ for every $s \in I$.

Corollary 1.2 *If p, q are two points in $\Omega(\gamma) \cap (\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$ and the linking number between the curve γ and the (positively oriented) chain C_p, polar to p, is different to the linking number between γ and the (positively oriented) chain C_q, polar to q, then p and q lie in distinct components of $\Omega(\gamma)$. In particular, the number of all possible linking numbers $\ell(\gamma, C)$ where C is any chain not intersecting the image of γ is smaller or equal to the number of components of $\Omega(\gamma)$.*

Corollary 1.3 *Let us assume that the image of the smooth curve $\gamma : S^1 \to \partial\mathbb{H}^2_{\mathbb{C}} \cong S^3$ is the intersection of S^3 and the algebraic curve $f(x, y) = 0$ with an isolated singularity at the origin* $(0, 0)$. *Moreover, assume that the intersection of the unit ball in $\mathbb{C}^2$ and $f^{-1}(0)$ is a topological disk, and γ has the orientation induced by the boundary of this set. If $\sigma : S^1 \to \partial\mathbb{H}^2_{\mathbb{C}} \cong S^3$ is a positive parametrization of a chain C not meeting $S^3 \cap f^{-1}(0)$, then*

$$\ell(\gamma, C) = \frac{1}{2\pi i} \int_{F\circ\sigma} \frac{dz}{z}, \tag{1}$$

where $F : (S^3 \setminus f^{-1}(0)) \to S^1$, $F(x, y) = \frac{f(x,y)}{|f(x,y)|}$ is the Milnor fibration. In particular, the number of all possible values of the integrals (1)*, where σ runs over all positive* *parametrizations of chains not meeting* $f^{-1}(0)$, *is smaller or equal to the number of components of* $\Omega(\gamma)$.

Theorem 1.4 *Let* γ *be the trefoil knot obtained as the intersection of the algebraic curve* $\{(x, y) \in \mathbb{C}^2 : y^2 = x^3\}$ *and* S^3*:*

$$\gamma : S^1 \to \partial\mathbb{H}^2_{\mathbb{C}} \cong S^3$$
$$\gamma(e^{2\pi i t}) = [(\sqrt{\alpha}e^{2\pi i t})^2 : (\sqrt{\alpha}e^{2\pi i t})^3 : 1],$$

where α *denotes the positive real root of the equation* $x^3 + x^2 - 1 = 0$.

(i) The set $\Omega(\gamma)$ *has four connected components,* Ω_j, *for* $j = 0, 1, 2, 3$ *where* Ω_j *consists of those points* $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ *such that the polynomial in the* ζ *variable given by* $p(\zeta) = \bar{y}\zeta^3 + \bar{x}\zeta^2 - \bar{z}$ *has precisely* j *roots within the open disk* $D(0, \sqrt{\alpha}) \subset \mathbb{C}$, *and there is no root with* $|\zeta|^2 = \alpha$.

(ii) The set

$$\Lambda(\gamma)\backslash S(\gamma)$$

is the union of three solid tori, where

$$S(\gamma) = \{q \in \Lambda(\gamma) : \{q\} = \ell_{p_1} \cap \ell_{p_2} \quad \text{for some} \quad p_1, p_2 \in \gamma(I), p_1 \neq p_2\}$$

is the subset of $\Lambda(\gamma)$ *consisting of all the points of intersection of any two distinct tangent lines to* $\partial\mathbb{H}^2_{\mathbb{C}}$, *at points in the image of* γ.

The Theorem 1.1 says, roughly speaking, that a continuous path in $\Omega(\gamma) \cap (\mathbb{P}^2_{\mathbb{C}} \backslash \overline{\mathbb{H}^2_{\mathbb{C}}})$ corresponds to a continuous homotopy of “positive” chains not meeting the knot γ. The main idea in the proof of Theorem 1.1 is the continuous correspondence between a point in $\mathbb{P}^2_{\mathbb{C}}$ and its polar chain with a suitable parametrization, and this idea is formalized in Lemma 3.2 and Proposition 3.3. All the results obtained in this paper are consequences of Theorem 1.1 in some way.

The Corollary 1.2 gives a “practical” way to estimate the number of components of $\Omega(\gamma)$ by means of computing all the possible linking numbers of chains not meeting γ. An outline for its proof is the following: The diameter of the chain polar to a point in $\mathbb{P}^2_{\mathbb{C}} \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}$ goes to zero as the point approaches to the boundary at infinity $\partial\mathbb{H}^2_{\mathbb{C}}$ (see Lemma 3.4). Hence it is natural to think on the component of $\Omega(\gamma)$ containing $\mathbb{H}^2_{\mathbb{C}}$ as the points in $\mathbb{H}^2_{\mathbb{C}}$ and the null-homotopic chains in $\partial\mathbb{H}^2_{\mathbb{C}} \backslash \gamma(S^1)$, including the degenerate chains in $\partial\mathbb{H}^2_{\mathbb{C}} \backslash \gamma(S^1)$ (see Lemma 3.6). Since the linking number is invariant under homotopy, Theorem 1.1 and Lemma 3.6 can be used to prove Corollary 1.2.

The Corollary 1.3 is a restatement of Corollary 1.2 for the case when γ can be obtained as the intersection of an algebraic curve and $\partial\mathbb{H}^2_{\mathbb{C}}$. In this case, the linking numbers between the knot and the chains not meeting the knot can be computed as a winding number using the Milnor’s fibration. For a detailed treatment on Milnor’s fibration see Milnor (1968) and Seade (2007).

Theorem 1.4 computes the number of components of $\Omega(\gamma)$ for the particular case when γ is a parametrization of the trefoil knot obtained as an intersection of the algebraic curve $y^2 = x^3$ and the 3-sphere $\partial\mathbb{H}^2_{\mathbb{C}}$. Moreover, it gives a topological description of the set $\Lambda(\gamma)$.

The points in $\Lambda(\gamma)$ can be characterized as those points $[x : y : z]$ whose polar line passes through a point in the knot, and this condition can be translated to the existence of a root of the equation in the ζ-variable, $p(\zeta) = \bar{y}\zeta^3 + \bar{x}\zeta^2 - \bar{z} = 0$, in a fixed circle (of center $0 \in \mathbb{C}$ and radius $\sqrt{\alpha}$). This gives a natural partition of $\Omega(\gamma)$ into four sets Ω_j, $j = 0, 1, 2, 3$ as described in Theorem 1.4. It is proved in Sect. 5.1 that each Ω_j is connected and that the linking number $\ell(\gamma, C_p)$ is equal to j for every $p \in \Omega_j \setminus \overline{\mathbb{H}^2_{\mathbb{C}}}$. Thus the proof of Theorem 1.4 i) follows from Corollary 1.2.

The proof of Theorem 1.4 (ii) is included in the Sect. 5.2. A brief outline is the following: We fix a point p_0 in the image of γ and we notice that the points lying in ℓ_{p_0} (the complex line tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at p_0) and the "singular set" $S(\gamma)$, form a closed curve (it is a Pascal's Limaçon up to change of coordinates by an element in PU(2, 1)) such that its complement in ℓ_{p_0} consists of three disjoint topological disks. Translating these three disks with the flow $\phi_t([x : y : z]) = [e^{2it}x : e^{3it}y : z]$, $t \in [0, 2\pi]$ we obtain all the set $\Lambda(\gamma)\setminus S(\gamma)$ and its topological description.

Finally, a section on open questions is included.

2 Preliminaries

2.1 Projective Geometry

The complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ is defined as

$$\mathbb{P}^2_{\mathbb{C}} := (\mathbb{C}^3\setminus\{0\})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^3\setminus\{0\}$ by the usual scalar multiplication. It is a well known fact that $\mathbb{P}^2_{\mathbb{C}}$ is a compact connected complex 2-dimensional manifold. Let $[\ ] : \mathbb{C}^3\setminus\{0\} \to \mathbb{P}^2_{\mathbb{C}}$ be the quotient map, if $\mathbf{v} = (x, y, z) \in \mathbb{C}^3\setminus\{0\}$ then we write $[\mathbf{v}] = [x : y : z]$. Also, $\ell \subset \mathbb{P}^2_{\mathbb{C}}$ is said to be a complex line if $[\ell]^{-1} \cup \{0\}$ is a complex linear subspace of dimension 2, so that ℓ is equal to a set of the form

$$\{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : Ax + By + Cz = 0\}$$

for some $A, B, C \in \mathbb{C}$ not all zero. Some useful properties are listed below:

- Any two distinct complex lines in $\mathbb{P}^2_{\mathbb{C}}$ intersect in one single point.
- Any two distinct points $p = [x : y : z]$, $q = [x' : y' : z']$ define a unique complex line containing p, q. In fact, this line can be described as the set:

$$\{[\lambda x + \mu x' : \lambda y + \mu y' : \lambda z + \mu z'] : \lambda, \mu \in \mathbb{C}, |\lambda|^2 + |\mu|^2 \neq 0\}$$

- Any complex line in $\mathbb{P}^2_{\mathbb{C}}$ is biholomorphic to the Riemann sphere S^2.

Consider the action of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ on $\mathrm{GL}(3, \mathbb{C})$ given by the usual scalar multiplication, then

$$\mathrm{PGL}(3, \mathbb{C}) = \mathrm{GL}(3, \mathbb{C})/\mathbb{C}^*$$

is a Lie group whose elements are called projective transformations. Let $[[\ ]] : \mathrm{GL}(3, \mathbb{C}) \to \mathrm{PGL}(3, \mathbb{C})$ be the quotient map. If $g \in \mathrm{PGL}(3, \mathbb{C})$ and $\mathbf{g} \in \mathrm{GL}(3, \mathbb{C})$, we say that $\mathbf{g}$ is a lift of g whenever $[[\mathbf{g}]] = g$. One can show that $\mathrm{PGL}(3, \mathbb{C})$ is a Lie group that acts transitively, effectively and by biholomorphisms on $\mathbb{P}^2_{\mathbb{C}}$ by $[[\mathbf{g}]]([\mathbf{v}]) = [\mathbf{gv}]$, where $\mathbf{v} \in \mathbb{C}^3\setminus\{0\}$ and $\mathbf{g} \in \mathrm{GL}(3, \mathbb{C})$. Notice that any element $g \in \mathrm{PGL}(3, \mathbb{C})$ maps complex lines to complex lines.

