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A Landesman–Lazer Local Condition for Semilinear Elliptic Problems

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Abstract

The objective of this paper is to study the existence, multiplicity and non existence of solutions for semilinear elliptic problems under a local Landesman–Lazer condition. There is no growth restriction at infinity on the nonlinear term and it may change sign. In order to establish the existence of solution we combine the Lyapunov–Schmidt reduction method with truncation and approximation arguments via bootstrap methods. In our applications we also consider the existence of a bifurcation point which may have multiple positive solutions for a fixed value of the parameters.

Keywords Semilinear elliptic problems · Variational methods · Lyapunov–Schmidt reduction method · Landesman–Lazer condition · Bifurcation point

Mathematics Subject Classification 35J20 · 35J61 · 58J55

1 Introduction and Main Results

In this paper we are concerned with the existence, non existence and multiplicity of weak solutions for the following problem

$$\begin{cases} -\Delta u = \lambda u + \mu h(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda > 0$; $\mu \neq 0$ is a real parameter and $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

One of the most celebrated articles in the area of Nonlinear Analysis is the paper due to Landesman et al. (1975) which, via Topological Degree Theory, establishes the existence or non existence of solution for resonant nonlinear elliptic problems. In Landesman et al. (1975), one of the most famous hypotheses for this class of problems was introduced, nowadays called the Landesman–Lazer condition.

We emphasize that, besides providing the basic tools for studying resonant problems, the article Landesman et al. (1975) has allowed researchers to consider the possibility of using different methods to study that class of problems—see (Ahmad et al. 1976; Amann 1979; Arcoya and Gámez 2001; Bartolo and Benci 1983; Rabinowitz 1986; Ruiz 2004; Shaw 1977; Silva 1991) and references therein. In particular we mention the paper by Rabinowitz (1978) where it was demonstrated one of the most important results on the minimax theory: the Saddle Point Theorem.

Supposing that $\lambda = \lambda_1$ and that h is of the form $h(x, s) = f(x) + g(s)$, with $f \in L^2(\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ a bounded continuous function with finite limits $g^{\pm} = \lim_{s \to \pm\infty} g(s)$, the Landesman–Lazer condition for Problem (1.1) may be written as

$$\left[\int_{\Omega} (f + g^{-})\varphi_1 dx\right]\left[\int_{\Omega} (f + g^{+})\varphi_1 dx\right] < 0, \tag{1.2}$$

where φ_1 is a positive eigenfunction associated with λ_1, the first eigenvalue of the operator $-\Delta$ under Dirichlet boundary conditions.

We observe that the Landesman–Lazer condition (1.2) implies that there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that either

(H_0^+) $\displaystyle\int_{\Omega} h(x, t_1\varphi_1)\varphi_1 dx > 0 > \int_{\Omega} h(x, t_2\varphi_1)\varphi_1 dx,$

or

(H_0^-) $\displaystyle\int_{\Omega} h(x, t_1\varphi_1)\varphi_1 dx < 0 < \int_{\Omega} h(x, t_2\varphi_1)\varphi_1 dx.$

The main objective of this work is to consider the existence, multiplicity and non existence of solutions for Problem (1.1) under the local version of the Landesman–Lazer condition $(H_0^{\pm})$ when the parameters μ and λ are close to zero and λ_1, respectively. One of the characteristics of our main results is that they do not impose any growth restriction at infinity on the nonlinear term h. It is worthwhile mentioning that in Landesman et al. (1975), and in the vast majority of subsequent results, it is imposed a growth restriction at infinity on the nonlinear term.

In order to establish our first result on the existence of solution for Problem (1.1) we assume h satisfies (H_0^+) and the technical hypothesis:

(H_1) h is locally $L^{\sigma}(\Omega)$-bounded, $\sigma > \{N/2, 1\}$, i. e., given $S > 0$, there is $f_S \in L^{\sigma}(\Omega)$ such that $|h(x, s)| \leq f_S(x)$ for every $|s| \leq S$, a. e. in Ω.

We observe that our results are proved via variational methods. However, since there is no global growth restriction on the nonlinearity h, the associated functional may not be well defined in $H_0^1(\Omega)$. We overcome this fact by combining the hypothesis

(H_1) with an appropriated truncation of the function h. The hypothesis (H_1) also plays an important role in the approximation method that we use to derive the existence of solution for Problem (1.1).

Now we may state:

Theorem 1.1 *Suppose h satisfies (H_0^+) and (H_1). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

For proving a corresponding result under the hypothesis (H_0^-), we suppose more regularity on the function h:

(H_2) h is locally $L^\sigma(\Omega)$-Lipschitz, $\sigma > \{N/2, 1\}$, i. e., given $S > 0$, there is $\zeta_S \in L^\sigma(\Omega)$ such that $|h(x, s_1) - h(x, s_2)| \leq \zeta_S(x)|s_1 - s_2|$, for every $|s_1|,\ |s_2| \leq S$, a. e. in Ω.

Theorem 1.2 *Suppose h satisfies (H_0^-), (H_1) and (H_2) . Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

It is important to note that in Theorems 1.1 and 1.2 the projections of the solutions u_μ on the direction of φ_1 are located between $t_1\varphi_1$ and $t_2\varphi_1$. As a direct consequence of this fact, we may establish multiple solutions for the Problem (1.1) under the following version of $(H_0^\pm)$:

(H_0) there exist $k \in \mathbb{N}$ and $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \ldots, k$ such that

$$\left[\int_\Omega h(x, t_i\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\varphi_1)\varphi_1 dx\right] < 0.$$

Theorem 1.3 *Suppose h satisfies (H_0), (H_1) and (H_2). Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has k weak solutions $u_i = \hat{t}_i\varphi_1 + v_i$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \ldots, k$.*

Remark 1.4 Note that the solutions provided by Theorems 1.1–1.3 are of class $C^{1,\gamma}(\overline{\Omega})$ if $N = 1$ and of class $C^{0,\gamma}(\overline{\Omega})$ if $N \geq 2$. If we assume (H_1) holds with $\sigma > N$, we may assert that those solutions are actually in $C^{1,\gamma}(\overline{\Omega})$. Using this fact we may verify that the solution u_μ given by Theorems 1.1 and 1.2 is positive or negative in Ω provided $t_1 \geq 0$ or $t_2 \leq 0$, respectively. Moreover, for $|\mu| > 0$ sufficiently small, the solutions of Theorem 1.3 are ordered, see Theorems 2.10 and 2.11 in Sect. 2.

The proofs of Theorems 1.1 and 1.2 are inspired by the Lyapunov–Schmidt Reduction Method, as presented in the articles (Castro and Lazer 1979; Castro 1982; Cossio 2004; Landesman et al. 1975). However, under the hypotheses of those theorems, that method can not be applied directly since we do not impose any global growth restriction on the nonlinear term h. In order to overcome such difficulty we combine the Lyapunov–Schmidt reduction method with a truncation argument and an approximation method based on the bootstrap technique.

Theorems 1.1 and 1.2 imply that the local Landesman–Lazer condition $(H_0^{\pm})$ is sufficient to provide the existence of solution u_μ for Problem (1.1) such that its orthogonal projection on the direction of φ_1 belongs to the interval $(t_1\varphi_1, t_2\varphi_1)$. It is natural to question if this condition is also necessary. Here we prove such result under the hypotheses (see also Theorem 2.9)

(H_3) there exists $f \in L^\sigma(\Omega)$, $\sigma > \{N/2, 1\}$, such that $|h(x,s)| \leq f(x)(1 + |s|)$, for every $s \in \mathbb{R}$, a. e. in Ω;

and

(H_4) there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that $\int_\Omega h(x, t\varphi_1)\varphi_1 dx \neq 0$, for every $t \in [t_1, t_2]$.

Theorem 1.5 *Suppose h satisfies (H_3) and (H_4). Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has no weak solution $u_\mu = t\varphi_1 + v$, with $t \in [t_1, t_2]$ and $v \in \langle\varphi_1\rangle^\perp$.*

It is important to note that, under our hypotheses, h may change sign in Ω. This characterizes the Problem (1.1) as indefinite. This class of problems has been object of an intense research in the last three decades since the articles by Alama and Tarantello (1993), Berestycki et al. (1994) and Ouyang (1991) (see Alama and Del Pino 1996; Alama and Tarantello 1996; Chang and Jiang 2004; Costa and Tehrani 2001; De Figueiredo et al. 2003, 2006; Medeiros et al. 2014; Silva and Silva 2013 and references there in). Here we present, as a consequence of our main theorems, results on the existence of solutions for this class of problems, relating the hypotheses assumed for such problems with the Landesman–Lazer condition. More specifically, we consider the existence of a solution for the following problem

$$\begin{cases} -\Delta u = \lambda u + \beta b_1(x) u^q + b_2(x) u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda, \beta > 0$; $p > q$, with $p \neq 1$, and $b_1, b_2 \in L^\sigma(\Omega)$, with $\sigma > N$.

Considering the nomenclature for elliptic problems used in the literature, Problem (1.3) is superlinear or sublinear at infinity if $p > 1$ or $0 < p < 1$ and it is superlinear, linear or sublinear at the origin if $q > 1$, $q = 1$ or $0 < q < 1$, respectively.

