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Approaches to Multi-Attribute Group Decision Making Based on Induced Interval-Valued Pythagorean Fuzzy Einstein Hybrid Aggregation Operators

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Abstract For the multi-attribute group decision-making problems where attribute values are the interval-valued Pythagorean fuzzy numbers, the group decision-making method based on induced Einstein averaging aggregation operators are developed. Firstly, induced interval-valued Pythagorean fuzzy Einstein ordered weighted averaging (I-IVPFEOWA) aggregation operator and induced interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging (I-IVPFEHWA) aggregation operator, were proposed. Some general properties of these operators, such as idempotency, commutativity, monotonicity and boundedness, were discussed, and some special cases in these operators were analyzed. Furthermore, the method for multi-attribute group decision-making problems based on these operators was developed, and the operational progressions were explained in detail. These methods provide more general, more accurate and precise results as compared to the existing methods. Therefore these methods play a vital role in daily life problems. At the end of the paper the proposed operators have been applied to decision making problems to show the weight, practicality and effectiveness of the new approach.

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1 Introduction

Multiple attribute group decision making problems are the important parts of modern decision theory. In real decision making, because the decision making problems are fuzzy and uncertain, the attribute values are not always expressed as real numbers, and some of them are more suitable to be denoted by fuzzy numbers. In order to handle the uncertainties, intuitionistic fuzzy set (Atanassov 1986) theory is one of the successful extensions of the fuzzy set theory (Zadeh 1965), which is characterized by the degree of membership and degree of non-membership has been presented. Later on, Atanassov and Gargov (1989) extended it to the interval-valued intuitionistic fuzzy sets, which is characterized by a membership degree and a non-membership degree, whose values are intervals rather than real numbers. Over the last four decades, the IFS and IVIFS have received more and more attention by introducing the various kinds of aggregation operators, information measures and employed them to solve the decision-making problems under the different environment (Garg 2016a; Rahman et al. 2018a; Su et al. 2011; Xu 2000; Wei and Wang 2007; Xu and Jain 2007; Wang et al. 2009; Xu 2010).

However, in day-to-day life, there are many situations where this condition is ruled out. For instance, if a person gives their preference in the form of membership and non-membership degrees towards a particular object is 0.8 and 0.6, and then clearly this situation is not handling with IFS. In order to resolve it, Yager et al. (2013), (Yager 2014) proposed the Pythagorean fuzzy set by relaxing this sum condition to its square sum less than one. For instance, corresponding to the above-considered example, we see that $(0.8)^2 + (0.6)^2 = 1$ and hence Pythagorean fuzzy set (PFS) is an extension of the existing intuitionistic fuzzy set (IFS). After their pioneer work, Yager and Abbasov (2013), studied the relationship between the Pythagorean numbers and the complex numbers. Rahman et al. (2016, 2017a, b, c, d, e) introduced some aggregation operators and applied them to group decision making. Zeng and Xu (2014) introduced the notion of TOPSIS method using Pythagorean fuzzy numbers. Peng and Yang (2015a) developed some important results for Pythagorean fuzzy sets. Garg (2016b, 2017a) used the Einstein sum and Einstein product and introduced the notion of Pythagorean fuzzy Einstein arithmetic aggregation operators and Pythagorean fuzzy Einstein geometric aggregation operators such as, Pythagorean fuzzy Einstein weighted averaging (PFEWA) operator, Pythagorean fuzzy Einstein ordered weighted averaging (PFEOWA) operator, generalized Pythagorean fuzzy Einstein weighted averaging (GPFEWA) operator, generalized Pythagorean fuzzy Einstein ordered weighted averaging (GPFEOWA) operator, Pythagorean fuzzy Einstein weighted geometric (PFEWG) operator, Pythagorean fuzzy Einstein ordered weighted geometric (PFEOWG) operator, generalized Pythagorean fuzzy Einstein weighted geometric (GPFEWG) operator, generalized Pythagorean fuzzy Einstein ordered weighted geometric (GPFEOWG) operator and also applied them to group decision making. But, in some real decision-making problems, due to insufficiency in available information, it may be difficult for decision makers to exactly quantify their opinions with a crisp number, but they can be represented by an interval number within [0,1]. Therefore it is so important to present the idea of interval-valued Pythagorean fuzzy sets (IVPFSs), which permit the membership degrees and non- membership degrees to a given set to have an interval value. Zhang (2016) introduced the concept of interval-valued Pythagorean fuzzy set. Peng and Yang (2015) introduced the notion of, interval-valued Pythagorean fuzzy weighted averaging (IVPFWA) operator, interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operator and also introduced some of their fundamental and important properties. Garg (2016c) presented an interval-valued Pythagorean fuzzy weighted average (IVPFWA) and interval-valued Pythagorean fuzzy weighted geometric (IVPFWG) operators for solving the decision-making problem under IVPFS environment. Also, a novel accuracy function has been defined in it for ranking the different interval-valued Pythagorean fuzzy numbers (IVPFNs). Now, in order to compare the interval numbers, some score, as well as accuracy function, have been taken for measurement and then applied to solve MCDM problems. Garg (2016d) defined the concepts of correlation and correlation coefficients of PFSs. Garg (2017b) also presented an improved accuracy functions under the IVPFS for solving the decision-making problems. Rahman et al. (2018b, c, d 2017f, g, h) introduced the notion of interval-valued Pythagorean fuzzy ordered weighted averaging (IVPFOWA) aggregation operators, interval-valued Pythagorean fuzzy hybrid weighted averaging (IVPFHWA) aggregation operator, interval-valued Pythagorean fuzzy ordered weighted geometric (IVPFOWG) aggregation operators, interval-valued Pythagorean fuzzy hybrid geometric (IVPFHG) aggregation operator, interval-valued Pythagorean fuzzy Einstein weighted averaging (IVPFEWA) aggregation operator, interval-valued Pythagorean fuzzy Einstein ordered weighted averaging (IVPFEOWA) aggregation operator, induced interval-valued Pythagorean fuzzy ordered weighted averaging (I-IVPFOWA) aggregation operator, induced interval-valued Pythagorean fuzzy hybrid averaging (I-IVPFHA) aggregation operator, induced Pythagorean fuzzy ordered weighted averaging (I-PFOWA) aggregation operator, induced Pythagorean fuzzy hybrid averaging (I-PFHA) aggregation operator, and applied them to multiple attribute group decision making. Peng et al. (Peng and Selvachandran 2017; Peng and Dai 2017; Peng et al. 2017; Peng 2018) introduced many aggregation operators and their applications using Pythagorean fuzzy information.

Thus in this paper, we introduce the notion of induced Einstein aggregation operator based on interval-valued Pythagorean fuzzy numbers such as, induced interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging operator. Moreover, we develop some of their basic properties and give an example. Obviously the operator and method proposed in this paper is more general, more accurate and more flexible as compared to the existing methods. At the last of the paper we present an application of the proposed operator.

The remainder of this article is structured as follows. In Sect. 2, we give some basic definitions and results which will be used in our later sections. In Sect. 3, we introduce some Einstein operations of interval-valued Pythagorean fuzzy numbers. In Sect. 4, we introduce the notion of induced interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging (I-IVPFEHWA) aggregation operator. We also discuss various properties of the proposed operator including idempotency, boundedness; monotonicity. In Sect. 5, we apply the proposed operator to multiple attribute group decision making problems, with Pythagorean fuzzy information. In Sect. 6, we develop a numerical example. In Sect. 7, we have conclusion.

2 Preliminaries

Definition 1 (Peng and Yang 2015b) Let K be a fixed set, then interval-valued Pythagorean fuzzy set (IVPFS), I in K can be defined as:

$$I = \{\langle k, \eta_I(k), \upsilon_I(k)\rangle \mid k \in K\}, \tag{1}$$

where

$$\eta_I(k) = \left[\eta_I^a(k), \eta_I^b(k)\right] \subset [0,\ 1], \tag{2}$$

and

$$\upsilon_I(k) = \left[\upsilon_I^a(k),\ \upsilon_I^b(k)\right] \subset [0,\ 1]\ . \tag{3}$$

Also

$$\eta_I^a(k) = \inf(\eta_I(k)), \tag{4}$$
$$\eta_I^b(k) = \sup(\eta_I(k)), \tag{5}$$
$$\upsilon_I^a(k) = \inf(\upsilon_I(k)), \tag{6}$$
$$\upsilon_I^b(k) = \sup(\upsilon_I(k)), \tag{7}$$

and

$$0 \le \left(\eta_I^b(k)\right)^2 + \left(\upsilon_I^b(k)\right)^2 \le 1, \tag{8}$$

If

$$\pi_I(k) = \left[\pi_I^a(k), \pi_I^b(k)\right], \text{ for all } k \in K, \tag{9}$$

then it is called the interval-valued Pythagorean fuzzy index of k to I, where

$$\pi_I^a(k) = \sqrt{1 - \left(\eta_I^b(k)\right)^2 - \left(\upsilon_I^b(k)\right)^2}, \tag{10}$$
$$\pi_I^b(k) = \sqrt{1 - \left(\eta_I^a(k)\right)^2 - \left(\upsilon_I^a(k)\right)^2}. \tag{11}$$

Definition 2 (Peng and Yang 2015b; Garg 2016c) Let $\lambda = ([\eta_\lambda, \upsilon_\lambda], [x_\lambda, y_\lambda])$ be the interval-valued Pythagorean fuzzy number (IVPFN), then the score function can be defined as:

$$S(\lambda) = \frac{1}{2}\left[(\eta_\lambda)^2 + (\upsilon_\lambda)^2 - (x_\lambda)^2 - (y_\lambda)^2\right]. \tag{12}$$

Definition 3 (Peng and Yang 2015b; Garg 2016c) Let $\lambda = ([\eta_\lambda, \upsilon_\lambda], [x_\lambda, y_\lambda])$ be the interval-valued Pythagorean fuzzy number (IVPFN), then the accurecy degree can be defined as:

$$H(\lambda) = \frac{1}{2}\left[(\eta_\lambda)^2 + (\upsilon_\lambda)^2 + (x_\lambda)^2 + (y_\lambda)^2\right]. \tag{13}$$

If $\lambda_j = \left(\left[\eta_{\lambda_j}, \upsilon_{\lambda_j}\right], \left[x_{\lambda_j}, y_{\lambda_j}\right]\right)$ $(j = 1, 2)$ be a collection of interval-valued Pythagorean fuzzy values (IVPFVs), then the following hold:

1. If $S(\lambda_1) \prec S(\lambda_2)$, then $\lambda_1 \prec \lambda_2$
2. If $S(\lambda_1) = S(\lambda_2)$, then we have the following three conditions.
   1. If $H(\lambda_1) = H(\lambda_2)$, then $\lambda_1 = \lambda_2$
   2. If $H(\lambda_1) \prec H(\lambda_2)$, then $\lambda_1 \prec \lambda_2$
   3. If $H(\lambda_1) \succ H(\lambda_2)$, then $\lambda_1 \succ \lambda_2$

Definition 4 (Rahman et al. 2018c) The induced interval-valued Pythagorean fuzzy ordered weighted averaging (I-IVPFOWA) aggregation operator can be defined as:

$$\begin{aligned} &\text{I - IVPFOWA}_w\left(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \langle u_3, \lambda_3\rangle, \ldots \langle u_n, \lambda_n\rangle\right) \\ &= \left(\begin{array}{l} \left[\sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\eta_{\lambda_{\sigma(j)}}\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\upsilon_{\lambda_{\sigma(j)}}\right)^2\right)^{w_j}}\right], \\ \left[\prod_{j=1}^{n}\left(x_{\lambda_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{n}\left(y_{\lambda_{\sigma(j)}}\right)^{w_j}\right] \end{array}\right), \end{aligned} \tag{14}$$

where $w = (w_1, w_2, w_3, \ldots w_n)^T$ be the weighted vector with some conditions such as, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Also $\lambda_{\sigma(j)}$ is the λ_j value of the interval-valued Pythagorean fuzzy ordered weighted averaging pair $\langle u_j, \lambda_j\rangle$ having the jth largest u_j and u_j in $\langle u_j, \lambda_j\rangle$ is referred to as the order inducing variable and λ_j as the Pythagorean fuzzy argument variable.