2.2 Complex Hyperbolic Geometry

Let $\mathbb{C}^{2,1}$ denote $\mathbb{C}^3$ equipped with the Hermitian form

$$H(\mathbf{z}, \mathbf{w}) = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3,$$

where $\mathbf{z} = (z_1, z_2, z_3)$, $\mathbf{w} = (w_1, w_2, w_3)$.

Denote by

$$\begin{aligned} V_- &= \{\mathbf{z} \in \mathbb{C}^{2,1} : H(\mathbf{z}, \mathbf{z}) < 0\}, \\ V_0 &= \{\mathbf{z} \in \mathbb{C}^{2,1}\setminus\{\mathbf{0}\} : H(\mathbf{z}, \mathbf{z}) = 0\}, \\ V_+ &= \{\mathbf{z} \in \mathbb{C}^{2,1} : H(\mathbf{z}, \mathbf{z}) > 0\}, \end{aligned}$$

the sets of negative, null and positive vectors in $\mathbb{C}^{2,1}\setminus\{\mathbf{0}\}$, respectively.

The projectivisation of the set of negative vectors,

$$\begin{aligned} [V_-] &= \{[z_1 : z_2 : z_3] \in \mathbb{P}^2_{\mathbb{C}} : |z_1|^2 + |z_2|^2 - |z_3|^2 < 0\} \\ &= \{[z_1 : z_2 : 1] \in \mathbb{P}^2_{\mathbb{C}} : |z_1|^2 + |z_2|^2 < 1\}, \end{aligned}$$

is a complex 2-dimensional open ball in $\mathbb{P}^2_{\mathbb{C}}$. Moreover, $[V_-]$ equipped with the quadratic form induced by the Hermitian form H is a model for the complex hyperbolic space $\mathbb{H}^2_{\mathbb{C}}$. The projectivisation of the set of null vectors,

$$\begin{aligned} [V_0] &= \{[z_1 : z_2 : z_3] \in \mathbb{P}^2_{\mathbb{C}} : |z_1|^2 + |z_2|^2 - |z_3|^2 = 0\} \\ &= \{[z_1 : z_2 : 1] \in \mathbb{P}^2_{\mathbb{C}} : |z_1|^2 + |z_2|^2 = 1\}, \end{aligned}$$

is a 3-sphere in $\mathbb{P}^2_{\mathbb{C}}$ and it is the boundary of $\mathbb{H}^2_{\mathbb{C}}$, denoted $\partial\mathbb{H}^2_{\mathbb{C}}$.

Finally, the projectivisation of the set of positive vectors,

$$[V_+] = \{[z_1 : z_2 : z_3] \in \mathbb{P}^2_{\mathbb{C}} : |z_1|^2 + |z_2|^2 - |z_3|^2 > 0\},$$

is the complement in $\mathbb{P}^2_{\mathbb{C}}$ of the complex 2-dimensional closed ball $\overline{\mathbb{H}^2_{\mathbb{C}}} := \mathbb{H}^2_{\mathbb{C}} \cup \partial\mathbb{H}^2_{\mathbb{C}}$.

The group of holomorphic isometries of $\mathbb{H}^2_{\mathbb{C}}$ is PU(2, 1), the projectivisation in PGL(3, $\mathbb{C}$) of the unitary group, $U(2, 1)$, respect to the Hermitian form H:

$$U(2,1) = \{\mathbf{g} \in \mathrm{GL}(3,\mathbb{C}) : H(\mathbf{gz}, \mathbf{gw}) = H(\mathbf{z}, \mathbf{w})\}.$$

The group PU(2, 1) acts transitively in $\mathbb{H}^2_{\mathbb{C}}$ and by diffeomorphisms in the boundary $\partial\mathbb{H}^2_{\mathbb{C}} \cong S^3$.

The disks obtained as the intersection of a complex line and $\mathbb{H}^2_{\mathbb{C}}$ are totally geodesic subspaces of $\mathbb{H}^2_{\mathbb{C}}$, they are called complex geodesics. The boundary at infinity of a complex geodesic is a circle obtained as the intersection of $\partial\mathbb{H}^2_{\mathbb{C}}$ and a complex line, these circles are called *chains* and they play an important role in this paper, more information on chains is given in Sect. 3. Another important fact we use along this paper is the following: Given any point $p = [w_1 : w_2 : w_3] \in \partial\mathbb{H}^2_{\mathbb{C}}$, there exists an unique complex line, denoted ℓ_p, tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at p. Moreover, ℓ_p is given by the set:

$$\{[z_1 : z_2 : z_3] \in \mathbb{P}^2_{\mathbb{C}} : z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3 = 0\}.$$

If we consider $\mathbb{C}^3$ with the Hermitian form

$$H_1(\mathbf{z}, \mathbf{w}) = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1,$$

where $\mathbf{z} = (z_1, z_2, z_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$, then we have that

$$H(C\mathbf{z}, C\mathbf{w}) = H_1(\mathbf{z}, \mathbf{w}),$$

where

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Hence $C(V_-)$, $C(V_0)$, $C(V_+)$ are the sets of negative, null and positive vectors for H_1, respectively. The projectivisation of $C(V_-)$ equipped with the Hermitian form H_1 is the Siegel model for complex hyperbolic space

$$\{[z_1 : z_2 : 1] \in \mathbb{P}^2_{\mathbb{C}} : 2\Re(z_1) + |z_2|^2 < 0\}$$

and its boundary is the set

$$\{[z_1 : z_2 : 1] \in \mathbb{P}^2_{\mathbb{C}} : 2\Re(z_1) + |z_2|^2 = 0\} \cup \{[1 : 0 : 0]\}.$$

Any finite point in this boundary, can be written in the form

$$[-|\zeta|^2 + iv : \sqrt{2}\zeta : 1],$$

for some $(\zeta, v) \in \mathbb{C} \times \mathbb{R}$. Hence there is a natural identification of this boundary set with the one point compactification of the Heisenberg space $\mathbb{C} \times \mathbb{R} = \mathcal{H}$. The vertical lines $\zeta = \zeta_0$ (constant) are chains in this representation. For more details on complex hyperbolic geometry, see the book (Goldman 1999).

2.3 Linking Number

This subsection is based on the books Rolfsen (2003) and Spivak (1999). Given $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ (or S^3) two disjoint piecewise smooth knots, the linking number between γ_1 and γ_2, denoted by

$$\ell(\gamma_1, \gamma_2),$$

is (roughly speaking) the number of turns that γ_1 gives around γ_2.

In this subsection we describe some equivalent methods for defining this integer, all of which turn out to be equivalent, up to a sign.

- (Linking number as an intersection number) Let $\gamma_1 : S^1 \to \mathbb{R}^3$ be an embedding such that $\gamma_1(S^1) = \partial M$ for some compact oriented 2-manifold with boundary M. Assume that $\gamma_2 : S^1 \to \mathbb{R}^3$ is transversal to M. The linking number $\ell(\gamma_1, \gamma_2)$ is the oriented intersection number of γ_2 and M.
- (Linking number as a degree) Define a map $F : S^1 \times S^1 \to S^2$ by the formula

  $$F(w, z) = \frac{\gamma_2(w) - \gamma_1(z)}{|\gamma_2(w) - \gamma_1(z)|},$$

  then

  $$\ell(\gamma_1, \gamma_2) = \deg(F)$$

- (Linking number as a Gauss integral) If we consider the knots γ_1 and γ_2 as functions from $[0, 1]$ to $\mathbb{R}^3$ then

  $$\ell(\gamma_1, \gamma_2) = -\frac{1}{4\pi} \int_0^1 \int_0^1 \frac{A(s, t)}{(r(s, t))^3}\, ds\, dt,$$

  where

  $$r(s, t) = |\gamma_2(t) - \gamma_1(s)|,$$
  $$A(s, t) = \det \begin{pmatrix} (\gamma_1^{(1)})'(s) & (\gamma_1^{(2)})'(s) & (\gamma_1^{(3)})'(s) \\ (\gamma_2^{(1)})'(t) & (\gamma_2^{(2)})'(t) & (\gamma_2^{(3)})'(t) \\ \gamma_2^{(1)}(t) - \gamma_1^{(1)}(s) & \gamma_2^{(2)}(t) - \gamma_1^{(2)}(s) & \gamma_2^{(3)}(t) - \gamma_1^{(3)}(s) \end{pmatrix}.$$

- (Linking number as an integer in the abelianization of a fundamental group) The knot γ_1 is a loop in $S^3 \backslash \gamma_2(S^1)$, hence it represents an element in the fundamental group $\pi_1(S^3 \backslash \gamma_2(S^1))$ with a suitable base point. This group abelianizes to $\mathbb{Z}$ and the loop γ_1 is carried to an integer, called $\ell(\gamma_1, \gamma_2)$.

It is important to remark that the linking number is symmetric: $\ell(\gamma_1, \gamma_2) = \ell(\gamma_2, \gamma_1)$, whenever γ_1, γ_2 are disjoint knots in S^3. Another useful property about linking numbers is the invariance under homotopy: If $H : [0, 1] \times S^1 \to S^3$ is a homotopy such that

$$
\begin{gathered}
H(0, z) = \gamma_0(z), \\
H(1, z) = \gamma_1(z), \\
\{H(s, z) : s \in [0, 1], z \in S^1\} \cap \{\gamma_2(s) : s \in [0, 1]\} = \varnothing,
\end{gathered}
$$

then

$$\ell(\gamma_0, \gamma_2) = \ell(\gamma_1, \gamma_2).$$

3 Chains

Definition 3.1 A *chain* is the intersection of a complex line in $\mathbb{P}^2_{\mathbb{C}}$ and $S^3 = \partial \mathbb{H}^2_{\mathbb{C}}$. The points in $S^3 = \partial \mathbb{H}^2_{\mathbb{C}}$ are considered as chains and we call them *degenerate chains*. We denote by $\mathcal{C}$ the space of all chains and we call it the *chains space*

If $p = [\mathbf{v}] \in \mathbb{P}^2_{\mathbb{C}} \backslash \mathbb{H}^2_{\mathbb{C}}$ then $\mathbf{v}$ is a positive vector, so the orthogonal complement, $\langle \mathbf{v} \rangle^{\perp}$, respect to the Hermitian form H (see Sect. 2.2) is a two dimensional subspace of $\mathbb{C}^{2,1}$ and it induces a complex line, ℓ_p, called the *polar line to* p. The chain, C_p, obtained as the intersection $\ell_p \cap \partial \mathbb{H}^2_{\mathbb{C}}$ is *the polar chain to* p.

