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Framed Surfaces in the Euclidean Space

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Abstract A framed surface is a smooth surface in the Euclidean space with a moving frame. The framed surfaces may have singularities. We treat smooth surfaces with singular points, that is, singular surfaces more directly. By using the moving frame, the basic invariants and curvatures of the framed surface are introduced. Then we show that the existence and uniqueness for the basic invariants of the framed surfaces. We give properties of framed surfaces and typical examples. Moreover, we construct framed surfaces as one-parameter families of Legendre curves along framed curves. We give a criteria for singularities of framed surfaces by using the curvature of Legendre curves and framed curves.

Keywords Framed surface · Frontal · Singular point · Basic invariant · Curvature

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1 Introduction

The geometry of smooth surfaces in the Euclidean space is a classical object. Recently, smooth surfaces with singular points are more important for differential geometry, differential equations and physics (for instance, Arnol'd 1990; Arnol'd et al. 1986; Bruce and Giblin 1992; Fujimori et al. 2008; Fukui 2017; Fukui and Hasegawa 2012; Gray et al. 2006; Ishikawa 2015; Izumiya and Saji 2010; Izumiya et al. 2015; Kokubu et al. 2005; Martins and Nuño-Ballesteros 2015; Martins and Saji 2016; Martins et al. 2016; Oset Sinha and Tari 2015, 2017; Saji 2017; Saji et al. 2009; Teramoto 2016). One of the idea to treat the smooth surfaces with singular points is that we consider the fronts or frontals as smooth surfaces with singular points (cf. Arnol'd 1990; Arnol'd et al. 1986; Martins and Saji 2016; Martins et al. 2016; Saji et al. 2009; Teramoto 2016).

In this paper, we give an other consideration of smooth surfaces with singular points. The idea is a generalisation of not only the Legendre curves (Fukunaga and Takahashi 2013) but also framed curves in the Euclidean space (Honda and Takahashi 2016). It is also related the Cartan's moving frame (cf. Ivey and Landsberg 2016).

A framed surface in the Euclidean space is a smooth surface with a moving frame. The framed surface is a generalisation of not only regular surfaces but also frontals at least locally. The framed surfaces may have singularities. We would like to treat the surfaces with singular points more directly. In fact, we introduce the basic invariants of the framed surface in Sect. 2. Then we give the existence and uniqueness theorems of the basic invariants for the framed surface in Sect. 3. We investigate properties of the framed surfaces. We give a curvature and a concomitant mapping of the framed surfaces in Sect. 4. These mappings are useful to recognize a Legendre immersion or a framed immersion. Moreover, we construct framed surfaces as one-parameter families of Legendre curves along framed curves in Sect. 5. As an application of the construction, we give a criterion that the framed surface is locally diffeomorphic to the cuspidal edge, swallowtail and cuspidal cross cap by using the curvatures of the Legendre curves and the framed curves. We give concrete examples in Sect. 6.

All mappings and manifolds considered here are differential of class C^∞.

2 Definitions and Notations

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$, where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of a is given by $|a| = \sqrt{a \cdot a}$ and the vector product is given by

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where $\{e_1, e_2, e_3\}$ is the canonical basis on $\mathbb{R}^3$. Let U be a simply connected domain of $\mathbb{R}^2$ and S^2 be the unit sphere in $\mathbb{R}^3$, that is, $S^2 = \{a \in \mathbb{R}^3 | |a| = 1\}$. We denote a 3-dimensional smooth manifold $\{(a, b) \in S^2 \times S^2 | a \cdot b = 0\}$ by Δ.

Definition 1 We say that $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ is a *framed surface* if $x_u(u, v) \cdot n(u, v) = 0, x_v(u, v) \cdot n(u, v) = 0$ for all $(u, v) \in U$, where $x_u(u, v) = (\partial x/\partial u)(u, v)$ and $x_v(u, v) = (\partial x/\partial v)(u, v)$. We say that $x : U \to \mathbb{R}^3$ is a *framed base surface* if there exists $(n, s) : U \to \Delta$ such that (x, n, s) is a framed surface.

We also say that $(x, n) : U \to \mathbb{R}^3 \times S^2$ is a *Legendre surface* (respectively, a *Legendre immersion*) if $x_u(u, v) \cdot n(u, v) = 0, x_v(u, v) \cdot n(u, v) = 0$ for all $(u, v) \in U$. We say that $x : U \to \mathbb{R}^3$ is a *frontal* (respectively, a *front*) if there exists $n : U \to S^2$ such that (x, n) is a Legendre surface (respectively, Legendre immersion). For definition and properties of frontals see Arnol'd (1990); Arnol'd et al. (1986).

Suppose that $x : U \to \mathbb{R}^3$ is a regular surface. Then $(x, n) : U \to \mathbb{R}^3 \times S^2$ is a Legendre immersion, where $n = x_u \times x_v/|x_u \times x_v|$. There exists a smooth mapping $s : U \to S^2$ such that (x, n, s) is a framed surface. Actually we may take $s = x_u/|x_u|$ or $s = x_v/|x_v|$.

By definition, the framed base surface is a frontal. On the other hand, the frontal is a framed base surface at least locally. In this paper, we consider framed base surfaces as singular surfaces. If we do not confuse in the sentence, we also say that x is a framed surface.

We denote $t(u, v) = n(u, v) \times s(u, v)$. Then $\{n(u, v), s(u, v), t(u, v)\}$ is a moving frame along $x(u, v)$. Thus, we have the following systems of differential equations:

$$\begin{pmatrix} x_u \\ x_v \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} n_u \\ s_u \\ t_u \end{pmatrix} = \begin{pmatrix} 0 & e_1 & f_1 \\ -e_1 & 0 & g_1 \\ -f_1 & -g_1 & 0 \end{pmatrix} \begin{pmatrix} n \\ s \\ t \end{pmatrix}, \quad \begin{pmatrix} n_v \\ s_v \\ t_v \end{pmatrix} = \begin{pmatrix} 0 & e_2 & f_2 \\ -e_2 & 0 & g_2 \\ -f_2 & -g_2 & 0 \end{pmatrix} \begin{pmatrix} n \\ s \\ t \end{pmatrix}, \tag{2}$$

where $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ are smooth functions and we call the functions *basic invariants* of the framed surface. We denote the above matrices by $\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the framed surface (x, n, s). Note that (u, v) is a singular point of x if and only if $\det \mathcal{G}(u, v) = 0$.

Since the integrability conditions $x_{uv} = x_{vu}$ and $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, the basic invariants should be satisfied the following conditions:

$$\begin{cases} a_{1,v} - b_1 g_2 = a_{2,u} - b_2 g_1, \\ b_{1,v} - a_2 g_1 = b_{2,u} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \end{cases} \tag{3}$$

$$\begin{cases} e_{1,v} - f_1 g_2 = e_{2,u} - f_2 g_1, \\ f_{1,v} - e_2 g_1 = f_{2,u} - e_1 g_2, \\ g_{1,v} - e_1 f_2 = g_{2,u} - e_2 f_1. \end{cases} \tag{4}$$

3 Properties of Framed Surfaces

We consider basic properties of framed surfaces. We give fundamental theorems for framed surfaces, that is, the existence and uniqueness theorems for the basic invariants of framed surfaces.

Definition 2 Let $(x, n, s), (\widetilde{x}, \widetilde{n}, \widetilde{s}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces. We say that (x, n, s) and $(\widetilde{x}, \widetilde{n}, \widetilde{s})$ are *congruent as framed surfaces* if there exist a constant rotation $A \in SO(3)$ and a translation $a \in \mathbb{R}^3$ such that

$$\widetilde{x}(u, v) = A(x(u, v)) + a, \widetilde{n}(u, v) = A(n(u, v)), \widetilde{s}(u, v) = A(s(u, v)),$$

for all $(u, v) \in U$.

The existence theorem of framed surfaces follows from the existence of solutions of partial differential equations.

Theorem 1 (The Existence Theorem for framed surfaces) *Let U be a simply connected domain in $\mathbb{R}^2$ and let $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ be smooth functions with the integrability conditions (3) and (4). Then there exists a framed surface $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants is $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2)$.*

Proof Since the integrability condition (4), there exists an orthonormal frame $\{n, s, t\}$ such that the condition (2) holds. Moreover, by the integrability condition (3), there exists a smooth mapping $x : U \to \mathbb{R}^3$ such that the condition (1) holds. Therefore, there exists a framed surface $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants is $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2)$. □

Theorem 2 (The Uniqueness Theorem for framed surfaces) *Let $(x, n, s), (\widetilde{x}, \widetilde{n}, \widetilde{s}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces with basic invariants $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2), (\widetilde{\mathscr{G}}, \widetilde{\mathscr{F}}_1, \widetilde{\mathscr{F}}_2)$, respectively. Then (x, n, s) and $(\widetilde{x}, \widetilde{n}, \widetilde{s})$ are congruent as framed surfaces if and only if the basic invariants $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2)$ and $(\widetilde{\mathscr{G}}, \widetilde{\mathscr{F}}_1, \widetilde{\mathscr{F}}_2)$ coincide.*

In order to prove the uniqueness theorem, we prepare the following two lemmas.

Lemma 1 *If (x, n, s) and $(\widetilde{x}, \widetilde{n}, \widetilde{s})$ are congruent as framed surfaces, then $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2) = (\widetilde{\mathscr{G}}, \widetilde{\mathscr{F}}_1, \widetilde{\mathscr{F}}_2)$.*

Proof By Definition 2 and a direct calculation, we obtain the lemma. □

Lemma 2 *If $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2) = (\widetilde{\mathscr{G}}, \widetilde{\mathscr{F}}_1, \widetilde{\mathscr{F}}_2)$ and $(x, n, s)(u_0, v_0) = (\widetilde{x}, \widetilde{n}, \widetilde{s})(u_0, v_0)$ for some point $(u_0, v_0) \in U$, then $(x, n, s) = (\widetilde{x}, \widetilde{n}, \widetilde{s})$.*

Proof Firstly, we show $(n, s, t) = (\widetilde{n}, \widetilde{s}, \widetilde{t})$, where $n \times s = t$ and $\widetilde{n} \times \widetilde{s} = \widetilde{t}$. We define a function $f : U \to \mathbb{R}$ by $f(u, v) = n(u, v) \cdot \widetilde{n}(u, v) + s(u, v) \cdot \widetilde{s}(u, v) + t(u, v) \cdot \widetilde{t}(u, v)$. By the definition of the basic invariants, we have

$$\begin{aligned} f_u = &(e_1 - \widetilde{e_1})(s \cdot \widetilde{n}) + (f_1 - \widetilde{f_1})(t \cdot \widetilde{n}) + (\widetilde{e_1} - e_1)(n \cdot \widetilde{s}) \\ &+ (\widetilde{f_1} - f_1)(n \cdot \widetilde{t}) + (g_1 - \widetilde{g_1})(t \cdot \widetilde{s}) + (\widetilde{g_1} - g_1)(s \cdot \widetilde{t}). \end{aligned}$$

By the assumption $\mathscr{F}_1 = \widetilde{\mathscr{F}}_1$, we have $f_u(u,v) = 0$ for all $(u,v) \in U$. Similarly, we also have $f_v(u,v) = 0$ for all $(u,v) \in U$. Moreover, by the assumption $(n,s)(u_0,v_0) = (\widetilde{n},\widetilde{s})(u_0,v_0)$, we have $f(u_0,v_0) = 3$. It conclude that $f(u,v) = 3$ for all $(u,v) \in U$. Hence, we have $n \cdot \widetilde{n} = s \cdot \widetilde{s} = t \cdot \widetilde{t} = 1$. It follows that $n = \widetilde{n}$, $s = \widetilde{s}$ and $t = \widetilde{t}$.

Next, we show $x = \widetilde{x}$. By the assumption $\mathscr{G}_1 = \widetilde{\mathscr{G}}_1$, we have $x_u = a_1 s + b_1 t = \widetilde{a}_1\widetilde{s} + \widetilde{b}_1\widetilde{t} = \widetilde{x}_u$ and $x_v = a_2 s + b_2 t = \widetilde{a}_2\widetilde{s} + \widetilde{b}_2\widetilde{t} = \widetilde{x}_v$. Then, we have $(x - \widetilde{x})_u = (x - \widetilde{x})_v = 0$. Since $x(u_0,v_0) = \widetilde{x}(u_0,v_0)$, we have $x(u,v) = \widetilde{x}(u,v)$ for all $(u,v) \in U$. Therefore, we have $(x,n,s) = (\widetilde{x},\widetilde{n},\widetilde{s})$. □

Proof of Theorem 2. The necessary part of the theorem is Lemma 1.