Definition 5 (Rahman et al. 2018c) The induced interval-valued Pythagorean fuzzy hybrid weighted averaging (I-IVPFHWA) aggregation operator can be defined as:

$$\begin{aligned} &\text{I - IVPFHWA}_{\omega, w}\left(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \langle u_3, \lambda_3\rangle, \ldots, \langle u_n, \lambda_n\rangle\right) \\ &= \left(\begin{array}{l} \left[\sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\eta_{\dot{\lambda}_{\sigma(j)}}\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\upsilon_{\dot{\lambda}_{\sigma(j)}}\right)^2\right)^{w_j}}\right], \\ \left[\prod_{j=1}^{n}\left(x_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}, \prod_{j=1}^{n}\left(y_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}\right] \end{array}\right) \end{aligned} \tag{15}$$

where $\dot{\lambda}_{\sigma(j)}$ is the jth largest of the weighted interval-valued Pythagorean fuzzy values $\dot{\lambda}_{\sigma(j)}$ $\left(\dot{\lambda}_{\sigma(j)} = n\omega_j \lambda_{\sigma(j)}\right)$, $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ be the weighted vector of I-IVPFHWA aggregation operator such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. $w = (w_1, w_2, w_3, \ldots, w_n)^T$ be the weighted vector of λ_j $(j = 1, 2, 3, \ldots, n)$ with some conditions, such that $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, where n is the balancing coefficient, which plays a role of balance. If the vector $\omega = (\omega_1, \omega_2, \omega_3, \ldots \omega_n)^T$ approaches to $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then $(n\omega_1\lambda_1, n\omega_2\lambda_2, n\omega_3\lambda_3, \ldots, n\omega_n\lambda_n)^T$ approaches $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)^T$.

3 Some Einstein Operations of Interval-Valued Pythagorean Fuzzy Sets

Definition 6 Let $P_j = \left\{\left\langle k, \eta_{P_j}(k), v_{P_j}(k)\right\rangle | k \in K\right\}$ $(j = 1, 2)$ are three IVPFSs, then some Einstein operations can be define as follows.

1. $P_1 \cup P_2 = \left\{\left\langle k, \left[S\left(\eta^a_{P_1}, \eta^a_{P_2}\right), S\left(\eta^b_{P_1}, \eta^b_{P_2}\right)\right], \left[T\left(v^a_{P_1}, v^a_{P_2}\right), T\left(v^b_{P_1}, v^b_{P_2}\right)\right]\right\rangle | k \in K\right\}.$
2. $P_1 \cap P_2 = \left\{\left\langle k, \left[T\left(\eta^a_{P_1}, \eta^a_{P_2}\right), T\left(\eta^b_{P_1}, \eta^b_{P_2}\right)\right], \left[S\left(v^a_{P_1}, v^a_{P_2}\right), S\left(v^b_{P_1}, v^b_{P_2}\right)\right]\right\rangle | k \in K\right\}.$
3. $$P_1^c = \left\{\left\langle k, v_{P_1}(k), \eta_{P_1}(k)\right\rangle | k \in K\right\}.$$
4. $$P_1 \subset P_2 \Leftrightarrow \eta^a_{P_1} \leq \eta^a_{P_2}, \eta^b_{P_1} \leq \eta^b_{P_2}, v^a_{P_1} \geq v^a_{P_2}, v^b_{P_1} \geq v^b_{P_2}.$$
5. $$P_1 \supset P_2 \Leftrightarrow \eta^a_{P_1} \geq \eta^a_{P_2}, \eta^b_{P_1} \geq \eta^b_{P_2}, v^a_{P_1} \leq v^a_{P_2}, v^b_{P_1} \leq v^b_{P_2}.$$

where T and S denotes t-norm and t-conorm respectively.

Definition 7 Let $P_j = \left\{\left\langle k, \eta_{P_j}(k), v_{P_j}(k)\right\rangle | k \in K\right\}$ $(j = 1, 2)$ are three IVPFSs, then

$$P_1 \oplus_\varepsilon P_2 = \left\{\left(\begin{array}{l} k, \left[\frac{\sqrt{\left(\eta^a_{P_1}\right)^2+\left(\eta^a_{P_2}\right)^2}}{\sqrt{1+\left(\eta^a_{P_1}\right)^2\left(\eta^a_{P_2}\right)^2}}, \frac{\sqrt{\left(\eta^b_{P_1}\right)^2+\left(\eta^b_{P_2}\right)^2}}{\sqrt{1+\left(\eta^b_{P_1}\right)^2\left(\eta^b_{P_2}\right)^2}}\right], \\ \left[\frac{v^a_{P_1} v^a_{P_2}}{\sqrt{1+\left(1-\left(v^a_{P_1}\right)^2\right)\left(1-\left(v^a_{P_2}\right)^2\right)}}, \frac{v^b_{P_1} v^b_{P_2}}{\sqrt{1+\left(1-\left(v^b_{P_1}\right)^2\right)\left(1-\left(v^b_{P_2}\right)^2\right)}}\right] \end{array}\right) | k \in K\right\}, \quad (16)$$

$$P_1 \otimes_\varepsilon P_2 = \left\{\left(\begin{array}{l} k, \left[\frac{\eta^a_{P_1} \eta^a_{P_2}}{\sqrt{1+\left(1-\left(\eta^a_{P_1}\right)^2\right)\left(1-\left(\eta^a_{P_2}\right)^2\right)}}, \frac{\eta^b_{P_1} \eta^b_{P_2}}{\sqrt{1+\left(1-\left(\eta^b_{P_1}\right)^2\right)\left(1-\left(\eta^b_{P_2}\right)^2\right)}}\right], \\ \left[\frac{\sqrt{\left(v^a_{P_1}\right)^2+\left(v^a_{P_2}\right)^2}}{\sqrt{1+\left(v^a_{P_1}\right)^2\left(v^a_{P_2}\right)^2}}, \frac{\sqrt{\left(v^b_{P_1}\right)^2+\left(v^b_{P_2}\right)^2}}{\sqrt{1+\left(v^b_{P_1}\right)^2\left(v^b_{P_2}\right)^2}}\right] \end{array}\right) | k \in K\right\}, \quad (17)$$

$$(P)^{\delta} = \left\{ \left(\begin{array}{c} k, \left[\frac{\sqrt{2\left((\eta_P^a)^2\right)^{\delta}}}{\sqrt{\left(2-(\eta_P^a)^2\right)^{\delta}+\left((\eta_P^a)^2\right)^{\delta}}}, \frac{\sqrt{2\left((\eta_P^b)^2\right)^{\delta}}}{\sqrt{\left(2-(\eta_P^b)^2\right)^{\delta}+\left((\eta_P^b)^2\right)^{\delta}}} \right], \\ \left[\frac{\sqrt{\left(1+(v_P^a)^2\right)^{\delta}-\left(1-(v_P^a)^2\right)^{\delta}}}{\sqrt{\left(1+(v_P^a)^2\right)^{\delta}+\left(1-(v_P^a)^2\right)^{\delta}}}, \frac{\sqrt{\left(1+(v_P^b)^2\right)^{\delta}-\left(1-(v_P^b)^2\right)^{\delta}}}{\sqrt{\left(1+(v_P^b)^2\right)^{\delta}+\left(1-(v_P^b)^2\right)^{\delta}}} \right] \end{array} \right) | k \in K \right\}, \quad (18)$$

$$\delta\,(P) = \left\{ \left(\begin{array}{c} k, \left[\frac{\sqrt{\left(1+(\eta_P^a)^2\right)^{\delta}-\left(1-(\eta_P^a)^2\right)^{\delta}}}{\sqrt{\left(1+(\eta_P^a)^2\right)^{\delta}+\left(1-(\eta_P^a)^2\right)^{\delta}}}, \frac{\sqrt{\left(1+(\eta_P^b)^2\right)^{\delta}-\left(1-(\eta_P^b)^2\right)^{\delta}}}{\sqrt{\left(1+(\eta_P^b)^2\right)^{\delta}+\left(1-(\eta_P^b)^2\right)^{\delta}}} \right], \\ \left[\frac{\sqrt{2\left((v_P^a)^2\right)^{\delta}}}{\sqrt{\left(2-(v_P^a)^2\right)^{\delta}+\left((v_P^a)^2\right)^{\delta}}}, \frac{\sqrt{2\left((v_P^b)^2\right)^{\delta}}}{\sqrt{\left(2-(v_P^b)^2\right)^{\delta}+\left((v_P^b)^2\right)^{\delta}}} \right] \end{array} \right) | k \in K \right\}. \quad (19)$$