Conversely, if ℓ is a complex line transversal to $\partial \mathbb{H}^2_{\mathbb{C}}$, then we can write $\ell = [L \backslash \{\mathbf{0}\}]$ where L is a two dimensional complex vector subspace of $\mathbb{C}^{2,1}$. Moreover, the orthogonal complement of L, respect to the Hermitian form H as in Sect. 2.2, is a one dimensional complex subspace of $\mathbb{C}^{2,1}$ which induces a point in $\mathbb{P}^2_{\mathbb{C}} \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}$, this point is called the *polar point to* the line ℓ.

There is a natural identification of a chain $\partial \mathbb{H}^2_{\mathbb{C}} \cap \ell$ with the polar point to ℓ. In fact, there is a bijection

$$P : \mathcal{C} \longleftrightarrow \mathbb{P}^2_{\mathbb{C}} \backslash \mathbb{H}^2_{\mathbb{C}}$$

between the space of chains and the complement of the complex hyperbolic space. We remark that for a degenerate chain $\{p\} = \partial \mathbb{H}^2_{\mathbb{C}} \cap \ell$ the corresponding point is $p \in \partial \mathbb{H}^2_{\mathbb{C}}$.

We equip the space of chains $\mathcal{C}$ with a topology by requiring that the bijection P be an homeomorphism. Moreover, we equip $\mathcal{C}$ with a structure of differentiable manifold with boundary by requiring that P be a diffeomorphism.

We parametrize the horizontal chain $C_0 = \{[z : 0 : 1] \in \mathbb{P}^2_{\mathbb{C}} : |z|^2 = 1\}$ as a curve in the following way

$$\sigma_0 : S^1 \to C_0$$
$$\sigma_0(e^{2\pi i t}) = [e^{2\pi i t} : 0 : 1],$$

and we say that C_0 is positively oriented with this parametrization. Moreover, given any non degenerate chain C there exists $g \in \mathrm{PU}(2,1)$ mapping C_0 to C. We say that $g \circ \sigma_0 : S^1 \to C$ is a positive parametrization of C. In what follows we use only positive parametrizations for the non degenerate chains.

Let $\gamma : S^1 \to S^3 \cong \partial\mathbb{H}^2_{\mathbb{C}}$ be a simple closed curve (perhaps piecewise differentiable).

We denote by $\Lambda(\gamma)$ the set

$$\bigcup_{p\in\gamma(S^1)} \ell_p,$$

where ℓ_p is the only complex line tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at p. The open set $\mathbb{P}^2_{\mathbb{C}}\setminus\Lambda(\gamma)$ is denoted by $\Omega(\gamma)$.

Lemma 3.2 *If* $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{C}^{2,1}$ *is a positive vector then the vectors*

$$\mathbf{v}(\mathbf{u}) = \left(\frac{|u_1|^2 + |u_2|^2}{|u_1|^2 + |u_2|^2 - |u_3|^2}\right)^{1/2} \left(\frac{\bar{u}_3\, u_1}{|u_1|^2 + |u_2|^2}, \frac{\bar{u}_3\, u_2}{|u_1|^2 + |u_2|^2}, 1\right),$$
$$\mathbf{w}(\mathbf{u}) = \frac{1}{(|u_1|^2 + |u_2|^2)^{1/2}} \left(-\bar{u}_2, \bar{u}_1, 0\right),$$

satisfy that:

(a) $\langle \mathbf{v}(\mathbf{u}), \mathbf{v}(\mathbf{u})\rangle = -1$,
(b) $\langle \mathbf{w}(\mathbf{u}), \mathbf{w}(\mathbf{u})\rangle = 1$,
(c) $\langle \mathbf{u}, \mathbf{v}(\mathbf{u})\rangle = \langle \mathbf{u}, \mathbf{w}(\mathbf{u})\rangle = \langle \mathbf{v}(\mathbf{u}), \mathbf{w}(\mathbf{u})\rangle = 0$.
(d) $\mathbf{v}(\mathbf{u})$ *and* $\mathbf{w}(\mathbf{u})$ *are smooth functions of* $\mathbf{u}$.
(e)

$$\sigma(e^{2\pi i t}) = [\mathbf{v}(\mathbf{u}) + e^{2\pi i t}\mathbf{w}(\mathbf{u})] \in \mathbb{P}^2_{\mathbb{C}}, \quad t \in [0, 1]$$

gives a positive parametrization of the polar chain to $[\mathbf{u}] \in \mathbb{P}^2_{\mathbb{C}}\setminus\overline{\mathbb{H}^2_{\mathbb{C}}}$.

Proof The proof of items (a) to (d) follows by straightforward computations and we omit it.

(e) First, by (c), we have that

$$\langle \mathbf{u}, \mathbf{v}(\mathbf{u}) + e^{2\pi i t}\mathbf{w}(\mathbf{u})\rangle = 0$$

for every $t \in [0, 1]$. Also, by (a), (b) and (c) we have that

$$\langle \mathbf{v}(\mathbf{u}) + e^{2\pi i t}\mathbf{w}(\mathbf{u}), \mathbf{v}(\mathbf{u}) + e^{2\pi i t}\mathbf{w}(\mathbf{u})\rangle = 0.$$

Hence $\sigma(e^{2\pi i t})$ gives a parametrization of the polar chain to $[\mathbf{u}]$.

Finally, the linear transformation, $\mathbf{g}$, such that

$$\begin{aligned}\mathbf{g}(\mathbf{e}_1) &= \mathbf{w}(\mathbf{u}),\\ \mathbf{g}(\mathbf{e}_2) &= \frac{\mathbf{u}}{\sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}},\\ \mathbf{g}(\mathbf{e}_3) &= \mathbf{v}(\mathbf{u}),\end{aligned}$$

lies in SU(2, 1) and the induced transformation $g = [\mathbf{g}]$ satisfies that

$$g^{-1} \circ \sigma(e^{2\pi i t}) = [\mathbf{e}_3 + e^{2\pi i t}\mathbf{e}_1] = [e^{2\pi i t} : 0 : 1].$$

Therefore $\sigma(e^{2\pi i t})$ gives a positive parametrization of the polar chain to $[\mathbf{u}]$. □

Proposition 3.3 *Let us assume that $p_0, p_1 \in (\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$ and C_{p_0}, C_{p_1} are the corresponding non degenerate polar chains. If $\nu : I \to (\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$ is a continuous path in $(\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$ joining p_0 and p_1, then there exists a continuous homotopy of chains between C_{p_0} and C_{p_1}. In other words, there is a continuous map*

$$H : I \times S^1 \to \partial \mathbb{H}^2_{\mathbb{C}}$$

such that $H(s, \cdot) : S^1 \to \partial \mathbb{H}^2_{\mathbb{C}}$ is a positive parametrization of the polar chain to $\nu(s)$, denoted $C_{\nu(s)}$, for every $s \in I$. The converse statement is also true.

Proof Let $\mathbf{u} : I \to \mathbb{C}^{2,1}$ be a *continuous* lift of ν. In other words, $\mathbf{u}$ is continuous and $\nu(s) = [\mathbf{u}(s)]$ for every $s \in I$ (there are many of such continuous lifts). Using the notation of Lemma 3.2, we define

$$\begin{aligned}H &: I \times S^1 \to \partial \mathbb{H}^2_{\mathbb{C}}\\ H(s, e^{2\pi i t}) &= [\mathbf{v}(\mathbf{u}(s)) + e^{2\pi i t}\mathbf{w}(\mathbf{u}(s))],\end{aligned}$$

and the proof follows from Lemma 3.2.

Conversely, given the homotopy, one can take the polars to the chains $C_{\nu(s)}$ and obtain a continuous path in $\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}}$ from p_0 to p_1. □

Proof of Theorem 1.1 If p_0 and p_1 can be joined by a continuous path $\nu : I \to \Omega(\gamma) \cap (\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$, then Proposition 3.3 implies that there exists a continuous map

$$H : I \times S^1 \to \partial \mathbb{H}^2_{\mathbb{C}}$$

such that $H(s, \cdot) : S^1 \to \partial \mathbb{H}^2_{\mathbb{C}}$ is a positive parametrization of the chain to $C_{\nu(s)}$ for every $s \in I$.

If we assume that $C_{\nu(s_0)} \cap \gamma(S^1) \neq \varnothing$ for some $s_0 \in I$, then

$$[\mathbf{v}(\mathbf{u}(s_0)) + e^{2\pi i t_1}\mathbf{w}(\mathbf{u}(s_0))] = \gamma(e^{2\pi i t_2})$$

for some $t_1, t_2 \in [0, 1]$. So $\nu(s_0) = [\mathbf{u}(s_0)] \in \ell_{\gamma(e^{2\pi i t_2})}$, because $\mathbf{u}(s_0)$ is orthogonal to the null vector $\mathbf{v}(\mathbf{u}(s_0)) + e^{2\pi i t_1}\mathbf{w}(\mathbf{u}(s_0))$. A contradiction to the hypothesis that ν lies in $\Omega(\gamma)$.

The converse statement is obtained by taking the path of polars to the chains $H(s, S^1)$, $s \in I$. $\square$

Notation. In what follows, $\ell(\gamma, C)$ denotes the linking number between the closed curve $\gamma : S^1 \to \partial\mathbb{H}^2_{\mathbb{C}}$ and the positively oriented chain C.

Lemma 3.4 *Let $\mathbf{u} \in \mathbb{C}^{2,1}$ be a positive vector,*

(i) the point $[\mathbf{v}(\mathbf{u})]$ is the Euclidean center of the polar chain to $[\mathbf{u}]$ (considered in the chart $\{[z_1 : z_2 : z_3] \mid z_3 \neq 0\} \cong \mathbb{C}^2$) and its Euclidean radius is equal to

$$\left(\frac{|u_1|^2 + |u_2|^2 - |u_3|^2}{|u_1|^2 + |u_2|^2}\right)^{1/2}$$

(ii) If $[\mathbf{u}]$ goes to $q \in \partial\mathbb{H}^2_{\mathbb{C}}$ then the polar chain to $[\mathbf{u}]$ goes to the degenerate chain q.