We prove the sufficient part of the theorem. Fixing a point $(u_0,v_0) \in U$, there exist $A \in SO(3)$ and $a \in \mathbb{R}^3$ such that $(x,n,s)(u_0,v_0) = (A\widetilde{x} + a, A\widetilde{n}, A\widetilde{s})(u_0,v_0)$. By Lemmas 1 and 2, we have $(x,n,s) = (A\widetilde{x}+a, A\widetilde{n}, A\widetilde{s})$, that is, (x,n,s) and $(\widetilde{x},\widetilde{n},\widetilde{s})$ are congruent as framed surfaces. □

Let $(x,n,s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2)$. We consider rotations and reflections of the vectors s, t. We denote

$$\begin{pmatrix} s^\theta(u,v) \\ t^\theta(u,v) \end{pmatrix} = \begin{pmatrix} \cos\theta(u,v) & -\sin\theta(u,v) \\ \sin\theta(u,v) & \cos\theta(u,v) \end{pmatrix} \begin{pmatrix} s(u,v) \\ t(u,v) \end{pmatrix},$$

where $\theta : U \to \mathbb{R}$ is a smooth function. Then $n \times s^\theta = t^\theta$ and $\{n, s^\theta, t^\theta\}$ is also a moving frame along x. It follows that (x,n,s^θ) is a framed surface. We call the frame $\{n, s^\theta, t^\theta\}$ a *rotation frame* by θ of the framed surface (x,n,s). We denote by $(\mathscr{G}^\theta, \mathscr{F}_1^\theta, \mathscr{F}_2^\theta)$ the basic invariants of (x,n,s^θ). Moreover, we consider a moving frame $\{n^r, s^r, t^r\} = \{-n, t, s\}$ along x and call it a *reflection frame* of the framed surface (x,n,s). We denote by $(\mathscr{G}^r, \mathscr{F}_1^r, \mathscr{F}_2^r)$ the basic invariants of (x,n^r,s^r).

By a direct calculation, we have the following.

Proposition 1 *Under the above notations, we have the relations between the basic invariants $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2)$ and $(\mathscr{G}^\theta, \mathscr{F}_1^\theta, \mathscr{F}_2^\theta)$, $(\mathscr{G}^r, \mathscr{F}_1^r, \mathscr{F}_2^r)$, respectively.*

(1) *For any smooth function $\theta : U \to \mathbb{R}$,*

$$\mathscr{G}^\theta = \mathscr{G}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a_1\cos\theta - b_1\sin\theta & a_1\sin\theta + b_1\cos\theta \\ a_2\cos\theta - b_2\sin\theta & a_2\sin\theta + b_2\cos\theta \end{pmatrix},$$

$$\mathscr{F}_1^\theta = \begin{pmatrix} 0 & e_1\cos\theta - f_1\sin\theta & e_1\sin\theta + f_1\cos\theta \\ -e_1\cos\theta + f_1\sin\theta & 0 & g_1 - \theta_u \\ -e_1\sin\theta - f_1\cos\theta & -g_1 + \theta_u & 0 \end{pmatrix},$$

$$\mathscr{F}_2^\theta = \begin{pmatrix} 0 & e_2\cos\theta - f_2\sin\theta & e_2\sin\theta + f_2\cos\theta \\ -e_2\cos\theta + f_2\sin\theta & 0 & g_2 - \theta_v \\ -e_2\sin\theta - f_2\cos\theta & -g_2 + \theta_v & 0 \end{pmatrix}.$$

(2)

$$\mathcal{G}^r = \mathcal{G}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}b_1 & a_1\\ b_2 & a_2\end{pmatrix}, \mathcal{F}_1^r = \begin{pmatrix}0 & -f_1 & -e_1\\ f_1 & 0 & -g_1\\ e_1 & g_1 & 0\end{pmatrix},$$
$$\mathcal{F}_2^r = \begin{pmatrix}0 & -f_2 & -e_2\\ f_2 & 0 & -g_2\\ e_2 & g_2 & 0\end{pmatrix}.$$

Especially, we have

$$\begin{pmatrix}e_i^\theta\\ f_i^\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e_i\\ f_i\end{pmatrix}, \ i = 1, 2.$$

We consider the integrability conditions (3) and (4) of (x, n, s^θ) and (x, n^r, s^r), respectively. Since

$$x_u = a_1 s + b_1 t = a_1^\theta s^\theta + b_1^\theta t^\theta = a_1^r s^r + b_1^r t^r, x_v = a_2 s + b_2 t = a_2^\theta s^\theta + b_2^\theta t^\theta$$
$$= a_2^r s^r + b_2^r t^r,$$

we also have

$$\begin{cases} a^\theta_{1,v} - b^\theta_1 g^\theta_2 = a^\theta_{2,u} - b^\theta_2 g^\theta_1,\\ b^\theta_{1,v} - a^\theta_2 g^\theta_1 = b^\theta_{2,u} - a^\theta_1 g^\theta_2,\\ a^\theta_1 e^\theta_2 + b^\theta_1 f^\theta_2 = a^\theta_2 e^\theta_1 + b^\theta_2 f^\theta_1, \end{cases}$$

for any $\theta : U \to \mathbb{R}$, and

$$\begin{cases} a^r_{1,v} - b^r_1 g^r_2 = a^r_{2,u} - b^r_2 g^r_1,\\ b^r_{1,v} - a^r_2 g^r_1 = b^r_{2,u} - a^r_1 g^r_2,\\ a^r_1 e^r_2 + b^r_1 f^r_2 = a^r_2 e^r_1 + b^r_2 f^r_1. \end{cases}$$

Proposition 2 *Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Then the following are equivalent for any smooth function $\theta : U \to \mathbb{R}$.*
(1) $\mathcal{F}_{2,u} - \mathcal{F}_{1,v} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$.
(2) $\mathcal{F}^\theta_{2,u} - \mathcal{F}^\theta_{1,v} = \mathcal{F}^\theta_1\mathcal{F}^\theta_2 - \mathcal{F}^\theta_2\mathcal{F}^\theta_1$.
(3) $\mathcal{F}^r_{2,u} - \mathcal{F}^r_{1,v} = \mathcal{F}^r_1\mathcal{F}^r_2 - \mathcal{F}^r_2\mathcal{F}^r_1$.

Proof We prove that (1) is equivalent to (2). We define matrices $R(\theta)$ and Θ by

$$R(\theta) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}, \ \Theta = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\theta\\ 0 & \theta & 0\end{pmatrix}.$$

Then we have $\mathscr{F}_1^\theta = \Theta_u + R(\theta)\mathscr{F}_1 R(-\theta)$ and $\mathscr{F}_2^\theta = \Theta_v + R(\theta)\mathscr{F}_2 R(-\theta)$ by Proposition 1 (1). By a direct calculation, we have

$$\begin{aligned}\mathscr{F}_{2,u}^\theta - \mathscr{F}_{1,v}^\theta = \Theta_{vu} &+ R(\theta)_u\mathscr{F}_2 R(-\theta) + R(\theta)\mathscr{F}_{2,u}R(-\theta) + R(\theta)\mathscr{F}_2 R(-\theta)_u \\ &- \Theta_{uv} - R(\theta)_v\mathscr{F}_1 R(-\theta) - R(\theta)\mathscr{F}_{1,v}R(-\theta) - R(\theta)\mathscr{F}_1 R(-\theta)_v.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathscr{F}_1^\theta\mathscr{F}_2^\theta - \mathscr{F}_2^\theta\mathscr{F}_1^\theta = \Theta_u R(\theta)\mathscr{F}_2 R(-\theta) &+ R(\theta)\mathscr{F}_1 R(-\theta)\Theta_v - \Theta_v R(\theta)\mathscr{F}_1 R(-\theta) \\ &- R(\theta)\mathscr{F}_2 R(-\theta)\Theta_u + R(\theta)(\mathscr{F}_1\mathscr{F}_2 - \mathscr{F}_2\mathscr{F}_1)R(-\theta).\end{aligned}$$

By using the relations $\Theta_u R(\theta) = R(\theta)_u$, $R(-\theta)\Theta_u = R(-\theta)_u$, $\Theta_v R(\theta) = R(\theta)_v$ and $R(-\theta)\Theta_v = R(-\theta)_v$, we have $R(\theta)(\mathscr{F}_{2,u} - \mathscr{F}_{1,v})R(-\theta) = R(\theta)(\mathscr{F}_1\mathscr{F}_2 - \mathscr{F}_2\mathscr{F}_1)R(-\theta)$. Since $R(\theta)$ and $R(-\theta)$ are invertible matrices, we conclude that (1) is equivalent to (2).

Next, we prove that (1) is equivalent to (3). We define a matrix R by

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then we have $\mathscr{F}_1^r = R\mathscr{F}_1 R$ and $\mathscr{F}_2^r = R\mathscr{F}_2 R$ by Proposition 1 (2). Thus, we have

$$\mathscr{F}_{2,u}^r - \mathscr{F}_{1,v}^r = R\mathscr{F}_{2,u}R - R\mathscr{F}_{1,v}R = R(\mathscr{F}_{2,u} - \mathscr{F}_{1,v})R.$$

On the other hand,

$$\mathscr{F}_1^r\mathscr{F}_2^r - \mathscr{F}_2^r\mathscr{F}_1^r = R\mathscr{F}_1 RR\mathscr{F}_2 R - R\mathscr{F}_2 RR\mathscr{F}_1 R = R(\mathscr{F}_1\mathscr{F}_2 - \mathscr{F}_2\mathscr{F}_1)R.$$

Note that R^2 is equal to the unit matrix. Since R is an invertible matrix, we conclude that (1) is equivalent to (3). □

Next we consider a parameter change of the domain U and a diffeomorphism of the target space $\mathbb{R}^3$.

Proposition 3 *Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2)$. Let $\phi : V \to U$, $(p, q) \mapsto \phi(p, q) = (u(p, q), v(p, q))$ be a parameter change, that is, a diffeomorphism of the domain. Then $(\widetilde{x}, \widetilde{n}, \widetilde{s}) = (x, n, s) \circ \phi : V \to \mathbb{R}^3 \times \Delta$ is a framed surface. Moreover, the basic invariants $(\widetilde{\mathscr{G}}, \widetilde{\mathscr{F}}_1, \widetilde{\mathscr{F}}_2)$ of $(\widetilde{x}, \widetilde{n}, \widetilde{s})$ is given by*

$$\begin{pmatrix} \widetilde{a}_1 & \widetilde{b}_1 \\ \widetilde{a}_2 & \widetilde{b}_2 \end{pmatrix}(p, q) = \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p, q)\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}(\phi(p, q))$$

$$\begin{pmatrix} \widetilde{e}_1 & \widetilde{f}_1 & \widetilde{g}_1 \\ \widetilde{e}_2 & \widetilde{f}_2 & \widetilde{g}_2 \end{pmatrix}(p, q) = \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p, q)\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix}(\phi(p, q)).$$

Proof By the chain rule, we have

$$\begin{aligned}
\widetilde{x}_p(p,q) &= x_u(\phi(p,q))u_p(p,q) + x_v(\phi(p,q))v_p(p,q)\\
&= \{a_1(\phi(p,q))s(\phi(p,q)) + b_1(\phi(p,q))t(\phi(p,q))\}u_p(p,q)\\
&\quad + \{a_2(\phi(p,q))s(\phi(p,q)) + b_2(\phi(p,q))t(\phi(p,q)))\}v_p(p,q)\\
&= \{a_1(\phi(p,q))u_p(p,q) + a_2(\phi(p,q))v_p(p,q)\}\widetilde{s}(p,q)\\
&\quad + \{b_1(\phi(p,q))u_p(p,q) + b_2(\phi(p,q))v_p(p,q)\}\widetilde{t}(p,q),\\
\widetilde{x}_q(p,q) &= x_u(\phi(p,q))u_q(p,q) + x_v(\phi(p,q))v_q(p,q)\\
&= \{a_1(\phi(p,q))s(\phi(p,q)) + b_1(\phi(p,q))t(\phi(p,q))\}u_q(p,q)\\
&\quad + \{a_2(\phi(p,q))s(\phi(p,q)) + b_2(\phi(p,q))t(\phi(p,q)))\}v_q(p,q)\\
&= \{a_1(\phi(p,q))u_q(p,q) + a_2(\phi(p,q))v_q(p,q)\}\widetilde{s}(p,q)\\
&\quad + \{b_1(\phi(p,q))u_q(p,q) + b_2(\phi(p,q))v_q(p,q)\}\widetilde{t}(p,q).
\end{aligned}$$

It follows that we have the first equation. The second equation in the proposition is proved similarly as the above by using the chain rule. □

Proposition 4 *Let $(x,n,s): U \to \mathbb{R}^3 \times \Delta$ be a framed surface. Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a diffeomorphism. Then there exists a smooth mapping $(n^\Phi, s^\Phi): U \to \Delta$ such that $(\Phi \circ x, n^\Phi, s^\Phi): U \to \mathbb{R}^3 \times \Delta$ is a framed surface.*

Proof We denote the Jacobian matrix of Φ at x by $D_\Phi(x)$. Since Φ is a diffeomorphism, $D_\Phi(x) \in GL(3,\mathbb{R})$. We define a mapping $(n^\Phi, s^\Phi): U \to \Delta$ by

$$(n^\Phi, s^\Phi)(u,v) = \left(\frac{n(u,v)\,{}^T(D_\Phi)^{-1}(x(u,v))}{|n(u,v)\,{}^T(D_\Phi)^{-1}(x(u,v))|}, \frac{s(u,v)D_\Phi(x(u,v))}{|s(u,v)D_\Phi(x(u,v))|}\right),$$

where ${}^T A$ is the transpose of the matrix A. Then we show that $(\Phi \circ x, n^\Phi, s^\Phi): U \to \mathbb{R}^3 \times \Delta$ is a framed surface. In fact, since $(d/du)(\Phi \circ x)(u,v) = x_u(u,v)D_\Phi \circ x(u,v)$ and $(d/dv)(\Phi \circ x)(u,v) = x_v(u,v)D_\Phi \circ x(u,v)$, we have

$$\begin{aligned}
\left(\frac{d}{du}(\Phi \circ x)\right) \cdot n^\Phi &= \frac{1}{|n^T(D_\Phi)^{-1} \circ x|} x_u(D_\Phi \circ x)((D_\Phi)^{-1} \circ x)^T n\\
&= \frac{1}{|n^T(D_\Phi)^{-1} \circ x|} x_u{}^T n = 0,\\
\left(\frac{d}{dv}(\Phi \circ x)\right) \cdot n^\Phi &= \frac{1}{|n^T(D_\Phi)^{-1} \circ x|} x_v(D_\Phi \circ x)((D_\Phi)^{-1} \circ x)^T n\\
&= \frac{1}{|n^T(D_\Phi)^{-1} \circ x|} x_v{}^T n = 0.
\end{aligned}$$

Note that all vectors in this proof are row vectors. Moreover, we have

$$n^{\Phi} \cdot s^{\Phi} = \frac{1}{|n\ {}^T(D_{\Phi})^{-1} \circ x||sD_{\Phi} \circ x|} n({}^T(D_{\Phi})^{-1} \circ x)({}^T D_{\Phi} \circ x)^T s$$
$$= \frac{1}{|n\ {}^T(D_{\Phi})^{-1} \circ x||sD_{\Phi} \circ x|}\ n^T s = 0.$$

Therefore, $(\Phi \circ x, n^{\Phi}, s^{\Phi}) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface. □

4 Curvatures of Framed Surfaces

Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathscr{G}, \mathscr{F}_1, \mathscr{F}_2)$.