Definition 8 (Rahman et al. 2017g) Let $\lambda = ([\eta, v], [x, y]), \lambda_1 = ([\eta_1, v_1], [x_1, y_1]), \lambda_2 = ([\eta_2, v_2], [x_2, y_2])$ are three IVPFNs and $\delta \succ 0$, then some Einstein operations for $\lambda, \lambda_1, \lambda_2$ can be define as follows:

$$\lambda_1 \oplus_{\varepsilon} \lambda_2 = \left(\left[\frac{\sqrt{\eta_1^2+\eta_2^2}}{\sqrt{1+\eta_1^2\eta_2^2}}, \frac{\sqrt{v_1^2+v_2^2}}{\sqrt{1+v_1^2v_2^2}} \right], \left[\frac{x_1x_2}{\sqrt{1+(1-x_1^2)(1-x_2^2)}}, \frac{y_1y_2}{\sqrt{1+(1-y_1^2)(1-y_2^2)}} \right] \right), \quad (20)$$

$$\lambda_1 \otimes_{\varepsilon} \lambda_2 = \left(\left[\frac{\eta_1\eta_2}{\sqrt{1+(1-\eta_1^2)(1-\eta_2^2)}}, \frac{v_1v_2}{\sqrt{1+(1-v_1^2)(1-v_2^2)}} \right], \left[\frac{\sqrt{x_1^2+x_2^2}}{\sqrt{1+x_1^2x_2^2}}, \frac{\sqrt{y_1^2+y_2^2}}{\sqrt{1+y_1^2y_2^2}} \right] \right), \quad (21)$$

$$\delta\lambda = \left(\left[\frac{\sqrt{(1+\eta^2)^{\delta}-(1-\eta^2)^{\delta}}}{\sqrt{(1+\eta^2)^{\delta}+(1-\eta^2)^{\delta}}}, \frac{\sqrt{(1+v^2)^{\delta}-(1-v^2)^{\delta}}}{\sqrt{(1+v^2)^{\delta}+(1-v^2)^{\delta}}} \right], \left[\frac{\sqrt{2(x^2)^{\delta}}}{\sqrt{(2-x^2)^{\delta}+(x^2)^{\delta}}}, \frac{\sqrt{2(y^2)^{\delta}}}{\sqrt{(2-y^2)^{\delta}+(y^2)^{\delta}}} \right] \right), \quad (22)$$

$$\lambda^{\delta} = \left(\left[\frac{\sqrt{2(\eta^2)^{\delta}}}{\sqrt{(2-\eta^2)^{\delta}+(\eta^2)^{\delta}}}, \frac{\sqrt{2(v^2)^{\delta}}}{\sqrt{(2-v^2)^{\delta}+(v^2)^{\delta}}} \right], \left[\frac{\sqrt{(1+x^2)^{\delta}-(1-x^2)^{\delta}}}{\sqrt{(1+x^2)^{\delta}+(1-x^2)^{\delta}}}, \frac{\sqrt{(1+y^2)^{\delta}-(1-y^2)^{\delta}}}{\sqrt{(1+y^2)^{\delta}+(1-y^2)^{\delta}}} \right] \right). \quad (23)$$

Remark 1 In the following, let us look $\delta\lambda$ and λ^{δ} some special cases of δ and λ.

1. If $\lambda = ([\eta, v], [x, y]) = ([1, 1], [0, 0])$ i. e,. $\eta = v = 1$ and $x = y = 0$, then

$$\lambda^{\delta} = \left(\left[\frac{\sqrt{2(\eta^2)^{\delta}}}{\sqrt{(2-\eta^2)^{\delta}+(\eta^2)^{\delta}}}, \frac{\sqrt{2(v^2)^{\delta}}}{\sqrt{(2-v^2)^{\delta}+(v^2)^{\delta}}} \right], \left[\frac{\sqrt{(1+x^2)^{\delta}-(1-x^2)^{\delta}}}{\sqrt{(1+x^2)^{\delta}+(1-x^2)^{\delta}}}, \frac{\sqrt{(1+y^2)^{\delta}-(1-y^2)^{\delta}}}{\sqrt{(1+y^2)^{\delta}+(1-y^2)^{\delta}}} \right] \right)$$
$$= ([1, 1], [0, 0]).$$

Thus $\lambda^{\delta} = ([1, 1], [0, 0])$ and $\delta\lambda = ([0, 0], [1, 1])$.

2. If $\lambda = ([\eta, v], [x, y]) = ([0, 0], [1, 1])$ i. e,. $\eta = v = 0$ and $x = y = 1$, then

$$\lambda^{\delta} = \left(\left[\frac{\sqrt{2(\eta^2)^{\delta}}}{\sqrt{(2-\eta^2)^{\delta}+(\eta^2)^{\delta}}}, \frac{\sqrt{2(v^2)^{\delta}}}{\sqrt{(2-v^2)^{\delta}+(v^2)^{\delta}}} \right], \left[\frac{\sqrt{(1+x^2)^{\delta}-(1-x^2)^{\delta}}}{\sqrt{(1+x^2)^{\delta}+(1-x^2)^{\delta}}}, \frac{\sqrt{(1+y^2)^{\delta}-(1-y^2)^{\delta}}}{\sqrt{(1+y^2)^{\delta}+(1-y^2)^{\delta}}} \right] \right)$$
$$= ([0, 0], [1, 1]).$$

Thus $\lambda^{\delta} = ([0,\ 0],\ [1,\ 1])$ and $\delta\lambda = ([1,\ 1],\ [0,\ 0])$.

3. If $\lambda = ([\eta,\ v],\ [x,\ y]) = ([0,\ 0],\ [0,\ 0])$ i. e,. $\eta = v = 0$ and $x = y = 0$, then

$$\lambda^{\delta} = \left(\left[\frac{\sqrt{2(\eta^2)^{\delta}}}{\sqrt{(2-\eta^2)^{\delta}+(\eta^2)^{\delta}}},\ \frac{\sqrt{2(v^2)^{\delta}}}{\sqrt{(2-v^2)^{\delta}+(v^2)^{\delta}}}\right],\left[\frac{\sqrt{(1+x^2)^{\delta}-(1-x^2)^{\delta}}}{\sqrt{(1+x^2)^{\delta}+(1-x^2)^{\delta}}},\ \frac{\sqrt{(1+y^2)^{\delta}-(1-y^2)^{\delta}}}{\sqrt{(1+y^2)^{\delta}+(1-y^2)^{\delta}}}\right]\right)$$
$$= ([0,\ 0],\ [0,\ 0]).$$

Thus $\lambda^{\delta} = ([0,\ 0],\ [0,\ 0])$ and $\delta\lambda = ([0,\ 0],\ [0,\ 0])$.

4. If $\delta \to 0$ and $0 \leq \eta,\ v,\ x,\ y \leq 1$, then

$$\lambda^{\delta} = \left(\left[\frac{\sqrt{2(\eta^2)^{\delta}}}{\sqrt{(2-\eta^2)^{\delta}+(\eta^2)^{\delta}}},\ \frac{\sqrt{2(v^2)^{\delta}}}{\sqrt{(2-v^2)^{\delta}+(v^2)^{\delta}}}\right],\left[\frac{\sqrt{(1+x^2)^{\delta}-(1-x^2)^{\delta}}}{\sqrt{(1+x^2)^{\delta}+(1-x^2)^{\delta}}},\ \frac{\sqrt{(1+y^2)^{\delta}-(1-y^2)^{\delta}}}{\sqrt{(1+y^2)^{\delta}+(1-y^2)^{\delta}}}\right]\right)$$
$$= ([1,\ 1],\ [0,\ 0]).$$

Thus $\lambda^{\delta} = ([1,\ 1],\ [0,\ 0])$ and $\delta\lambda = ([0,\ 0],\ [1,\ 1])$.

5. If $\delta \to +\infty$ and $0 \leq \eta,\ v,\ x,\ y \leq 1$, then

$$\lambda^{\delta} = \left(\left[\frac{\sqrt{2(\eta^2)^{\delta}}}{\sqrt{(2-\eta^2)^{\delta}+(\eta^2)^{\delta}}},\ \frac{\sqrt{2(v^2)^{\delta}}}{\sqrt{(2-v^2)^{\delta}+(v^2)^{\delta}}}\right],\left[\frac{\sqrt{(1+x^2)^{\delta}-(1-x^2)^{\delta}}}{\sqrt{(1+x^2)^{\delta}+(1-x^2)^{\delta}}},\ \frac{\sqrt{(1+y^2)^{\delta}-(1-y^2)^{\delta}}}{\sqrt{(1+y^2)^{\delta}+(1-y^2)^{\delta}}}\right]\right)$$
$$= ([[0,\ 0],\ 1,\ 1]).$$

Thus $\lambda^{\delta} = ([[0,\ 0],\ 1,\ 1])$ and $\delta\lambda = ([1,\ 1],\ [0,\ 0])$.

6. If $\delta = 1$ and $0 \leq \eta,\ v,\ x,\ y \leq 1$, then

$$\lambda^{\delta} = \left(\left[\frac{\sqrt{2(\eta^2)^{\delta}}}{\sqrt{(2-\eta^2)^{\delta}+(\eta^2)^{\delta}}},\ \frac{\sqrt{2(v^2)^{\delta}}}{\sqrt{(2-v^2)^{\delta}+(v^2)^{\delta}}}\right],\left[\frac{\sqrt{(1+x^2)^{\delta}-(1-x^2)^{\delta}}}{\sqrt{(1+x^2)^{\delta}+(1-x^2)^{\delta}}},\ \frac{\sqrt{(1+y^2)^{\delta}-(1-y^2)^{\delta}}}{\sqrt{(1+y^2)^{\delta}+(1-y^2)^{\delta}}}\right]\right)$$
$$= \lambda.$$

Thus $\lambda^{\delta} = \lambda$ and $\delta\lambda = \lambda$.

4 Some Induced Interval-Valued Pythagorean Fuzzy Einstein Averaging Aggregation Operators

In this section, we develop two induced interval-valued Einstein averaging aggregation operators such as, induced interval-valued Pythagorean fuzzy Einstein ordered weighted averaging (I-IVPFEOWA) aggregation operator and induced interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging (I-IVPFEHWA) aggregation operator. We also discuss some desirable properties of the propose operators such as, idempotency, boundedness, commutatively and monotonicity.

4.1 Induced Interval-Valued Pythagorean Fuzzy Einstein Ordered Weighted Averaging Aggregation Operator

Definition 9 (Rahman et al. 2018) Let $\langle u_{j,}\lambda_j\rangle$ $(j = 1, 2, 3, \ldots, n)$ be a collection of 2-tuples, then the induced interval-valued Pythagorean fuzzy Einstein ordered weighted averaging aggregation operator can be define as:

$$\text{I - IVPFEOWA}_w\left(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \langle u_3, \lambda_3\rangle, \ldots, \langle u_n, \lambda_n\rangle\right)$$
$$= \left(\begin{array}{c}\left[\frac{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\eta^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\eta^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^{n}\left(1+v^2_{\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-v^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+v^2_{\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-v^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}\right], \\ \left[\frac{\sqrt{2\prod_{j=1}^{n}\left(x^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-x^2_{\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(x^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(y^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-y^2_{\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(y^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}\right]\end{array}\right), \tag{24}$$

where $w = (w_1, w_2, w_3, \ldots, w_n)^T$ be the weighted vector of $\lambda_j\ (j = 1, 2, 3, \ldots, n)$ with conditions $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Also $\lambda_{\sigma(j)}$ is the λ_j value of the IVPFEOWA pairs $\langle u_j, \lambda_j\rangle$ having the jth largest u_j and u_j in $\langle u_j, \lambda_j\rangle$ is referred to as the order inducing variable and λ_j as the Pythagorean fuzzy argument variable.