Setting

$$r_1 := \int_\Omega b_1 \varphi_1^{q+1} dx \quad \text{and} \quad r_2 := \int_\Omega b_2 \varphi_1^{p+1} dx,$$

for the problem linear or superlinear at the origin and superlinear at infinity we establish the following result:

Proposition 1.6 *Suppose $p > q \geq 1$ and $r_1 r_2 < 0$. Then there exist positive constants β^* and ν^* such that Problem (1.3) has a positive weak solution for every $\beta \in (0, \beta^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}}\nu^*$.*

For the problem sublinear at the origin and superlinear or sublinear at infinity we may state:

Proposition 1.7 *Suppose $r_1 > 0 > r_2$. Then*

(i) *if $0 < q < 1 < p$, there exist positive constants β_1^* and ν_1^* such that Problem (1.3) has a positive weak solution for every $\beta \in (0, \beta_1^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_1^*$.*
(ii) *if $0 < q < p < 1$, there exist positive constants β_2^* and ν_2^* such that Problem (1.3) has a positive weak solution for every $\beta \in (\beta_2^*, \infty)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_2^*$.*

We emphasize that in Proposition 1.6 and in item (i) of Proposition 1.7, we do not assume the restriction $p < (N+2)/(N-2)$ to derive the existence of a solution for (1.3). Furthermore, in Proposition 1.6 it is worthwhile mentioning that for the subcritical problem with a nonlinear term superlinear at the origin and infinity ($1 < q < p < (N+2)/(N-2)$), we are not able to apply a minimax theorem Rabinowitz (1986) to find a positive solution for Problem (1.3) since we do not know if the associated functional satisfies any version of the Palais–Smale compactness condition. Note that, under our hypothesis, the nonlinear term $f(x, s) = \beta b_1(x) s^q + b_2(x) s^p$ does not satisfy $\lim_{s\to\infty} |f(x, s) - b_2(x) s^p|/s = 0$ and we do not suppose that Problem (1.3) has a thick zero set (see e.g. Alama and Tarantello 1993; Medeiros et al. 2014).

For the problem sublinear at the origin we refer the reader to the papers by Ambrosetti et al. (1994) and De Figueiredo et al. (2003).

We also note that, as an application of Theorem 1.3, we establish the existence of multiple solutions for Problem (1.1) with an indefinite nonlinear term $h(x, s)$ that is a polynomial function with respect to the second variable (see Proposition 3.1).

Inspired by the work of Landesman et al. (1975), we consider applications of our main results for Problem (1.1) with h of the form $h(x) = f(x) + g(s)$. More specifically, we are interested in the existence of solution for the problem:

$$\begin{cases} -\Delta u = \lambda u + \mu(f(x) + g(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda, \mu > 0$ and $f \in L^\sigma(\Omega)$, with $\sigma > \{N/2, 1\}$.

In our first result on existence of a solution for Problem (1.4) we suppose

(G_1) $g : \mathbb{R} \to \mathbb{R}$ a continuous function and there exists $M > 0$ such that

$$g(s) \geq -M \text{ if } s \leq 0 \text{ and } g(s) \leq M \text{ if } s \geq 0.$$

Denoting by $g_i^- := \liminf_{s\to-\infty} g(s)$ and $g_s^+ := \limsup_{s\to+\infty} g(s)$, we assume:

(LL^+)
$$\int_\Omega (f + g_i^-)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+)\varphi_1 dx.$$

Proposition 1.8 *Suppose g satisfies (G_1) and (LL^+). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.4) has a weak solution.*

Next, supposing

$(\hat{G}_1)$ $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz function and there exists $M > 0$ such that

$$g(s) \leq M \text{ if } s \leq 0 \text{ and } g(s) \geq -M \text{ if } s \geq 0$$

and

$$(LL^-) \qquad \int_\Omega (f + g_s^-)\varphi_1 dx < 0 < \int_\Omega (f + g_i^+)\varphi_1 dx,$$

where $g_s^- := \limsup_{s\to-\infty} g(s)$ and $g_i^+ := \liminf_{s\to+\infty} g(s)$, we obtain

Proposition 1.9 *Suppose g satisfies $(\hat{G}_1)$ and (LL^-). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.4) has a weak solution.*

We observe that the above results allow us to consider g such that $g_i^- = +\infty$ and $g_s^+ = -\infty$ or $g_s^- = -\infty$ and $g_i^+ = +\infty$, respectively. Moreover g may have unbounded oscillatory behavior.

In our last application we consider the existence of a bifurcation point for the solutions of the problem

$$\begin{cases} -\Delta u = \lambda u + f(\alpha, x, u) + b(x)u^p & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, $\lambda \in \mathbb{R}$, $p > 1$, $\alpha \in \mathbb{R}^m$, $m \geq 0$, and $f : \mathbb{R}^m \times \Omega \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(\alpha, x, s) = \sum_{i=1}^m \alpha_i b_i(x) s^{p_i}, \text{ for every } \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m, s \in \mathbb{R}, \text{ a. e. in } \Omega \tag{1.6}$$

with $1 < p_i < p$, $p_i \neq p_j$, $i \neq j \in \{1, \ldots, m\}$. For $m = 0$, we take $\mathbb{R}^m = \{0\}$ and $f \equiv 0$.

We also suppose that $b_1, \ldots, b_m, b \in L^\sigma(\Omega)$, $\sigma > N$, and that b satisfies

$$r = \int_\Omega b(x)\varphi_1^{p+1} dx \neq 0. \tag{1.7}$$

Denoting by $\mathbb{H}$ the Hilbert space $\mathbb{R} \times \mathbb{R}^m \times H_0^1(\Omega)$, a solution of Problem (1.5) is a point $(\lambda, \alpha, u) \in \mathbb{H}$ such that u solves (1.5) for the parameters (λ, α). The elements of the subspace of $\mathbb{H}$ formed by the solutions $\{(\lambda, \alpha, 0)/(\lambda, \alpha) \in \mathbb{R} \times \mathbb{R}^m\}$ will be referred to as the trivial solutions of Problem (1.5). We recall that $(\lambda_0, \alpha_0, 0)$ is a bifurcation point for the solutions of (1.5) if $(\lambda_0, \alpha_0, 0)$ is in the closure of the set of nontrivial solutions of (1.5) or, equivalently, if every neighborhood of $(\lambda_0, \alpha_0, 0)$ in $\mathbb{H}$ contains a nontrivial solution of (1.5).

Our main objective here is to find hypotheses on f that guarantee that $(\lambda_1, 0, 0)$ is a bifurcation point such that close to it we may find multiple nontrivial positive solutions associated with the same parameters (λ, α).

We say that $(\lambda_0, \alpha_0, 0) \in \mathbb{H}$ is a bifurcation point of multiplicity k for the positive solutions of (1.5) if every neighborhood of $(\lambda_0, \alpha_0, 0)$ in $\mathbb{H}$ possesses k distinct nontrivial solutions $(\lambda, \alpha, u_1), \dots, (\lambda, \alpha, u_k)$ with $u_1, \dots, u_k$ positive in Ω.

Considering $m \geq 1$, we define $r_i = \int_\Omega b_i(x)\varphi_1^{p_i+1} dx$, $1 \leq i \leq m$, and we set

$$J := \{i \in \{1, \dots, m\}; r_i \neq 0\}.$$

We denote by k_J the number of elements of J. We also set $k_J = 0$ if $m = 0$.

Proposition 1.10 *Suppose b satisfies (1.7). Then $(\lambda_1, 0, 0)$ is a bifurcation point of multiplicity $k_J + 1$ for the positive solutions of (1.5).*

We note that for the particular case in which $f \equiv 0$ (corresponding to $m = 0$) we have the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u + b(x)u^p, & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

As a consequence of Proposition 1.10 and its proof, we may assert that if r, given by (1.7), is positive then there exists $\underline{\lambda} < \lambda_1$ such that Problem (1.8) has a positive solution for every $\underline{\lambda} < \lambda < \lambda_1$. On the other hand, if r is negative then there exists $\overline{\lambda} > \lambda_1$ such that Problem (1.8) has a positive solution for every $\lambda_1 < \lambda < \overline{\lambda}$.

Problem (1.8) for dimensions $N \geq 3$ with critical exponent, $p = (N+2)/(N-2)$, and $b \equiv 1$ has been considered by Brézis and Nirenberg (1983). Applying variational methods, those authors proved that Problem (1.8) has a positive has for every $\lambda \in (0, \lambda_1)$ if $N \geq 4$. They also proved that if Ω is a ball and $N = 3$, Problem (1.8) has a positive solution if and only if $\lambda \in (\lambda_1/4, \lambda_1)$. We also observe that, assuming more regularity on b, as a consequence of the results derived by Rabinowitz (1971) via bifurcation theory, we may assert that Problem (1.8) possesses a component $\mathcal{C}$ of positive solutions which meets $(\lambda_1, 0)$ and it is unbounded in $\mathbb{R} \times H_0^1(\Omega)$. In view of the above observation we may expect that the projection of $\mathcal{C}$ on the λ-axis contains the interval $(\underline{\lambda}, \lambda_1)$ or $(\lambda_1, \overline{\lambda})$ provided r is either positive or negative, respectively.