Definition 3 We define a smooth mapping $C_F = (J_F, K_F, H_F) : U \to \mathbb{R}^3$ by

$$J_F = \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, K_F = \det\begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix},$$
$$H_F = -\frac{1}{2}\left\{\det\begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} - \det\begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}\right\}.$$

We call $C_F = (J_F, K_F, H_F)$ a *curvature of the framed surface*.

Remark 1 By the integrability condition (4), we have $K_F = g_{1,v} - g_{2,u}$.

For concrete examples of curvatures of framed surfaces, see Sect. 6.

Suppose that $x : U \to \mathbb{R}^3$ is a regular surface. Then there exists $(n, s) : U \to \Delta$ such that (x, n, s) is a framed surface, see Sect. 2. Let $E = x_u \cdot x_u$, $F = x_u \cdot x_v$, $G = x_v \cdot x_v$ be the coefficients of the first fundamental form and $L = -x_u \cdot n_u$, $M = -x_u \cdot n_v$, $N = -x_v \cdot n_v$ be the coefficients of the second fundamental form. The relationship between the first, second fundamental invariants and the basic invariant is as follows:

$$E = a_1^2 + b_1^2, \quad F = a_1 b_1 + a_2 b_2, \quad G = a_2^2 + b_2^2,$$
$$L = -a_1 e_1 - b_1 f_1, \quad M = -a_1 e_2 - b_1 f_2, \quad N = -a_2 e_2 - b_2 f_2.$$

By the integrability condition (3), we have $M = -a_2 e_1 - b_2 f_1$. We denote the Gauss curvature and the mean curvature of the regular surface x by K and H. Then

$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN - 2FM + GL}{2(EG - F)^2}.$$

By a direct calculation, we give a relationship between the Gauss curvature, the mean curvature and the curvature of the framed surface (x, n, s) as follows.

Proposition 5 *Under the above notation, we have* $K = K_F/J_F$ *and* $H = H_F/J_F$.

Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. Note that the condition $H_F^2(u, v) - J_F(u, v)K_F(u, v) \geq 0$ holds for all $(u, v) \in U$.

We give a relation between the curvature of the framed surface and the framed surfaces which given by a rotation frame and a reflection frame. We denote the curvatures $C_F^\theta = (J_F^\theta, K_F^\theta, H_F^\theta)$ of the framed surface (x, n, s^θ) and $C_F^r = (J_F^r, K_F^r, H_F^r)$ of the framed surface (x, n^r, s^r), respectively.

Proposition 6 *Under the above notation, we have the following.*

(1) $(J_F^\theta, K_F^\theta, H_F^\theta) = (J_F, K_F, H_F)$ *for any smooth function* $\theta : U \to \mathbb{R}$.
(2) $(J_F^r, K_F^r, H_F^r) = (-J_F, -K_F, H_F)$.

Proof (1) By Proposition 1 (1), we have

$$J_F^\theta = \det \begin{pmatrix} a_1^\theta & b_1^\theta \\ a_2^\theta & b_2^\theta \end{pmatrix} = \det \left\{ \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\} = J_F,$$

$$K_F^\theta = \det \begin{pmatrix} e_1^\theta & f_1^\theta \\ e_2^\theta & f_2^\theta \end{pmatrix} = \det \left\{ \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\} = K_F.$$

We show $H_F^\theta = H_F$. By Proposition 1 (1), we also have

$$\begin{pmatrix} a_1^\theta & f_1^\theta \\ a_2^\theta & f_2^\theta \end{pmatrix} = \begin{pmatrix} a_1\cos\theta - b_1\sin\theta & e_1\sin\theta + f_1\cos\theta \\ a_2\cos\theta - b_2\sin\theta & e_2\sin\theta + f_2\cos\theta \end{pmatrix},$$

$$\begin{pmatrix} b_1^\theta & e_1^\theta \\ b_2^\theta & e_2^\theta \end{pmatrix} = \begin{pmatrix} a_1\sin\theta + b_1\cos\theta & e_1\cos\theta - f_1\sin\theta \\ a_2\sin\theta + b_2\cos\theta & e_2\cos\theta - f_2\sin\theta \end{pmatrix}.$$

It follows that

$$\begin{aligned}
\det \begin{pmatrix} a_1^\theta & f_1^\theta \\ a_2^\theta & f_2^\theta \end{pmatrix} &= a_1e_2\cos\theta\sin\theta - b_1f_2\sin\theta\cos\theta + a_1f_2\cos^2\theta - b_1e_2\sin^2\theta \\
&\quad -e_1a_2\cos\theta\sin\theta + f_1b_2\cos\theta\sin\theta + e_1b_2\sin^2\theta - f_1a_2\cos^2\theta, \\
\det \begin{pmatrix} b_1^\theta & e_1^\theta \\ b_2^\theta & e_2^\theta \end{pmatrix} &= a_1e_2\cos\theta\sin\theta - b_1f_2\cos\theta\sin\theta - a_1f_2\sin^2\theta + b_1e_2\cos^2\theta \\
&\quad - e_1a_2\cos\theta\sin\theta + f_1b_2\sin\theta\cos\theta - e_1b_2\cos^2\theta + f_1a_2\sin^2\theta.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
H_F^\theta &= -\frac{1}{2}\left\{ \det \begin{pmatrix} a_1^\theta & f_1^\theta \\ a_2^\theta & f_2^\theta \end{pmatrix} - \det \begin{pmatrix} b_1^\theta & e_1^\theta \\ b_2^\theta & e_2^\theta \end{pmatrix} \right\} \\
&= -\frac{1}{2}(a_1f_2\cos^2\theta - b_1e_2\sin^2\theta + e_1b_2\sin^2\theta - f_1a_2\cos^2\theta \\
&\quad + a_1f_2\sin^2\theta - b_1e_2\cos^2\theta + e_1b_2\cos^2\theta - f_1a_2\sin^2\theta) \\
&= -\frac{1}{2}(a_1f_2 - f_1a_2 - b_1e_2 + e_1b_2) = H_F.
\end{aligned}$$

(2) By Proposition 1 (2), we have

$$J_F^r = \det\begin{pmatrix} a_1^r & b_1^r \\ a_2^r & b_2^r \end{pmatrix} = \det\left\{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} = -J_F,$$
$$K_F^r = \det\begin{pmatrix} e_1^r & f_1^r \\ e_2^r & f_2^r \end{pmatrix} = \det\left\{\begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} = -K_F.$$

Moreover,

$$\begin{aligned} H_F^r &= -\frac{1}{2}\left\{\det\begin{pmatrix} a_1^r & f_1^r \\ a_2^r & f_2^r \end{pmatrix} - \det\begin{pmatrix} b_1^r & e_1^r \\ b_2^r & e_2^r \end{pmatrix}\right\} \\ &= -\frac{1}{2}\left\{\det\begin{pmatrix} b_1 & -e_1 \\ b_2 & -e_2 \end{pmatrix} - \det\begin{pmatrix} a_1 & -f_1 \\ a_2 & -f_2 \end{pmatrix}\right\} = H_F. \end{aligned}$$

□

Let $\phi : V \to U, (p, q) \mapsto \phi(p, q) = (u(p, q), v(p, q))$ be a parameter change. By Proposition 3, $(\widetilde{x}, \widetilde{n}, \widetilde{s}) = (x, n, s) \circ \phi : V \to \mathbb{R}^3 \times \Delta$ is a framed surface with basic invariants $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$. We denote the curvature of the framed surface $(\widetilde{x}, \widetilde{n}, \widetilde{s})$ by $(\widetilde{J}_F, \widetilde{K}_F, \widetilde{H}_F)$.

Proposition 7 *Under the above notation, the curvature* $(\widetilde{J}_F, \widetilde{K}_F, \widetilde{H}_F) : V \to \mathbb{R}^3$ *is given by*

$$\begin{aligned} &(\widetilde{J}_F(p, q), \widetilde{K}_F(p, q), \widetilde{H}_F(p, q)) \\ &\quad = (J_\phi(p, q)J_F(\phi(p, q)), J_\phi(p, q)K_F(\phi(p, q)), J_\phi(p, q)H_F(\phi(p, q))), \end{aligned}$$

where J_ϕ *is the Jacobian of the parameter change* ϕ.

Proof We have $\widetilde{J}_F(p, q) = J_\phi(p, q)J_F(\phi(p, q))$ and $\widetilde{K}_F(p, q) = J_\phi(p, q)K_F(\phi(p, q))$ by Proposition 3. Since

$$\begin{pmatrix} \widetilde{a}_1 & \widetilde{f}_1 \\ \widetilde{a}_2 & \widetilde{f}_2 \end{pmatrix}(p, q) = \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p, q)\begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix}(\phi(p, q)),$$
$$\begin{pmatrix} \widetilde{b}_1 & \widetilde{e}_1 \\ \widetilde{b}_2 & \widetilde{e}_2 \end{pmatrix}(p, q) = \begin{pmatrix} u_p & v_p \\ u_q & v_q \end{pmatrix}(p, q)\begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}(\phi(p, q)),$$

we have $\widetilde{H}_F(p, q) = J_\phi(p, q)H_F(\phi(p, q))$. □

The curvature is useful to recognize that the framed base surface is a front or not.

Proposition 8 *Let* $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ *be a framed surface and* $p \in U$. *Then* $(x, n) : U \to \mathbb{R}^3 \times S^2$ *is a Legendre immersion around* p *if and only if* $C_F(p) \neq 0$.

Proof We show the necessarily part of the proposition, that is, if $C_F(p) = 0$, then $(x, n) : U \to \mathbb{R}^3 \times S^2$ is not a Legendre immersion at p. Since $J_F(p) = 0$, there exist $k_1, k_2 \in \mathbb{R}$ such that $k_1^2 + k_2^2 \neq 0$ and $k_1(a_1, a_2) + k_2(b_1, b_2) = 0$ at p.

Moreover, since $K_F(p) = 0$, there exist $h_1, h_2 \in \mathbb{R}$ such that $h_1^2 + h_2^2 \neq 0$ and $h_1(e_1, e_2) + h_2(f_1, f_2) = 0$ at p. We divide into the following four cases: $k_1h_1 \neq 0$, $k_2h_1 \neq 0$, $k_1h_2 \neq 0$ and $k_2h_2 \neq 0$.

Suppose that $k_1h_1 \neq 0$. In this case, we have $(a_1, a_2) = -(k_2/k_1)(b_1, b_2)$ and $(e_1, e_2) = -(h_2/h_1)(f_1, f_2)$ at p. Thus,

$$\begin{pmatrix} x_u & n_u \\ x_v & n_v \end{pmatrix}(p) = \begin{pmatrix} b_1w_1 & f_1w_2 \\ b_2w_1 & f_2w_2 \end{pmatrix}(p),$$

where $w_1 = -(k_2/k_1)s + t$ and $w_2 = -(h_2/h_1)s + t$. Since w_1 and w_2 are non-zero vectors, rank $\begin{pmatrix} x_u & n_u \\ x_v & n_v \end{pmatrix}(p) < 2$ if and only if det $\begin{pmatrix} b_1 & f_1 \\ b_2 & f_2 \end{pmatrix}(p) = 0$.

Now suppose that det $\begin{pmatrix} b_1 & f_1 \\ b_2 & f_2 \end{pmatrix}(p) \neq 0$. By the assumption $H_F(p) = 0$, we have

$$0 = \det \begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix}(p) - \det \begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}(p) = \left(-\frac{k_2}{k_1} + \frac{h_2}{h_1}\right) \det \begin{pmatrix} b_1 & f_1 \\ b_2 & f_2 \end{pmatrix}(p).$$

It follows that

$$-\frac{k_2}{k_1} + \frac{h_2}{h_1} = 0. \tag{5}$$

On the other hand, by the integrability condition (4),

$$0 = \det \begin{pmatrix} a_1 & e_1 \\ a_2 & e_2 \end{pmatrix}(p) + \det \begin{pmatrix} b_1 & f_1 \\ b_2 & f_2 \end{pmatrix}(p) = \left(\frac{h_2k_2}{h_1k_1} + 1\right) \det \begin{pmatrix} b_1 & f_1 \\ b_2 & f_2 \end{pmatrix}(p).$$

Hence, we have

$$\frac{h_2k_2}{h_1k_1} + 1 = 0. \tag{6}$$

By the Eqs. (5) and (6), we have $h_2^2/h_1^2 + 1 = 0$, and this is a contradiction. Therefore, we conclude det $\begin{pmatrix} b_1 & f_1 \\ b_2 & f_2 \end{pmatrix}(p) = 0$. It follows that (x, n) is not an immersion at p. The other cases are also proved similarly.