Example 1 Let

$$\begin{aligned}\langle u_1, \lambda_1\rangle &= \langle 0.5, ([0.3, 0.6], [0.3, 0.5])\rangle \\ \langle u_2, \lambda_2\rangle &= \langle 0.7, ([0.2, 0.6], [0.3, 0.6])\rangle \\ \langle u_3, \lambda_3\rangle &= \langle 0.3, ([0.4, 0.7], [0.2, 0.6])\rangle \\ \langle u_4, \lambda_4\rangle &= \langle 0.9, ([0.3, 0.4], [0.5, 0.7])\rangle,\end{aligned}$$

and let $w = (0.1, 0.2, 0.3, 0.4)^T$ be the weighted vector of $\lambda_j\ (j = 1, 2, 3, 4)$. If we give ordered to the interval-valued Pythagorean fuzzy ordered weighted averaging pairs according to the first element, then

$$\begin{aligned}\langle u_4, \lambda_4\rangle &= \langle 0.9, ([0.3, 0.4], [0.5, 0.7])\rangle \\ \langle u_2, \lambda_2\rangle &= \langle 0.7, ([0.2, 0.6], [0.3, 0.6])\rangle \\ \langle u_3, \lambda_3\rangle &= \langle 0.5, ([0.3, 0.6], [0.3, 0.5])\rangle \\ \langle u_1, \lambda_1\rangle &= \langle 0.3, ([0.4, 0.7], [0.2, 0.6])\rangle.\end{aligned}$$

Hence

$$\begin{aligned}\langle u_{\sigma(1)}, \lambda_{\sigma(1)}\rangle &= \langle 0.9, ([0.3, 0.4], [0.5, 0.7])\rangle \\ \langle u_{\sigma(2)}, \lambda_{\sigma(2)}\rangle &= \langle 0.7, ([0.2, 0.6], [0.3, 0.6])\rangle\end{aligned}$$

$$\left\langle u_{\sigma(3)},\ \lambda_{\sigma(3)}\right\rangle = \langle 0.5,\ ([0.3,\ 0.6],\ [0.3,\ 0.5])\rangle$$
$$\left\langle u_{\sigma(4)},\ \lambda_{\sigma(4)}\right\rangle = \langle 0.3,\ ([0.4,\ 0.7],\ [0.2,\ 0.6])\rangle .$$

Thus

$$\begin{aligned}
&\text{I - IVPFEOWA}_w\left(\langle u_1,\ \lambda_1\rangle,\ \langle u_2,\ \lambda_2\rangle,\ \langle u_3,\ \lambda_3\rangle,\ \langle u_4,\ \lambda_4\rangle\right)\\
&=\left(\begin{array}{c}\left[\dfrac{\sqrt{\prod_{j=1}^{4}\left(1+\eta^2_{\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{4}\left(1-\eta^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(1+\eta^2_{\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{4}\left(1-\eta^2_{\lambda_{\sigma(j)}}\right)^{w_j}}},\ \dfrac{\sqrt{\prod_{j=1}^{4}\left(1+v^2_{\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{4}\left(1-v^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(1+v^2_{\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{4}\left(1-v^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}\right],\\
\left[\dfrac{\sqrt{2\prod_{j=1}^{4}\left(x^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(2-x^2_{\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{4}\left(x^2_{\lambda_{\sigma(j)}}\right)^{w_j}}},\ \dfrac{\sqrt{2\prod_{j=1}^{4}\left(y^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(2-y^2_{\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{4}\left(y^2_{\lambda_{\sigma(j)}}\right)^{w_j}}}\right]\end{array}\right)\\
&=([0.32,\ 0.62],\ [0.26,\ 0.57]).
\end{aligned}$$

4.2 Induced interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator

Definition 10 Let $\left\langle u_{j,}\lambda_j\right\rangle$ $(j = 1,\ 2,\ 3,\ldots,\ n)$ be a collection of 2-tuples, then the induced interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging aggregation operator can be define as:

$$\begin{aligned}
&\text{I - IVPFEHWA}_{\omega,w}\left(\langle u_1, \lambda_1\rangle,\ \langle u_2, \lambda_2\rangle,\ \langle u_3,\ \lambda_3\rangle,\ldots,\ \langle u_n,\ \lambda_n\rangle\right)\\
&=\left(\begin{array}{c}\left[\dfrac{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}},\ \dfrac{\sqrt{\prod_{j=1}^{n}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}\right],\\
\left[\dfrac{\sqrt{2\prod_{j=1}^{n}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}},\ \dfrac{\sqrt{2\prod_{j=1}^{n}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}\right]\end{array}\right),
\end{aligned}\tag{25}$$

where $\dot\lambda_{\sigma(j)}$ is the jth largest of the weighted interval-valued Pythagorean fuzzy values $\dot\lambda_{\sigma(j)}\left(\dot\lambda_{\sigma(j)} = n\omega_j\lambda_j\right)$, $w = (w_1,\ w_2,\ w_3,\ldots,\ w_n)^T$ be the weighted vector of I-IVPFEHWA operator such that, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ be the weighted vector of $\lambda_j (j = 1, 2, 3 \ldots, n)$ such that $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient, which plays a role of balance. If the vector $(w_1, w_2, w_3, \ldots, w_n)^T$ approaches $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then $(n\omega_1\lambda_1, n\omega_2\lambda_2, n\omega_3\lambda_3, \ldots, n\omega_n\lambda_n)^T$ approaches $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)^T$.

Example 2 Let

$$\begin{aligned}
\langle u_1, \lambda_1\rangle &= \langle 0.5, ([0.3, 0.6], [0.5, 0.6])\rangle \\
\langle u_2, \lambda_2\rangle &= \langle 0.8, ([0.4, 0.7], [0.2, 0.5])\rangle \\
\langle u_3, \lambda_3\rangle &= \langle 0.4, ([0.4, 0.5], [0.4, 0.8])\rangle \\
\langle u_4, \lambda_4\rangle &= \langle 0.6, ([0.4, 0.6], [0.3, 0.4])\rangle,
\end{aligned}$$

and let $\omega = (0.1, 0.2, 0.3, 0.4)^T$, be the weighted vector of λ_j $(j = 1, 2, 3, 4)$, then

$$\begin{aligned}
\dot{\lambda}_1 &= ([0.19, 0.40], [0.75, 0.81]) \\
\dot{\lambda}_2 &= ([0.36, 0.64], [0.27, 0.57]) \\
\dot{\lambda}_3 &= ([0.43, 0.54], [0.33, 0.76]) \\
\dot{\lambda}_4 &= ([0.49, 0.71], [0.14, 0.23]).
\end{aligned}$$

Performing the ordering of the IVPFOWA pairs with respect to the first element, then

$$\begin{aligned}
\langle u_2, \lambda_2\rangle &= \langle 0.8, ([0.4, 0.7], [0.2, 0.5])\rangle \\
\langle u_4, \lambda_4\rangle &= \langle 0.6, ([0.4, 0.6], [0.3, 0.4])\rangle \\
\langle u_1, \lambda_1\rangle &= \langle 0.5, ([0.3, 0.6], [0.5, 0.6])\rangle \\
\langle u_3, \lambda_3\rangle &= \langle 0.4, ([0.4, 0.5], [0.4, 0.8])\rangle.
\end{aligned}$$

This ordering includes the ordered interval-valued Pythagorean fuzzy arguments, then

$$\begin{aligned}
\dot{\lambda}_{\sigma(1)} &= ([0.36, 0.64], [0.27, 0.57]) \\
\dot{\lambda}_{\sigma(2)} &= ([0.49, 0.71], [0.14, 0.23]) \\
\dot{\lambda}_{\sigma(3)} &= ([0.19, 0.40], [0.75, 0.81]) \\
\dot{\lambda}_{\sigma(4)} &= ([0.43, 0.54], [0.33, 0.76]).
\end{aligned}$$

Now using I-IVPFEHWA operator, where $w = (0.1, 0.2, 0.3, 0.4)^T$, then we have

$$\text{I - IVPFEHWA}_{\omega, w}\left(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \langle u_3, \lambda_3\rangle, \langle u_4, \lambda_4\rangle\right)$$
$$= \left(\begin{array}{c} \left[\frac{\sqrt{\prod_{j=1}^{4}\left(1+\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{4}\left(1-\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(1+\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{4}\left(1-\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^{4}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{4}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{4}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}} \right], \\ \left[\frac{\sqrt{2\prod_{j=1}^{4}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(2-x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{4}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{4}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(2-y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{4}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}} \right] \end{array} \right)$$
$$= ([0.38, 0.56], [0.36, 0.61]).$$

Theorem 1 *Let $\langle u_j, \lambda_j\rangle$ $(j = 1, 2, 3, \ldots, n)$ be a collection of 2-tuples , then their aggregated value by using I-IVPFEHWA operator is also IVPFV , and*

$$\text{I - IVPFEHWA}_{\omega, w}\left(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \langle u_3, \lambda_3\rangle, \ldots, \langle u_n, \lambda_n\rangle\right)$$
$$= \left(\begin{array}{c} \left[\frac{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\eta^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^{n}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{n}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-v^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}} \right], \\ \left[\frac{\sqrt{2\prod_{j=1}^{n}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(x^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(y^2_{\dot\lambda_{\sigma(j)}}\right)^{w_j}}} \right] \end{array} \right). \quad (26)$$

Proof We prove Eq. (26) by using mathematical induction on n.