We emphasize that Proposition 1.10 presents us the possibility of having a bifurcation for which we may find multiple positive solutions associated with the same value of the parameters. We also note that, considering $m \geq 1$ and taking appropriated functions $\alpha(\lambda)$, $\lambda \in \mathbb{R}$, in the expression of the function f given by (1.6), we may find nonlinear eigenvalue problems for which $(\lambda_1, 0)$ is a bifurcation point of multiplicity greater than one for the positive solutions of the problem (see Proposition 3.3).

As mentioned previously we do not suppose that h satisfies any global growth restriction, which prevents a direct application of variational methods since the associated functional is not well defined. To overcome such difficulty in our proof of Theorem 1.1, firstly we establish a version of this theorem assuming that $|h|$ is bounded by a function in $L^{\sigma}(\Omega)$. In this case we use a minimization argument to find a solution of class $C^{0,\gamma}(\overline{\Omega})$ $u_{\mu} = t\varphi_1 + v, t \in (t_1, t_2)$ and $v = v(\mu) \in \langle\varphi_1\rangle^{\perp}$. Next, considering an appropriated truncation of h, we apply this version of Theorem 1.1 and a bootstrap argument to verify that $\|v\|_{\infty} \to 0$ as $\mu \to 0^+$. This allows us to show that the solution we have found is actually a solution of Theorem 1.1 whenever $\mu > 0$ is sufficiently small.

In order to prove Theorem 1.2, we first establish a version of it by supposing that $|h|$ is bounded by a function in $L^{\sigma}(\Omega)$ and that h is Lipschitz with respect to the second variable, with the constant of Lipschitz given by a function in $L^{\sigma}(\Omega)$. We note that the proof of this version is based on the Liapunov–Schmidt Reduction Method. Next, as in the proof of Theorem 1.1, we complete the proof of Theorem 1.2 by truncating h and by using an approximation method.

This paper is organized as follows: in the Sect. 2 we present the proofs of the main results: we reserve the Sects. 2.1, 2.2 and 2.3 for the proofs of Theorems 1.1, 1.2–1.3 and 1.5, respectively. In Sect. 2.4 we establish the results mentioned in Remark 1.4 on positivity, negativity and ordering of the solutions of Problem (1.1). In Sect. 3 we proof Propositions 1.6–1.9 and 1.10. There, as an application of Theorem 1.3, we establish a result on the existence of multiple solutions when $h(x, s)$ is a polynomial function with respect to the variable s. Moreover we also present a result, for a version of Problem (1.8), that states that $(\lambda_1, 0)$ is a bifurcation point with multiple positive solutions associated with the same value of the parameter λ.

Throughout this work, we denote by

$$\|u\| = \left(\int_{\Omega} |\nabla u|^2 dx\right)^{\frac{1}{2}}, \quad \|u\|_{k,q} = \left(\sum_{0\le|\alpha|\le k} \|D^{\alpha}u\|_q^q\right)^{\frac{1}{q}}, \quad \text{and}$$

$$\|u\|_q = \left(\int_{\Omega} |u|^q dx\right)^{\frac{1}{q}},$$

the norms of the spaces $H_0^1(\Omega)$, $W^{k,q}(\Omega)$ and $L^q(\Omega)$, with $q \in [1, \infty)$, respectively. Moreover, we consider $C^{1,\gamma}(\overline{\Omega})$ equipped with its usual norm $\|\cdot\|_{1,\gamma}$. The symbols A, d_q, K and K_i, $i = 1, 2, \ldots$, represent positive constants, reserving d_q for the imbedding constant of $H_0^1(\Omega)$ in $L^q(\Omega)$, $q \in [1, 2^*]$.

2 Proofs of the Main Results

We reserve this section for our proofs of Theorems 1.1–1.3 and 1.5. Here we also verify the assertion made in Remark 1.4 on the positivity/negativity of the solution provided by Theorems 1.1 and 1.2 and on the ordering of the solutions given by Theorem 1.3.

Henceforth in this section we suppose without loss of generality that there exists $\bar{\lambda} > 0$ such that $0 < \lambda < \bar{\lambda}$, with $\bar{\lambda} < \lambda_2$, the second eigenvalue of the operator $-\Delta$ under the Dirichlet boundary conditions.

2.1 Proof of Theorem 1.1

We begin by proving a version of Theorem 1.1 with $|h|$ bounded by a function in $L^\sigma(\Omega)$. More specifically we suppose that

(H_1^*) there exists $f \in L^\sigma(\Omega), \sigma > \{N/2, 1\}$, such that $|h(x,s)| \le f(x)$, for every $s \in \mathbb{R}$, a. e. in Ω.

Theorem 2.1 *Suppose h satisfies (H_0^+) and (H_1^*). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

We note that, under the hypothesis (H_1^*), the associated functional $I_\mu : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_\mu(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 - \mu \int_\Omega H(x,u)dx, \tag{2.1}$$

where $H(x,t) = \int_0^t h(x,s)ds$, is well defined and it is of class C^1. In addition, the critical points of I_μ are weak solutions of Problem (1.1).

As mentioned in the introduction, our proof of Theorem 2.1 is based on a minimization argument. More specifically, we shall verify that for $\mu > 0$ sufficiently small the infimum of I_μ on the set $C := \{u = \tau\varphi_1 + v; \tau \in [t_1, t_2], v \in \langle\varphi_1\rangle^\perp\}$ is attained on an interior point of C. On this direction, we prove:

Lemma 2.2 *Suppose h satisfies (H_1^*). Then, for every $\mu > 0$, the functional I_μ is bounded from below and coercive on C.*

Proof From $\lambda < \overline{\lambda} < \lambda_2$, (H_1^*), the Hölder inequality and the Sobolev Imbedding Theorem, for every $u = \tau\varphi_1 + v$, $v \in \langle\varphi_1\rangle^\perp$, we get

$$I_\mu(u) \ge \frac{-(\overline{\lambda} - \lambda_1)}{2\lambda_1}\tau^2 + \frac{(\lambda_2 - \overline{\lambda})}{2\lambda_2}\|v\|^2 - \mu d_{\sigma'}\|f\|_\sigma\|v\|.$$

Since $\tau \in [t_1, t_2]$ whenever $u \in C$, the above inequality implies that I_μ is bounded from below and coercive on C. The lemma is proved. □

Now we define

$$m_C := \inf\{I_\mu(u); u \in C\} \tag{2.2}$$

and, for every $\tau \in [t_1, t_2]$,

$$m_\tau := \inf\{I_\mu(u); u = \tau\varphi_1 + v, v \in \langle\varphi_1\rangle^\perp\}. \tag{2.3}$$

It follows frow Lemma 2.2 that $m_\tau \ge m_C > -\infty$ for every $\tau \in [t_1, t_2]$. Actually, we may also obtain:

Corollary 2.3 *For every $\mu > 0$, there exists $u_\mu \in C$ such that $I_\mu(u_\mu) = m_C$. Moreover, for every $\mu > 0$ and $\tau \in [t_1, t_2]$, there exists $v_\mu \in \langle\varphi_1\rangle^\perp$ such that $I_\mu(\tau\varphi_1 + v_\mu) = m_\tau$.*

Proof Noting that (H_1^*) implies that I_μ is weakly lower semicontinuous, we may invoke Lemma 2.2 and the fact that C is a closed convex set to conclude that, for every $\mu > 0$, there exists $u_\mu \in C$ such that $I_\mu(u_\mu) = m_C$. An analogous argument implies that for every $\mu > 0$ and $\tau \in [t_1, t_2]$, we may find $v_\mu \in \langle\varphi_1\rangle^\perp$ such that $I_\mu(\tau\varphi_1 + v_\mu) = m_\tau$. The proof of the corollary is complete. □

For every $\tau \in [t_1, t_2]$ and $\mu > 0$, we set

$$S_\tau := \{v \in \langle\varphi_1\rangle^\perp; I_\mu(\tau\varphi_1 + v) = m_\tau\}.$$

In view of Corollary 2.3, $S_\tau \neq \emptyset$ for every $\mu > 0$ and $\tau \in [t_1, t_2]$. Considering the extreme points of the interval $[t_1, t_2]$, we obtain:

Lemma 2.4 *Suppose h satisfies (H_1^*). Then, given $\delta > 0$, there exists $\mu_1 > 0$ such that $\|v\| < \delta$, for every $v \in S_{t_1} \cup S_{t_2}$.*

Proof Without loss of generality we suppose that $v \in S_{t_1}$. Using Corollary 2.3, (H_1^*), the Hölder Inequality and the Sobolev Imbedding Theorem, we obtain

$$\|v\|^2 = \lambda\|v\|_2^2 + \mu\int_\Omega h(x, t_1\varphi_1 + v)v dx \leq \overline{\lambda}\|v\|^2/\lambda_2 + \mu d_{\sigma'}\|f\|_\sigma\|v\|.$$

Hence $\|v\| \leq \lambda_2\mu d_{\sigma'}\|f\|_\sigma/(\lambda_2 - \overline{\lambda})$. Taking $0 < \mu_2 < \min\{\mu_1, \delta(\lambda_2 - \overline{\lambda})/\lambda_2 d_{\sigma'}\|f\|_\sigma\}$, we complete the proof of Lemma 2.4. □

Now we may present:

Proof of Theorem 2.1 By Corollary 2.3, for every $\mu > 0$, there exists $u_\mu = t\varphi_1 + v \in C$ such that $I(u_\mu) = m_C$, m_C given by (2.2). It is clear that $m_C = m_t$ and, consequently, $v \in S_t$. In order to prove Theorem 2.1, it suffices to find $\mu^* > 0$ and $\nu^* > 0$ such that $u_\mu \in \text{int}(C)$ for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$.