Conversely, if rank $\begin{pmatrix} x_u & n_u \\ x_v & n_v \end{pmatrix}(p) < 2$, then there exist $k_1, k_2 \in \mathbb{R}$ such that $k_1^2 + k_2^2 \neq 0$ and $k_1(a_1, b_1, e_1, f_1) + k_2(a_2, b_2, e_2, f_2) = 0$ at p. By substituting this relations into C_F, we have $C_F(p) = 0$. □

Remark 2 By Propositions 5 and 8, if (x, n) is a Legendre immersion around $p \in U$ and p is a singular point of x, then the Gauss curvature K or the mean curvature H must be divergence at the point p.

By Proposition 8, if $C_F(p) = 0$, then x is not a front but a frontal at the point, that is, (x, n) is not an immersion. How about the condition that the framed surface is an immersion or not? Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$. We define a smooth mapping $I_F : U \to \mathbb{R}^8$ by

$$I_F = \left(C_F, \det \begin{pmatrix} a_1 & g_1 \\ a_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}, \det \begin{pmatrix} a_1 & e_1 \\ a_2 & e_2 \end{pmatrix} \right).$$

We call the mapping $I_F : U \to \mathbb{R}^8$ a *concomitant mapping* of the framed surface (x, n, s). We say that $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ is a *framed immersion* if (x, n, s) is an immersion.

Proposition 9 *Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface and $p \in U$. Then (x, n, s) is a framed immersion around p if and only if $I_F(p) \neq 0$.*

Proof We show the necessarily part of the proposition, that is, if $I_F(p) = 0$, then (x, n, s) is not a framed immersion at p. It is enough to show that

$$\operatorname{rank} \begin{pmatrix} x_u & n_u & s_u \\ x_v & n_v & s_v \end{pmatrix} (p) < 2.$$

The above condition is equivalent to the following conditions,

$$\operatorname{rank} \begin{pmatrix} x_u & n_u \\ x_v & n_v \end{pmatrix} (p), \ \operatorname{rank} \begin{pmatrix} x_u & s_u \\ x_v & s_v \end{pmatrix} (p), \ \operatorname{rank} \begin{pmatrix} n_u & s_u \\ n_v & s_v \end{pmatrix} (p) < 2.$$

By the assumption $C_F(p) = 0$ and Proposition 8, $\operatorname{rank} \begin{pmatrix} x_u & n_u \\ x_v & n_v \end{pmatrix} (p) < 2$.

We show $\operatorname{rank} \begin{pmatrix} x_u & s_u \\ x_v & s_v \end{pmatrix} (p) < 2$. By the definition of the basic invariants, we have

$$\begin{pmatrix} x_u & s_u \\ x_v & s_v \end{pmatrix} = \begin{pmatrix} a_1 s + b_1 t & -e_1 n + g_1 t \\ a_2 s + b_2 t & -e_2 n + g_2 t \end{pmatrix}.$$

Since $J_F(p) = 0$ and $\det \begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix} (p) = 0$, there exist $k_1, k_2 \in \mathbb{R}$ such that $k_1^2 + k_2^2 \neq 0$ and $k_1(a_1, a_2) + k_2(b_1, b_2) = 0$ at p. Moreover, there exist $h_1, h_2 \in \mathbb{R}$ such that $h_1^2 + h_2^2 \neq 0$ and $h_1(e_1, e_2) + h_2(g_1, g_2) = 0$ at p. We divide into the following four cases: $k_1 h_1 \neq 0$, $k_2 h_1 \neq 0$, $k_1 h_2 \neq 0$ and $k_2 h_2 \neq 0$.

Suppose that $k_1 h_1 \neq 0$. In this case, we have $(a_1, a_2) = -(k_2/k_1)(b_1, b_2)$ and $(e_1, e_2) = -(h_2/h_1)(g_1, g_2)$ at p. Thus,

$$\begin{pmatrix} x_u & s_u \\ x_v & s_v \end{pmatrix} (p) = \begin{pmatrix} b_1 w_1 & g_1 w_2 \\ b_2 w_1 & g_2 w_2 \end{pmatrix} (p),$$

where $w_1 = -(k_2/k_1)s + t$ and $w_2 = (h_2/h_1)n + t$. Since w_1 and w_2 are non-zero vectors, rank $\begin{pmatrix} x_u & s_u \\ x_v & s_v \end{pmatrix}(p) < 2$ if and only if $\det \begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix}(p) = 0$. By the assumption $I_F(p) = 0$, we have $\det \begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix}(p) = 0$. Therefore, rank $\begin{pmatrix} x_u & s_u \\ x_v & s_v \end{pmatrix}(p) < 2$. The other cases are also proved similarly.

Next, we show rank $\begin{pmatrix} n_u & s_u \\ n_v & s_v \end{pmatrix}(p) < 2$. By the definition of the basic invariants, we have

$$\begin{pmatrix} n_u & s_u \\ n_v & s_v \end{pmatrix}(p) = \begin{pmatrix} e_1 s + f_1 t & -e_1 n + g_1 t \\ e_2 s + f_2 t & -e_2 n + g_2 t \end{pmatrix}(p).$$

Since we assume $K_F(p) = 0$ and $\det \begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix}(p) = 0$, there exist $k_1, k_2, h_1, h_2 \in \mathbb{R}$ such that $k_1^2 + k_2^2 \neq 0$, $h_1^2 + h_2^2 \neq 0$, $k_1(e_1, e_2) + k_2(f_1, f_2) = 0$ and $h_1(e_1, e_2) + h_2(g_1, g_2) = 0$ at p. We divide into the following four cases: $k_1 h_1 \neq 0$, $k_2 h_1 \neq 0$, $k_1 h_2 \neq 0$ and $k_2 h_2 \neq 0$.

Suppose that $k_1 h_1 \neq 0$. In this case, we have $(e_1, e_2) = -(k_2/k_1)(f_1, f_2)$ and $(e_1, e_2) = -(h_2/h_1)(g_1, g_2)$ at p. Thus,

$$\begin{pmatrix} n_u & s_u \\ n_v & s_v \end{pmatrix}(p) = \begin{pmatrix} f_1 w_1 & g_1 w_2 \\ f_2 w_1 & g_2 w_2 \end{pmatrix}(p),$$

where $w_1 = -(k_2/k_1)s + t$ and $w_2 = (h_2/h_1)n + t$. Since w_1 and w_2 are non-zero vectors, rank $\begin{pmatrix} n_u & s_u \\ n_v & s_v \end{pmatrix}(p) < 2$ if and only if $\det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}(p) = 0$. By the assumption $I_F(p) = 0$, we have $\det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}(p) = 0$. Therefore, rank $\begin{pmatrix} n_u & s_u \\ n_v & s_v \end{pmatrix}(p) < 2$. The other cases are also proved similarly. Therefore, (x, n, s) is not an immersion at p.

Conversely, if rank $\begin{pmatrix} x_u & n_u & s_u \\ x_v & n_v & s_v \end{pmatrix}(p) < 2$, then there exist $k_1, k_2 \in \mathbb{R}$ such that $k_1^2 + k_2^2 \neq 0$ and $k_1(a_1, b_1, e_1, f_1, g_1) + k_2(a_2, b_2, e_2, f_2, g_2) = 0$ at p. By substituting this relations into I_F, we have $I_F(p) = 0$. □

As a summary, we have the following result.

Corollary 1 *Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface and $p \in U$.*

(1) *x is an immersion (a regular surface) around p if and only if $J_F(p) \neq 0$.*
(2) *(x, n) is a Legendre immersion around p if and only if $C_F(p) \neq 0$.*
(3) *(x, n, s) is a framed immersion around p if and only if $I_F(p) \neq 0$.*

Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with I_F. We denote $I_F = (I_{F,1}, \ldots, I_{F,8})$ and $C_F = (J_F, K_F, H_F) = (I_{F,1}, I_{F,2}, I_{F,3})$. Let $\phi : V \to U, (p, q) \mapsto \phi(p, q) = (u(p, q), v(p, q))$ be a parameter change of the domain. We denote the concomitant mapping of the framed surface $(\widetilde{x}, \widetilde{n}, \widetilde{s}) = (x, n, s) \circ \phi : V \to \mathbb{R}^3 \times \Delta$ by $\widetilde{I_F}$. By Proposition 3, we have the following proposition.

Proposition 10 *Under the above notation, the concomitant mapping $\widetilde{I}_F : V \to \mathbb{R}^8$ is given by*

$$(\widetilde{I}_{F,1}(p,q), \dots, \widetilde{I}_{F,8}(p,q)) = (J_\phi(p,q) I_{F,1}(\phi(p,q)), \dots, J_\phi(p,q) I_{F,8}(\phi(p,q))).$$

Remark 3 We denote the concomitant mapping of the framed surface which given by a rotation frame (respectively, a reflection frame) by I_F^θ (respectively I_F^r). By Proposition 1 (1) and (2), we have the following.

$$\begin{aligned}
I^\theta_{F,4} &= \det\begin{pmatrix} a_1^\theta & g_1^\theta \\ a_2^\theta & g_2^\theta \end{pmatrix} = I_{F,4}\cos\theta - I_{F,5}\sin\theta - \det\begin{pmatrix} a_1 & \theta_u \\ a_2 & \theta_v \end{pmatrix}\cos\theta \\
&\quad + \det\begin{pmatrix} b_1 & \theta_u \\ b_2 & \theta_v \end{pmatrix}\sin\theta, \\
I^\theta_{F,5} &= \det\begin{pmatrix} b_1^\theta & g_1^\theta \\ b_2^\theta & g_2^\theta \end{pmatrix} = I_{F,4}\sin\theta + I_{F,5}\cos\theta - \det\begin{pmatrix} a_1 & \theta_u \\ a_2 & \theta_v \end{pmatrix}\sin\theta \\
&\quad - \det\begin{pmatrix} b_1 & \theta_u \\ b_2 & \theta_v \end{pmatrix}\cos\theta, \\
I^\theta_{F,6} &= \det\begin{pmatrix} e_1^\theta & g_1^\theta \\ e_2^\theta & g_2^\theta \end{pmatrix} = I_{F,6}\cos\theta - I_{F,7}\sin\theta - \det\begin{pmatrix} e_1 & \theta_u \\ e_2 & \theta_v \end{pmatrix}\cos\theta \\
&\quad + \det\begin{pmatrix} f_1 & \theta_u \\ f_2 & \theta_v \end{pmatrix}\sin\theta, \\
I^\theta_{F,7} &= \det\begin{pmatrix} f_1^\theta & g_1^\theta \\ f_2^\theta & g_2^\theta \end{pmatrix} = I_{F,6}\sin\theta + I_{F,7}\cos\theta - \det\begin{pmatrix} e_1 & \theta_u \\ e_2 & \theta_v \end{pmatrix}\sin\theta \\
&\quad - \det\begin{pmatrix} f_1 & \theta_u \\ f_2 & \theta_v \end{pmatrix}\cos\theta, \\
I^\theta_{F,8} &= \det\begin{pmatrix} a_1^\theta & e_1^\theta \\ a_2^\theta & e_2^\theta \end{pmatrix} = (\cos^2\theta - \sin^2\theta) I_{F,8} \\
&\quad - \cos\theta\sin\theta\left\{\det\begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} + \det\begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}\right\},
\end{aligned}$$

and

$$\begin{aligned}
I^r_{F,4} &= \det\begin{pmatrix} a_1^r & g_1^r \\ a_2^r & g_2^r \end{pmatrix} = \det\begin{pmatrix} b_1 & -g_1 \\ b_2 & -g_2 \end{pmatrix} = -\det\begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix}, \\
I^r_{F,5} &= \det\begin{pmatrix} b_1^r & g_1^r \\ b_2^r & g_2^r \end{pmatrix} = \det\begin{pmatrix} a_1 & -g_1 \\ a_2 & -g_2 \end{pmatrix} = -\det\begin{pmatrix} a_1 & g_1 \\ a_2 & g_2 \end{pmatrix}, \\
I^r_{F,6} &= \det\begin{pmatrix} e_1^r & g_1^r \\ e_2^r & g_2^r \end{pmatrix} = \det\begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix},
\end{aligned}$$

$$I^r_{F,7} = \det\begin{pmatrix} f^r_1 & g^r_1 \\ f^r_2 & g^r_2 \end{pmatrix} = \det\begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix},$$

$$I^r_{F,8} = \det\begin{pmatrix} a^r_1 & e^r_1 \\ a^r_2 & e^r_2 \end{pmatrix} = \det\begin{pmatrix} b_1 & -f_1 \\ b_2 & -f_2 \end{pmatrix} = -\det\begin{pmatrix} b_1 & f_1 \\ b_2 & f_2 \end{pmatrix} = \det\begin{pmatrix} a_1 & e_1 \\ a_2 & e_2 \end{pmatrix},$$

that is, $I^r_F = (-J_F, -K_F, H_F, -I_{F,5}, -I_{F,4}, I_{F,7}, I_{F,6}, I_{F,8})$.

