# Complete Parallel Mean Curvature Surfaces in Two-Dimensional Complex Space-Forms

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**Abstract** The purpose of this article is to determine explicitly the complete surfaces with parallel mean curvature vector, both in the complex projective plane and the complex hyperbolic plane. The main results are as follows: when the curvature of the ambient space is positive, there exists a unique such surface up to rigid motions of the target space. On the other hand, when the curvature of the ambient space is negative, there are 'non-trivial' complete parallel mean curvature surfaces generated by Jacobi elliptic functions and they exhaust such surfaces.

**Keywords** Mean curvature vector · Parallel mean curvature surface · Jacobi elliptic function · Complex space-form

**Mathematics Subject Classification** Primary 53C42; Secondary 53C55

## 1 Introduction

Recently, the study of non-zero parallel mean curvature surfaces with codimension two has progressed in the various ambient spaces (see Manzano et al. 2017 for references). Such surfaces in complex space-forms are already classified when the Kaehler angle is constant by Chen (1998), and finally by Hirakawa (2006). If the Kaehler angles of these surfaces are not constant, Ogata (1995) was the first to study such surfaces in non-flat complex space-forms, and later in Kenmotsu and Zhou (2000), Kenmotsu (2016), Hirakawa (2006), Ferreira and Tribuzy (2014), and Fetcu (2012), they continued the research. In particular, Kenmotsu and Zhou (2000) and Kenmotsu (2016) proved that

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the first and second fundamental forms of such a surface in non-flat complex space-forms are determined by a harmonic function and five real constants. We shall note that a parallel mean curvature surface in a flat complex space-form with codimension two is contained in a totally umbilical real hypersurface of the ambient space as a minimal or constant mean curvature surface (Chen 1973; Yau 1974), and also if the mean curvature vector vanishes identically, there are many articles studying such surfaces (see, for instance, Eschenburg et al. 1985).

In this article, by the refinement of our method developed in Kenmotsu and Ogata (2015) and Kenmotsu (2018), we prove that any parallel mean curvature surface in the complex projective plane $\mathbb{C}P^2$ with the Fubini–Study metric must have constant Kaehler angle, thus it is congruent to the flat torus if the surface is complete. For parallel mean curvature surfaces in the complex hyperbolic plane $\mathbb{C}H^2$, with its canonical structure of Kaehler surface, the situation is different from $\mathbb{C}P^2$-case. There is a two-parameter family of those surfaces, say $x_{t,p}$, $(0 \le t \le \pi, 0 < p \, (\neq 1/4) < \infty)$, such that any parallel mean curvature surface of a general type in $\mathbb{C}H^2$ is locally congruent to some $x_{t,p}$. Moreover, there exists the limit of $x_{t,p}$, as $p$ tends to 0, that is Hirakawa surface, and also letting $p \to 1/4+$ and $p \to 1/4-$, there are the limits of $x_{t,p}$ which are the same ones presented in Section 4 of Kenmotsu (2016).

In Sect. 2, we refine Lemma 3.3 of Kenmotsu and Zhou (2000), because its statement is incomplete. In Sect. 3, the key result of this article is proved by using our previous results (Kenmotsu 2016, 2018) that is Lemma 3.4: a constant appeared in the second fundamental form in Kenmotsu (2018) must be zero if the ambient space is non-flat. In Sect. 4, we will concentrate on the study in $\mathbb{C}H^2$-case and prove that any parallel mean curvature surface of a general type in $\mathbb{C}H^2$ is determined by two real constants, one of which is the parameter of the associated family, and the other is the modulus of a Jacobi elliptic function of the first kind. The moduli space of the complete parallel mean curvature surfaces in $\mathbb{C}H^2$ is presented at the end of this section. In Sect. 5 we state some open questions related to this article.

## 2 Parallel Mean Curvature Surfaces of a Special Type

This section establishes the notation that will be used in this article, recalls the structure equations of parallel mean curvature surfaces, and defines two families of parallel mean curvature surfaces as a refinement of Lemma 3.3 of Kenmotsu and Zhou (2000).