From the continuity of the functional $J : H_0^1(\Omega) \to \mathbb{R}$, given by

$$J(u) = \int_\Omega h(x, u)\varphi_1 dx, \quad \text{for every } u \in H_0^1(\Omega),$$

and the hypothesis (H_0^+), we find $a > 0$ and $\delta > 0$ such that, for every $u \in H_0^1(\Omega)$, with $\|u\| < \delta$, we have

$$\int_\Omega h(x, t_1\varphi_1 + u)\varphi_1 dx > a > 0 > -a > \int_\Omega h(x, t_2\varphi_1 + u)\varphi_1 dx. \tag{2.4}$$

Considering the above value of δ, we take $\mu^* = \mu_1$, μ_1 given by Lemma 2.4. Taking $\mu \in (0, \mu^*)$, by Lemma 2.4 and (2.4), for every $v_1 \in S_{t_1}$ and $v_2 \in S_{t_2}$, we obtain

$$\langle I_\mu'(t_1\varphi_1 + v_1), \varphi_1\rangle < -\mu\left[\frac{(\lambda - \lambda_1)t_1}{\lambda_1\mu}\|\varphi_1\|^2 + a\right]$$

and

$$\langle I_{\mu}^{'}(t_2\varphi_1 + v_2), \varphi_1\rangle > -\mu\left[\frac{(\lambda - \lambda_1)t_2}{\lambda_1\mu}\|\varphi_1\|^2 - a\right].$$

Hence, taking $0 < \nu^* < (\lambda_1 a)/(1 + |t_1| + |t_2|)\|\varphi_1\|^2$, for every $0 < \mu < \mu^*$ and $|\lambda - \lambda_1| < \mu\nu^*$, we get

$$\langle I_{\mu}^{'}(t_1\varphi_1 + v_1), \varphi_1\rangle < 0 < \langle I_{\mu}^{'}(t_2\varphi_1 + v_2), \varphi_1\rangle, \tag{2.5}$$

whenever $v_1 \in S_{t_1}$ and $v_2 \in S_{t_2}$. Supposing, for example, that $u_\mu = t_1\varphi_1 + v$, it follows from (2.5) that there is $\tau \in (t_1, t_2)$ such that $m_C \le m_\tau \le I_\mu(\tau\varphi_1 + v) < I_\mu(t_1\varphi_1 + v) = I_\mu(u_\mu) = m_C$. This contradiction implies that we may not have $t = t_1$. A similar argument also implies that we may not have $t = t_2$. Consequently $u_\mu \in \mathrm{int}(C)$. The proof of Theorem 2.1 is complete. □

Before proving Theorem 1.1 we state the following estimate:

Lemma 2.5 *Suppose h satisfies* (H_1^*). *Consider* μ^* *and* ν^* *the positive constants given by Theorem* 2.1. *Then there exists* $b_\sigma > 0$ *such that* $\|v\|_{2,\sigma} \le b_\sigma\mu$ *for every* $\mu \in (0, \mu^*)$, $|\lambda - \lambda_1| < \mu\nu^*$ *and* $u_\mu = t\varphi_1 + v$ *a weak solution of Problem* (1.1) *with* $t \in (t_1, t_2)$ *and* $v \in \langle\varphi_1\rangle^\perp$.

Proof of Lemma 2.5 We present a proof of the lemma for $N \ge 3$ since if $N = 1$ or $N = 2$ the proof may be easily adapted by using the fact that $H_0^1(\Omega)$ is continuously imbedded in $L^p(\Omega)$, for every $p \ge 1$.

First of all we claim that there exists $\hat{b}_\sigma > 0$ such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$,

$$\|v\|_\sigma \le \hat{b}_\sigma\mu. \tag{2.6}$$

As $u = t\varphi_1 + v$ is a weak solution of Problem (1.1) we obtain that v is weak solution of the problem

$$\begin{cases} -\Delta v = (\lambda - \lambda_1)t\varphi_1 + \lambda v + \mu h(x, t\varphi_1 + v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, using $0 < \lambda < \overline{\lambda} < \lambda_2$, (H_1^*), the Hölder Inequality and the Sobolev Imbedding Theorem, we obtain

$$\|v\| \le \frac{\lambda_2}{\lambda_2 - \overline{\lambda}} d_{\sigma'}\|f\|_\sigma\mu. \tag{2.7}$$

Taking $p_0 = 2^* = 2N/(N-2)$, by (2.7) and the continuity of the Sobolev Imbedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, we find $K_0 > 0$ such that

$$\|v\|_{p_0} \le K_0\mu. \tag{2.8}$$

Next, we define $\hat{h}(x) = (\lambda - \lambda_1)t\varphi_1(x) + \lambda v(x) + \mu h(x, t\varphi_1(x) + v(x))$, for every $x \in \Omega$, and we consider $\hat{p_0} = \min\{\sigma, p_0\}$. From (H_1^*), $|\lambda - \lambda_1| < \mu\nu^*$, (2.8) and the fact that $L^\sigma(\Omega)$ and $L^{p_0}(\Omega)$ are continuously imbedded in $L^{\hat{p_0}}(\Omega)$, we have that there exists $\hat{K_0} > 0$ such that

$$\|\hat{h}\|_{p_0} \le \hat{K_0}\mu. \tag{2.9}$$

Thus, by Agmon–Douglis–Nirenberg Theorem (Agmon et al. 1959), we find $\hat{K} > 0$ such that

$$\|v\|_{2,\hat{p_0}} \le \hat{K}\hat{K_0}\mu. \tag{2.10}$$

If $p_0 \ge N/2$, we have that $\hat{p_0} \ge N/2$. In this case the estimate (2.6) is a direct consequence of (2.10) and the continuity of the Sobolev Imbedding $W^{2,N/2}(\Omega) \hookrightarrow L^\sigma(\Omega)$. On the other hand, if $p_0 < N/2$, we have that $\hat{p_0} = p_0$. In this case, from (2.10) and the fact that $W^{2,p_0}(\Omega)$ is continuously imbedded in $L^{p_1}(\Omega)$, $p_1 = Np_0/(N-p_0)$, we find $K_1 > 0$ such that $\|v\|_{p_1} \le K_1\mu$.

Arguing as above we have that either (2.6) holds or there exist $0 < p_0 < p_1 < \cdots < p_m < N/2$ and $K_1, K_2, \ldots, K_m > 0$ such that $p_i = Np_{i-1}/(N-2p_{i-1})$, $i = 1, \ldots, m$, and $\|v\|_{p_i} \le K_i\mu$, $i = 0, 1, \ldots, m$. Noting that those relations imply that $\lim_{m\to\infty} p_m = \infty$, we conclude that (2.6) must hold. The claim is proved.

From the above claim and the arguments used to derive (2.9) and (2.10), we find $b_\sigma > 0$ such that $\|v\|_{2,\sigma} \le b_\sigma\mu$ for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$. The proof of Lemma 2.5 is complete. □

Now we are in condition of presenting a

Proof of Theorem 1.1 We consider the truncated function h_R defined by

$$h_R(x, s) = \chi(s)h(x, s), \text{ for every } x \in \overline{\Omega},\ s \in \mathbb{R}, \tag{2.11}$$

where $R > \max\{|t_1|, |t_2|\}\|\varphi_1\|_\infty > 0$ and $\chi \in C^\infty(\mathbb{R}, [0, 1])$ is a function satisfiying $\chi(s) \equiv 1$, if $|s| \le R + 1$, and $\chi(s) \equiv 0$, if $|s| \ge R + 2$. Associated with h_R, we consider the truncaded problem

$$\begin{cases} -\Delta u = \lambda u + \mu h_R(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

As $\|t_i\varphi_1\|_\infty < R$, $i = 1, 2$, it follows from (2.11) and (H_0^+) that h_R satisfies the hypothesis (H_0^+). Moreover, from (H_1) we may assert that h_R satisfies (H_1^*) with $f = f_{R+2}$. Applying Theorem 2.1 we find $\hat{\mu}$ and ν^* such that, for every $\mu \in (0, \hat{\mu})$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (2.12) has a weak solution $u_\mu = t_\mu\varphi_1 + v_\mu$, with $t_\mu \in (t_1, t_2)$ and $v_\mu \in \langle\varphi_1\rangle^\perp$. Since $\sigma > N/2$, we may apply the Sobolev Imbedding Theorem and Lemma 2.5 to conclude that $\|v_\mu\|_\infty \to 0$ as $\mu \to 0^+$. Consequently there exists $\mu^* \in (0, \hat{\mu})$ such that $\|v_\mu\|_\infty < 1$ for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$. Thus $\|u_\mu\|_\infty \le |t_\mu|\|\varphi_1\|_\infty + \|v_\mu\|_\infty < R + 1$. Hence $\chi(u_\mu) = 1$ and $h_R(x, u_\mu) =$

$h(x, u_\mu)$, almost everywhere in Ω. It follows that u_μ is actually a solution of Problem (1.1). This concludes the proof of Theorem 1.1. □

2.2 Proofs of Theorems 1.2 and 1.3

As in Sect. 2.1, we firstly prove a version of Theorem 1.2 with h satisfying (H_1^*) and a hypothesis stronger than (H_2):

(H_2^*) there exists $\zeta \in L^\sigma(\Omega)$, $\sigma > \{N/2, 1\}$, such that $|h(x, s_2) - h(x, s_1)| \leq \zeta(x)|s_2 - s_1|$, for every $s_1, s_2 \in \mathbb{R}$, a. e. in Ω.