Proof First, we notice that $[\mathbf{v}(\mathbf{u})]$ and $\sigma(e^{2\pi i t}) = [\mathbf{v}(\mathbf{u}) + e^{2\pi i t}\mathbf{w}(\mathbf{u})]$ in the chart $\{[z_1 : z_2 : z_3] \mid z_3 \neq 0\} \cong \mathbb{C}^2$ are equal to

$$\left(\frac{\bar{u}_3 u_1}{a^2}, \frac{\bar{u}_3 u_2}{a^2}\right) \quad \text{and} \quad \left(\frac{\bar{u}_3 u_1 - b\bar{u}_2 e^{2\pi i t}}{a^2}, \frac{\bar{u}_3 u_2 + b\bar{u}_1 e^{2\pi i t}}{a^2}\right),$$

where $a = \left(|u_1|^2 + |u_2|^2\right)^{1/2}$ and $b = \left(|u_1|^2 + |u_2|^2 - |u_3|^2\right)^{1/2}$. By straightforward computations, the Euclidean distance between these two points is constant and equal to $\frac{b}{a}$. Thus, (i) follows. The proof of (ii) follows from (i). $\square$

Lemma 3.5 *If $p \in \Omega(\gamma) \cap (\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$ can be joined by a continuous path in $\Omega(\gamma)$ to a point in $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \gamma(S^1)$ then there is a homotopy of chains between C_p and a degenerate chain in $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \gamma(S^1)$. In particular*

$$\ell(\gamma, C_p) = 0.$$

Proof We can assume that there is a path $\nu : [0, 1] \to \mathbb{P}^2_{\mathbb{C}} \setminus \mathbb{H}^2_{\mathbb{C}}$ such that

- $\nu(0) = p \in \Omega(\gamma) \cap (\mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}})$,
- $\nu(1) = q \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \gamma(S^1)$,
- $\nu([0, 1)) \subset \mathbb{P}^2_{\mathbb{C}} \setminus \overline{\mathbb{H}^2_{\mathbb{C}}}$.

Since $q \in \partial\mathbb{H}^2_{\mathbb{C}} \setminus \gamma(S^1)$, there is an Euclidean ball, $B(q, \epsilon)$ centered at q not meeting $\Lambda(\gamma)$. By Lemma 3.4, there exists $t_0 \in [0, 1)$ such that $C_{\nu(t_0)}$, the polar chain to $\nu(t_0)$, is contained in $B(q, \epsilon)$. Hence, there is a homotopy of chains between $C_{\nu(t_0)}$ and a degenerate chain in $\partial\mathbb{H}^2_{\mathbb{C}} \setminus \gamma(S^1)$. Finally, there is a homotopy of chains from C_p to $C_{\nu(t_0)}$, by Theorem 1.1. $\square$

We remark that there are smooth versions of Proposition 3.3, Theorem 1.1 and Lemma 3.5. Roughly speaking, a smooth path in the complement of complex hyperbolic plane can be translated into a smooth homotopy of chains and vice versa.

Corollary 3.6 *If Ω_0 denotes the component of $\Omega(\gamma)$ containing $\mathbb{H}^2_{\mathbb{C}}$, then*

$$\ell(\gamma, C_p) = 0, \text{ for every } p \in \Omega_0 \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}.$$

Proof First, we notice that $\overline{\mathbb{H}^2_{\mathbb{C}}} \backslash \gamma(S^1)$ is contained in Ω_0 because $\mathbb{H}^2_{\mathbb{C}} \subset \Omega_0$. Now, if $p \in \Omega_0 \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}$, then there is a continuous path in $\Omega(\gamma)$ joining p and a point in $\partial \mathbb{H}^2_{\mathbb{C}} \backslash \gamma(S^1)$. It follows by Lemma 3.5 that $\ell(\gamma, C_p) = 0$. □

The converse of Corollary 3.6 is not necessarily true, for example, when the image of the curve γ and a chain C form a Whitehead link, then $\ell(\gamma, C) = 0$ and the polar of the chain C does not lie in the connected component of $\Omega(\gamma)$ containing $\mathbb{H}^2_{\mathbb{C}}$, otherwise, C is nullhomotopic in $\partial \mathbb{H}^2_{\mathbb{C}} \backslash \gamma(S^1)$ by Lemma 3.5, and it is a known fact that it is not possible.

Proof of Corollary 1.2 If p and q are in the same component of $\Omega(\gamma)$ then there is a (smooth) path $\nu : I \to \Omega(\gamma)$ joining p and q.

If $\nu(I) \cap \overline{\mathbb{H}^2_{\mathbb{C}}} = \varnothing$ then by Theorem 1.1 there is a homotopy of positively oriented chains between C_p and C_q whose image is contained in $\partial \mathbb{H}^2_{\mathbb{C}} \backslash \gamma(S^1)$. Hence,

$$\ell(\gamma, C_p) = \ell(\gamma, C_q)$$

and it contradicts the hypothesis.

If $\nu(I) \cap \overline{\mathbb{H}^2_{\mathbb{C}}} \neq \varnothing$ then by Lemma 3.5

$$\ell(\gamma, C_p) = 0 = \ell(\gamma, C_q)$$

and it contradicts the hypothesis. □

The number of all possible linking numbers $\ell(\gamma, C)$ where C is a chain not intersecting $\gamma(S^1)$ is not necessarily equal to the number of components of $\Omega(\gamma)$, as shown in the example of the whitehead link above.

4 Applications of Chains Homotopy and Linking Numbers

In this section, we include two examples for the description of the components of $\Omega(\gamma)$ when γ is a chain and an $\mathbb{R}$-circle. The main difference with the approach given in Cano et al. (2016), is the use of Theorem 1.1 and its Corollary 1.2 to describe each component as a set of points where the linking number of the polar chain and the curve γ is constant.

4.1 Chain

First, we consider the case when the image of the curve $\gamma : S^1 \to \partial\mathbb{H}^2_{\mathbb{C}}$ is a chain. It is proved in Cano et al. (2016) that $\Lambda(\gamma)$ is a complex cone over a circle, in other words, there exists $q \in \Lambda(\gamma)$ such that $\Lambda(\gamma)\backslash\{q\}$ is diffeomorphic to $S^1 \times \mathbb{C}$. Moreover, $\Omega(\gamma)$ has two components each one diffeomorphic to $D \times \mathbb{C}$, where $D \subset \mathbb{C}$ is an open disk.

Let us assume that the image of γ is the chain

$$C_0 = \{[\zeta : 0 : 1] \in \partial\mathbb{H}^2_{\mathbb{C}} : |\zeta| = 1\}.$$

Since $\zeta\bar{x} - \bar{z} = 0$ is the equation of the complex line tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at the point $[\zeta : 0 : 1]$, we have that

$$\begin{aligned}\Lambda(\gamma) &= \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : \zeta\bar{x} - \bar{z} = 0 \text{ for some } \zeta \in S^1\}\\ &= \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : |x| = |z|\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\Omega_0 &= \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : |x| < |z|\},\\ \Omega_1 &= \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : |x| > |z|\}\end{aligned}$$

are the components of $\Omega(\gamma)$.

Now, let us assume that γ is a positive parametrization of C_0. If C_p is the polar chain to a point $p \in \mathbb{P}^2_{\mathbb{C}}\backslash\overline{\mathbb{H}^2_{\mathbb{C}}}$ then

$$\ell(\gamma, C_p) = 0 \text{ or } 1$$

according to whether $p \in \Omega_0\backslash\overline{\mathbb{H}^2_{\mathbb{C}}}$ or in $p \in \Omega_1$.

4.2 $\mathbb{R}$-circle

Now, we consider the case when

$$\begin{aligned}&\gamma : S^1 \to \partial\mathbb{H}^2_{\mathbb{C}}\\ &\gamma(e^{2\pi i t}) = [\sin 2\pi t : \cos 2\pi t : 1]\end{aligned} \tag{2}$$

is a parametrization of the $\mathbb{R}$-circle

$$\partial\mathbb{H}^2_{\mathbb{R}} = \partial\mathbb{H}^2_{\mathbb{C}} \cap \mathbb{P}^2_{\mathbb{R}},$$

where $\mathbb{P}^2_{\mathbb{R}} = \{[r_1 : r_2 : r_3] \in \mathbb{P}^2_{\mathbb{C}} \,|\, r_1, r_2, r_3 \in \mathbb{R}\}$.

If $p=[x:y:z]\in\mathbb{P}^2_{\mathbb{C}}$, let us denote by

$$\ell_p=\{[u:v:w]\in\mathbb{P}^2_{\mathbb{C}}\mid x\bar{u}+y\bar{v}-z\bar{w}=0\}$$

the polar line to p (If $p\in\partial\mathbb{H}^2_{\mathbb{C}}$ then ℓ_p is the only complex line tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at p).

The problem on computing the number of components of $\Omega(\gamma)$ for an $\mathbb{R}$-circle was solved in Cano et al. (2016). We restate some of these results in Cano et al. (2016) in order to show our approach which gives a generalization.

Proposition 4.1 *If γ is the $\mathbb{R}$-circle given by (2), then*

(i) The set $\Lambda(\gamma)$ is described as

$$\begin{aligned}\Lambda(\gamma)&=\{p\in\mathbb{P}^2_{\mathbb{C}}:\ell_p\cap\partial\mathbb{H}^2_{\mathbb{R}}\neq\varnothing\}\\&=\{[x:y:z]\in\mathbb{P}^2_{\mathbb{C}}\mid xr_1+yr_2-zr_3=0\ \textit{for some}\ [r_1:r_2:r_3]\in\partial\mathbb{H}^2_{\mathbb{R}}\}.\end{aligned}$$

(ii) Moreover, $\Omega(\gamma)$ has three components:

$$\begin{aligned}\Omega_0&=\{p\in\mathbb{P}^2_{\mathbb{C}}:\ell_p\cap\mathbb{P}^2_{\mathbb{R}}\subset\mathbb{P}^2_{\mathbb{R}}\setminus\overline{\mathbb{H}^2_{\mathbb{R}}}\},\\\Omega_1^-&=\left\{p=[x:y:z]\in\mathbb{P}^2_{\mathbb{C}}:\ell_p\cap\mathbb{P}^2_{\mathbb{R}}\ \textit{is a point in}\ \mathbb{H}^2_{\mathbb{R}}\ \textit{and}\ \begin{vmatrix}\Re(x)&\Re(y)\\\Im(x)&\Im(y)\end{vmatrix}<0\right\},\\\Omega_1^+&=\left\{p=[x:y:z]\in\mathbb{P}^2_{\mathbb{C}}:\ell_p\cap\mathbb{P}^2_{\mathbb{R}}\ \textit{is a point in}\ \mathbb{H}^2_{\mathbb{R}}\ \textit{and}\ \begin{vmatrix}\Re(x)&\Re(y)\\\Im(x)&\Im(y)\end{vmatrix}>0\right\}.\end{aligned}$$

Proof (i) By definition, $p=[x:y:z]\in\Lambda(\gamma)$ if and only if there exists $q=[r_1:r_2:r_3]\in\partial\mathbb{H}^2_{\mathbb{R}}$ such that $p\in\ell_q$ and it happens if and only if $q\in\ell_p$.