Let $\overline{M}[4\rho]$ be a complex two-dimensional complex space-form with constant holomorphic sectional curvature $4\rho$, and let $M$ be an oriented and connected real two-dimensional Riemannian manifold with Gaussian curvature $K$ and $x : M \longrightarrow \overline{M}[4\rho]$ be an isometric immersion, with Kaehler angle $\alpha$ such that the mean curvature vector $H$ is nonzero and parallel for the normal connection on the normal bundle of the immersion. Since the length of the mean curvature vector is constant, we write $|H| = 2b > 0$. Hereafter, we call such an immersion as a parallel mean curvature surface.

Let $M_0 = \{p \in M \mid x$ is neither holomorphic nor anti-holomorphic at $p\}$. $M_0$ is an open dense subset of $M$. Because all of the calculations and formulas on $M_0$ presented in Ogata (1995) are valid until page 400, according to a remark in Hirakawa

(2006), there exists a local field of unitary coframes $\{w_1, w_2\}$ on $M_0$ such that, by restricting it to $x$, the Riemannian metric $ds^2$ on $M_0$ can be written as $ds^2 = \phi\bar{\phi}$, where $\phi = \cos\alpha/2 \cdot \omega_1 + \sin\alpha/2 \cdot \bar{\omega}_2$. Let $a$ and $c$ be the complex-valued functions on $M_0$ that determine the second fundamental form of $x$. Then the Kaehler angle $\alpha$ and the complex 1-form $\phi$ satisfy

$$d\alpha = (a+b)\phi + (\bar{a}+b)\bar{\phi}, \quad \text{and} \quad d\phi = (\bar{a}-b)\cot\alpha\,\phi\wedge\bar{\phi}. \quad (2.1)$$

Equations (2.4), (2.5), and (2.6) in Ogata (1995) are

$$K = -4(|a|^2 - b^2) + 6\rho\cos^2\alpha, \quad (2.2)$$

$$da \wedge \phi = -\left(2a(\bar{a}-b)\cot\alpha + \frac{3}{2}\rho\sin\alpha\cos\alpha\right)\phi\wedge\bar{\phi}, \quad (2.3)$$

$$dc \wedge \bar{\phi} = 2c(a-b)\cot\alpha\,\phi\wedge\bar{\phi}, \quad (2.4)$$

$$|c|^2 = |a|^2 + \frac{\rho}{2}(-2+3\sin^2\alpha), \quad (2.5)$$

where (2.2) is the Gauss equation, (2.3) and (2.4) are the Codazzi–Mainardi equations, and (2.5) is the Ricci equation of $x$.

*Remark 2.1* (1) The immersion $x$ is holomorphic (resp. anti-holomorphic) at $p \in M$ if and only if $\alpha = 0$ (resp. $\alpha = \pi$) at $p$. Hence, $\sin\alpha \neq 0$ on $M_0$.

(2) The unitary coframes $\{w_1, w_2\}$ appeared in (2.1)–(2.5) are uniquely determined up to orientations of both $\overline{M}[4\rho]$ and $M_0$ (see Ogata 1995), hence the complex one-form $\phi$ on $M_0$ is unique up to sign and conjugacy. Thus, for the immersion $x$ satisfying (2.1)–(2.5), the data $\{\psi := \bar{\phi}, \alpha, \bar{a}, \bar{c}\}$, which satisfy the conjugated structure equations, define the same surface as $x$.