Our initial goal is to prove

Theorem 2.6 *Suppose h satisfies (H_0^-), (H_1^*) and (H_2^*). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem* (1.1) *has a solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.*

As in Sect. 2.1, we prove Theorem 2.6 by finding a critical point of the functional I_μ, given by (2.1). In order to establish the existence of such critical point we use the Lyapunov-Schmidt Reduction Method. For the sake of completeness we announce the version of that method that we will be using in our proof of Theorem 1.2 (see Castro 1982).

Theorem 2.7 *Let Y and Z be closed subspaces of a real Hilbert X such that $X = Y \oplus Z$. Let $\Phi : X \to \mathbb{R}$ be a functional of class C^1. If there exists an increasing function $\phi : (0, \infty) \to (0, \infty)$ such that $\phi(s) \to \infty$ as $s \to \infty$ and*

$$\begin{aligned}&\langle \Phi^{'}(y + z_1) - \Phi^{'}(y + z_2), z_1 - z_2\rangle \\ &\quad \geq \|z_1 - z_2\|\phi(\|z_1 - z_2\|), \ \textit{for every } y \in Y, z_1, z_2 \in Z.\end{aligned}$$

Then,

(i) *There exists a continuous function $\psi : Y \to Z$ such that $\Phi(y + \psi(y)) = \min_{z\in Z} \Phi(y + z)$. Moreover, $\psi(y)$ is the only element of Z such that $\langle \Phi^{'}(y + \psi(y)), z\rangle = 0$, for every $z \in Z$.*
(ii) *The function $\hat{\Phi} : Y \to \mathbb{R}$ defined by $\hat{\Phi}(y) = \Phi(y + \psi(y))$ is of class C^1 and*

$$\langle \hat{\Phi}^{'}(y_1), y_2\rangle = \langle \Phi^{'}(y_1 + \psi(y_1)), y_2\rangle, \ \textit{for every } \ y_1, y_2 \in Y.$$

(iii) *$y \in Y$ is a critical point of $\hat{\Phi}$ if and only if $y + \psi(y)$ is a critical point of Φ.*

We are ready to prove Theorem 2.6.

Proof of Theorem 2.6 From (H_1^*), (H_2^*), the Hölder Inequality and the Sobolev Imbedding Theorem, for every $\tau \in \mathbb{R}$ and $v_1, v_2 \in \langle\varphi_1\rangle^\perp$, we have that

$$
\begin{aligned}
&\langle I_{\mu}^{'}(\tau\varphi_1+v_1)-I_{\mu}^{'}(\tau\varphi_1+v_2), v_1-v_2\rangle\\
&\geq \|v_1-v_2\|^2-\frac{\overline{\lambda}}{\lambda_2}\|v_1-v_2\|^2-\mu\int_{\Omega}[\,h(x,\tau\varphi_1+v_1)\\
&\quad -h(x,\tau\varphi_1+v_2)\,](v_1-v_2)dx\\
&\geq \frac{\lambda_2-\overline{\lambda}}{\lambda_2}\|v_1-v_2\|^2-\mu\|\zeta\|_{\sigma}\|v_1-v_2\|_{2\sigma'}^2\\
&\geq \left[\frac{\lambda_2-\overline{\lambda}}{\lambda_2}-\mu\|\zeta\|_{\sigma}d_{2\sigma'}^2\right]\|v_1-v_2\|^2.
\end{aligned}
$$

Therefore, taking $0<\mu_1<(\lambda_2-\overline{\lambda})/2\lambda_2\|\zeta\|_{\sigma}d_{2\sigma'}^2$ we obtain that I satisfies the hypothesis of Theorem 2.7, with $Y=\langle\varphi_1\rangle,\ Z=\langle\varphi_1\rangle^{\perp}$ and $\phi(s)=(\lambda_2-\overline{\lambda})s/2\lambda_2$, for every $s\in(0,\infty)$. Hence, by Theorem 2.7-(i), there exists a continuous function $\psi:\langle\varphi_1\rangle\to\langle\varphi_1\rangle^{\perp}$ such that $\psi(\tau\varphi_1)$, with $\tau\in\mathbb{R}$, is the only element of the space $\langle\varphi_1\rangle^{\perp}$ that satisfies

$$
\langle I_{\mu}^{'}(\tau\varphi_1+\psi(\tau\varphi_1)), v\rangle=0,\ \text{for every } \tau\in\mathbb{R},\ v\in\langle\varphi_1\rangle^{\perp}. \tag{2.13}
$$

Moreover, from Theorem 2.7-(ii), the functional $\hat{I}:\langle\varphi_1\rangle\to\mathbb{R}$, given by $\hat{I}_{\mu}(\tau\varphi_1)=I_{\mu}(\tau\varphi_1+\psi(\tau\varphi_1))$, is of class C^1 and

$$
\langle\hat{I}_{\mu}^{'}(\tau\varphi_1),\varphi_1\rangle=-\mu\left[\frac{\lambda-\lambda_1}{\lambda_1\mu}\|\varphi_1\|^2\tau+\int_{\Omega}h(x,\tau\varphi_1+\psi(\tau\varphi_1))\varphi_1dx\right]. \tag{2.14}
$$

On the other hand, from (H_1^*) and (H_0^-) we may work as in (2.4) to conclude that there exists $\delta>0$ such that, for every $u\in H_0^1(\Omega)$, with $\|u\|<\delta$, we have

$$
\begin{aligned}
\int_{\Omega}h(x,t_1\varphi_1+u)\varphi_1dx&<\frac{1}{2}\int_{\Omega}h(x,t_1\varphi_1)\varphi_1dx\\
&<0<\frac{1}{2}\int_{\Omega}h(x,t_2\varphi_1)\varphi_1dx<\int_{\Omega}h(x,t_2\varphi_1+u)\varphi_1dx.
\end{aligned} \tag{2.15}
$$

Taking $v=\psi(\tau\varphi_1)$ in (2.13), it follows from (H_1^*), the Hölder Inequality and the Sobolev Imbedding Theorem that

$$
\|\psi(\tau\varphi_1)\|^2\leq\frac{\overline{\lambda}}{\lambda_2}\|\psi(\tau\varphi_1)\|^2+\mu d_{\sigma'}\|f\|_{\sigma}\|\psi(\tau\varphi_1)\|.
$$

Hence $\|\psi(\tau\varphi_1)\|\leq\lambda_2/(\lambda_2-\overline{\lambda})\mu d_{\sigma'}\|f\|_{\sigma}$, for every $\mu\in(0,\mu_1)$, $\tau\in\mathbb{R}$. Consequently, taking $\mu^*<\min\{\mu_1,(\lambda_2-\overline{\lambda})\delta/\lambda_2 d_{\sigma'}\|f\|_{\sigma}\}$, where $\delta>0$ is given in (2.15), we obtain that $\|\psi(\tau\varphi_1)\|<\delta$, for every $\tau\in\mathbb{R}$ and $\mu\in(0,\mu^*)$. Consequently, from (2.14) and (2.15),

$$
\langle\hat{I}_{\mu}^{'}(t_1\varphi_1),\varphi_1\rangle>-\mu\left[\frac{|\lambda-\lambda_1|}{\lambda_1\mu}\|\varphi_1\|^2|t_1|+\frac{1}{2}\int_{\Omega}h(x,t_1\varphi_1)\varphi_1dx\right]
$$

and

$$\langle \hat{I}'_\mu(t_2\varphi_1), \varphi_1\rangle < -\mu\Big[-\frac{|\lambda-\lambda_1|}{\lambda_1\mu}\|\varphi_1\|^2|t_2| + \frac{1}{2}\int_\Omega h(x, t_2\varphi_1)\varphi_1 dx\Big].$$

Therefore, there exists $\nu^* > 0$ such that $\langle \hat{I}'_\mu(t_1\varphi_1), \varphi_1\rangle > 0$ and $\langle \hat{I}'_\mu(t_2\varphi_1), \varphi_1\rangle < 0$, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$. By the Intermediate Value Theorem, there exists $t \in (t_1, t_2)$ such that $\langle \hat{I}'_\mu(t\varphi_1), \varphi_1\rangle = 0$. Thus, by Theorem 2.7-(iii), $u_\mu = t\varphi_1 + \psi(t\varphi_1)$ is a critical point of the functional I_μ. This concludes the proof of Theorem 2.6. □