(ii) Now, if $p=[x:y:z]\in\mathbb{P}^2_{\mathbb{C}}$ then the polar line ℓ_p meets the real projective space $\mathbb{P}^2_{\mathbb{R}}$ in a real projective line or a point, according to whether the space of solutions of the system of equations with real variables r_1,r_2,r_3:

$$\begin{aligned}\Re(x)r_1+\Re(y)r_2-\Re(z)r_3&=0,\\\Im(x)r_1+\Im(y)r_2-\Im(z)r_3&=0,\end{aligned}\tag{3}$$

is two dimensional or one dimensional (over $\mathbb{R}$).

In other words, $\ell_p\cap\mathbb{P}^2_{\mathbb{R}}$ is equal to:

- A real projective line in $\mathbb{P}^2_{\mathbb{R}}$ if and only if $p=[x:y:z]\in\mathbb{P}^2_{\mathbb{R}}$.
- A point in $\mathbb{P}^2_{\mathbb{R}}$ if and only if there is a non-zero 2×2 determinant of the system (3).

In the case when $p=[x:y:z]\notin\mathbb{P}^2_{\mathbb{R}}$, the non-zero real solution for the system (3) given by:

$$\mathbf{v}=\begin{pmatrix}i(z\bar{y}-\bar{z}y)\\i(x\bar{z}-\bar{x}z)\\i(x\bar{y}-\bar{x}y)\end{pmatrix}\tag{4}$$

is proposed in Cano et al. (2016).
If $p \in \Omega_0$ then we have two possibilities:

- $\ell_p \cap \mathbb{P}^2_{\mathbb{R}}$ is a point in $\mathbb{P}^2_{\mathbb{R}} \setminus \overline{\mathbb{H}^2_{\mathbb{R}}}$ if and only if $\mathbf{v}$ is a positive vector.
- $\ell_p \cap \mathbb{P}^2_{\mathbb{R}}$ is a real projective line contained in $\mathbb{P}^2_{\mathbb{R}} \setminus \overline{\mathbb{H}^2_{\mathbb{R}}}$ if and only if $p \in \mathbb{H}^2_{\mathbb{R}}$.

Therefore, Ω_0 is equal to the component of $\Omega(\gamma)$ containing $\mathbb{H}^2_{\mathbb{C}}$ (see Cano et al. 2016).
Now, we notice that $p \in \Omega_1^-$ if and only if $\mathbf{v}$ given by (4) is a negative vector and its last coordinate is a negative real number.
Finally, $p \in \Omega_1^+$ if and only if $\mathbf{v}$ given by (4) is a negative vector and its last coordinate is a positive real number. It follows from Cano et al. (2016) that Ω_1^- and Ω_1^+ are components of $\Omega(\gamma)$.

It is proved in Cano et al. (2016) that $\Lambda(\gamma)$ is a 3-dimensional semi-algebraic set and the Mobius strip, $\mathcal{M} = \Lambda(\gamma) \cap \mathbb{P}^2_{\mathbb{R}}$, is its singular set. Also, $\Lambda(\gamma) \setminus \mathcal{M}$ is a disjoint union of two solid torus $S^1 \times \mathbb{R}^2$. Moreover, each component $\Omega_0, \Omega_1^-, \Omega_1^+$ is diffeomorphic to an open 4-ball.

Notice that $\mathbb{H}^2_{\mathbb{C}} \subset \Omega_0$, so $\ell(\gamma, C_p) = 0$ for every $p \in \Omega_0 \setminus \overline{\mathbb{H}^2_{\mathbb{C}}}$, because of Lemma 3.6. On the other hand,

$$\ell(\gamma, C_p) = \begin{cases} -1 & \text{if } p \in \Omega_1^-, \\ +1 & \text{if } p \in \Omega_1^+. \end{cases}$$

In order to prove this statement, we notice that the points $p^- = [i : 1 : 0]$ and $p^+ = [1 : i : 0]$ lie in Ω_1^- and Ω_1^+, respectively. The corresponding polar chains C_{p^-}, C_{p^+} have positive parametrizations (see Lemma 3.2) given by:

$$\sigma^-(e^{2\pi i t}) = \left[-\frac{e^{2\pi i t}}{\sqrt{2}} : -\frac{i e^{2\pi i t}}{\sqrt{2}} : 1\right], \quad t \in [0, 1],$$
$$\sigma^+(e^{2\pi i t}) = \left[\frac{i e^{2\pi i t}}{\sqrt{2}} : \frac{e^{2\pi i t}}{\sqrt{2}} : 1\right], \quad t \in [0, 1].$$

The curves obtained by composing σ^-, σ^+ with stereographic projection $S^3 \setminus \{(0, i)\} \to \mathbb{R}^3$ are respectively

$$e^{2\pi i t} \mapsto \left(-\frac{\cos(2\pi t)}{\sqrt{2} + \cos(2\pi t)}, -\frac{\sin(2\pi t)}{\sqrt{2} + \cos(2\pi t)}, \frac{\sin(2\pi t)}{\sqrt{2} + \cos(2\pi t)}\right)$$

and

$$e^{2\pi i t} \mapsto \left(\frac{-\sin(2\pi t)}{\sqrt{2} - \sin(2\pi t)}, \frac{\cos(2\pi t)}{\sqrt{2} - \sin(2\pi t)}, \frac{\cos(2\pi t)}{\sqrt{2} - \sin(2\pi t)}\right),$$

and it is not hard to check that the linking number of these two curves with the curve obtained by composing γ with stereographic projection is as claimed.

5 The Trefoil Knot

Let us consider the trefoil knot obtained as the intersection of the algebraic curve $y^2 = x^3$ with $\partial\mathbb{H}^2_{\mathbb{C}} \cong S^3$. If α denotes the positive real root of the equation $x^3 + x^2 - 1 = 0$, then the knot can be parametrized in the following way:

$$\gamma : S^1 \to \mathbb{H}^2_{\mathbb{C}} \cong S^3$$
$$\gamma(e^{2\pi it}) = [(\sqrt{\alpha}e^{2\pi it})^2 : (\sqrt{\alpha}e^{2\pi it})^3 : 1].$$

The image of γ is the set:

$$\left\{[\zeta^2 : \zeta^3 : 1] \in \mathbb{P}^2_{\mathbb{C}} \mid |\zeta|^2 = \alpha\right\},$$

and $\Lambda(\gamma)$ can be described as those points $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\bar{x}\zeta^2 + \bar{y}\zeta^3 - \bar{z} = 0. \tag{5}$$

for some $\zeta \in \mathbb{C}$ such that $|\zeta|^2 = \alpha$.

5.1 The Components of $\Omega(\gamma)$

If $[x : y : z] \in \Omega(\gamma) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda(\gamma)$, then the solutions of the (at most cubic) equation (5)

$$\bar{y}\zeta^3 + \bar{x}\zeta^2 - \bar{z} = 0$$

satisfy that $|\zeta|^2 \neq \alpha$. The main purpose of this subsection is the proof of Theorem 1.4 i).

For readers convenience we provide a guide for this section: Lemma 5.2 says, roughly speaking, that a continuous path in $\Omega(\gamma)$, thought of as a continuous path of cubic equations, can be thought of as a continuous path in $\mathbb{C}^3$ of the corresponding roots of these cubic equations, and conversely.

The hypothesis of Lemma 5.2 is not satisfied in the line $y = 0$. In fact, the equation (5) has degree less than three in this line. However Lemma 5.3, shows that the intersection of $\Omega(\gamma)$ with the line $y = 0$ consists of the complement of a circle. An analogous result is obtained in Lemma 5.4 for the intersection of $\Omega(\gamma)$ with the line at infinity $z = 0$.

The Lemmas 5.5, 5.6, 5.7 and 5.8 show that each Ω_j is path connected for $j = 0, 1, 2, 3$.

The Lemmas 5.9 and 5.10 show that the linking number of the trefoil knot γ and a chain polar to a point in $\Omega_j \setminus \overline{\mathbb{H}^2_{\mathbb{C}}}$ (for $j = 0, 1, 2, 3$) is equal to j. Finally, a proof of the Theorem 1.4(i) is given.

First, we state the following lemmas:

Lemma 5.1 *If* $[x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ *and* $t \in \mathbb{R}$ *then the set of roots of the polynomial equation in* ζ,

$$\overline{(e^{2it}x)}\zeta^2 + \overline{(e^{3it}y)}\zeta^3 - \overline{z} = 0$$

is obtained from the set of roots of the equation

$$\overline{x}\zeta^2 + \overline{y}\zeta^3 - \overline{z} = 0$$

by multiplying by e^{it}. *Hence, the smooth flow*

$$\phi : \mathbb{R} \times \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}},$$
$$\phi_t[x : y : z] = [e^{2it}x : e^{3it}y : z]$$

preserves $\Lambda(\gamma)$, $\Omega(\gamma)$, Ω_0, Ω_1, Ω_2 *and* Ω_3.

The proof of Lemma 5.1 is straightforward and we omit it. The following Lemma can be generalized to the case where the polynomial in question comes from the (p, q) torus knot, by using the fact that the roots of a polynomial depend continuously on its coefficients. Here we include this version for reader's convenience.