Proposition 11 *Let $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface with basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$.*

(1) *Suppose that $(g_1, g_2) \neq (0, 0)$ at $p \in U$. If*

$$\det\begin{pmatrix} a_1 & g_1 \\ a_2 & g_2 \end{pmatrix} = \det\begin{pmatrix} b_1 & g_1 \\ b_2 & g_2 \end{pmatrix} = \det\begin{pmatrix} e_1 & g_1 \\ e_2 & g_2 \end{pmatrix} = \det\begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix} = 0$$

at p, then $I_F(p) = 0$.

(2) *Suppose that $(g_1, g_2) = (0, 0)$ at $p \in U$. If $C_F(p) = 0$, then $I_F(p) = 0$.*

Proof (1) By the assumptions, there exist $k_i \in \mathbb{R}$, $i = 1, \ldots, 4$ such that

$$(a_1, a_2) = k_1(g_1, g_2), (b_1, b_2) = k_2(g_1, g_2),$$
$$(e_1, e_2) = k_3(g_1, g_2), (f_1, f_2) = k_4(g_1, g_2)$$

at $p \in U$. It follows that $I_F(p) = 0$.

(2) Since $C_F(p) = 0$ and Proposition 8, (x, n) is not an immersion at $p \in U$. It follows that $\det\begin{pmatrix} a_1 & e_1 \\ a_2 & e_2 \end{pmatrix} = 0$. Hence we have $I_F(p) = 0$. □

Next, we consider parallel surfaces of framed surfaces. For a framed surface $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$, we define a parallel surface $x^\lambda : U \to \mathbb{R}^3$ of the framed surface by $x^\lambda(u, v) = x(u, v) + \lambda n(u, v)$, where $\lambda \in \mathbb{R}$.

Proposition 12 *Under the above notations, x^λ is a framed base surface. Indeed, $(x^\lambda, n, s) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.*

Proof By definition,

$$x^\lambda_u = x_u + \lambda n_u = (a_1 + \lambda e_1)s + (b_1 + \lambda f_1)t,$$
$$x^\lambda_v = x_v + \lambda n_v = (a_2 + \lambda e_2)s + (b_2 + \lambda f_2)t.$$

Thus, $x^\lambda_u \cdot n = x^\lambda_v \cdot n = 0$. Since (x, n, s) is a framed surface, we have $n \cdot s = 0$. Therefore, (x^λ, n, s) is a framed surface. □

By a direct calculation, we have the following proposition.

Proposition 13 *Let* $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ *be a framed surface with basic invariants* $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *and the concomitant mapping* I_F. *Then, the basic invariant* $(\mathcal{G}^\lambda, \mathcal{F}_1^\lambda, \mathcal{F}_2^\lambda)$ *and the concomitant mapping* I_F^λ *of the parallel surface* (x^λ, n, s) *are given by*

$$\mathcal{G}^\lambda = \mathcal{G} + \lambda \begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix}, \ \mathcal{F}_1^\lambda = \mathcal{F}_1, \ \mathcal{F}_2^\lambda = \mathcal{F}_2,$$
$$J_F^\lambda = J_F - 2H_F\lambda + K_F\lambda^2, \ K_F^\lambda = K_F, \ H_F^\lambda = H_F - K_F\lambda,$$
$$I_{F,4}^\lambda = I_{F,4} + \lambda I_{F,6}, \ I_{F,5}^\lambda = I_{F,5} + \lambda I_{F,7}, \ I_{F,6}^\lambda = I_{F,6}, \ I_{F,7}^\lambda = I_{F,7}, \ I_{F,8}^\lambda = I_{F,8}.$$

5 Framed Surfaces as One-Parameter Families of Legendre Curves Along Framed Curves

We consider a framed curve in the Euclidean space (Honda and Takahashi 2016) and a one-parameter family of Legendre curves (Fukunaga and Takahashi 2013; Takahashi 2017). We construct framed surfaces as one-parameter families of Legendre curves along the framed curves. The idea is a cut off the surface by a plane of a special direction along a space curve.

Let $I, J \subset \mathbb{R}$ be intervals with parameters u, v, respectively. For $a, b \in \mathbb{R}^3$, we denote the orthonormal plane of a through b by $\langle a \rangle_b^\perp$, that is,

$$\langle a \rangle_b^\perp = \{x \in \mathbb{R}^3 | a \cdot (x - b) = 0\}.$$

If b is the origin, then we denote $\langle a \rangle_0^\perp$ by $\langle a \rangle^\perp$ briefly.

Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature (ℓ, m, n, α), see Appendix A (cf. Honda and Takahashi 2016). We denote $\mu(u) = \nu_1(u) \times \nu_2(u)$. For each $u \in I$, we consider a Legendre curve $(x(u, \cdot), \nu^L(u, \cdot)) : J \to \langle \mu(u) \rangle_{\gamma(u)}^\perp \times (S^2 \cap \langle \mu(u) \rangle^\perp)$, that is, $x_v(u, v) \cdot \nu^L(u, v) = 0$ for all $(u, v) \in I \times J$. We identify the Euclidean plane $\mathbb{R}^2$ and the plane $\langle \mu(u) \rangle_{\gamma(u)}^\perp$ via $(a_1, a_2) \mapsto \gamma(u) + a_1\nu_1(u) + a_2\nu_2(u)$, and S^1 and $S^2 \cap \langle \mu(u) \rangle^\perp$ via $(b_1, b_2) \mapsto b_1\nu_1(u) + b_2\nu_2(u)$. We consider induced inner product on $\langle \mu(u) \rangle^\perp$ by $(a_1\nu_1(u) + a_2\nu_2(u)) \cdot (b_1\nu_1(u) + b_2\nu_2(u)) = a_1b_1 + a_2b_2$. Under the identification, $(x(u, \cdot), \nu^L(u, \cdot))$ is a Legendre curve in the sense of Appendix B (cf. Fukunaga and Takahashi 2013). The curvature of the Legendre curve $(x(u, \cdot), \nu^L(u, \cdot))$ is denoted by $(\ell^L(u, \cdot), \beta^L(u, \cdot))$. By definition, there exist functions $x_1, x_2 : I \times J \to \mathbb{R}$ such that $x : I \times J \to \mathbb{R}^3$ is given by $x(u, v) = \gamma(u) + x_1(u, v)\nu_1(u) + x_2(u, v)\nu_2(u)$. We assume that x_1 and x_2 are smooth functions, namely, x is a smooth surface. We denote $\nu^L(u, v) = \nu_1^L(u, v)\nu_1(u) + \nu_2^L(u, v)\nu_2(u)$ and $\mu^L(u, v) = -\nu_2^L(u, v)\nu_1(u) + \nu_1^L(u, v)\nu_2(u)$. We also assume that ν_1^L and ν_2^L are smooth functions. It follows that the curvature of the Legendre curve $(\ell^L, \beta^L) : I \times J \to \mathbb{R}^2$ is a smooth mapping.

Theorem 3 *Under the above notations, suppose that there exists a smooth function* $\theta : I \times J \to \mathbb{R}$ *such that* $x_u(u, v) \cdot n(u, v) = 0$ *for all* $(u, v) \in I \times J$, *where* $n(u, v) = \cos\theta(u, v)\nu^L(u, v) + \sin\theta(u, v)\mu(u)$. *We define* $s : I \times J \to S^2$ *by*

$s(u,v) = -\mu^L(u,v)$. Then $(x,n,s) : I \times J \to \mathbb{R}^3 \times \Delta$ is a framed surface with basic invariants,

$$
\begin{aligned}
a_1(u,v) &= (x_{1u}(u,v) - x_2(u,v)\ell(u))\nu_2^L(u,v) - (x_{2u}(u,v) + x_1(u,v)\ell(u))\nu_1^L(u,v),\\
b_1(u,v) &= \sin\theta(u,v)\Big((x_{1u}(u,v) - x_2(u,v)\ell(u))\nu_1^L(u,v)\\
&\quad + (x_{2u}(u,v) + x_1(u,v)\ell(u))\nu_2^L(u,v)\Big)\\
&\quad - \cos\theta(u,v)(\alpha(u) + x_1(u,v)m(u) + x_2(u,v)n(u)),\\
a_2(u,v) &= -\beta^L(u,v),\\
b_2(u,v) &= 0,\\
e_1(u,v) &= \sin\theta(u,v)(n(u)\nu_1^L(u,v) - m(u)\nu_2^L(u,v))\\
&\quad + \cos\theta(u,v)(\nu_{1u}^L(u,v)\nu_2^L(u,v) - \nu_{2u}^L(u,v)\nu_1^L(u,v) - \ell(u)),\\
f_1(u,v) &= -\theta_u(u,v) - m(u)\nu_1^L(u,v) - n(u)\nu_2^L(u,v),\\
g_1(u,v) &= \sin\theta(u,v)(\nu_{2u}^L(u,v)\nu_1^L(u,v) - \nu_{1u}^L(u,v)\nu_2^L(u,v) + \ell(u))\\
&\quad + \cos\theta(u,v)(n(u)\nu_1^L(u,v) - m(u)\nu_2^L(u,v)),\\
e_2(u,v) &= -\cos\theta(u,v)\ell^L(u,v),\\
f_2(u,v) &= -\theta_v(u,v),\\
g_2(u,v) &= \sin\theta(u,v)\ell^L(u,v).
\end{aligned}
$$

Proof By definition, we have $n(u,v)\cdot s(u,v) = 0$ for all $(u,v) \in I \times J$. It follows that $(n,s) \in \Delta$. By the assumption, we have $x_u(u,v)\cdot n(u,v) = 0$ for all $(u,v) \in I \times J$. Since $x_v(u,v)\cdot \nu^L(u,v) = 0$, we have

$$
\begin{aligned}
x_v(u,v)\cdot n(u,v) &= (x_{1v}(u,v)\nu_1(u) + x_{2v}\nu_2(u))\\
&\quad\cdot(\cos\theta(u,v)\nu^L(u,v) + \sin\theta(u,v)\mu(u))\\
&= \cos\theta(u,v)(x_{1v}(u,v)\nu_1^L(u,v) + x_{2v}(u,v)\nu_2^L(u,v)) = 0
\end{aligned}
$$

for all $(u,v) \in I \times J$. Hence $(x,n,s) : I \times J \to \mathbb{R}^3 \times \Delta$ is a framed surface. We omit (u,v) and u below. By a direct calculation, we have

$$
\begin{aligned}
x_u &= (x_{1u} - x_2\ell)\nu_1 + (x_{2u} + x_1\ell)\nu_2 + (\alpha + x_1 m + x_2 n)\mu,\\
x_v &= x_{1v}\nu_1 + x_{2v}\nu_2,\\
n &= \cos\theta\nu_1^L\nu_1 + \cos\theta\nu_2^L\nu_2 + \sin\theta\mu,\\
s &= \nu_2^L\nu_1 - \nu_1^L\nu_2,\\
t &= n \times s = \sin\theta\nu_1^L\nu_1 + \sin\theta\nu_2^L\nu_2 - \cos\theta\mu,\\
n_u &= (-\theta_u\sin\theta\nu_1^L + \cos\theta\nu_{1u}^L - \cos\theta\nu_2^L\ell - \sin\theta m)\nu_1\\
&\quad + (-\theta_u\sin\theta\nu_2^L + \cos\theta\nu_1^L\ell + \cos\theta\nu_{2u}^L - \sin\theta n)\nu_2\\
&\quad + \cos\theta(\nu_1^L m + \nu_2^L n + \theta_u)\mu,\\
s_u &= (\nu_{2u}^L + \nu_1^L\ell)\nu_1 + (-\nu_{1u}^L + \nu_2^L\ell)\nu_2 + (\nu_2^L m - \nu_1^L n)\mu,
\end{aligned}
$$

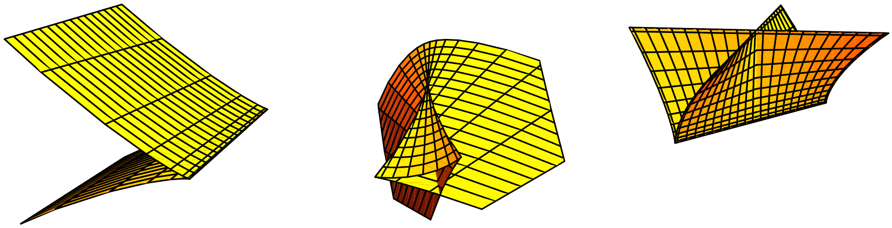

Fig. 1 Cuspidal edge, swallowtail and cuspidal cross cap, respectively

$$n_v = -\theta_v \sin\theta \nu^L + \cos\theta \nu^L_v + \theta_v \cos\theta \mu,$$
$$s_v = \ell^L \nu^L.$$

It follows that we have the basic invariants as the above. □

By a direct calculation, we have the following condition:

$$\begin{aligned} x_u(u,v)\cdot n(u,v) &= (x_{1u}(u,v) - x_2(u,v)\ell(u))\cos\theta(u,v)\nu^L_1(u,v) \\ &\quad + (x_{2u}(u,v) + x_1(u,v)\ell(u))\cos\theta(u,v)\nu^L_2(u,v) \\ &\quad + (\alpha(u) + x_1(u,v)m(u) + x_2(u,v)n(u))\sin\theta(u,v) \\ &= 0 \end{aligned}$$

for all $(u, v) \in I \times J$.