For $n = 2$,

$$w_1 \oplus_\varepsilon \dot\lambda_1 = \left(\begin{array}{c} \left[\frac{\sqrt{\left(1+\eta^2_{\dot\lambda_1}\right)^{w_1} - \left(1-\eta^2_{\dot\lambda_1}\right)^{w_1}}}{\sqrt{\left(1+\eta^2_{\dot\lambda_1}\right)^{w_1} + \left(1-\eta^2_{\dot\lambda_1}\right)^{w_1}}}, \frac{\sqrt{\left(1+v^2_{\dot\lambda_1}\right)^{w_1} - \left(1-v^2_{\dot\lambda_1}\right)^{w_1}}}{\sqrt{\left(1+v^2_{\dot\lambda_1}\right)^{w_1} + \left(1-v^2_{\dot\lambda_1}\right)^{w_1}}} \right], \\ \left[\frac{\sqrt{2\left(x^2_{\dot\lambda_1}\right)^{w_1}}}{\sqrt{\left(2-x^2_{\dot\lambda_1}\right)^{w_1} + \left(x^2_{\dot\lambda_1}\right)^{w_1}}}, \frac{\sqrt{2\left(y^2_{\dot\lambda_1}\right)^{w_1}}}{\sqrt{\left(2-y^2_{\dot\lambda_1}\right)^{w_1} + \left(y^2_{\dot\lambda_1}\right)^{w_1}}} \right] \end{array} \right),$$

and

$$w_2 \oplus_\varepsilon \dot{\lambda}_2 = \left(\begin{array}{c} \left[\frac{\sqrt{\left(1+\eta_{\dot{\lambda}_2}^2\right)^{w_2} - \left(1-\eta_{\dot{\lambda}_2}^2\right)^{w_2}}}{\sqrt{\left(1+\eta_{\dot{\lambda}_2}^2\right)^{w_2} + \left(1-\eta_{\dot{\lambda}_2}^2\right)^{w_2}}}, \frac{\sqrt{\left(1+v_{\dot{\lambda}_2}^2\right)^{w_2} - \left(1-v_{\dot{\lambda}_2}^2\right)^{w_2}}}{\sqrt{\left(1+v_{\dot{\lambda}_2}^2\right)^{w_2} + \left(1-v_{\dot{\lambda}_2}^2\right)^{w_2}}} \right], \\ \left[\frac{\sqrt{2\left(x_{\dot{\lambda}_2}^2\right)^{w_2}}}{\sqrt{\left(2-x_{\dot{\lambda}_2}^2\right)^{w_1} + \left(x_{\dot{\lambda}_2}^2\right)^{w_2}}}, \frac{\sqrt{2\left(y_{\dot{\lambda}_2}^2\right)^{w_2}}}{\sqrt{\left(2-y_{\dot{\lambda}_2}^2\right)^{w_1} + \left(y_{\dot{\lambda}_2}^2\right)^{w_2}}} \right] \end{array} \right).$$

Then

I - IVPFEHWA$_{\omega,\, w}$ $(\langle u_1,\ \lambda_1\rangle,\ \langle u_2,\ \lambda_2\rangle)$

$$= \left(\begin{array}{c} \left[\frac{\sqrt{\prod_{j=1}^{2}\left(1+\eta_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^{2}\left(1-\eta_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(1+\eta_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{2}\left(1-\eta_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^{2}\left(1+v_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^{2}\left(1-v_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(1+v_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{2}\left(1-v_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}} \right], \\ \left[\frac{\sqrt{2\prod_{j=1}^{2}\left(x_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(2-x_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{2}\left(x_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{2}\left(y_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(2-y_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{2}\left(y_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}} \right] \end{array} \right).$$

Thus the result is true for $n = 2$. Now we assume that Eq. (26) holds for $n = k$, then

I - IVPFEHWA$_{\omega,\, w}$ $(\langle u_1,\ \lambda_1\rangle,\ \langle u_2,\ \lambda_2\rangle,\ \langle u_3,\ \lambda_3\rangle, \ldots,\ \langle u_k,\ \lambda_k\rangle)$

$$= \left(\begin{array}{c} \left[\frac{\sqrt{\prod_{j=1}^{k}\left(1+\eta_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^{k}\left(1-\eta_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\eta_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{k}\left(1-\eta_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^{k}\left(1+v_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^{k}\left(1-v_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+v_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{k}\left(1-v_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}} \right], \\ \left[\frac{\sqrt{2\prod_{j=1}^{k}\left(x_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-x_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{k}\left(x_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{k}\left(y_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-y_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{k}\left(y_{\dot{\lambda}_{\sigma(j)}}^2\right)^{w_j}}} \right] \end{array} \right).$$

Suppose Eq. (26) holds for $n = k$, then we show that Eq. (26) holds for $n = k + 1$, then

$$\text{I - IVPFEHWA}_{\omega,w}\left(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \langle u_3, \lambda_3\rangle, \ldots, \langle u_{k+1}, \lambda_{k+1}\rangle\right)$$
$$= \left(\begin{array}{c} \left[\frac{\sqrt{\prod_{j=1}^{k}\left(1+\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}} \right], \\ \left[\frac{\sqrt{2\prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}} \right] \end{array} \right)$$
$$\oplus_{\varepsilon} \left(\begin{array}{c} \left[\frac{\sqrt{\left(1+\eta^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} - \left(1-\eta^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(1+\eta^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} + \left(1-\eta^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}, \frac{\sqrt{\left(1+v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} - \left(1-v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(1+v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} + \left(1-v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}} \right], \\ \left[\frac{\sqrt{2\left(x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(2-x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} + \left(x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}, \frac{\sqrt{2\left(y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(2-y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} + \left(y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}} \right] \end{array} \right). \quad (27)$$

Let

$$t_1 = \sqrt{\prod_{j=1}^{k}\left(1+\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{k}\left(1-\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$
$$t_2 = \sqrt{\prod_{j=1}^{k}\left(1+\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$
$$p_1 = \sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$
$$p_2 = \sqrt{\prod_{j=1}^{k}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$
$$w_1 = \sqrt{\left(1+\eta^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} - \left(1-\eta^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$
$$w_2 = \sqrt{\left(1+\eta^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} + \left(1-\eta^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$
$$a_1 = \sqrt{\left(1+v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} - \left(1-v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

$$a_2 = \sqrt{\left(1+v^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} + \left(1-v^2_{\dot{\lambda}_{k+1}}\right)^{wk+1}}$$

$$r_2 = \sqrt{\prod_{j=1}^{k}\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}},$$

$$s_1 = \sqrt{2\prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}},\ r_1 = \sqrt{2\prod_{j=1}^{k}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$b_2 = \sqrt{\left(2-x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} + \left(x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

$$c_2 = \sqrt{\left(2-y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}} + \left(y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}}$$

$$s_2 = \sqrt{\prod_{j=1}^{k}\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}$$

$$b_1 = \sqrt{2\left(x^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}},\ c_1 = \sqrt{2\left(y^2_{\dot{\lambda}_{k+1}}\right)^{w_{k+1}}},$$

Now putting these values in Eq. (27), we have

$$\begin{aligned}
&\text{I - IVPFEHWA}_{\omega,w}\left(\langle u_1,\lambda_1\rangle,\langle u_2,\lambda_2\rangle,\langle u_3,\lambda_3\rangle,\ldots,\langle u_{k+1},\lambda_{k+1}\rangle\right)\\
&=\left(\left[\frac{t_1}{t_2},\frac{p_1}{p_2}\right],\left[\frac{r_1}{r_2},\frac{s_1}{s_2}\right]\right)\oplus_\varepsilon\left(\left[\frac{w_1}{w_2},\frac{a_1}{a_2}\right],\left[\frac{b_1}{b_2},\frac{c_1}{c_2}\right]\right)\\
&=\left(\left[\frac{\sqrt{\left(\frac{t_1}{t_2}\right)^2+\left(\frac{w_1}{w_2}\right)^2}}{\sqrt{1+\left(\frac{t_1}{t_2}\right)^2\left(\frac{w_1}{w_2}\right)^2}},\frac{\sqrt{\left(\frac{p_1}{p_2}\right)^2+\left(\frac{a_1}{a_2}\right)^2}}{\sqrt{1+\left(\frac{p_1}{p_2}\right)^2\left(\frac{a_1}{a_2}\right)^2}}\right],\left[\frac{\left(\frac{r_1}{r_2}\right)\left(\frac{b_1}{b_2}\right)}{\sqrt{1+\left(1-\left(\frac{r_1}{r_2}\right)^2\right)\left(1-\left(\frac{b_1}{b_2}\right)^2\right)}},\frac{\left(\frac{s_1}{s_2}\right)\left(\frac{c_1}{c_2}\right)}{\sqrt{1+\left(1-\left(\frac{s_1}{s_2}\right)\right)^2\left(1-\left(\frac{c_1}{c_2}\right)\right)^2}}\right]\right)\\
&=\left(\left[\frac{\sqrt{(t_1w_2)^2+(t_2w_1)^2}}{\sqrt{(t_2w_2)^2+(t_1w_1)^2}},\frac{\sqrt{(p_1a_2)^2+(a_1p_2)^2}}{\sqrt{(p_2a_2)^2+(p_1a_1)^2}}\right],\left[\frac{r_1b_1}{\sqrt{2r_2^2b_2^2+r_1^2b_1^2-r_2^2b_1^2-r_1^2b_2^2}},\frac{s_1c_1}{\sqrt{2s_2^2c_2^2+s_1^2c_1^2-s_2^2c_1^2-s_1^2c_2^2}}\right]\right).
\end{aligned}\tag{28}$$

Again putting the values of $(t_1w_2)^2+(t_2w_1)^2$, $(t_2w_2)^2+(t_1w_1)^2$, $(p_1a_2)^2+(a_1p_2)^2$, $(t_1w_2)^2+(t_2w_1)^2$, $(t_2w_2)^2+(t_1w_1)^2$, $(p_1a_2)^2+(a_1p_2)^2$, $2s_2^2c_2^2+s_1^2c_1^2-s_2^2c_1^2-s_1^2c_2^2$, in Eq. (28), then we have

$$\begin{aligned}
&\text{I - IVPFEHWA}_{\omega,w}\left(\langle u_1,\lambda_1\rangle,\langle u_2,\lambda_2\rangle,\langle u_3,\lambda_3\rangle,\ldots,\langle u_{k+1},\lambda_{k+1}\rangle\right)\\
&=\left(\begin{array}{c}\left[\frac{\sqrt{\prod_{j=1}^{k+1}\left(1+\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k+1}\left(1-\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k+1}\left(1-\eta^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}},\frac{\sqrt{\prod_{j=1}^{k+1}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k+1}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k+1}\left(1-v^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right],\\
\left[\frac{\sqrt{2\prod_{j=1}^{k+1}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k+1}\left(x^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}},\frac{\sqrt{2\prod_{j=1}^{k+1}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k+1}\left(y^2_{\dot{\lambda}_{\sigma(j)}}\right)^{w_j}}}\right]\end{array}\right).
\end{aligned}$$

Hence Eq. (26) also holds for $n=k+1$. Thus Eq. (26) holds for all n.