Lemma 5.2 *Let* $p_0 = [x_0 : y_0 : z_0]$, $p_1 = [x_1 : y_1 : z_1] \in \Omega(\gamma)$, *such that* $y_0 \neq 0$ *and* $y_1 \neq 0$. *There exists a continuous path from* p_0 *to* p_1, $g : [0, 1] \to \Omega(\gamma)$, $g(t) = [g_1(t) : g_2(t) : g_3(t)]$ *such that* $g_2(t) \neq 0$ *for all* $t \in [0, 1]$ *if and only if there exists a continuous path* $\mathbf{r} : [0, 1] \to \mathbb{C}^3$, $\mathbf{r}(t) = (r_1(t), r_2(t), r_3(t))$, *such that:*

(i) The sets of roots of the polynomial equations $\zeta^3 + \left(\frac{\overline{x_0}}{\overline{y_0}}\right)\zeta^2 - \left(\frac{\overline{z_0}}{\overline{y_0}}\right) = 0$ *and* $\zeta^3 + \left(\frac{\overline{x_1}}{\overline{y_1}}\right)\zeta^2 - \left(\frac{\overline{z_1}}{\overline{y_1}}\right) = 0$ *are respectively equal to* $\{r_1(0), r_2(0), r_3(0)\}$ *and* $\{r_1(1), r_2(1), r_3(1)\}$.
(ii) $r_1(t)r_2(t) + r_2(t)r_3(t) + r_3(t)r_1(t) = 0$, *for every* $t \in [0, 1]$,
(iii) $|r_1(t)|^2 \neq \alpha$, $|r_2(t)|^2 \neq \alpha$, $|r_3(t)|^2 \neq \alpha$ *for every* $t \in [0, 1]$.

Proof First, if we assume that $g : [0, 1] \to \Omega(\gamma)$ is a continuous path joining p_0 and p_1, then The Tartaglia–Cardano formulas for a cubic equation, imply that the roots of the cubic equation

$$\zeta^3 + \left(\frac{\overline{g_1(t)}}{\overline{g_2(t)}}\right)\zeta^2 - \left(\frac{\overline{g_3(t)}}{\overline{g_2(t)}}\right) = 0$$

are continuous functions of its coefficients, hence continuous functions of t. If $r_1(t)$, $r_2(t)$, $r_3(t)$ denote such roots, then the path $\mathbf{r} : I \to \mathbb{C}^3$, $\mathbf{r}(t) = (r_1(t), r_2(t), r_3(t))$ satisfies (i) by construction. Item (ii) is satisfied by Viète identities, and finally, (iii) is satisfied because $g(t) \in \Omega(\gamma)$ for every $t \in [0, 1]$.

Conversely, if there exists a continuous path

$$\mathbf{r} : [0, 1] \to \mathbb{C}^3,$$

$$\mathbf{r}(t) = (r_1(t), r_2(t), r_3(t)),$$

satisfying the conditions (i) to (iii), then it is not hard to check that the continuous path $g : [0, 1] \to \mathbb{P}^2_{\mathbb{C}}$ given by

$$g(t) = \left[-(\overline{r_1(t)} + \overline{r_2(t)} + \overline{r_3(t)}) : 1 : \overline{r_1(t)}\ \overline{r_2(t)}\ \overline{r_3(t)} \right]$$

satisfies the required conditions.

Lemma 5.3 *The set* $\Omega(\gamma) \cap \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : y = 0\}$ *has two connected components:*

$$C_0 = \{[x : 0 : z] \mid \alpha|x| < |z|\},$$
$$C_2 = \{[x : 0 : z] \mid \alpha|x| > |z|\}.$$

Moreover, $C_0 \subset \Omega_0$ *and* $C_2 \subset \Omega_2$.

Proof We notice that a point $[x : 0 : z] \in \Lambda(\gamma)$ if and only if the equation $\bar{x}\zeta^2 - \bar{z} = 0$ has a solution ζ such that $|\zeta|^2 = \alpha$, and it happens if and only if

$$\alpha|x| = |z|.$$

This equation represents a circle in the line $\{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : y = 0\}$. Hence its complement has two connected components defined as

$$C_0 = \{[x : 0 : z] \mid \alpha|x| < |z|\},$$
$$C_2 = \{[x : 0 : z] \mid \alpha|x| > |z|\}.$$

Moreover, if $[x : 0 : z] \in C_0$ then the inequality $\alpha|x| < |z|$ implies that the equation $\bar{x}\zeta^2 - \bar{z} = 0$ has no root within the closed disk $\overline{D(0, \sqrt{\alpha})}$. Therefore $C_0 \subset \Omega_0$. An analogous argument proves that $C_2 \subset \Omega_2$. □

Lemma 5.4 *The set* $\Omega(\gamma) \cap \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : z = 0\}$ *has two connected components:*

$$B_2 = \{[x : y : 0] \mid |x| > \sqrt{\alpha}\,|y|\},$$
$$B_3 = \{[x : y : 0] \mid |x| < \sqrt{\alpha}\,|y|\}.$$

Moreover, $B_2 \subset \Omega_2$ *and* $B_3 \subset \Omega_3$.

The proof of this Lemma is analogous to the proof of Lemma 5.3 and we omit it.

Lemma 5.5 *The set* Ω_0 *is a connected subset of* $\Omega(\gamma)$. *In fact,* Ω_0 *is a contractible space.*

Proof The point $[0 : 0 : 1]$ lies in Ω_0 because the equation

$$-1 = 0$$

has no roots in the open disk $D(0, \sqrt{\alpha})$.

Let $[x : y : z]$ be a point in Ω_0, we give a continuous path in Ω_0 joining $[0 : 0 : 1]$ to the point $[x : y : z]$. First, we notice that $z \neq 0$, otherwise $\zeta = 0$ is a root of the equation

$$\bar{x}\zeta^2 + \bar{y}\zeta^3 - \bar{z} = 0,$$

contradicting the assumption that $[x : y : z] \in \Omega_0$. Now, the path

$$\begin{aligned} f &: [0, 1] \to \mathbb{P}^2_{\mathbb{C}} \\ f(t) &= [t^2 x : t^3 y : z] \end{aligned}$$

is smooth, $f(0) = [0 : 0 : z] = [0 : 0 : 1]$ and $f(1) = [x : y : z]$. Also, for every $1 > t > 0$, the set of roots of the polynomial equation in ζ:

$$\overline{t^2 x}\zeta^2 + \overline{t^3 y}\zeta^3 - \bar{z} = 0$$

is obtained from the set of roots of the equation

$$\bar{x}\zeta^2 + \bar{y}\zeta^3 - \bar{z} = 0$$

by multiplying by $\frac{1}{t} > 1$. Hence $f(t) \in \Omega_0$ for every $t \in [0, 1]$.

Moreover, Ω_0 is contractible because the function

$$\begin{aligned} \psi &: [0, 1] \times \Omega_0 \to \Omega_0 \\ \psi_t[x : y : z] &= [(1 - t)^2 x : (1 - t)^3 y : z] \end{aligned}$$

is a deformation retraction. □

Lemma 5.6 *The set Ω_1 is a connected subset of $\Omega(\gamma)$.*

Proof Let $[x : y : z] \in \Omega_1$ then necessarily (5) is a cubic polynomial equation. Let us denote by r_1, r_2, r_3 the roots of the equation (5). We can assume that $r_1 \in D(0, \sqrt{\alpha})$. By Lemma 5.1 we can assume that $r_1 \in \mathbb{R}$ and $0 < r_1 < \sqrt{\alpha}$.

We denote by T_r the Möbius transformation $T_r(z) = -\frac{rz}{z+r}$.

Claim 1. $r_2, r_3 \notin \overline{D(0, \sqrt{\alpha})} \cup T_{r_1}(\overline{D(0, \sqrt{\alpha})})$.

By hypothesis, $r_2 \notin \overline{D(0, \sqrt{\alpha})}$. If we assume that $r_2 \in T_{r_1}(\overline{D(0, \sqrt{\alpha})})$ then $r_3 = T_{r_1}(r_2) \in T_{r_1}(T_{r_1}(\overline{D(0, \sqrt{\alpha})})) = \overline{D(0, \sqrt{\alpha})}$ contradicting the hypothesis that $[x : y : z] \in \Omega_1$. The proof for r_3 is analogous and we omit it.

Claim 2. The complement of the set $\overline{D(0, \sqrt{\alpha})} \cup T_{r_1}(\overline{D(0, \sqrt{\alpha})})$ in $\mathbb{C}$ is the intersection of the interior of the disk with diameter the points $T_{r_1}(-\sqrt{\alpha})$ and $T_{r_1}(\sqrt{\alpha})$, and the exterior of the closed disk $\overline{D(0, \sqrt{\alpha})}$. See the Fig. 1.

Notice that

$$T_{r_1}(-\sqrt{\alpha}) = \frac{r_1\sqrt{\alpha}}{-\sqrt{\alpha} + r_1}$$

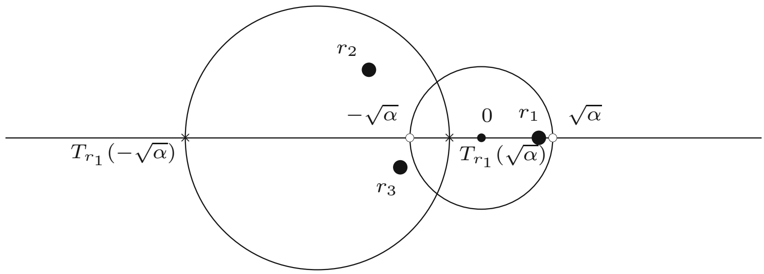

Fig. 1 The set $D(0, \sqrt{\alpha}) \cup T_{r1}(D(0, \sqrt{\alpha}))$

$$T_{r_1}(\sqrt{\alpha}) = -\frac{r_1\sqrt{\alpha}}{\sqrt{\alpha}+r_1}$$
$$T_{r_1}(0) = 0.$$

Claim 3. r_1 lies in the interval $(\frac{\sqrt{\alpha}}{2}, \sqrt{\alpha})$.

If $0 \leq r_1 \leq \frac{\sqrt{\alpha}}{2}$ then $-\sqrt{\alpha} \leq T_{r_1}(-\sqrt{\alpha}) \leq T_{r_1}(\sqrt{\alpha})$, so the complement of the set $\overline{D(0, \sqrt{\alpha})} \cup T_{r_1}(\overline{D(0, \sqrt{\alpha})})$ in $\mathbb{C}$ is empty.