For a parallel mean curvature surface $x$ satisfying $a = \bar{a}$ on $M_0$, it is proved in Kenmotsu and Zhou (2000) and Kenmotsu and Ogata (2015) that if $\rho > 0$, then $x$ is totally real and the image is a part of the flat torus and in the case of $\rho < 0$, if $x$ has a constant Kaehler angle, then it is totally real, flat and $-2b^2 \leq \rho < 0$, or it is non-totally real, $K = \text{constant} = -2b^2$, and $\rho = -3b^2$ (see also Theorem 1.1 of Hirakawa 2006). If the Kaehler angle is not constant, then Lemma 3.3 of Kenmotsu and Zhou (2000) tells us that there are 'non-trivial ones'. But, the statement of Lemma 3.3 is incomplete, so we give here the correction as follows:

**Theorem 2.2** *Let* $x : M \longrightarrow \overline{M}[4\rho]$ *be a parallel mean curvature surface with* $|H| = 2b \,(> 0)$. *Suppose that the Kaehler angle is not constant and* $a = \bar{a}$ *on* $M_0$. *If* $\rho < 0$, *then* $\rho = -3b^2$ *and* $x$ *is determined by a real constant and the amplitude of a Jacobi elliptic function with the modulus either* $2\sqrt{2}/3$ *or* $1/3$.

*Proof* All formulas and results in Kenmotsu and Zhou (2000) are valid under the assumption of Theorem 2.2 (Kenmotsu and Ogata 2015). The first claim of Theorem 2.2 is proved by the first paragraph of page 302 in Kenmotsu and Zhou (2000). Formula (2.5) and Theorem 4.1, both in Kenmotsu and Zhou (2000), imply $\lambda(u)^4(a(u)^2+b^2(3-9/2\sin^2\alpha(u))) = m^2$, for some non-zero real constant $m$, where

$\theta(u)$ in Kenmotsu and Zhou (2000) is replaced by $\alpha(u)$ and $\phi = \lambda(u)(du+idv)$. Since $\lambda(u)$ is real-valued and defined up to a real multiplicative constant, we assume that $m = 9b^3$. Formula (2.10) and Theorem 4.1, both in Kenmotsu and Zhou (2000), yield $a(u) = b(1 - 9/4\sin^2\alpha(u))$. These two formulas imply $\lambda(u)^2(8 - 9\sin^2\alpha(u)) = \pm 36b^2$. By (2.3) of Kenmotsu and Zhou (2000), $\alpha$ satisfies the ODE $(d\alpha/du)^2 = \pm 9b^4(8 - 9\sin^2\alpha(u))$. Since we may assume $\alpha(u) > 0$, two cases to be studied arise:

$$\begin{array}{lll} \text{(i)} & \frac{d\alpha}{du} = 3b^2\sqrt{-8 + 9\sin^2\alpha(u)}, & -8 + 9\sin^2\alpha(u) > 0, \\ \text{(ii)} & \frac{d\tilde\alpha}{du} = 3b^2\sqrt{8 - 9\sin^2\tilde\alpha(u)}, & 8 - 9\sin^2\tilde\alpha(u) > 0. \end{array}$$

For case (i), put $\sin\gamma(u) = -3\cos\alpha(u)$. The function $\gamma(u)$ satisfies the ODE $d\gamma/du = 9b^2\sqrt{1 - 1/9\sin^2\gamma}$, i.e., $\gamma(u) = am(9b^2u, 1/3)$, where $am(\cdot, p)$ denotes the amplitude of a Jacobi elliptic function with the modulus $p$. Hence,

$$\cos\alpha(u) = -\frac{1}{3}\sin\left(am\left(9b^2u, \frac{1}{3}\right)\right), \quad -\frac{K(1/3)}{9b^2} < u < \frac{K(1/3)}{9b^2}, \qquad (2.6)$$

where $K(1/3)$ denotes the complete integral of the first kind of the Jacobi elliptic function with the modulus $1/3$. For case (ii), put $\sin\tilde\gamma(u) = 3/(2\sqrt{2})\sin\tilde\alpha(u)$. Then the function $\tilde\gamma(u)$ satisfies the ODE $d\tilde\gamma/du = 9b^2\sqrt{1 - 8/9\sin^2\tilde\gamma}$, which means $\tilde\gamma(u) = am(9b^2u, 2\sqrt{2}/3)$. Hence,

$$\sin\tilde\alpha(u) = \frac{2\sqrt{2}}{3}\sin\left(am\left(9b^2u, \frac{2\sqrt{2}}{3}\right)\right), \quad -\frac{K(2\sqrt{2}/3)}{9b^2} < u < \frac{K(2\sqrt{2}/3)}{9b^2}. \tag{2.7}$$