Lemma 1 [7] *Let* $\lambda_j \succ 0,\ w_j \succ 0\ (j=1,\ 2,\ 3,\ldots,n)$ *and* $\sum_{j=1}^{n} w_j = 1$, *then*

$$\prod_{j=1}^{n} \left(\lambda_j\right)^{w_j} \leq \sum_{j=1}^{n} w_j \lambda_j. \tag{29}$$

Theorem 2 *Let* $\left\langle u_{j,} \lambda_j \right\rangle$ $(j=1,\ 2,\ 3,\ldots,\ n)$ *be a collection of 2-tuples , then*

$$\begin{aligned} &\text{I - IVPFEHWA}_{\omega,\,w}\left(\langle u_1,\ \lambda_1\rangle,\ \langle u_2,\ \lambda_2\rangle,\ \langle u_3,\ \lambda_3\rangle,\ldots,\ \langle u_n,\ \lambda_n\rangle\right) \\ &\leq \text{I - IVPFHWA}_{\omega,\,w}\left(\langle u_1,\ \lambda_1\rangle,\ \langle u_2,\ \lambda_2\rangle,\ \langle u_3,\ \lambda_3\rangle,\ldots,\ \langle u_n,\ \lambda_n\rangle\right). \end{aligned} \tag{30}$$

Theorem 3 Idempotency: *Let* $\left\langle u_{j,} \lambda_j \right\rangle$ $(j=1,\ 2,\ 3,\ldots,\ n)$ *be a collection of 2-tuples, if* $\dot{\lambda}_{\sigma(j)} = \dot{\lambda}$ *for all* j, then

$$\text{I - IVPFEHWA}_{\omega,w}\left(\langle u_1,\lambda_1\rangle,\ \langle u_2,\lambda_2\rangle,\langle u_3,\lambda_3\rangle,\ldots,\langle u_n,\lambda_n\rangle\right) = \dot{\lambda}. \tag{31}$$

Proof

$$\text{I - IVPFEHWA}_{\omega,w}\left(\langle u_1,\ \lambda_1\rangle,\ \langle u_2,\ \lambda_2\rangle,\ \langle u_3,\ \lambda_3\rangle,\ldots,\ \langle u_n,\ \lambda_n\rangle\right)$$

$$= \left(\begin{array}{c} \left[\dfrac{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j} - \prod_{j=1}^{n}\left(1-\eta^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\eta^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j}}}, \dfrac{\sqrt{\prod_{j=1}^{n}\left(1+v^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j} - \prod_{j=1}^{n}\left(1-v^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+v^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-v^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j}}} \right], \\ \left[\dfrac{\sqrt{2\prod_{j=1}^{n}\left(x^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-x^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j} + \prod_{j=1}^{n}\left(x^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j}}}, \dfrac{\sqrt{2\prod_{j=1}^{n}\left(y^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-y^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j} + \prod_{j=1}^{n}\left(y^2_{\dot{\lambda}_{\sigma(n)}}\right)^{w_j}}} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\dfrac{\sqrt{\left(1+\eta^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j} - \left(1-\eta^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j}}}{\sqrt{\left(1+\eta^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j} + \left(1-\eta^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j}}}, \dfrac{\sqrt{\left(1+v^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j} - \left(1-v^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j}}}{\sqrt{\left(1+v^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j} + \left(1-v^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j}}} \right], \\ \left[\dfrac{\sqrt{2\left(x^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j}}}{\sqrt{\left(2-x^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j} + \left(x^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j}}}, \dfrac{\sqrt{2\left(y^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j}}}{\sqrt{\left(2-y^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j} + \left(y^2_{\dot{\lambda}}\right)^{\sum_{j=1}^{n} w_j}}} \right] \end{array} \right)$$

$$= \left(\left[\frac{\sqrt{\left(1+\eta^2_{\dot{\lambda}}\right)-\left(1-\eta^2_{\dot{\lambda}}\right)}}{\sqrt{\left(1+\eta^2_{\dot{\lambda}}\right)+\left(1-\eta^2_{\dot{\lambda}}\right)}}, \frac{\sqrt{\left(1+v^2_{\dot{\lambda}}\right)-\left(1-v^2_{\dot{\lambda}}\right)}}{\sqrt{\left(1+v^2_{\dot{\lambda}}\right)+\left(1-v^2_{\dot{\lambda}}\right)}} \right], \left[\frac{\sqrt{2\left(x^2_{\dot{\lambda}}\right)}}{\sqrt{\left(2-x^2_{\dot{\lambda}}\right)+\left(x^2_{\dot{\lambda}}\right)}}, \frac{\sqrt{2\left(y^2_{\dot{\lambda}}\right)}}{\sqrt{\left(2-y^2_{\dot{\lambda}}\right)+\left(y^2_{\dot{\lambda}}\right)}} \right] \right) = \dot{\lambda}.$$

Theorem 4 Boundedness: *Let* $\left\langle u_j,\ \lambda_j \right\rangle$ $(j=1,\ 2,\ 3,\ldots,\ n)$ *be a collection of 2-tuples , then*

$$\dot{\lambda}_{\min} \leq \text{I - IVPFHWA}_{\omega,w}\left(\langle u_1, \lambda_1\rangle, \langle u_2, \lambda_2\rangle, \langle u_3, \lambda_3\rangle, \ldots, \langle u_n, \lambda_n\rangle\right) \leq \dot{\lambda}_{\max}, \tag{32}$$

for all w_j and

$$\dot{\lambda}_{\max} = \left(\max_j\left[\eta_{\dot{\lambda}_j}, v_{\dot{\lambda}_j}\right], \min_j\left[x_{\dot{\lambda}_j}, y_{\dot{\lambda}_j}\right]\right), \tag{33}$$

$$\dot{\lambda}_{\min} = \left(\min_j\left[\eta_{\dot{\lambda}_j}, v_{\dot{\lambda}_j}\right], \max_j\left[x_{\dot{\lambda}_j}, y_{\dot{\lambda}_j}\right]\right). \tag{34}$$

Proof Since

$$\dot{\lambda}_{\max} = \left(\max_j\left[\eta_{\dot{\lambda}_j}, v_{\dot{\lambda}_j}\right], \min_j\left[x_{\dot{\lambda}_j}, y_{\dot{\lambda}_j}\right]\right),$$

and

$$\dot{\lambda}_{\min} = \left(\min_j\left[\eta_{\dot{\lambda}_j}, v_{\dot{\lambda}_j}\right], \max_j\left[x_{\dot{\lambda}_j}, y_{\dot{\lambda}_j}\right]\right),$$

Let

$$\text{I - IVPFHWA}_{\omega,\, w}\left(\langle u_1,\, \lambda_1\rangle, \langle u_2,\, \lambda_2\rangle, \langle u_3,\, \lambda_3\rangle, \ldots, \langle u_n,\, \lambda_n\rangle\right) = \dot{\lambda}. \tag{35}$$

Then

$$\left(\left[\eta_{\min}, v_{\min}\right], \left[x_{\max}, y_{\max}\right]\right) \leq \left(\left[\eta_{\dot{\lambda}_j}, v_{\dot{\lambda}_j}\right], \left[x_{\dot{\lambda}_j}, y_{\dot{\lambda}_j}\right]\right), \tag{36}$$

$$\left(\left[\eta_{\max}, v_{\max}\right], \left[x_{\min}, y_{\min}\right]\right) \geq \left(\left[\eta_{\dot{\lambda}_j}, v_{\dot{\lambda}_j}\right], \left[x_{\dot{\lambda}_j}, y_{\dot{\lambda}_j}\right]\right). \tag{37}$$

By the score function, we have

$$S\left(\dot{\lambda}_{\min}\right) \leq S\left(\text{I - IVPFHWA}\right), \tag{38}$$

$$S\left(\dot{\lambda}_{\max}\right) \geq S\left(\text{I - IVPFHWA}\right). \tag{39}$$

Thus from Eqs. (38) and (39), we have

$$\dot{\lambda}_{\min} \leq \text{I - IVPFHWA}_{\omega,\, w}\left(\langle u_1,\, \lambda_1\rangle, \langle u_2,\, \lambda_2\rangle, \langle u_3,\, \lambda_3\rangle, \ldots, \langle u_n,\, \lambda_n\rangle\right) \leq \dot{\lambda}_{\max}.$$

The proof is completed.

Theorem 5 Monotonicity: *If $\langle u_j, \lambda_j\rangle$ and $\left\langle u_j, \lambda_j^*\right\rangle$ $(j = 1, 2, 3, \ldots, n)$ be two set of 2-tuples, where $\dot{\lambda}_{\sigma(j)} \prec \dot{\lambda}^*_{\sigma(j)}$ for all j, then*

$$\begin{aligned} &\text{I - IVPFEHWA}_{\omega,\,w}\left(\langle u_1,\,\lambda_1\rangle,\,\langle u_2,\,\lambda_2\rangle,\,\langle u_3,\,\lambda_3\rangle,\ldots,\,\langle u_n,\,\lambda_n\rangle\right) \\ &\leq \text{I - IVPFEHWA}_{\omega,\,w}\left(\left\langle u_1,\,\lambda_1^*\right\rangle,\,\left\langle u_2,\,\lambda_2^*\right\rangle,\,\left\langle u_3,\,\lambda_3^*\right\rangle,\ldots,\,\left\langle u_n,\,\lambda_n^*\right\rangle\right). \end{aligned} \tag{40}$$

Proof Since

$$\begin{aligned} &\text{I - IVPFEHWA}_{\omega,\,w}\left(\langle u_1,\,\lambda_1\rangle,\,\langle u_2,\,\lambda_2\rangle,\,\langle u_3,\,\lambda_3\rangle,\ldots,\,\langle u_n,\,\lambda_n\rangle\right) \\ &\quad = w_1\dot{\lambda}_{\sigma(1)} \oplus_\varepsilon w_2\dot{\lambda}_{\sigma(2)} \oplus_\varepsilon w_3\dot{\lambda}_{\sigma(3)} \oplus_\varepsilon \ldots \oplus_\varepsilon w_n\dot{\lambda}_{\sigma(n)}, \end{aligned} \tag{41}$$

and

$$\begin{aligned} &\text{I - IVPFEHWA}_{\omega,\,w}\left(\left\langle u_1,\,\lambda_1^*\right\rangle,\,\left\langle u_2,\,\lambda_2^*\right\rangle,\,\left\langle u_3,\,\lambda_3^*\right\rangle,\ldots,\,\left\langle u_n,\,\lambda_n^*\right\rangle\right) \\ &\quad = w_1\dot{\lambda}^*_{\sigma(1)} \oplus_\varepsilon w_2\dot{\lambda}^*_{\sigma(2)} \oplus_\varepsilon w_3\dot{\lambda}^*_{\sigma(3)} \oplus_\varepsilon \ldots \oplus_\varepsilon w_n\dot{\lambda}^*_{\sigma(n)}. \end{aligned} \tag{42}$$

As $\dot{\lambda}_{\sigma(j)} \leq \lambda^*_{\sigma(j)}$ for all j, thus Eq. (40) always holds.