Claim 4. If $r_1 \in (\frac{\sqrt{\alpha}}{2}, \sqrt{\alpha})$ then the point $-2r_1 \in (\overline{D(0, \sqrt{\alpha})} \cup T_{r_1}(\overline{D(0, \sqrt{\alpha})}))^c$. Since the region $(\overline{D(0, \sqrt{\alpha})} \cup T_{r_1}(\overline{D(0, \sqrt{\alpha})}))^c$ is connected, we can assume that $r_2 = -2r_1$. Hence $r_3 = -2r_1$.

Claim 5. The path $\mathbf{r} : [0, 1] \to \mathbb{C}^3$ given by

$$\mathbf{r}(t) = (1-t)(r_1, -2r_1, -2r_1) + t\left(\frac{1}{2}, -1, -1\right)$$

satisfies conditions (i), (ii) and (iii) of Lemma 5.2. Therefore, there exists a path from $[x : y : z]$ to $[3 : 2 : 1]$ in Ω_1. □

Lemma 5.7 *The set* Ω_2 *is a connected subset of* $\Omega(\gamma)$.

Proof If $[x : y : z] \in \Omega_2$ then we have two cases:

Case 1. $y \neq 0$ then (5) is a cubic polynomial equation. Let us denote by r_1, r_2, r_3 the roots of the equation (5). We can assume that $r_1, r_2 \in D(0, \sqrt{\alpha})$. Moreover, by Lemma 5.1, we can assume that $r_1 \in \mathbb{R}$ and $0 \leq r_1 < \sqrt{\alpha}$.

Subcase 1. First, we assume that $r_1 \neq 0$.

We denote by T_r the Möbius transformation $T_r(z) = -\frac{rz}{z+r}$.

Claim 1.

$$r_2 \in D(0, \sqrt{\alpha}) \cap (T_{r_1}(\overline{D(0, \sqrt{\alpha})}))^c,$$
$$r_3 \in (\overline{D(0, \sqrt{\alpha})})^c \cap T_{r_1}(D(0, \sqrt{\alpha})).$$

If $r_2 \in T_{r_1}(\overline{D(0, \sqrt{\alpha})})$ then

$$r_3 = T_{r_1}(r_2) \in T_{r_1}(T_{r_1}(\overline{D(0, \sqrt{\alpha})})) = \overline{D(0, \sqrt{\alpha})},$$

and it is a contradiction. The second statement follows analogously.

Claim 2. The set $(\overline{D(0, \sqrt{\alpha})})^c \cap T_{r_1}(D(0, \sqrt{\alpha}))$ is the intersection of the exterior of the disk with diameter the segment determined by the points $T_{r_1}(-\sqrt{\alpha})$ and $T_{r_1}(\sqrt{\alpha})$, and the exterior of the disk $D(0, \sqrt{\alpha})$.

Claim 3. The set $(\overline{D(0, \sqrt{\alpha})})^c \cap T_{r_1}(D(0, \sqrt{\alpha}))$ is connected and contains the point 1 for every r_1 with $0 < r_1 < \sqrt{\alpha}$. So we can assume that $r_3 = 1$.

Claim 4. The path $\mathbf{r} : [0, 1] \to \mathbb{C}^3$ given by

$$\mathbf{r}(t) = \left((1-t)r_1 + \frac{t}{2}, -\frac{(1-t)r_1 + \frac{t}{2}}{(1-t)r_1 + \frac{t}{2} + 1}, 1 \right)$$

satisfies conditions (i), (ii) and (iii) of Lemma 5.2. Therefore, there exists a path from $[x : y : z]$ to $[7 : -6 : 1]$ in Ω_2.

Subcase 2. If we assume that $r_1 = 0$, then $z = 0$ and $r_2 = 0$. It follows from Lemma 5.4 that $[x : y : 0] \in B_2$. Also, we can assume that $[x : y : 0] = [-1 : 1 : 0]$.

The continuous path $g : [0, \epsilon] \to \mathbb{P}^2_{\mathbb{C}}$ given by

$$g(t) = [3t^2 + 3t + 1 : -(2t + 1) : t^2(1 + t)^2],$$

satisfies that $g(0) = [1 : -1 : 0]$ and the roots of the polynomial equation in ζ given by

$$-(2t + 1)\zeta^3 + (3t^2 + 3t + 1)\zeta^2 - t^2(1 + t)^2 = 0$$

are precisely $t, 1+t, -\frac{t(1+t)}{2t+1}$). Therefore the path is contained in Ω_2 for all $0 \le t \le \epsilon$ for ϵ small.

We have shown that $[x : y : z]$ can be joined to a point in Ω_2 with $r_1 \neq 0$ by a path contained in Ω_2. By the subcase 1 above, $[x : y : z]$ can be joined to the point $[7 : -6 : 1]$ by a continuous path in Ω_2.

Case 2. $y = 0$.

By Lemma 5.3, we can assume that $[x : y : z] = [1 : 0 : 0]$.

The continuous path $g : [0, 1] \to \mathbb{P}^2_{\mathbb{C}}$ given by

$$g(t) = [-(t^4 + t^2 + 1) : t(t^2 + 1) : -t^2],$$

satisfies that $g(0) = [1 : 0 : 0]$ and for $t > 0$ the roots of the polynomial equation in ζ given by

$$t(t^2 + 1)\zeta^3 - (t^4 + t^2 + 1)\zeta^2 + t^2 = 0$$

are precisely $t, -\frac{t}{t^2+1}, \frac{1}{t}$. Therefore the path is contained in Ω_2 for all $t > 0$ sufficiently small.

We have shown that $[1 : 0 : 0]$ can be joined to a point in Ω_2 with $y \neq 0$ by a path contained in Ω_2. The proof follows from Case 1 above. □

Lemma 5.8 *The set Ω_3 is a connected subset of $\Omega(\gamma)$. Moreover, Ω_3 is contractible.*

Proof Let $[x : y : z]$ be a point in Ω_3. First, we notice that $y \neq 0$, otherwise, the polynomial equation in the variable ζ,

$$\bar{x}\zeta^2 - \bar{z} = 0$$

cannot have three roots within the open disk $D(0, \sqrt{\alpha})$.

The path given by

$$f : [0, 1] \to \mathbb{P}^2_{\mathbb{C}}$$
$$f(t) = [tx : y : t^3 z],$$

is a smooth path and satisfies: $f(0) = [0 : y : 0] = [0 : 1 : 0] \in \Omega_3$, $f(1) = [x : y : z] \in \Omega_3$. Also, for $0 < t < 1$, the set of roots of the polynomial equation in ζ,

$$\overline{tx}\zeta^2 + \bar{y}\zeta^3 - \overline{t^3 z} = 0,$$

is obtained from the set of roots of the equation

$$\bar{x}\zeta^2 + \bar{y}\zeta^3 - \bar{z} = 0,$$

by multiplying by $0 < t < 1$. Hence $f(t) \in \Omega_3$ for every $t \in [0, 1]$ and we have shown that Ω_3 is path connected.

Moreover, the function

$$\psi : [0, 1] \times \Omega_3 \to \Omega_3$$
$$\psi_t[x : y : z] = [(1-t)^2 x : y : (1-t)^3 z]$$

is a deformation retraction, thus Ω_3 is contractible. □

Lemma 5.9 *If $p_1 = [1 : 1 : 1]$, $p_2 = [1 : 0 : 0]$, $p_3 := [0 : 1 : 0]$ then*

$$\ell(\gamma, C_{p_j}) = j, \quad \textit{for } j = 1, 2, 3,$$

where C_{p_j} denotes the polar chain to p_j, $j = 1, 2, 3$.

Proof In order to simplify the notation in this proof, we consider the chains and the curve γ as parametrized in the interval $I = [0, 1]$.

$$t \mapsto \left[\frac{1 - e^{2\pi i t}}{2} : \frac{1 + e^{2\pi i t}}{2} : 1\right], \quad t \in [0, 1]$$

is a parametrization of the polar chain to p_1. In Heisenberg coordinates, this curve is parametrized by:

$$\zeta(t) = -1, \quad v(t) = \frac{2\sin(2\pi t)}{1 + \cos(2\pi t)}, \quad t \in [0, 1], \quad t \neq 1/2.$$

Thus it parametrizes a vertical line passing through the point $(-1, 0)$. On the other hand,

$$\zeta(t) = \frac{\alpha^{3/2} e^{6\pi i t}}{\alpha e^{4\pi i t} - 1}, \quad v(t) = \frac{-\alpha \sin(4\pi t)}{|\alpha e^{4\pi i t} - 1|^2}, \quad t \in [0, 1] \tag{6}$$

is a parametrization of the trefoil knot in Heisenberg coordinates.

It follows that $\ell(\gamma, C_{p_1})$ is equal to the winding number of the curve

$$\zeta(t) = \frac{\alpha^{3/2} e^{6\pi i t}}{\alpha e^{4\pi i t} - 1}, \quad t \in [0, 1]$$

respect to the point -1. In other words,

$$\begin{aligned} \ell(\gamma, C_{p_1}) &= \frac{1}{2\pi i} \int_{\zeta(t)} \frac{dz}{z+1} \\ &= \frac{1}{2\pi i} \int_{w(t)} \frac{w^2(w^2-3)dw}{(w^2-1)(w^3+w^2-1)} \end{aligned}$$

where $w(t) = \sqrt{\alpha}\, e^{2\pi i t}$, $t \in [0, 1]$. Finally, using the fact that α is the only pole of $\frac{w^2(w^2-3)}{(w^2-1)(w^3+w^2-1)}$ in the disk $D(0, \sqrt{\alpha})$, we obtain that

$$\ell(\gamma, C_{p_1}) = \frac{\alpha^2(\alpha^2-3)}{(\alpha^2-1)(3\alpha^2+2\alpha)} = 1.$$

The curve

$$t \mapsto [0 : e^{2\pi i t} : 1], \quad t \in [0, 1]$$

is a parametrization of the polar chain to p_2. Applying the canonical stereographic projection $S^3 \setminus \{N\} \to \mathbb{R}^3$, we obtain the curve

$$t \mapsto \left(0, 0, \frac{\cos(2\pi t)}{1 - \sin(2\pi t)}\right), \quad t \in [0, 1],$$

which is a parametrization of the z-axis. Also, the stereographic projection of the trefoil knot is parametrized by

$$t \mapsto \left(\frac{\alpha \cos(4\pi t)}{1 - \alpha^{\frac{3}{2}} \sin(6\pi t)}, \frac{\alpha \sin(4\pi t)}{1 - \alpha^{\frac{3}{2}} \sin(6\pi t)}, \frac{\alpha^{\frac{3}{2}} \cos(6\pi t)}{1 - \alpha^{\frac{3}{2}} \sin(6\pi t)}\right), \quad t \in [0, 1].$$

It follows that $\ell(\gamma, C_{p_2})$ is equal to the winding number of the curve

$$t \mapsto \left(\frac{\alpha \cos(4\pi t)}{1 - \alpha^{\frac{3}{2}} \sin(6\pi t)}, \frac{\alpha \sin(4\pi t)}{1 - \alpha^{\frac{3}{2}} \sin(6\pi t)}\right), \quad t \in [0, 1],$$

respect to the origin and such winding number is equal to two.