Now we will find the expression of $c$ in terms of $\sin\alpha$: By (2.5), $c$ is written as $c = b|8 - 9\sin^2\alpha|/4\exp(it(u,v))$ for some real-valued function $t(u,v)$. Equation (2.4) implies $t(u,v) = $ constant. The first and second fundamental forms of $x$ are determined by the Kaehler angle $\alpha(u)$ or $\tilde\alpha(u)$ and a real number $t \in [0,\pi]$ as follows:

$$\begin{cases} \lambda(u) = \frac{6b}{\sqrt{|8-9\sin^2\alpha(u)|}}, & 8 - 9\sin^2\alpha(u) \neq 0, \\ a(u) = b\left(1 - \frac{9}{4}\sin^2\alpha(u)\right), & c(u) = \frac{b}{4}|8 - 9\sin^2\alpha(u)|e^{it}, \end{cases} \tag{2.8}$$

where in the above formulas, $\alpha(u)$ is replaced by $\tilde\alpha(u)$ if $8 - 9\sin^2\alpha(u) > 0$. Conversely, given $b > 0$, $t \in [0,\pi]$, and the amplitude $\gamma(u)$ with a modulus $1/3$ or its conjugate $\tilde\gamma(u)$, we define $\alpha(u)$ by (2.6) or (2.7), and $\lambda(u)$, $a(u)$ and $c(u)$ by (2.8). Then it is straightforward to prove that these data satisfy the structure equations (2.1)–(2.5) under the condition $\rho = -3b^2$, proving Theorem 2.2.                                        □

*Remark 2.3* Case (i) above is missed in Lemma 3.3 of Kenmotsu and Zhou (2000).

We denote both surfaces obtained in Theorem 2.2 as $x_{t,1/4+}$ if it is determined by $\gamma(u)$ and $x_{t,1/4-}$ if determined by $\tilde\gamma(u)$. The surface $x_{t,1/4\pm}$ is isometric to $x_{0,1/4\pm}$ and both surfaces have the same length of the mean curvature vector. Consequently, the

set $\{x_{t,1/4\pm} \mid t \in [0, \pi]\}$ is the associated family of $x_{0,1/4\pm}$. We note that in general $\sin \alpha(u) \neq \sin \tilde{\alpha}(u)$, hence surfaces $x_{t,1/4+}$ and $x_{t,1/4-}$ are not isometric to each other. Let $\mathbb{D}_{\pm}$ be the domain in $\mathbb{R}^2$ given by

$$\mathbb{D}_{\pm} = \left\{ (u, v) \in \mathbb{R}^2 \mid -\frac{K(q)}{9b^2} < u < \frac{K(q)}{9b^2}, \ -\infty < v < \infty \right\},$$

where the plus sign and the minus one of $\mathbb{D}_{\pm}$ are taken for $q = 1/3$ and $q = 2\sqrt{2}/3$, respectively. We will prove that the immersions $x_{t,1/4\pm}$ are complete.