Next we present the proofs of Theorems 1.2 and 1.3:

Proof of Theorem 1.2 We consider h_R defined by (2.11) and the associated truncated problem (2.12). As in the proof of Theorem 1.1, we have that h_R satisfies the hypothesis (H_0^-) and (H_1^*) with $f = f_{R+2}$. Furthermore, from the definition of h, (H_1) and (H_2), we obtain that h_R satisfies (H_2^*) with $\zeta = \|\chi'\|_\infty f_{R+2} + \zeta_{R+2} \in L^\sigma(\Omega)$, $\sigma > \{N/2, 1\}$, where f_{R+2} and ζ_{R+2} are given by (H_1) and (H_2), respectively. These facts allow us to apply Theorem 2.6, finding positive constants $\hat{\mu}$ and ν^* such that, for every $\mu \in (0, \hat{\mu})$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (2.12) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$. Next, using Lemma 2.5 and arguing as in the proof of Theorem 1.1, we obtain $\mu^* \in (0, \hat{\mu})$ such that u_μ is a solution of Problem (1.1) for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3 Taking $-h$, whenever $\mu < 0$, we may apply Theorems 1.1 or 1.2, for every $i \in \{1, \dots, k\}$, to find positive constants μ_i and ν_i such that, for every $0 < |\mu| < \mu_i$ and $|\lambda - \lambda_1| < |\mu|\nu_i$, Problem (1.1) has a weak solution $u_i = \hat{t}_i\varphi_1 + v_i$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$. The proof of Theorem 1.3 is completed by taking $0 < \mu^* < \min\{\mu_i; i = 1, \dots, k\}$ and $0 < \nu^* < \min\{\nu_i; i = 1, \dots, k\}$. □

Remark 2.8 We observe that, by Lemma 2.5 and the proofs of Theorems 1.1 and 1.2, we may assert that there exists $b_\sigma > 0$ such that $\|v\|_{2,\sigma} \le b_\sigma\mu$ for the solutions $u_\mu = t\varphi_1 + v$, given by these theorems. In particular, since $\sigma > N/2$, we obtain that $v_\mu \to 0$ in $C(\overline{\Omega})$ as $\mu \to 0^+$. We also note that a similar remark may be made for the k solutions obtained in Theorem 1.3.

2.3 Proof of Theorem 1.5

Here we present a proof of Theorem 1.5. We also establish a version of this theorem for solutions uniformly bounded in $H_0^1(\Omega)$ under the hypothesis that h has subcritical growth.

Proof of Theorem 1.5 Arguing by contradiction, we suppose there exist $(\mu_k) \subset \mathbb{R}\backslash\{0\}$, $(\hat{\lambda}_k) \subset \mathbb{R}$ and $(u_k) \subset H_0^1(\Omega)$ such that $|\mu_k| \to 0$, $|\hat{\lambda}_k - \lambda_1| < |\mu_k|/k$ and $u_k = \tau_k\varphi_1 + v_k$, with $\tau_k \in [t_1, t_2]$ and $v_k \in \langle\varphi_1\rangle^\perp$, a weak solution of Problem (1.1), with parameters $\mu = \mu_k$ and $\lambda = \hat{\lambda}_k$. Then, using (H_3), the Hölder inequality and the

Sobolev Imbedding Theorem, we find $K > 0$ such that

$$\|v_k\|^2 = \hat{\lambda}_k \|v_k\|_2^2 + \mu_k \int_\Omega h(x, u_k) v_k dx$$
$$\leq \left(\lambda_1 + \frac{|\mu_k|}{K}\right) \frac{\|v_k\|^2}{\lambda_2} + |\mu_k| K (\|f\|_\sigma d_{\sigma'} \|v_k\| + \|f\|_\sigma d_{2\sigma'}^2 \|v_k\|^2).$$

Consequently,

$$\left[1 - \left(\frac{|\mu_k|}{K\lambda_2} + \frac{\lambda_1}{\lambda_2}\right) - |\mu_k| K \|f\|_\sigma d_{2\sigma'}^2\right] \|v_k\| \leq |\mu_k| K \|f\|_\sigma d_{\sigma'}.$$

Since $\mu_k \to 0$, we conclude that $\|v_k\| \to 0$. Hence, from the compactness of the Sobolev Imbedding $H_0^1(\Omega) \hookrightarrow L^{2\sigma'}(\Omega)$, we may suppose that $u_k \to \tau_0 \varphi_1$, a. e. in Ω, and that there exists $g \in L^{2\sigma'}(\Omega)$ such that, for every $k \in \mathbb{N}$, $|u_k| \leq g$, a. e. in Ω.

Next, invoking (H_3), (H_4) and the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_\Omega h(x, u_k) \varphi_1 dx \to \int_\Omega h(x, \tau_0 \varphi_1) \varphi_1 dx \neq 0.$$

Consequently, from $|\hat{\lambda}_k - \lambda_1||\tau_k|/|\mu_k| \to 0$, we get

$$\langle I'_{\mu_k}(u_k), \varphi_1 \rangle = -\mu_k \left[\frac{\hat{\lambda}_k - \lambda_1}{\mu_k \lambda_1} \|\varphi_1\|^2 \tau_k + \int_\Omega h(x, u_k) \varphi_1 dx\right] \neq 0,$$

for every $k \in \mathbb{N}$ sufficiently large. However this contradicts the fact that u_k is a solution of Problem (1.1). The proof of Theorem 1.5 is complete. □

Supposing

$(\hat{H}_3)$ there exist $a > 0$ and $1 < p < \infty$ ($p < (N+2)(N-2)$ if $N \geq 3$), such that $|h(x, s)| \leq a(1 + |s|^p)$, for every $s \in \mathbb{R}$, a. e. in Ω,

we may state a version of Theorem 1.5 for solutions of Problem (1.1) which are uniformly bounded in $H_0^1(\Omega)$:

Theorem 2.9 *Suppose h satisfies $(\hat{H}_3)$ and (H_4). Then, given $M > 0$, there exist positive constants μ^* and ν^* such that for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has no weak solution $u_\mu = t\varphi_1 + v$, with $t \in [t_1, t_2]$ and $v \in \langle \varphi_1 \rangle^\perp$, with $\|u_\mu\| \leq M$.*

Proof Arguing by contradiction, we suppose that there exist $M > 0$, $(\mu_k) \subset \mathbb{R}\backslash\{0\}$, $(\hat{\lambda}_k) \subset \mathbb{R}$ and $(u_k) \subset H_0^1(\Omega)$ such that $|\mu_k| \to 0$, $|\hat{\lambda}_k - \lambda_1| < |\mu_k|/k$ and $u_k = \tau_k \varphi_1 + v_k$, with $\tau_k \in [t_1, t_2]$, $v_k \in \langle \varphi_1 \rangle^\perp$ and $\|u_k\| \leq M$, a weak solution of Problem (1.1), with parameters $\mu = \mu_k$ and $\lambda = \hat{\lambda}_k$.

Invoking $(\hat{H}_3)$, we may write

$$|h(x, u_k(x)| \leq a(1 + |u_k(x)|^{p-1})(1 + |u_k(x)|), \text{ a.e. in } \Omega.$$

Next, using the above inequality, $\|u_k\| \leq M$ and the Sobolev Imbedding Theorem, we find $f \in L^{\sigma}(\Omega)$, $\sigma > \{1, N/2\}$, such that

$$|h(x, u_k(x)| \leq f(x)(1 + |u_k(x)|), \text{ a.e. in } \Omega.$$

Now, following the argument used in the proof of Theorem 1.5, we derive a contradiction with the fact that u_k is a solution of Problem (1.1). The theorem is proved. □

2.4 Positivity, Negativity and Ordering of the Solutions

Here we verify the assertions made in Remark 1.4. We denote by $(\hat{H}_1)$ the hypothesis (H_1) with $\sigma > N$.

Theorem 2.10 *Suppose h satisfies (H_0^+) and $(\hat{H}_1)$ or (H_0^-), $(\hat{H}_1)$ and (H_2). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a positive or negative solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$, provided $t_1 \geq 0$ or $t_2 \leq 0$, respectively.*

Proof Since the proves of the other cases are similar we just present a proof of Theorem 2.10 for h satisfying (H_0^+) and $(\hat{H}_1)$ with $t_1 \geq 0$. Using the continuity of the function $\Phi(t) = \int_\Omega h(x, t\varphi_1)\varphi_1 dx,\ t \in \mathbb{R}$, we may assume without loss of generality that $t_1 > 0$. Then, applying Theorem 1.1, we find $\mu_1 > 0$ and $\nu^* > 0$ such that, for every $0 < \mu < \mu_1$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a solution $u_\mu = t_\mu\varphi_1 + v_\mu$, with $t_\mu \in (t_1, t_2)$ and $v_\mu \in \langle\varphi_1\rangle^\perp$. Furthermore, from $\sigma > N$, Lemma 2.5 and the Sobolev Imbedding Theorem, we obtain that $v_\mu \to 0$ in $C^1(\overline{\Omega})$ as $\mu \to 0^+$.