The curve

$$t \mapsto \left[-e^{2\pi i t} : 0 : 1\right], \quad t \in [0,1]$$

is a parametrization of the polar chain to p_3. In Heisenberg coordinates, this curve is parametrized by:

$$\zeta(t) = 0, \quad v(t) = \frac{\sin(2\pi t)}{1+\cos(2\pi t)}, \quad t \in [0,1], \quad t \neq 1/2.$$

Thus it parametrizes the v-axis. It follows that $\ell(\gamma, C_{p_1})$ is equal to the winding number of the projection of the curve (6) to the ζ-plane, with respect to the origin. By straightforward computations this winding number is equal to 3. □

Lemma 5.10 *If $p \in \Omega_j \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}$ (where $j = 0, 1, 2, 3$) then*

$$\ell(\gamma, C_p) = j,$$

where C_p is the polar chain to p.

Proof If $p \in \Omega_0 \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}$ then p can be joined, by a continuous path contained in Ω_0, to the point $[0 : 0 : 1] \in \Omega_0 \cap \mathbb{H}^2_{\mathbb{C}}$ (because Ω_0 is path connected). So p can be joined by a continuous path contained in Ω_0 to a point in $\partial \mathbb{H}^2_{\mathbb{C}}$. Lemma 3.5 implies that $\ell(\gamma, C_p) = 0$.

If $p \in \Omega_1 \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}$ then p can be joined by a continuous path contained in Ω_1 to the point $p_1 = [1 : 1 : 1] \in \Omega_1 \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}$ (because Ω_1 is path connected). By Lemma 3.5, this continuous path is contained in $\mathbb{P}^2_{\mathbb{C}} \backslash \overline{\mathbb{H}^2_{\mathbb{C}}}$, otherwise $\ell(\gamma, C_{p_1}) = 0$, which is a contradiction to Lemma 5.9. Therefore

$$\ell(\gamma, C_p) = \ell(\gamma, C_{p_1}) = 1.$$

The proof for the other cases is analogous. □

Finally, we prove the first item in Theorem 1.4.

Proof of 1.4 (i) Lemmas 5.5, 5.6, 5.7, 5.8 show that Ω_j is path connected for every $j = 0, 1, 2, 3$. Finally, Corollary 1.2 and Lemma 5.10 imply that $\Omega_0, \Omega_1, \Omega_2, \Omega_3$ are the components of the open set $\Omega(\gamma)$. □

The proof of the following Corollary is straightforward and we omit it.

Corollary 5.11 *Assuming the hypothesis of Theorem 1.4, we have that*

(i) $\mathbb{H}^2_{\mathbb{C}} \subset \Omega_0$.
(ii) *Every Ω_j is open in $\mathbb{P}^2_{\mathbb{C}}$ for $j = 0, 1, 2, 3$.*
(iii) $\partial \Omega_j \subset \Lambda(\gamma)$, *for $j = 0, 1, 2, 3$.*

5.2 The Set $\Lambda(\gamma)$

The purpose of this subsection is to describe the topology of the set $\Lambda(\gamma)$ when γ is the trefoil knot given by:

$$\gamma : S^1 \to \mathbb{H}^2_{\mathbb{C}} \cong S^3$$
$$\gamma(e^{2\pi i t}) = [(\sqrt{\alpha}e^{2\pi i t})^2 : (\sqrt{\alpha}e^{2\pi i t})^3 : 1].$$

In other words, we prove Theorem 1.4(ii).

Proof of Theorem 1.4(ii) First, we claim that $S(\gamma)$ is ϕ_t invariant, where $\phi_t[x : y : z] = [e^{2it}x : e^{3it}y : z]$, $t \in \mathbb{R}$. In fact, if $q \in S(\gamma)$, then there exist $p_1, p_2 \in \gamma(S^1)$ such that $\{q\} = \ell_{p_1} \cap \ell_{p_2}$. We see that $\phi_t(\ell_p) = \ell_{\phi_t(p)}$ for every $t \in \mathbb{R}$ and $p \in \partial\mathbb{H}^2_{\mathbb{C}}$, because $\phi_t \in \mathrm{PU}(2, 1)$. It follows that

$$\begin{aligned}\phi_t(\{q\}) &= \phi_t(\ell_{p_1} \cap \ell_{p_2})\\ &= \phi_t(\ell_{p_1}) \cap \phi_t(\ell_{p_2})\\ &= \ell_{\phi_t(p_1)} \cap \ell_{\phi_t(p_2)}.\end{aligned}$$

Hence, the claim is proved by the fact that $\gamma(S^1)$ is invariant under ϕ_t.

Since the action of the flow ϕ_t, $t \in \mathbb{R}$, is transitive on the set $\gamma(S^1)$, for any fixed $p_0 \in \gamma(S^1)$, we have that

$$\Lambda(\gamma) = \bigcup_{t\in[0,2\pi]} \ell_{\phi_t(p_0)} = \bigcup_{t\in[0,2\pi]} \phi_t(\ell_{p_0}),$$

and

$$\Lambda(\gamma)\backslash S(\gamma) = \bigcup_{t\in[0,2\pi]} \phi_t(\ell_{p_0}\backslash S(\gamma)). \tag{7}$$

In particular, we choose $p_0 = [\alpha : \alpha^{3/2} : 1] \in \gamma(S^1)$, and we see that

$$\ell_{p_0} = \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} : \alpha\bar{x} + \alpha^{3/2}\bar{y} - \bar{z} = 0\}$$

It follows that $\ell_{p_0} \cap S(\gamma)$ consists of those points $q = [x : y : z] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\alpha\bar{x} + \alpha^{3/2}\bar{y} - \bar{z} = 0, \text{ and}$$
$$\alpha e^{2it}\bar{x} + \alpha^{3/2}e^{3it}\bar{y} - \bar{z} = 0, \quad \text{for some } t \in [0, 2\pi].$$

By solving this system of equations, we can describe $\ell_{p_0} \cap S(\gamma)$ as the set of points $[x(t) : y(t) : z(t)]$, $t \in [0, 2\pi]$, such that

$$x(t) = \frac{1 + e^{2it} + e^{4it}}{\alpha},$$

$$y(t) = -\frac{e^{2it} + e^{4it}}{\alpha^{3/2}},$$
$$z(t) = 1.$$

Now, we make a change of coordinates by the element $g \in \mathrm{PU}(2,1)$ induced by the matrix

$$\mathbf{g} = \begin{pmatrix} -\alpha^{3/2} & \alpha & 0 \\ \alpha & \alpha^{3/2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so ℓ_{p_0} corresponds to the horizontal line given in inhomogeneous coordinates by

$$\{(w, 1) \in \mathbb{C}^2 : w \in \mathbb{C}\},$$

and $\ell_{p_0} \cap S(\gamma)$ corresponds to the curve

$$w(t) = -\sqrt{\alpha} - \left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}\right)(e^{2it} + e^{4it}),\ \ t \in [0, 2\pi]$$

which is a limaçon of Pascal satisfying that $\ell_{p_0} \backslash S(\gamma)$ consists of three topological disks. Finally, the proof follows from (7). □

If $S^1(\sqrt{\alpha}) \subset \mathbb{C}$ denotes the circle of center 0 and radius $\sqrt{\alpha}$ then $\Lambda(\gamma)$ can be parametrized by the function

$$L : S^1(\sqrt{\alpha}) \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$$
$$L(\zeta, [\mu_1 : \mu_2]) = [\mu_1(\zeta^2, \zeta^3, 1) + \mu_2(-\overline{\zeta}, 1, 0)] = [\mu_1\zeta^2 - \mu_2\overline{\zeta} : \zeta^3 + \mu_2 : \mu_1],$$

because the point $[-\overline{\zeta} : 1 : 0]$ lies on the complex line tangent to $\partial\mathbb{H}^2_{\mathbb{C}}$ at the point $[\zeta^2 : \zeta^3 : 1]$. Notice that the points in $S(\gamma)$ are those points $q \in \Lambda(\gamma)$ for which the inverse image $L^{-1}(q)$ consists of more than one point.

6 Open Questions

Let $C_1, \ldots, C_n$ be n distinct vertical chains in the Heisenberg space $\mathcal{H}$, one can build a smooth trivial knot $\gamma : S^1 \to \mathcal{H}$ so that $\ell(\gamma, C_k) = k$ for $k = 1, \ldots, n$. Thus $\Omega(\gamma)$ has at least $n + 1$ components by Corollary 1.2. Hence, there exist trivial knots $\gamma : S^1 \to \partial\mathbb{H}^2_{\mathbb{C}} \cong S^3$ such that the number of components of $\Omega(\gamma)$ is arbitrarily large. On the other hand, a natural question is the following: What is the minimum number of components of $\Omega(\gamma)$, when γ runs over all smooth trivial knots? It is not hard to see that this minimum number is equal to two and it is achieved in the case when the image of γ is a chain.

Some more general questions are the following:

(i) What is the minimum number of components of $\Omega(\gamma)$, when γ runs over all smooth knots in the same class of isotopy?
(ii) Is it possible to characterize all the curves in the same isotopy class such that $\Omega(\gamma)$ achieves its minimum number of components?
(iii) Other questions in the same style are obtained by replacing "number of components of $\Omega(\gamma)$" by "number of all distinct linking numbers $\ell(\gamma, C)$ for all chains C not meeting the image of γ" in the two questions above.

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