Theorem 6 *IVPFEWA aggregation operator is a special case of the I-IVPFEHWA aggregation operator.*

Proof Let $w = \left(\frac{1}{n},\,\frac{1}{n},\,\frac{1}{n},\ldots,\,\frac{1}{n},\right)^T$, then we have

$$\begin{aligned} &\text{I - IVPFEHWA}_{\omega,\,w}\left(\langle u_1,\,\lambda_1\rangle,\,\langle u_2,\,\lambda_2\rangle,\,\langle u_3,\,\lambda_3\rangle,\ldots,\,\langle u_n,\,\lambda_n\rangle\right) \\ &= w_1\dot{\lambda}_{\sigma(1)} \oplus_\varepsilon w_2\dot{\lambda}_{\sigma(2)} \oplus_\varepsilon w_3\dot{\lambda}_{\sigma(3)} \oplus_\varepsilon \ldots \oplus_\varepsilon w_n\dot{\lambda}_{\sigma(n)} \\ &= \frac{1}{n}\left(\dot{\lambda}_{\sigma(1)} \oplus_\varepsilon \dot{\lambda}_{\sigma(2)} \oplus_\varepsilon \dot{\lambda}_{\sigma(3)} \oplus_\varepsilon \ldots \oplus_\varepsilon \dot{\lambda}_{\sigma(n)}\right) \\ &= \omega_1\lambda_1 \oplus_\varepsilon \omega_2\lambda_2 \oplus_\varepsilon \omega_3\lambda_3 \oplus_\varepsilon \ldots \oplus_\varepsilon \omega_n\lambda_n \\ &= \text{IVPFEWA}_\omega\left(\lambda_1,\,\lambda_2,\,\lambda_3,\ldots,\,\lambda_n\right). \end{aligned}$$

The proof is completed.

Theorem 7 *I-IVPFEOWA aggregation operator is a special case of the I-IVPFEHWA aggregation operator.*

Proof Let $\omega = \left(\frac{1}{n},\,\frac{1}{n},\,\frac{1}{n},\ldots,\,\frac{1}{n},\right)^T$, and $\dot{\lambda}_{\sigma(j)} = n\omega_j\lambda_{\sigma(j)} = n\left(\frac{1}{n}\lambda_j\right) = \lambda_{\sigma(j)}$, then we have

$$\begin{aligned} &\text{I - IVPFEHWA}_{\omega,\,w}\left(\langle u_1,\,\lambda_1\rangle,\,\langle u_2,\,\lambda_2\rangle,\,\langle u_3,\,\lambda_3\rangle,\ldots,\,\langle u_n,\,\lambda_n\rangle\right) \\ &= w_1\dot{\lambda}_{\sigma(1)} \oplus_\varepsilon w_2\dot{\lambda}_{\sigma(2)} \oplus_\varepsilon w_3\dot{\lambda}_{\sigma(3)} \oplus_\varepsilon \ldots \oplus_\varepsilon w_n\dot{\lambda}_{\sigma(n)} \\ &= w_1\lambda_{\sigma(1)} \oplus_\varepsilon w_2\lambda_{\sigma(2)} \oplus_\varepsilon w_3\lambda_{\sigma(3)} \oplus_\varepsilon \ldots \oplus_\varepsilon w_n\lambda_{\sigma(n)} \\ &= \text{I - IVPFEOWA}_w\left(\langle u_1,\,\lambda_1\rangle,\,\langle u_2,\,\lambda_2\rangle,\,\langle u_3,\,\lambda_3\rangle,\ldots,\,\langle u_n,\,\lambda_n\rangle\right). \end{aligned}$$

The proof is completed

5 An Approach to Group Decision Making Based on Induced Interval-Valued Pythagorean Fuzzy Einstein Hybrid Weighted Averaging Aggregation Operator

Let $X = \{X_1, X_2, X_3, \ldots, X_n\}$ be a finite set of n alternatives, and $C = \{C_1, C_2, C_3, \ldots, C_m\}$ be a finite set of m attributes. Suppose the grade of the alternatives $X_j(j = 1, 2, 3, \ldots, n)$ on attributes $C_i(i = 1, 2, 3, \ldots, m)$ given by decision makers is interval-valued Pythagorean fuzzy numbers. Let $E = \{E_1, E_2, E_3, \ldots, E_k\}$ be the set of k decision makers. Let $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ be the weighted vector of the attributes $X_j\ (j = 1, 2, 3, \ldots, n)$, such that $\omega_i \in [0, 1]$, $\sum_{i=1}^{m} \omega_i = 1$, and let $\varpi = (\varpi_1, \varpi_2, \varpi_3, \ldots, \varpi_k)^T$ be the weighted vector of the decision makers $E^s\ (s = 1, 2, 3, \ldots, k)$, such that $\varpi_s \in [0, 1]$ and $\sum_{s=1}^{k} \varpi_s = 1$. Let $E = \langle u_{ij}, a_{ij}\rangle = \langle u_j, ([\eta_{ij}, \nu_{ij}], [x_{ij}, y_{ij}])\rangle$, where $[\eta_{ij}, \nu_{ij}]$ indicates the interval degree that the alternative X_j satisfies the attribute C_i and $[x_{ij}, y_{ij}]$ indicates the interval degree that the alternative X_j does not satisfy the attribute C_j. And also $[\eta_{ij}, \nu_{ij}] \in [0, 1]$, $[x_{ij}, y_{ij}] \in [0, 1]$, with condition $0 \leq (\nu_{ij})^2 + (y_{ij})^2 \leq 1 (i = 1, 2, 3, \ldots, m,\ j = 1, 2, 3, \ldots, n)$.

Algorithm

Step 1 Construct decision matrices, $E^s_{m\times n} = \left[a^{(s)}_{ij}\right]_{m\times n}$ for decision.

Step 2 If the criteria have two types such as, benefit criteria and cost criteria, then the interval-valued Pythagorean fuzzy decision matrices $E^s_{m\times n} = \left[a^{(s)}_{ij}\right]_{m\times n}$ can be converted into the normalized interval-valued Pythagorean fuzzy decision matrices, $R^s_{m\times n} = \left[r^{(s)}_{ij}\right]_{m\times n}$, where

$$r^{(s)}_{ij} = \begin{cases} \alpha^{(s)}_{ij}, & \text{for benefit criteria } C_i \\ \bar{a}^{(s)}_{ij}, & \text{for cost criteria } C_i, \end{cases} \quad \begin{pmatrix} j = 1, 2, 3, .., n \\ i = 1, 2, 3, .., m \end{pmatrix},$$

and $\bar{a}^{(s)}_{ij}$ is the complement of α^s_{ij}. If all the criteria have the same type, then there is no need of normalization.

Step 3 Utilize the I-IVPFEOWA aggregation operator to aggregate all the individual decision matrices, $R^s_{m\times n} = \left[r^{(s)}_{ij}\right]_{m\times n}$ into a single interval-valued Pythagorean fuzzy decision matrix, $R_{m\times n} = [r_{ij}]_{m\times n}$, where $r_{ij} = \langle[\eta_{ij}, \upsilon_{ij}], [x_{ij}, y_{ij}]\rangle\ (i = 1, 2, 3, \ldots, m,\ j = 1, 2, 3, \ldots, n)$.

Step 4 Calculate $\dot{r}_{ji} = n\omega_j r_{ij}$.

Step 5 Calculate the scores function of $\dot{r}_{ij}(i = 1, 2, 3, \ldots, m,\ j = 1, 2, 3, \ldots, n)$. If there is no difference between two or more than two scores then we have must to find out the accuracy degrees of the collective overall preference values

Step 6 Utilize the I-IVPFEHWA aggregation operator to aggregate all preference values

Table 1 Interval-valued Pythagorean fuzzy decision matrix of E^1

	X_1	X_2	X_3
C_1	$\langle 0.7, ([0.5, 0.8], [0.3, 0.4])\rangle$	$\langle 0.6, ([0.6, 0.7], [0.3, 0.6])\rangle$	$\langle 0.8, ([0.3, 0.7], [0.3, 0.5])\rangle$
C_2	$\langle 0.5, ([0.3, 0.5], [0.6, 0.7])\rangle$	$\langle 0.5, ([0.3, 0.7], [0.2, 0.6])\rangle$	$\langle 0.6, ([0.3, 0.6], [0.4, 0.7])\rangle$
C_3	$\langle 0.4, ([0.5, 0.7], [0.3, 0.7])\rangle$	$\langle 0.4, ([0.5, 0.6], [0.3, 0.7])\rangle$	$\langle 0.5, ([0.2, 0.6], [0.3, 0.7])\rangle$
C_4	$\langle 0.3, ([0.3, 0.6], [0.6, 0.7])\rangle$	$\langle 0.3, ([0.6, 0.5], [0.2, 0.7])\rangle$	$\langle 0.4, ([0.3, 0.4], [0.5, 0.6])\rangle$

Table 2 Interval-valued Pythagorean fuzzy decision matrix of E^2

	X_1	X_2	X_3
C_1	$\langle 0.5, ([0.4, 0.5], [0.3, 0.8])\rangle$	$\langle 0.4, ([0.5, 0.7], [0.3, 0.6])\rangle$	$\langle 0.5, ([0.2, 0.6], [0.3, 0.8])\rangle$
C_2	$\langle 0.3, ([0.3, 0.6], [0.5, 0.7])\rangle$	$\langle 0.6, ([0.3, 0.8], [0.2, 0.6])\rangle$	$\langle 0.4, ([0.3, 0.5], [0.5, 0.7])\rangle$
C_3	$\langle 0.8, ([0.5, 0.6], [0.3, 0.5])\rangle$	$\langle 0.7, ([0.5, 0.7], [0.3, 0.6])\rangle$	$\langle 0.9, ([0.2, 0.8], [0.3, 0.4])\rangle$
C_4	$\langle 0.6, ([0.3, 0.4], [0.6, 0.8])\rangle$	$\langle 0.3, ([0.3, 0.4], [0.2, 0.8])\rangle$	$\langle 0.7, ([0.3, 0.6], [0.3, 0.7])\rangle$

Step 7 Arrange the scores of the all alternatives in the form of descending order and select that alternative which has the highest score function

6 Illustrative Example

Suppose in an industry, the manager of the industry wants to develop a new system for information. For this purpose he constructs a committee to develop a best information system. There are three experts E^s $(s = 1, 2, 3)$ in the committee to act as decision makers, whose weight vector is $\varpi = (0.2, 0.3, 0.5)^T$. In the first selection, there are only three X_j $(j = 1, 2, 3)$ alternatives have been short listed for further process. There are several factors that must be considered while selecting the most suitable system, but here, we have consider only the following four criteria, whose weighted vector is $\omega = (0.1, 0.2, 0.3, 0.4)^T$.

1. C_1 : Expenditure of the new system
2. C_2 : Funding of the industry
3. C_3 : Struggle to transform the old system into new system
4. C_4 : Outsourcing software developer reliability

where C_1 and C_3, are cost type criteria and C_2, C_4, are benefit type criteria i.e., the attributes have two types criteria, thus we have must to change the cost type criteria into benefit type criteria.

Step 1 Construct decision matrices
Tables 1, 2, 3.

Step 2 Construct normalized decision matrices
Tables 4, 5, 6.