**Proposition 2.4** *For each $t \in [0, \pi]$, $x_{t,1/4\pm} : \mathbb{D}_{\pm} \longrightarrow \overline{M}[-12b^2]$ are complete.*

*Proof* For case (i), the Riemannian metric $ds^2$ on $x_{t,1/4+}$ is estimated as follows:

$$\frac{1}{36b^2} ds^2 = \frac{du^2 + dv^2}{-8 + 9\sin^2 \alpha} = \frac{du^2 + dv^2}{\cos^2 \gamma(u)}, \quad \left( -\frac{\pi}{2} < \gamma < \frac{\pi}{2} \right)$$

$$= \frac{1}{\cos^2 \gamma} \left( \frac{d\gamma^2}{9(1 - 1/9\sin^2 \gamma)} + dv^2 \right) \geq \frac{1}{9} \frac{1}{\cos^2 \gamma} (d\gamma^2 + d(3v)^2).$$

Since the metric in the right hand is complete (see, for instance, Lemma 3.13 of Hirakawa 2006), the considered metric is also complete. The proof in the second case is similar and we conclude the proof. □

*Remark 2.5* In the third line of the proof of Lemma 3.13 in Hirakawa (2006), arctan $u$ should be read as tan $u$.

A parallel mean curvature surface is of *a general type* if it satisfies $d\alpha \neq 0$ and $a \neq \bar{a}$ at a point of $M_0$. Now, we will recall results in Kenmotsu (2016) and Kenmotsu (2018) which are used in the next section: if $x$ is of a general type, then $a$ is a function of $\alpha$, say $a = a(\alpha)$, satisfying the first order complex ODE (see (3.1) of Kenmotsu 2016)

$$\frac{da}{d\alpha} = \frac{\cot \alpha}{\bar{a} + b} \left( -2ba + 2|a|^2 + \frac{3\rho}{2} \sin^2 \alpha \right), \quad (a + b \neq 0). \tag{2.9}$$

Let us consider isothermal coordinates for the Riemannian metric $\phi\bar{\phi}$ on $M_0$ that makes $M_0$ a Riemann surface with a local complex coordinate $z$. By Lemma 2.4 of Kenmotsu (2018), there is a complex analytic transformation $w = w(z)$ on the Riemann surface $M_0$ such that in $\phi = \mu dw$, $\mu$ is a complex-valued function of a single real variable $(w + \bar{w})/2$. Let $w = u + iv$, $(u, v \in \mathbb{R})$. Then $\alpha$ is a function of $u$ only, satisfying the ODE

$$\frac{d\alpha}{du} = 2 \exp \left( \int F(\alpha) d\alpha \right), \tag{2.10}$$

where $F(\alpha) = \cot \alpha (|a - b|^2 + 3\rho/2 \sin^2 \alpha)/|a + b|^2$ and $\mu$ is expressed as

$$\mu = \frac{\exp \left( \int F(\alpha) d\alpha \right)}{a + b}, \tag{2.11}$$

where $\mu$ is a complex-valued function of $\alpha$ defined up to sign and a real multiplicative constant.

We will also need the following fact proved in Lemma 2.5 of Kenmotsu (2018): there are real numbers $k$ and $t$ such that

$$c = \left(|a|^2 + \frac{\rho}{2}(-2 + 3\sin^2\alpha)\right)^{1/2} \frac{(\bar{a}+b)}{a+b} e^{-i(kv+t)}. \tag{2.12}$$

*Remark 2.6* In (2.14) of Kenmotsu (2018), $exp(-it)$, $(0 \leq t \leq \pi)$, is missed, because $c$ is unique up to a complex multiplicative constant with the absolute value equal to one.

## 3 Main Lemma

In this section, we prove that the constant $k$ in (2.12) is zero, which is the key result of this article.

**Lemma 3.1** *Let $x : M \longrightarrow \overline{M}[4\rho]$ be a parallel mean curvature surface with $|H| = 2b (> 0)$ of a general type. Then*

$$|a+b|^2 = \frac{(8b^2 + 3\rho\sin^2\alpha)}{4b}(\Re a + b), \tag{3.1}$$

$$(\Im a)^2 = (\Re a + b)\left(\frac{(8b^2 + 3\rho\sin^2\alpha)}{4b} - (\Re a + b)\right). \tag{3.2}$$

*where $\Re a$ and $\Im a$ denote the real and imaginary parts of $a$ respectively.*