By the above construction, we say that the framed surface (x, n, s) is *a one-parameter family of Legendre curves along a framed curve*.

As an application of Theorem 3, we give a condition that the surface x is diffeomorphic to the cuspidal edge, the swallowtail and the cuspidal cross cap, see Figure 1 and Examples 1, 2 and 3 of Sect. 6 for definitions (Fig. 1).

We recall the criteria for singularities of frontals stated in Fujimori et al. (2008), Kokubu et al. (2005) (see also, Izumiya and Saji 2010). Let $x : U \to \mathbb{R}^3$ be the frontal of a Legendre surface (x, n). We define a function $\lambda : U \to \mathbb{R}$ by $\lambda(u, v) = \det(x_u, x_v, n)(u, v)$ where (u, v) is a coordinate system on U. We call λ a *discriminant function* (or, a *signed area density function*). When a singular point p of x is non-degenerate, that is, $d\lambda(p) \neq 0$, there exists a smooth parametrization $\delta(t) : (-\varepsilon, \varepsilon) \to U$, $\delta(0) = p$ of the singular set $S(x)$. We call the curve $\delta(t)$ the singular curve of x. Moreover, there exists a smooth vector field $\eta(t)$ along δ satisfying that $\eta(t)$ generates $\ker dx_{\delta(t)}$. Now we define a function $\phi_x(t)$ on $(-\varepsilon, \varepsilon)$ by $\phi_x(t) = \det((x \circ \delta)', n \circ \delta, dn(\eta))(t)$. By using these notations, we have the following theorem.

Theorem 4 (Fujimori et al. 2008; Kokubu et al. 2005) *Let $(x, n) : U \to \mathbb{R}^3 \times S^2$ be a Legendre surface and $p \in U$ be a non-degenerate singular point of x. Then the following assertions hold.*

(1) *If $\eta\lambda(p) \neq 0$, then x is a front near p if and only if $\phi_x(0) \neq 0$ holds.*
(2) *The map germ x at p is $\mathscr{A}$-equivalent to the cuspidal edge if and only if x is a front near p and $\eta\lambda(p) \neq 0$ hold.*

(3) *The map germ x at p is $\mathscr{A}$-equivalent to the swallowtail if and only if x is a front near p and $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$ hold.*
(4) *The map germ x at p is $\mathscr{A}$-equivalent to the cuspidal cross cap if and only if $\eta\lambda(p) \neq 0$, $\phi_x(0) = 0$ and $\phi'_x(0) \neq 0$ hold.*

Here, $\eta\lambda : U \to \mathbb{R}$ means the directional derivative of λ by the vector field $\tilde{\eta}$, where $\tilde{\eta}$ is an extended vector field of η to U.

In this paper, if there is no confusion, we denote $\tilde{\eta}$ by η. By using the above theorem, we give criteria of singular points of the framed base surface which is given by a one-parameter family of Legendre curves along a framed curve.

Theorem 5 *Let $(x, n, s) : I \times J \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of Legendre curves along a framed curve. Suppose that $x(u, 0) = \gamma(u)$, the set of singular points of γ is dense in I and $(0, 0)$ is a non-degenerate singular point of x. Then we have the following two cases.*

(A) Suppose that $\beta^L(0, 0) = 0$ and $\alpha(0) \neq 0$.

(1) x at $(0, 0)$ is $\mathscr{A}$-equivalent to the cuspidal edge if and only if $\beta^L_v(0, 0) \neq 0$ and $\ell^L(0, 0) \neq 0$.

(2) x at $(0, 0)$ is $\mathscr{A}$-equivalent to the swallowtail if and only if $\beta^L_v(0, 0) = 0$, $\beta^L_{vv}(0, 0) \neq 0$, $\beta^L_u(0, 0) \neq 0$ and $\ell^L(0, 0) \neq 0$.

(3) x at $(0, 0)$ is $\mathscr{A}$-equivalent to the cuspidal cross cap if and only if $\beta^L_v(0, 0) \neq 0$, $\ell^L(0, 0) = 0$ and $(\ell^L \circ \delta)'(0) \neq 0$.

(B) Suppose that $\beta^L(0, 0) \neq 0$ and $\alpha(0) = 0$.

(1) x at $(0, 0)$ is $\mathscr{A}$-equivalent to the cuspidal edge if and only if $\alpha'(0) \neq 0$ and $\nu^L_1(0, 0)m(0) + \nu^L_2(0, 0)n(0) \neq 0$.

(2) x at $(0, 0)$ is $\mathscr{A}$-equivalent to the swallowtail if and only if $\alpha'(0) = 0$, $\alpha''(0) \neq 0$, $\nu^L_2(0, 0)m(0) - \nu^L_1(0, 0)n(0) \neq 0$ and $\nu^L_1(0, 0)m(0) + \nu^L_2(0, 0)n(0) \neq 0$.

(3) x at $(0, 0)$ is $\mathscr{A}$-equivalent to the cuspidal cross cap if and only if $\alpha'(0) \neq 0$, $\nu^L_1(0, 0)m(0) + \nu^L_2(0, 0)n(0) = 0$ and $((\beta^L(\nu^L_1 m + \nu^L_2 n + \theta_u) + a_1\theta_v) \circ \delta)'(0) \neq 0$.

Here δ is a singular curve of x.

Proof Let $x(u, v) = \gamma(u) + x_1(u, v)\nu_1(u) + x_2(u, v)\nu_2(u)$. By the assumption $\gamma(u) = x(u, 0)$, we have $x_1(u, 0) = x_2(u, 0) = 0$ for all $u \in I$. Moreover, since the set of singular points of γ is dense in I and $x_u(u, v) \cdot n(u, v) = 0$, we have $\sin\theta(u, 0) = 0$ and hence $\cos\theta(u, 0) = \pm 1$. By $b_2(u, v) = 0$ in Theorem 3, we have $\lambda(u, v) = -b_1(u, v)a_2(u, v) = \beta^L(u, v)b_1(u, v)$. Since $(0, 0)$ is a non-degenerate singular point of x, we divide two cases: (A) $\beta^L(0, 0) = 0$ and $b_1(0, 0) \neq 0$, (B) $\beta^L(0, 0) \neq 0$ and $b_1(0, 0) = 0$. Moreover, we have $\lambda_u(0, 0) \neq 0$ or $\lambda_v(0, 0) \neq 0$. By the integrability condition of $a_1e_2 + b_1f_2 = a_2e_1 + b_2f_1$, we have $\alpha\theta_v = -\beta^L(\nu^L_{1u}\nu^L_2 - \nu^L_{2u}\nu^L_1 - \ell)$ at $(0, 0)$. The other integrability conditions automatically hold at $(0, 0)$.

First we consider the case (A). By Theorem 3, $b_1(0, 0) \neq 0$ if and only if $\alpha(0) \neq 0$. Moreover, $b_1(u, 0) = \pm\alpha(u) \neq 0$ around $0 \in I$. Therefore, γ is a regular curve around $0 \in I$. In this case, (u, v) is a singular point of x if and only if $\beta^L(u, v) = 0$. Since $dx = x_u du + x_v dv = (a_1 s + b_1 t)du + a_2 s dv$ and $a_2(u, v) = -\beta^L(u, v)$, the null vector field η is given by $\partial/\partial v$. Therefore, the condition $\eta\lambda(0, 0) \neq 0$ is equivalent to $\beta^L_v(0, 0) \neq 0$, and the conditions $\eta\lambda(0, 0) = 0$ and $\eta\eta\lambda(0, 0) \neq 0$ are equivalent to $\beta^L_v(0,0) = 0$ and $\beta^L_{vv}(0,0) \neq 0$. Since $(0,0)$ is a non-degenerate singular point of x, we have $\beta^L_u(0,0) \neq 0$ or $\beta^L_v(0,0) \neq 0$. By the integrability condition, we have $\theta_v(0,0) = 0$. By a direct calculation, we have $K_F = -\ell^L(\nu^L_1 m + \nu^L_2 n)$ and $H_F = \alpha\ell^L$ at $(0,0)$. It follows that x is a front around $(0,0)$ if and only if $\ell^L(0,0) \neq 0$ by Proposition 8. Therefore, by Theorem 4, x at $(0,0)$ is $\mathscr{A}$-equivalent to the cuspidal edge (respectively, the swallowtail) if and only if $\beta^L_v(0,0) \neq 0$ and $\ell^L(0,0) \neq 0$ (respectively, $\beta^L_v(0,0) = 0$, $\beta^L_{vv}(0,0) \neq 0$, $\beta^L_u(0,0) \neq 0$ and $\ell^L(0,0) \neq 0$).

We now consider the condition for the cuspidal cross cap. Since $\eta\lambda(0,0) = \beta^L_v(0,0) \neq 0$, the singular curve δ is given by the form $\delta(t) = (t, v(t))$, where v is a smooth function with $v(0) = 0$. By a direct calculation,

$$\begin{aligned}
(x\circ\delta)' &= (\alpha + x_1 m + x_2 n)\mu + (x_{1u} - \beta^L\nu^L_2 v' - x_2\ell)\nu_1 + (x_{2u} + \beta^L\nu^L_1 v' + x_1\ell)\nu_2\\
n\circ\delta &= \cos\theta(\nu^L_1\nu_1 + \nu^L_2\nu_2) + \sin\theta\mu\\
dn(\eta) &= (-\theta_v\sin\theta\nu^L_1 - \cos\theta\ell^L\nu^L_2)\nu_1 + (-\theta_v\sin\theta\nu^L_2 + \cos\theta\ell^L\nu^L_1)\nu_2 + \theta_v\cos\theta\mu.
\end{aligned}$$

By straightforward calculations, we have

$$\begin{aligned}
\phi_x &= \det((x\circ\delta)', n\circ\delta, dn(\eta))\\
&= (\alpha + x_1 m + x_2 n)\ell^L + (x_{1u} - \beta^L\nu^L_2 v' - x_2\ell)(\theta_v\nu^L_2 - \sin\theta\cos\theta\ell^L\nu^L_1)\\
&\quad + (x_{2u} + \beta^L\nu^L_1 v' + x_1\ell)(-\theta_v\nu^L_1 - \sin\theta\cos\theta\ell^L\nu^L_2).
\end{aligned}$$

It follows that $\phi_x(0) = \alpha(0)\ell^L(0,0)$ and $\phi'_x(0) = \alpha(0)(\ell^L\circ\delta)'(0)$ under the condition $\phi_x(0) = 0$. Therefore, by Theorem 5, x at $(0,0)$ is $\mathscr{A}$-equivalent to the cuspidal cross cap if and only if $\beta^L_v(0,0) \neq 0$, $\ell^L(0,0) = 0$ and $(\ell^L\circ\delta)'(0) \neq 0$.

Second we consider the case (B). Since $b_1(0,0) = \mp\alpha(0) = 0$, 0 is a singular point of γ. In this case, (u,v) is a singular point of x if and only if $b_1(u,v) = 0$. Since $dx = x_u du + x_v dv = (a_1 s + b_1 t)du + a_2 s dv = a_1 s du - \beta^L s dv$ on the singular set of x, the null vector field η is given by $\beta^L(u,v)\partial/\partial u + a_1(u,v)\partial/\partial v$. Note that we have $a_1(u,0) = 0$ for all $u \in I$. Therefore, the condition $\eta\lambda(0,0) \neq 0$ is equivalent to $\alpha'(0) \neq 0$, and the conditions $\eta\lambda(0,0) = 0$ and $\eta\eta\lambda(0,0) \neq 0$ are equivalent to $\alpha'(0) = 0$ and $\alpha''(0) \neq 0$. Since $(0,0)$ is a non-degenerate singular point of x, we have $b_{1u}(0,0) \neq 0$ or $b_{1v}(0,0) \neq 0$, that is, $\alpha'(0) \neq 0$ or $\nu^L_2(0,0)m(0) - \nu^L_1(0,0)n(0) \neq 0$. By a direct calculation and the integrability condition, we have $K_F = -\ell^L(\nu^L_1 m + \nu^L_2 n)$ and $H_F = (1/2)\beta^L(\nu^L_1 m + \nu^L_2 n)$ at $(0,0)$. It follows that x is a front around $(0,0)$ if and only if $\nu^L_1(0,0)m(0) + \nu^L_2(0,0)n(0) \neq 0$ by Proposition 8. Therefore, by Theorem 4, x at $(0,0)$ is $\mathscr{A}$-equivalent to the cuspidal edge (respectively, the swallowtail) if and only if $\alpha'(0) \neq 0$ and $\nu^L_1(0,0)m(0) + \nu^L_2(0,0)n(0) \neq 0$ (respectively, $\alpha'(0) = 0$, $\alpha''(0) \neq 0$, $\nu^L_2(0,0)m(0) - \nu^L_1(0,0)n(0) \neq 0$ and $\nu^L_1(0,0)m(0) + \nu^L_2(0,0)n(0) \neq 0$).