Table 3 Interval-valued Pythagorean fuzzy decision matrix of E[3]

	X_1	X_2	X_3
C_1	⟨0.8, ([0.3, 0.8], [0.5, 0.6])⟩	⟨0.7, ([0.3, 0.5], [0.5, 0.7])⟩	⟨0.4, ([0.2, 0.8], [0.4, 0.6])⟩
C_2	⟨0.7, ([0.5, 0.7], [0.3, 0.4])⟩	⟨0.5, ([0.4, 0.6], [0.5, 0.8])⟩	⟨0.3, ([0.5, 0.6], [0.3, 0.5])⟩
C_3	⟨0.5, ([0.3, 0.6], [0.4, 0.6])⟩	⟨0.4, ([0.3, 0.5], [0.5, 0.6])⟩	⟨0.8, ([0.2, 0.4], [0.5, 0.7])⟩
C_4	⟨0.4, ([0.5, 0.7], [0.3, 0.4])⟩	⟨0.3, ([0.5, 0.7], [0.2, 0.4])⟩	⟨0.5, ([0.5, 0.7], [0.2, 0.5])⟩

Table 4 Normalized interval-valued Pythagorean fuzzy decision matrix R[1]

	X_1	X_2	X_3
C_1	⟨0.7, ([0.3, 0.4], [0.5, 0.8])⟩	⟨0.6, ([0.3, 0.6], [0.6, 0.7])⟩	⟨0.8, ([0.3, 0.5], [0.3, 0.7])⟩
C_2	⟨0.5, ([0.3, 0.5], [0.6, 0.7])⟩	⟨0.5, ([0.3, 0.7], [0.2, 0.6])⟩	⟨0.6, ([0.3, 0.6], [0.4, 0.7])⟩
C_3	⟨0.4, ([0.3, 0.7], [0.5, 0.7])⟩	⟨0.4, ([0.3, 0.7], [0.5, 0.6])⟩	⟨0.5, ([0.3, 0.7], [0.2, 0.6])⟩
C_4	⟨0.3, ([0.3, 0.6], [0.6, 0.7])⟩	⟨0.3, ([0.6, 0.5], [0.2, 0.7])⟩	⟨0.4, ([0.3, 0.4], [0.5, 0.6])⟩

Table 5 Normalized interval-valued Pythagorean fuzzy decision matrix R[2]

	X_1	X_2	X_3
C_1	⟨0.5, ([0.3, 0.8], [0.4, 0.5])⟩	⟨0.4, ([0.3, 0.6], [0.5, 0.7])⟩	⟨0.5, ([0.3, 0.8], [0.2, 0.6])⟩
C_2	⟨0.3, ([0.3, 0.6], [0.5, 0.7])⟩	⟨0.6, ([0.3, 0.8], [0.2, 0.6])⟩	⟨0.4, ([0.3, 0.5], [0.5, 0.7])⟩
C_3	⟨0.8, ([0.3, 0.5], [0.5, 0.6])⟩	⟨0.7, ([0.3, 0.6], [0.5, 0.7])⟩	⟨0.9, ([0.3, 0.4], [0.2, 0.8])⟩
C_4	⟨0.6, ([0.3, 0.4], [0.6, 0.8])⟩	⟨0.3, ([0.3, 0.4], [0.2, 0.8])⟩	⟨0.7, ([0.3, 0.6], [0.3, 0.7])⟩

Table 6 Normalized interval-valued Pythagorean fuzzy decision matrix R[3]

	X_1	X_2	X_3
C_1	⟨0.8, ([0.5, 0.6], [0.3, 0.8])⟩	⟨0.7, ([0.5, 0.7], [0.3, 0.5])⟩	⟨0.3, ([0.5, 0.6], [0.3, 0.5])⟩
C_2	⟨0.7, ([0.5, 0.7], [0.3, 0.4])⟩	⟨0.5, ([0.4, 0.6], [0.5, 0.8])⟩	⟨0.5, ([0.5, 0.7], [0.2, 0.5])⟩
C_3	⟨0.5, ([0.4, 0.6], [0.3, 0.6])⟩	⟨0.4, ([0.5, 0.6], [0.3, 0.5])⟩	⟨0.8, ([0.5, 0.7], [0.2, 0.4])⟩
C_4	⟨0.4, ([0.5, 0.7], [0.3, 0.4])⟩	⟨0.3, ([0.5, 0.7], [0.2, 0.4])⟩	⟨0.4, ([0.4, 0.6], [0.2, 0.8])⟩

Step 3 Utilize the I-IVPFEOWA aggregation operator to aggregate all the individual interval-valued Pythagorean fuzzy decision matrices, $R^s = \left[r_{ij}^{(s)}\right]_{m\times n}$ into a single interval-valued Pythagorean fuzzy decision matrix, $R = \left[r_{ij}\right]_{m\times n}$, where $\varpi = (0.2,\ 0.3,\ 0.5)^T$.
Table 7.

Table 7 Collective interval-valued Pythagorean fuzzy decision matrix R

	X_1	X_2	X_3
C_1	([0.41, 0.53], [0.38, 0.73])	([0.41, 0.65], [0.40, 0.59])	([0.41, 0.59], [0.21, 0.56])
C_2	([0.41, 0.59], [0.42, 0.56])	([0.35, 0.69], [0.32, 0.69])	([0.41, 0.65], [0.25, 0.59])
C_3	([0.35, 0.69], [0.36, 0.58])	([0.41, 0.62], [0.38, 0.57])	([0.35, 0.69], [0.20, 0.69])
C_4	([0.41, 0.65], [0.40, 0.53])	([0.47, 0.59], [0.20, 0.56])	([0.41, 0.53], [0.38, 0.57])

Step 4 Utilize the I-IVPFEOWA aggregation operator to derive the collective overall preference values, where $w = (0.1, 0.2, 0.3, 0.4)^T$.

$$r_1 = ([0.39, 0.64], [0.39, 0.57])$$
$$r_2 = ([0.42, 0.61], [0.28, 0.59])$$
$$r_3 = ([0.39, 0.61], [0.27, 0.61]).$$

Step 5 Calculate the score functions

$$S(r_1) = 0.042,\ S(r_2) = 0.069,\ S(r_3) = 0.043.$$

Step 6 Hence $S(r_2) \succ S(r_3) \succ S(r_1)$, thus the best option for selection is A_2

For I-IVPFEHWA Aggregation Operator.

Step 1 Calculate $\dot{\lambda}_{ij} = n\omega_i \lambda_{ij}$, where $\omega = (0.1, 0.2, 0.3, 0.4)^T$.

$$\dot{\lambda}_{11} = ([0.26, 0.34], [0.73, 0.89]),\ \dot{\lambda}_{21} = ([0.37, 0.53], [0.52, 0.64])$$
$$\dot{\lambda}_{31} = ([0.38, 0.74], [0.28, 0.51]),\ \dot{\lambda}_{41} = ([0.51, 0.78], [0.20, 0.32])$$
$$\dot{\lambda}_{12} = ([0.26, 0.42], [0.74, 0.83]),\ \dot{\lambda}_{22} = ([0.31, 0.62], [0.42, 0.75])$$
$$\dot{\lambda}_{32} = ([0.45, 0.66], [0.30, 0.50]),\ \dot{\lambda}_{42} = ([0.59, 0.72], [0.06, 0.35])$$
$$\dot{\lambda}_{13} = ([0.26, 0.38], [0.60, 0.82]),\ \dot{\lambda}_{23} = ([0.37, 0.59], [0.35, 0.67])$$
$$\dot{\lambda}_{33} = ([0.38, 0.74], [0.13, 0.63]),\ \dot{\lambda}_{43} = ([0.51, 0.66], [0.18, 0.37]).$$

Step 2 Calculate the score functions

$$S\left(\dot{\lambda}_{11}\right) = -0.57,\ S\left(\dot{\lambda}_{21}\right) = -0.13,\ S\left(\dot{\lambda}_{31}\right) = 0.18$$
$$S\left(\dot{\lambda}_{41}\right) = 0.37,\ S\left(\dot{\lambda}_{12}\right) = -0.50,\ S\left(\dot{\lambda}_{22}\right) = -0.12$$
$$S\left(\dot{\lambda}_{32}\right) = 0.15,\ S\left(\dot{\lambda}_{42}\right) = 0.37,\ S\left(\dot{\lambda}_{13}\right) = -0.41$$
$$S\left(\dot{\lambda}_{23}\right) = -0.04,\ S\left(\dot{\lambda}_{33}\right) = 0.13,\ S\left(\dot{\lambda}_{43}\right) = 0.26.$$

Step 3 Construct a collective hybrid decision matrix
Table 8.

Table 8 Collective interval-valued Pythagorean fuzzy decision matrix R

	x_1	x_2	x_3
C_1	([0.51, 0.78], [0.20, 0.32])	([0.59, 0.72], [0.06, 0.35])	([0.51, 0.66], [0.18, 0.37])
C_2	([0.38, 0.74], [0.28, 0.51])	([0.45, 0.66], [0.30, 0.50])	([0.38, 0.74], [0.13, 0.63])
C_3	([0.37, 0.53], [0.52, 0.64])	([0.31, 0.62], [0.42, 0.75])	([0.37, 0.59], [0.35, 0.67])
C_4	([0.26, 0.34], [0.73, 0.89])	([0.26, 0.42], [0.74, 0.83])	([0.26, 0.38], [0.60, 0.82])

Step 4 Utilize the I-IVPFEHWA aggregation operator to derive the collective overall preference values, $w = (0.1, 0.2, 0.3, 0.4)^T$.

$$r_1 = ([0.35,\ 0.56],\ [0.55,\ 0.67])$$
$$r_2 = ([0.36,\ 0.58],\ [0.42,\ 0.68])$$
$$r_3 = ([0.35,\ 0.57],\ [0.45,\ 0.69])\,.$$

Step 5 Calculate the score functions

$$S\,(r_1) = -0.154,\ S\,(r_1) = -0.088,\ S\,(r_1) = -0.096.$$

Step 6 Hence $S\,(r_2) \succ S\,(r_3) \succ S\,(r_1)$, thus the best option for selection is A_2.

7 Conclusion

In this paper, we have explore the induced Einstein hybrid aggregation operator based interval-valued Pythagorean fuzzy numbers and applied them to the multi-attribute group decision making problems where attribute values are the interval-valued Pythagorean fuzzy numbers. Firstly, induced interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging (I-IVPFEHWA) aggregation operator, was proposed. Some general properties of them, such as idempotency, monotonicity and boundedness, were studied, and some special cases of them were analyzed. Furthermore, a method to multi-criteria decision group making based on the proposed operator was developed, and the operational processes were illustrated in detail. Finally, an illustrative example is given to show the decision steps of the proposed methods and to demonstrate their effectiveness.

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