We claim that

$$\lim_{\mu \to 0^+} \frac{|v_\mu(x)|}{d(x, \partial\Omega)} = 0, \text{ uniformly for } x \in \Omega.$$

Supposing the claim, we use that $t \geq t_1 > 0$ and the fact that there exists $K > 0$ such that $\varphi_1(x) \geq Kd(x, \partial\Omega)$, for every $x \in \Omega$, to get $\mu^* \in (0, \mu_1)$ such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, $u_\mu(x) \geq \frac{t_1 K}{2} d(x, \partial\Omega) > 0$, for every $x \in \Omega$.

In order to conclude the proof of the theorem it remains to prove the claim. Arguing by contradiction, we suppose there exists $\epsilon > 0$, $(x_n) \subset \Omega$ and $(\mu_n) \subset \mathbb{R}^+ \backslash \{0\}$ such that $|v_{\mu_n}(x_n)| \geq \epsilon d(x_n, \partial\Omega)$, for every $n \in \mathbb{N}$ and $\mu_n \to 0$ as $n \to \infty$. Taking $(y_n) \subset \partial\Omega$ such that $d(x_n, \partial\Omega) = |x_n - y_n|$, from $v_{\mu_n}(y_n) = 0$ and the Mean Value Theorem, for every $n \in \mathbb{N}$, we find $\theta_n \in (0, 1)$ such that

$$\epsilon \leq \frac{|v_{\mu_n}(x_n)|}{d(x_n, \partial\Omega)} = \frac{|v_{\mu_n}(x_n) - v_{\mu_n}(y_n)|}{|y_n - x_n|}$$
$$= |\langle \nabla v_{\mu_n}(x_n + \theta_n(y_n - x_n)), \frac{y_n - x_n}{|y_n - x_n|}\rangle| \leq \|\nabla v_{\mu_n}\|_\infty.$$

However this contradicts $v_{\mu_n} \to 0$ in $C^1(\overline{\Omega})$ as $n \to \infty$. The proof of Theorem 2.10 is complete. □

Arguing as in the proof of Theorem 2.10, we may establish the ordering of the solutions given by Theorem 1.3.

Theorem 2.11 *Suppose h satisfies* (H_0), $(\hat{H}_1)$ *and* (H_2) *with* $k \geq 2$. *Then there exist positive constants* μ^* *and* ν^* *such that, for every* $0 < |\mu| < \mu^*$ *and* $|\lambda - \lambda_1| < |\mu|\nu^*$, *Problem* (1.1) *has k solutions* $u_i = \hat{t}_i\varphi_1 + v_i$, *with* $\hat{t}_i \in (t_i, t_{i+1})$ *and* $v_i \in \langle\varphi_1\rangle^\perp$, *such that* $u_i < u_{i+1}$, $i = 1, \ldots, k-1$.

Proof Arguing as in the proof of Theorem 2.10, for every $i \in \{1, \ldots, k\}$ we find $\tilde{t}_i \in (t_i, t_{i+1})$ such that

$$\left[\int_\Omega h(x, t_i\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, \tilde{t}_i\varphi_1)\varphi_1 dx\right] < 0, \ i = 1, \ldots, k.$$

Hence, applying Theorems 1.1 and 1.2, we find $\mu_1 > 0$ and $\nu^* > 0$ such that, for every $0 < |\mu| < \mu_1$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$, with $\hat{t}_i \in (t_i, \tilde{t}_i) \subset (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \ldots, k$. Furthermore, $v_i \to 0$ in $C^1(\overline{\Omega})$ as $\mu \to 0$. Noting that $u_{i+1} - u_i = (\hat{t}_{i+1} - \hat{t}_i)\varphi_1 + (v_{i+1} - v_i)$, with $\hat{t}_{i+1} - \hat{t}_i \geq t_{i+1} - \tilde{t}_i \geq d > 0$, for some $d > 0$, we may argue as in the proof of Theorem 2.10 to verify that there exists $\mu^* > 0$ such that the thesis of Theorem 2.11 holds for every $i = 1, \ldots, k-1$. The proof of the theorem is complete. □

3 Applications of the Main Results

The goal of this section is to present the proofs of Propositions 1.6–1.9. We also present an application of Theorem 1.3, deriving the existence of multiple solutions for Problem (1.1) when $h(x, s)$ is a polynomial function with respect to the second variable.

Proofs of Propositions 1.6 and 1.7 Since the argument used in the proof of Proposition 1.7 is analogous, we shall only present the proof of Proposition 1.6. Considering the rescaling $u = \beta^{\frac{1}{p-q}} w$, $\beta > 0$, we obtain that u is a positive solution of Problem 1.3 if, and only if, w is a positive solution of the problem

$$\begin{cases} -\Delta w = \lambda w + \mu(b_1 w^q + b_2 w^p) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $\mu = \beta^{\frac{p-1}{p-q}}$. Defining $h : \Omega \times \mathbb{R} \to \mathbb{R}$ by $h(x, s) = 0$, for $s \le 0$, and $h(x, s) = b_1(x)s^q + b_2(x)s^p$, for $s \ge 0$, we have that h is a Carathéodory function. Furthermore, since $\sigma > N$ and $p, q \ge 1$, h satisfies $(\hat{H}_1)$ and (H_2). Considering the function $\Phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(t) = \int_\Omega h(x, t\varphi_1)\varphi_1 dx, \text{ for every } t \ge 0, \tag{3.2}$$

we obtain that $\Phi(t) = r_1 t^q + r_2 t^p$, for every $t \ge 0$. Hence, using that $p > q > 0$ and $r_1 r_2 < 0$, we may find $0 < t_1 < t_2$ such that h satisfies either (H_0^+) or (H_0^-). Next, applying Theorem 2.10, we find $\mu^*, \nu^* > 0$ such that Problem (3.1) possesses a positive solution for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$. The proof of Proposition 1.6 follows directly from this result by taking $\beta^* = (\mu^*)^{\frac{p-q}{p-1}}$. □

Proofs of Propositions 1.8 and 1.9 We present only the proof of Proposition 1.8 since the argument used here may be easily adapted to prove Proposition 1.9. Since $h(x, s) := f(x) + g(s)$ satisfies (H_1), in order to derive the existence of a solution for Problem (1.4) via Theorem 1.1 we just need to verify that h satisfies (H_0^+): since, by (G_1), $g(s) + M \ge 0$, for every $s \le 0$, we may apply Fatou's Lemma and (LL^+) to get

$$\begin{aligned} \liminf_{t\to-\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx &= \int_\Omega f\varphi_1 dx \\ &+ \liminf_{t\to-\infty} \int_\Omega g(t\varphi_1)\varphi_1 dx \ge \int_\Omega (f + g_i^-)\varphi_1 dx > 0. \end{aligned}$$

Analogously, from $M - g(s) \ge 0$, for every $s \ge 0$, Fatou's Lemma and (LL^+), we obtain

$$\begin{aligned} \limsup_{t\to\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx &= \int_\Omega f\varphi_1 dx \\ &+ \limsup_{t\to\infty} \int_\Omega g(t\varphi_1)\varphi_1 dx \le \int_\Omega (f + g_s^+)\varphi_1 dx < 0. \end{aligned}$$

From the above relations, we find real numbers $t_1 < 0 < t_2$ such that the condition (H_0^+) is valid for these values. Proposition 1.8 is proved. □

We conclude this section by presenting an application of Theorem 1.3 when $h(x, s)$ is a polynomial function in the variable s, i.e., when h is given by

$$h(x, s) = \sum_{i=0}^{m} \alpha_i(x)s^i, \text{ for every } s \in \mathbb{R} \text{ a. e. in } \Omega, \tag{3.3}$$

with $\alpha_i \in L^\sigma(\Omega)$, $\sigma > \{N/2, 1\}$. In this case, the associated function Φ, given by (3.2), is also a polynomial function in the variable t. More specifically, we have that

$$\Phi(t) = \sum_{i=0}^{m} d_i t^i, \text{ with } d_i = \int_\Omega \alpha_i(x)\varphi_1^{i+1} dx, \ i = 1, \ldots, m.$$

As a consequence of Theorem 1.3, we establish the existence of multiple solutions for Problem (1.1) in function of the number of roots of odd multiplicity of Φ:

Proposition 3.1 *Suppose $h(x, s)$ is a polynomial function in the variable s. If the function Φ has k roots of odd multiplicity, $\tau_1, \ldots, \tau_k$, then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has k weak solutions $u_1, \ldots, u_k$. Moreover, if $|\lambda - \lambda_1|/\mu \to 0$, as $\mu \to 0$, the solution u_i converges in $C(\overline{\Omega})$ to $\tau_i\varphi_1$ as $\mu \to 0$, $i = 1, \ldots, k$.*

Proof Without loss of generality, we may suppose that $\tau_1 < \tau_2 < \cdots < \tau_k$. From the hypothesis of Proposition 3.1, we may write

$$\Phi(t) = (t - \tau_1)^{2n_1 - 1} \ldots (t - \tau_k)^{2n_k - 1}(t - c_1)^{2z_1} \ldots (t - c_l)^{2z_l} p(t),$$

with $n_1, \ldots, n_k, z_1, \ldots, z_l \in \mathbb{N}$ and $p(t)$ a product of irreducible quadratic polynomials. As a direct consequence of above expression, we may find $t_1, \ldots, t_{k+1}$ such that $t_1 \in (-\infty, \tau_1)$, $t_{k+1} \in (\tau_k, \infty)$ and $t_i \in (\tau_{i-1}, \tau_i)$, $i = 2, \ldots, k$, $c_j \notin (t_i, t_{i+1})$, for every $i = 1, \ldots, k$ and $j = 1, \ldots, l$; and

$$\Phi(t_i)\Phi(t_{i+1}) = \left[\int_\Omega h(x, t_i\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\varphi_1)\varphi_1 dx\right] < 0, \ i = 1, \ldots, k.$$

Hence h satisfies (H_0). Noting that h also satisfies (H_1) and (H_2), we may apply Theorem 1.3 to find positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \ldots, k$.