We now consider the condition for the cuspidal cross cap. Since $\eta\lambda(0,0) \neq 0$ is equivalent to $\alpha'(0) \neq 0$, the singular curve δ is given by the form $\delta(t) = (u(t), t)$, where u is a smooth function with $u(0) = 0$. By a direct calculation and $b_1(u(t), t) = 0$,

$$
\begin{aligned}
(x\circ\delta)' &= (\alpha + x_1 m + x_2 n)u'\mu + (x_{1u}u' - \beta^L\nu_2^L - x_2\ell u')\nu_1 + (x_{2u}u' \\
&\quad + \beta^L\nu_1^L + x_1\ell u')\nu_2 \\
&= \tan\theta((x_{1u} - x_2\ell)\nu_1^L + (x_{2u} + x_1\ell)\nu_2^L)u'\mu \\
&\quad + (x_{1u}u' - \beta^L\nu_2^L - x_2\ell u')\nu_1 + (x_{2u}u' + \beta^L\nu_1^L + x_1\ell u')\nu_2 \\
n\circ\delta &= \cos\theta(\nu_1^L\nu_1 + \nu_2^L\nu_2) + \sin\theta\mu \\
dn(\eta) &= (\sin\theta(-\theta_u\beta^L\nu_1^L - \beta^L m - \theta_v a_1\nu_1^L) \\
&\quad + \cos\theta(\beta^L\nu_{1u}^L - \beta^L\nu_2\ell - a_1\ell^L\nu_2^L))\nu_1 \\
&\quad + (\sin\theta(-\theta_u\beta^L\nu_2^L - \beta^L n - \theta_v a_1\nu_2^L) + \cos\theta(\beta^L\nu_{2u}^L \\
&\quad + \beta^L\nu_1\ell + a_1\ell^L\nu_1^L))\nu_2 \\
&\quad + \cos\theta(\beta^L(\nu_1^L m + \nu_2 n + \theta_u) + a_1\theta_v)\mu.
\end{aligned}
$$

By straightforward calculations, we have

$$
\begin{aligned}
\phi_x &= \det((x\circ\delta)', n\circ\delta, dn(\eta)) \\
&= \sin\theta\left((x_{1u} - x_2\ell)\nu_1^L + (x_{2u} + x_1\ell)\nu_2^L\right)u' \\
&\quad \times\left(\sin\theta\beta^L(-\nu_1^L n + \nu_2^L m) + \cos\theta(\beta^L\nu_1^L\nu_{2u}^L - \beta^L\nu_2^L\nu_{1u}^L + \beta^L\ell + a_1\ell^L)\right) \\
&\quad + (x_{1u}u' - \beta^L\nu_2^L - x_2\ell u')\Big(\cos^2\theta\nu_2^L(\beta^L(\nu_1^L m + \nu_2^L n + \theta_u) + a_1\theta_v) \\
&\quad - \sin\theta(\sin\theta(-\theta_u\beta^L\nu_2^L - \beta^L n - \theta_v a_1\nu_2^L) \\
&\quad + \cos\theta(\beta^L\nu_{2u}^L + \beta^L\nu_1^L\ell + a_1\ell^L\nu_1^L))\Big) \\
&\quad + (x_{2u}u' + \beta^L\nu_1^L + x_1\ell u')\Big(-\cos^2\theta\nu_1^L(\beta^L(\nu_1^L m + \nu_2^L n + \theta_u) + a_1\theta_v) \\
&\quad + \sin\theta(\sin\theta(-\theta_u\beta^L\nu_1^L - \beta^L m - \theta_v a_1\nu_1^L) \\
&\quad + \cos\theta(\beta^L\nu_{1u}^L - \beta^L\nu_2^L\ell - a_1\ell^L\nu_2^L))\Big).
\end{aligned}
$$

It follows that $\phi_x(0) = -(\beta^L(0,0))^2(\nu_1^L(0,0)m(0) + \nu_2^L(0,0)n(0))$, and $\phi_x'(0) = (\beta^L(\nu_1^L m + \nu_2^L n + \theta_u) + a_1\theta_v)\circ\delta)'(0)$ under the condition $\phi_x(0) = 0$. Therefore, by Theorem 5, x at $(0,0)$ is $\mathscr{A}$-equivalent to the cuspidal cross cap if and only if $\alpha'(0) \neq 0$, $\nu_1^L(0,0)m(0) + \nu_2^L(0,0)n(0) = 0$ and $(\beta^L(\nu_1^L m + \nu_2^L n + \theta_u) + a_1\theta_v)\circ\delta)'(0) \neq 0$. This complete the proof of the Theorem. □

Remark 4 Under the same assumptions in Theorem 5, if $\gamma(u)$ is the image of the singular curve of x, then it holds that the singular set is $S(x) = \{(u,0) | u \in I\}$ and one has the case (A). Since the null vector field η and the singular direction δ' are linearly independent at $(0,0)$, the singular point $(0,0)$ can not be the swallowtail.

Remark 5 The conditions $\nu_2^L(0,0)m(0) - \nu_1^L(0,0)n(0) \neq 0$, $\nu_1^L(0,0)m(0) + \nu_2^L(0,0)n(0) \neq 0$ in Theorem 5 (B) (2) is equivalent to the condition $(m(0), n(0)) \neq (0,0)$.

Corollary 2 *Let $(x, n, s) : I \times J \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of Legendre curves along a framed curve. Suppose that $\gamma : I \to \mathbb{R}^3$ is a regular curve, $x(u, \cdot) : J \to \langle \mu(u)\rangle^{\perp}_{\gamma(u)}$ is diffeomorphic to the 3/2-cusp at $0 \in J$ and $x(u, 0) = \gamma(u)$ for all $u \in I$. Then $x : I \times J \to \mathbb{R}^3$ is a front around $(u, 0)$. More precisely, $(x, n) : I \times J \to \mathbb{R}^3 \times S^2$ is a Legendre immersion around $(u, 0)$. Moreover, x is diffeomorphic to the cuspidal edge at $(u, 0)$.*

Proof Since γ is a regular curve, we have $\alpha(u) \neq 0$ for all $u \in I$. Moreover, $x(u, \cdot)$ is diffeomorphic to the 3/2-cusp at $0 \in J$ if and only if $x_v(u, 0) = 0$ and $\det(x_{vv}(u, 0), x_{vvv}(u, 0)) \neq 0$, for all $u \in I$ (cf. Bruce and Giblin 1992; Fukunaga and Takahashi 2014; Ishikawa 2007). By the definition of the curvature $(\ell^L(u, v), \beta^L(u, v))$ of the Legendre curve $(x(u, \cdot), \nu^L(u, \cdot))$, we have

$$\begin{aligned}
x_v(u, v) &= \beta^L(u, v)\mu^L(u, v),\\
x_{vv}(u, v) &= \beta^L_v(u, v)\mu^L(u, v) - \beta^L(u, v)\ell^L(u, v)\nu^L(u, v)\\
x_{vvv}(u, v) &= (\beta^L_{vv}(u, v) - \beta^L(u, v)(\ell^L(u, v))^2)\mu^L(u, v)\\
&\quad -2\beta^L_v(u, v)\ell^L(u, v)\nu^L(u, v).
\end{aligned}$$

It follows that $\beta^L(u, 0) = 0, \beta^L_v(u, 0) \neq 0$ and $\ell^L(u, 0) \neq 0$ for all $u \in I$. Since $x(u, 0) = \gamma(u)$, we have $x_1(u, 0) = x_2(u, 0) = 0$ for all $u \in I$. Therefore $x_{1u}(u, 0) = x_{2u}(u, 0) = 0$. Moreover, by the condition $x_u(u, v) \cdot n(u, v) = 0$ for all $(u, v) \in I \times J$, we have $\alpha(u) \sin\theta(u, 0) = 0$ and hence $\sin\theta(u, 0) = 0$. Then $a_1(u, 0) = 0, b_1(u, 0) = -\cos\theta(u, 0)\alpha(u), a_2(u, 0) = -\beta^L(u, 0), b_2(u, 0) = 0, e_2(u, 0) = -\cos\theta(u, 0)\ell^L(u, 0), f_2(u, 0) = -\theta_v(u, 0), g_2(u, 0) = 0$. It follows that $H_F(u, 0) = (1/2)\cos^2\theta(u, 0)\alpha(u)\ell^L(u, 0) \neq 0$ for all $u \in I$. By Proposition 8, (x, n) is a Legendre immersion around $(u, 0)$. Hence, x is a front around $(u, 0)$. Moreover, by Theorem 5 (A) (1), x is diffeomorphic to the cuspidal edge at $(u, 0)$. □

We also have the following result.

Theorem 6 *Suppose that $x : U \to \mathbb{R}^3$ is diffeomorphic to the cuspidal edge at $0 \in U$. Then there exist a parameter change $\phi : I \times J \to U$ around 0 and a smooth mapping $(n, s) : I \times J \to \Delta$ such that the framed surface $(x \circ \phi, n, s) : I \times J \to \mathbb{R}^3 \times \Delta$ is given by a one-parameter family of 3/2-cusp at $0 \in J$ along a regular curve $\gamma : I \to \mathbb{R}^3$ around $0 \in I$.*

Proof The normal form of cuspidal edge by using coordinate transformations on the source and isometries on the target is given by Martins and Saji (2016). Since the property of one-parameter family of Legendre curves along a framed curve are invariant as isometries on the target, there exists a parameter change $\phi : I \times J \to U$ around 0 such that $\widetilde{x} = x \circ \phi$ is given by the following form around $(0, 0) \in I \times J$:

$$\widetilde{x}(u, v) = \left(u, a(u) + \frac{v^2}{2}, b(u) + v^2 b_2(u) + v^3 b_3(u, v)\right),$$

where $a(0) = \dot{a}(0) = b(0) = \dot{b}(0) = b_2(0) = 0$ and $b_3(0,0) \neq 0$, by the proof of Theorem 3.1 in Martins and Saji (2016). Here we relabelled the coefficient functions.

We define a regular curve $\gamma : I \to \mathbb{R}^3$, $\gamma(u) = (u, 0, 0)$. If we take $(\nu_1, \nu_2) : I \to \Delta$ by $\nu_1(u) = (0, 1, 0)$, $\nu_2(u) = (0, 0, 1)$, then $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a framed curve. By $\widetilde{x}_v(u, v) = (0, v, 2vb_2(u)+3v^2b_3(u, v)+v^3b_{3v}(u, v))$, we have $\nu^L(u, v) = \nu_1^L(u, v)\nu_1(u) + \nu_2^L(u, v)\nu_2(u)$ and $\mu^L(u, v) = -\nu_2^L(u, v)\nu_1(u) + \nu_1^L(u, v)\nu_2(u)$, where

$$\nu_1^L(u, v) = -\frac{2b_2(u) + 3vb_3(u, v) + v^2b_{3v}(u, v)}{\sqrt{(2b_2(u) + 3vb_3(u, v) + v^2b_{3v}(u, v))^2 + 1}},$$
$$\nu_2^L(u, v) = \frac{1}{\sqrt{(2b_2(u) + 3vb_3(u, v) + v^2b_{3v}(u, v))^2 + 1}}.$$

It follows that the curvature of the Legendre curve $(\widetilde{x}(u, \cdot), \nu^L(u, \cdot))$ is given by

$$\ell^L(u, v) = \frac{3b_3(u, v) + 5vb_{3v}(u, v) + v^2b_{3vv}(u, v)}{(2b_2(u) + 3vb_3(u, v) + v^2b_{3v}(u, v))^2 + 1},$$
$$\beta^L(u, v) = -v\sqrt{(2b_2(u) + 3vb_3(u, v) + v^2b_{3v}(u, v))^2 + 1}.$$

We denote

$$\varphi(u, v) = \frac{a'(u)(2b^2 + 3vb_3(u, v) + v^2b_{3v}(u, v)) + b'(u) + v^2b_2'(u) + v^3b_{3u}(u, v)}{\sqrt{(2b_2(u) + 3vb_3(u, v) + v^2b_{3v}(u, v))^2 + 1}}.$$

Then we define a smooth mapping $(n, s) : I \times J \to \Delta$ by

$$n(u, v) = \frac{1}{\sqrt{1 + \varphi^2(u, v)}}\nu^L(u, v) - \frac{\varphi(u, v)}{\sqrt{1 + \varphi^2(u, v)}}\mu(u), s(u, v) = -\mu^L(u, v).$$

Since $\widetilde{x}_u(u, v) = (1, a'(u), b'(u) + v^2b_2'(u) + v^3b_{3u}(u, v))$, we have $\widetilde{x}_u(u, v) \cdot n(u, v) = 0$ for all $(u, v) \in I \times J$. It follows from Theorem 3 that $(\widetilde{x}, n, s)$ is a framed surface. Moreover, since $x_1(u, v) = a(u) + v^2/2$ and $x_2(u, v) = b(u) + v^2b_2(u, v) + v^3b_3(u, v)$, we have

$$(x_1, x_2)_v(u, v) = (v, 2vb_2(u) + 3v^2b_3(u, v) + v^3b_{3v}(u, v)),$$
$$(x_1, x_2)_{vv}(u, v) = (1, 2b_2(u) + 6vb_3(u, v) + 6v^2b_{3v}(u, v) + v^3b_{3vv}(u, v)),$$
$$(x_1, x_2)_{vvv}(u, v) = (0, 6b_3(u, v) + 18vb_{3v}(u, v) + 9v^2b_{3vv}(u, v) + v^3b_{3vvv}(u, v)).$$

It follows that $(x_1, x_2)_v(u, 0) = 0$ and $\det((x_1, x_2)_{vv}(u, 0), (x_1, x_2)_{vvv}(u, 0)) = 6b_3(u, 0) \neq 0$ around $(0, 0) \in I \times J$. Therefore, $(u, 0)$ is a 3/2-cusp of $\widetilde{x}(u, \cdot)$ around $0 \in I$. □

The singularities of the swallowtail and of the cuspidal cross cap are more complicated (cf. Fukui 2017; Oset Sinha and Saji 2017; Saji 2017). The corresponding results for

Corollary 2 and Theorem 6 of the swallowtail and the cuspidal cross cap (and other singularities) are future problems (cf. Fukunaga and Takahashi 2018).

6 Examples

We give typical examples of singularities of smooth surfaces. We detect the basic invariants and curvatures of framed surfaces.

Example 1 (cuspidal edge) A singular point $p \in U$ of a mapping $x : U \to \mathbb{R}^3$ is called a *cuspidal edge* if the map germ x at p is $\mathscr{A}$-equivalent (right-left equivalent) to the $(u, v) \mapsto (u, v^2, v^3)$ at 0. Let $x : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $x(u, v) = (u, v^2, v^3)$. If we take $(n, s) : U \to \Delta$, $n(u, v) = (1/\sqrt{9v^2+4})(0, -3v, 2)$, $s(u, v) = (1, 0, 0)$, then $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface. Since $t(u, v) = (1/\sqrt{9v^2+4})(0, 2, 3v)$, we have the following basic invariants.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & v\sqrt{9v^2+4} \end{pmatrix}, \quad \begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6/(9v^2+4) & 0 \end{pmatrix}.$$

It follows that the curvature C_F of (x, n, s) is given by

$$J_F(u, v) = v\sqrt{9v^2+4}, \; K_F(u, v) = 0, \; H_F(u, v) = \frac{3}{9v^2+4}.$$

Example 2 (swallowtail) A singular point $p \in U$ of a mapping $x : U \to \mathbb{R}^3$ is called a *swallowtail* if the map germ x at p is $\mathscr{A}$-equivalent to the $(u, v) \mapsto (3u^4 + u^2v, -4u^3 - 2uv, v)$ at 0. Let $x : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $x(u, v) = (3u^4 + u^2v, -4u^3 - 2uv, v)$. If we take $(n, s) : U \to \Delta, n(u, v) = (1/\sqrt{1+u^2+u^4})(1, u, u^2)$, $s(u, v) = (1/\sqrt{1+u^2})(u, -1, 0)$, then $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.

Since $t(u, v) = (1/\sqrt{1+u^2+u^4}\sqrt{1+u^2})(u^2, u^3, -1-u^2)$, we have the following basic invariants.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} (12u^2+2v)\sqrt{1+u^2} & 0 \\ \frac{u(2+u^2)}{\sqrt{1+u^2}} & -\frac{\sqrt{1+u^2+u^4}}{\sqrt{1+u^2}} \end{pmatrix},$$
$$\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{1+u^2+u^4}\sqrt{1+u^2}} & -\frac{u(2+u^2)}{(1+u^2+u^4)\sqrt{1+u^2}} & \frac{u^2}{(1+u^2)\sqrt{1+u^2+u^4}} \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that the curvature C_F of (x, n, s) is given by

$$J_F(u, v) = 2(6u^2+v)\sqrt{1+u^2+u^4}, \; K_F(u, v) = 0,$$
$$H_F(u, v) = -\frac{1+5u^2+5u^4+u^6}{2(1+u^2+u^4)(1+u^2)}.$$

Example 3 (cuspidal cross cap) A singular point $p \in U$ of a mapping $x : U \to \mathbb{R}^3$ is called a *cuspidal cross cap* if the map germ x at p is $\mathscr{A}$-equivalent to the $(u, v) \mapsto$

(u, v^2, uv^3) at 0. Let $x : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $x(u, v) = (u, v^2, uv^3)$. If we take $(n, s) : U \to \Delta$,

$$n(u, v) = \frac{1}{\sqrt{4v^6 + 9u^2v^2 + 4}}(-2v^3, -3uv, 2),\ s(u, v) = \frac{1}{\sqrt{1 + v^6}}(1, 0, v^3),$$

then $(x, n, s) : U \to \mathbb{R}^3 \times \Delta$ is a framed surface.

Since $t(u, v) = (1/\sqrt{4v^6 + 9u^2v^2 + 4}\sqrt{1 + v^6})(-3uv^4, 2v^6 + 2, 3uv)$, we have the following basic invariants.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1 + v^6} & 0 \\ \frac{3uv^5}{\sqrt{1+v^6}} & \frac{v\sqrt{4v^6+9u^2v^2+4}}{\sqrt{1+v^6}} \end{pmatrix},$$

$$\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{6v\sqrt{1+v^6}}{4v^6+9u^2v^2+4} & 0 \\ -\frac{6v^2}{\sqrt{4v^6+9u^2v^2+4}\sqrt{1+v^6}} & \frac{6u(2v^6-1)}{(4v^6+9u^2v^2+4)\sqrt{1+v^6}} & \frac{9uv^3}{\sqrt{4v^6+9u^2v^2+4}(1+v^6)} \end{pmatrix}.$$

It follows that the curvature C_F of (x, n, s) is given by $J_F(u, v) = v\sqrt{4v^6 + 9u^2v^2 + 4}$,

$$K_F(u, v) = -\frac{36v^3}{(4v^6 + 9u^2v^2 + 4)^{3/2}},\ H_F(u, v) = -\frac{3u(5v^6 - 1)}{4v^6 + 9u^2v^2 + 4}.$$

Example 4 (cross cap) A singular point $p \in U$ of a mapping $x : U \to \mathbb{R}^3$ is called a *cross cap* if the map germ x at p is $\mathscr{A}$-equivalent to the $(u, v) \mapsto (u, v^2, uv)$ at 0. Let $x : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $x(u, v) = (u, v^2, uv)$. Then it is well-known that the cross cap is not a frontal. However, if we consider the polar coordinate $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$, $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$, then $x \circ \phi$ is a frontal and the images are the same (cf. Fukui and Hasegawa 2012). Note that ϕ is not diffeomorphic but surjective. We rewrite $x \circ \phi$ as $x : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$, $x(r, \theta) = (r\cos\theta, r^2\sin\theta, r^2\cos\theta\sin\theta)$. In this case, if we take $(n, s) : \mathbb{R} \times \mathbb{R} \to \Delta$,

$$n(r, \theta) = \frac{1}{\sqrt{4r^2\sin^4\theta + 3\sin^2\theta + 1}}(-2r\sin^2\theta, -\cos\theta, 2\sin\theta),$$
$$s(r, \theta) = \frac{1}{\sqrt{3\sin^2\theta + 1}}(0, 2\sin\theta, \cos\theta),$$

then $(x, n, s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ is a framed surface. Since

$$t(r, \theta) = \frac{1}{\sqrt{(4r^2\sin^4\theta + 3\sin^2\theta + 1)(3\sin^2\theta + 1)}}(-(3\sin^2\theta + 1), 2r\sin^2\theta\cos\theta,$$
$$-4r\sin^3\theta),$$

we have the following basic invariants.

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \frac{2r\sin\theta(\sin^2\theta+1)}{\sqrt{3\sin^2\theta+1}} & \frac{-\cos\theta\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}}{\sqrt{3\sin^2\theta+1}} \\ r^2\cos\theta\sqrt{3\sin^2\theta+1} & \frac{r\sin\theta\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}}{\sqrt{3\sin^2\theta+1}} \end{pmatrix},$$

$$\begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \end{pmatrix} =$$
$$\begin{pmatrix} 0 & \frac{2\sin^2\theta\sqrt{3\sin^2\theta+1}}{4r^2\sin^4\theta+3\sin^2\theta+1} & 0 \\ \frac{2}{\sqrt{(4r^2\sin^4\theta+3\sin^2\theta+1)(3\sin^2\theta+1)}} & \frac{2r\sin\theta\cos\theta(3\sin^2\theta+2)}{(4r^2\sin^4\theta+3\sin^2\theta+1)\sqrt{3\sin^2\theta+1}} & \frac{4r\sin^2\theta}{\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}(3\sin^2\theta+1)} \end{pmatrix}$$

It follows that the curvature C_F of (x, n, s) is given by

$$J_F(r,\theta) = \frac{r^2(2\sin\theta(\sin^2\theta+1)+\cos^2\theta+1)\sqrt{4r^2\sin^4\theta+3\sin^2\theta+1}}{3\sin^2\theta+1},$$
$$K_F(r,\theta) = -\frac{2\sin^2\theta}{(4r^2\sin^4\theta+3\sin^2\theta+1)^{2/3}},$$
$$H_F(r,\theta) = -\frac{2\cos\theta(-3r^2\sin^6\theta+8r^2\sin^4\theta+3r^2\sin\theta+3\sin^2\theta+2)}{(4r^2\sin^4\theta+3\sin^2\theta+1)(2\sin^2\theta+1)}.$$

Especially, $C_F(r,\theta) \neq 0$ for any $(r,\theta) \in \mathbb{R}\times\mathbb{R}$, that is, x is a front by Proposition 8.

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A Framed Curves in the Euclidean Space

We quickly review on the theory of framed curves in the Euclidean space, see detail Honda and Takahashi (2016).

A framed curve in the Euclidean space is a smooth curve with a moving frame. We say that $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ is a *framed curve* if $\dot\gamma(t)\cdot\nu_1(t) = 0$ and $\dot\gamma(t)\cdot\nu_2(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^3$ is a *framed base curve* if there exists $(\nu_1, \nu_2) : I \to \Delta$ such that (γ, ν_1, ν_2) is a framed curve.

We put $\mu(t) = \nu_1(t) \times \nu_2(t)$. Then $\{\nu_1(t), \nu_2(t), \mu(t)\}$ is a moving frame along the framed base curve $\gamma(t)$ in $\mathbb{R}^3$ and we have the Frenet–Serret type formula,

$$\begin{pmatrix} \dot\nu_1(t) \\ \dot\nu_2(t) \\ \dot\mu(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \nu_1(t) \\ \nu_2(t) \\ \mu(t) \end{pmatrix}, \ \dot\gamma(t) = \alpha(t)\mu(t)$$

where $\ell(t) = \dot\nu_1(t)\cdot\nu_2(t)$, $m(t) = \dot\nu_1(t)\cdot\mu(t)$, $n(t) = \dot\nu_2(t)\cdot\mu(t)$ and $\alpha(t) = \dot\gamma(t)\cdot\mu(t)$. We call the functions (ℓ, m, n, α) *the curvature of the framed curve*. Note that t_0 is a singular point of γ if and only if $\alpha(t_0) = 0$.

Definition 4 Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves. We say that (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are *congruent as framed curves* if there exist a constant rotation $A \in SO(3)$ and a translation $a \in \mathbb{R}^3$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + a$, $\widetilde{\nu_1}(t) = A(\nu_1(t))$ and $\widetilde{\nu_2}(t) = A(\nu_2(t))$ for all $t \in I$.

Theorem 7 (The Existence Theorem for framed curves, Honda and Takahashi 2016) *Let $(\ell, m, n, \alpha) : I \to \mathbb{R}^4$ be a smooth mapping. There exists a framed curve $(\gamma, \nu_1, \nu_2) : I \to \mathbb{R}^3 \times \Delta$ whose curvature of the framed curve is (ℓ, m, n, α).*

Theorem 8 (The Uniqueness Theorem for framed curves, Honda and Takahashi 2016) *Let (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2) : I \to \mathbb{R}^3 \times \Delta$ be framed curves with the curvature (ℓ, m, n, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$, respectively. Then (γ, ν_1, ν_2) and $(\widetilde{\gamma}, \widetilde{\nu}_1, \widetilde{\nu}_2)$ are congruent as framed curves if and only if the curvatures (ℓ, m, n, α) and $(\widetilde{\ell}, \widetilde{m}, \widetilde{n}, \widetilde{\alpha})$ coincide.*

B Legendre Curves in the Euclidean Plane

We quickly review on the theory of Legendre curves in the unit tangent bundle over $\mathbb{R}^2$, see detail Fukunaga and Takahashi (2013).

We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ over $\mathbb{R}^2$ (cf. Arnol'd 1990; Arnol'd et al. 1986). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. We say that $\gamma : I \to \mathbb{R}^2$ is a frontal if there exists $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve. Examples of Legendre curves see Ishikawa (2007), Ishikawa (2015). We denote $J(a) = (-a_2, a_1)$ the anticlockwise rotation by $\pi/2$ of a vector $a = (a_1, a_2) \in \mathbb{R}^2$. We put $\mu(t) = J(\nu(t))$. Then $\{\nu(t), \mu(t)\}$ is a moving frame of a frontal $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet type formula,

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\mu}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \mu(t) \end{pmatrix}, \quad \dot{\gamma}(t) = \beta(t)\mu(t),$$

where $\ell(t) = \dot{\nu}(t) \cdot \mu(t)$ and $\beta(t) = \dot{\gamma}(t) \cdot \mu(t)$. We call the pair (ℓ, β) *the curvature of the Legendre curve*.

Definition 5 Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exist a constant rotation $A \in SO(2)$ and a translation $a \in \mathbb{R}^2$ such that $\widetilde{\gamma}(t) = A(\gamma(t)) + a$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$.

Theorem 9 (The Existence Theorem for Legendre curves, Fukunaga and Takahashi 2013) *Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose curvature of the Legendre curve is (ℓ, β).*

Theorem 10 (The Uniqueness Theorem for Legendre curves, Fukunaga and Takahashi 2013) *Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves with the curvatures of Legendre curves (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$, respectively. Then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are congruent as Legendre curves if and only if the curvatures (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$ coincide.*

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