Next we fix $i \in \{1, \ldots, k\}$. By Remark 2.8, $v_i = v_i(\mu) \to 0$ in $C(\overline{\Omega})$ as $\mu \to 0$. Hence, considering sequences $(\mu_m) \subset \mathbb{R}\backslash\{0\}$ and $(\lambda_m) \subset \mathbb{R}$ such that $\mu_m \to 0$ and $|\lambda_m - \lambda_1|/\mu_m \to 0$ as $m \to \infty$, and taking a subsequence, if necessary, we may suppose that $u_{i,m} = u_i(\mu_m) = \hat{t}_i(\mu_m)\varphi_1 + v_i(\mu_m) \to \hat{t}_0\varphi_1$ in $C(\overline{\Omega})$ as $m \to \infty$. Consequently, from

$$\begin{aligned} 0 &= -\frac{1}{\mu_m}\langle I'(\hat{t}_i(\mu_m)\varphi_1 + v_i(\mu_m)), \varphi_1\rangle \\ &= \frac{\lambda_m - \lambda_1}{\mu_m}\|\varphi_1\|_2^2\hat{t}_i(\mu_m) + \int_\Omega h(x, \hat{t}_i(\mu_m)\varphi_1 + v_i(\mu_m))\varphi_1 dx, \end{aligned}$$

and the Lebesgue Dominated Convergence Theorem, we obtain that $\Phi(\hat{t}_0) = \int_\Omega h(x, \hat{t}_0\varphi_1)\varphi_1 dx = 0$. As, by construction, τ_i is the only root of Φ in the interval $[t_i, t_{i+1}]$, we obtain that $u_i(\mu_m) \to \tau_i\varphi_1$ in $C(\overline{\Omega})$ as $m \to \infty$. The proof of Proposition 3.1 is complete. □

Remark 3.2 Assuming that $\alpha_i \in L^{\sigma}(\Omega)$, with $\sigma > N$, by Theorem 2.11 we may assert that the solutions $u_1, \ldots, u_k$ provided by Proposition 3.1, are ordered and of class $C^{1,\gamma}(\overline{\Omega})$.

Proof of Proposition 1.10 For simplicity of notation we set $k = k_J$. Moreover, reordering the terms in the expression of f we may suppose that $J = \{1, \ldots, k\}$. Since the argument for k even or zero is similar we shall prove Proposition 1.10 for k odd. First we consider that $r > 0$.

Given $a_1, \ldots, a_k, a \in \mathbb{R}$, we define the function $g : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(t) = \frac{t}{\lambda_1} + a_1 t^{p_1} + \cdots + a_k t^{p_k} + a t^p, \text{ for every } t \geq 0.$$

Since $1 < p_i < p$ and $p_i \neq p_j$, for $i \neq j \in \{1, \ldots, k\}$, we may find $a_1, \ldots, a_k \in \mathbb{R}\backslash\{0\}$, $a > 0$ and $0 < t_1 < t_2 < \cdots < t_{k+2}$ such that

$$g(t_1) > 0, \quad g(t_l)g(t_{l+1}) < 0, \text{ for every } 1 \leq l \leq k+1. \tag{3.4}$$

Next, considering $\lambda > \lambda_1$, we take $\mu = \lambda - \lambda_1 > 0$, and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ defined by

$$\begin{cases} \alpha_i = \frac{a_i}{r_i} \left(\frac{r}{a}\right)^{\frac{p_i-1}{p-1}} \mu^{\frac{p-p_i}{p-1}}, & \text{for every } 1 \leq i \leq k; \\ \alpha_i = \left(\frac{r}{a}\right)^{\frac{p_i-1}{p-1}} \mu^{\frac{p-p_i}{p-1}}, & \text{for every } k+1 \leq i \leq m. \end{cases} \tag{3.5}$$

We may assert that (λ, α, u), u positive in Ω, is a solution of Problem (1.5), with $\lambda > \lambda_1$ if, and only if, $w = u/\beta(\lambda)$, $\beta(\lambda) = (a\mu/r)^{\frac{1}{p-1}}$, is a positive solution of the problem

$$\begin{cases} -\Delta w = \lambda_1 w + \mu h(x, w), & \text{in } \Omega; \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where, for every $(x, s) \in \Omega \times \mathbb{R}$,

$$h(x, s) = s + \sum_{i=1}^{k} \frac{a_i}{r_i} b_i(x) s^{p_i} + \sum_{i=k+1}^{m} b_i(x) s^{p_i} + \frac{a}{r} b(x) s^p.$$

Considering the function Φ defined by (3.2) and the above expression of h, we obtain that $\Phi(t) = g(t)$, for every $t \geq 0$. Hence, by (3.4), Theorem 1.3 and Remark 1.4, we find $\mu^* > 0$ such that, for every $0 < \mu < \mu^*$, Problem (3.6) possesses $k+1$ positive solutions, $w_1, \ldots, w_{k+1} \in H_0^1(\Omega)$. Moreover, there exists $M > 0$ such that $\|w_i\| \leq M$, for every $1 \leq i \leq k+1$. Consequently, taking $\overline{\lambda} = \lambda_1 + \mu^*$, we obtain that for every $\lambda \in (\lambda_1, \overline{\lambda})$ and $\alpha(\lambda)$, given by (3.5), Problem (1.5) has $k+1$ solutions $\{(\lambda, \alpha(\lambda), u_i); u_i = \beta(\lambda) w_i,\ w_i \text{ positive in } \Omega,\ 1 \leq i \leq k+1\}$. Since $\|w_i\| \leq M$, $1 \leq i \leq k+1$, and $\alpha(\lambda) \to 0$ and $\beta(\lambda) \to 0$ as $\lambda \to \lambda_1^+$, we have proved that

$(\lambda_1, 0, 0)$ is a point of bifurcation of multiplicity $k + 1$ for the positive solutions of Problem (1.5).

If $r < 0$, we may argue as above to find $\underline{\lambda} < \lambda_1$, $\alpha \in \mathbb{R}^m$, and $u_i \in H_0^1(\Omega)$, u_i positive in Ω, $1 \leq i \leq k + 1$, such that, for every $\lambda \in (\underline{\lambda}, \lambda_1)$, $(\lambda, \alpha(\lambda), u_i)$ is a solution of (1.5), $1 \leq i \leq k+1$. Furthermore $\alpha(\lambda) \to 0$ in $\mathbb{R}^m$, $u_i(\lambda) \to 0$ in $H_0^1(\Omega)$, $1 \leq i \leq k + 1$, as $\lambda \to \lambda_1^-$. We conclude that $(\lambda_1, 0, 0)$ is a point of bifurcation of multiplicity $k+1$ for the positive solutions of Problem (1.5). The proof of Proposition 1.10 is complete. □

Fixed $a_1, \ldots, a_m \in \mathbb{R}$, we may consider $\hat{f}(\lambda, x, s) = f(\alpha(\lambda), x, s)$, with f given by (1.6), and $\alpha(\lambda) = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}$ defined by

$$\alpha_i = a_i |\lambda - \lambda_1|^{\frac{p-p_i}{p-1}}, \; 1 \leq i \leq m. \tag{3.7}$$

Next we consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u + \hat{f}(\lambda, x, u) + b(x)u^p & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Associated with this nonlinear eigenvalue problem we have the functions

$$\Phi_{\pm}(t) = \pm \frac{t}{\lambda_1} + \sum_{i=1}^{m} a_i r_i t^{p_i} + r t^p, \; \text{for every } t \geq 0.$$

Denoting by k_+ and k_- the numbers of times Φ_+ and Φ_- change sign on $(0, \infty)$, respectively, as a consequence of our argument (see the proof of Proposition 1.10) we may state:

Proposition 3.3 *Suppose $\hat{f}(\lambda, x, s) = f(\alpha(\lambda), x, s)$ with f and $\alpha(\lambda)$ given by (1.6) and (3.7), respectively. Then $(\lambda_1, 0)$ is a bifurcation point of multiplicity k_+ and k_- for the positive solutions of the Problem (3.8) on the interval (λ_1, ∞) and $(0, \lambda_1)$, respectively.*

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