*Proof* Formula (2.17) of Kenmotsu (2018) yields $\mu^2(8ba - 3\rho\sin^2\alpha) = c_1$, where the constant $c_1 (\in \mathbb{C})$ is not zero, because the immersion $x$ is of a general type. By a transformation: $\mu \to e^{i\xi}\mu$, $(\xi \in \mathbb{R})$, $c_1$ can be assumed to be real. By (2.11), $\mu \cdot (a+b)$ is also real-valued. Coupling these together, one obtains

$$\mu^2 \cdot (a+b)^2 = \frac{c_1(a+b)^2}{8ba - 3\rho\sin^2\alpha} = \overline{\mu^2 \cdot (a+b)^2} = \frac{c_1(\bar{a}+b)^2}{8b\bar{a} - 3\rho\sin^2\alpha}.$$

The simplification of the above formula proves (3.1). Formula (3.2) is another expression of (3.1), proving Lemma 3.1.                                                                                            □

Since we are now studying a parallel mean curvature surface of a general type, $\alpha$ is not constant, hence $8b^2 + 3\rho\sin^2\alpha \neq 0$. Then the real part of $a$ satisfies a linear ODE:

**Lemma 3.2** *Let $x : M \longrightarrow \overline{M}[4\rho]$ be a parallel mean curvature surface of a general type. Then $\Re a$ satisfies the first order linear ODE*

$$\frac{d}{d\alpha}\Re a + 2\cot\alpha \frac{(8b^2 - 3\rho\sin^2\alpha)}{8b^2 + 3\rho\sin^2\alpha}\Re a - 2b\cot\alpha \frac{(8b^2 + 9\rho\sin^2\alpha)}{8b^2 + 3\rho\sin^2\alpha} = 0. \tag{3.3}$$

*Proof* It follows from (2.9) and (3.2) that

$$\frac{d}{d\alpha}\Re a + \frac{(\Re a + b)}{2b|a+b|^2}\left\{(8b^2 - 3\rho\sin^2\alpha)\Re a - b(8b^2 + 9\rho\sin^2\alpha)\right\}\cot\alpha = 0.$$

We conclude by coupling this with (3.1).

If $\rho \neq 0$, then the general solution of (3.3) is given by

$$\Re a = -b + \frac{1}{3b\rho\sin^2\alpha}(8b^2 + 3\rho\sin^2\alpha)(-2b^2 + c_3(8b^2 + 3\rho\sin^2\alpha)), \quad (c_3 \in \mathbb{R}),$$
(3.4)

and if $\rho = 0$, then $\Re a = (c_3 - b/2\cos 2\alpha)/\sin^2\alpha$.

By (3.1) and (3.4),

$$|a + b|^2 = \frac{(8b^2 + 3\rho\sin^2\alpha)^2}{12b^2\rho\sin^2\alpha}(-2b^2 + c_3(8b^2 + 3\rho\sin^2\alpha)), \quad (\rho \neq 0), \quad (3.5)$$

in particular,

$$\rho(-2b^2 + c_3(8b^2 + 3\rho\sin^2\alpha)) \geq 0. \tag{3.6}$$

The formula $|a - b|^2 = |a + b|^2 - 4b\Re a$, (3.1), and (3.4) yield

$$|a - b|^2 + \frac{3\rho}{2}\sin^2\alpha = \frac{(8b^2 + 3\rho\sin^2\alpha)(16b^4 - c_3(64b^4 - 9\rho^2\sin^4\alpha))}{12b^2\rho\sin^2\alpha}.$$

By (3.4) and the above formula, the function $F(\alpha)$ in (2.10) can be expressed as

$$F(\alpha) = \frac{(16b^4 - 64b^2c_3 + 9c_3\rho^2\sin^4\alpha)}{(8b^2 + 3\rho\sin^2\alpha)(-2b^2 + c_3(8b^2 + 3\rho\sin^2\alpha))}\cot\alpha.$$

Integrating, we obtain, up to a positive multiplicative constant,

$$e^{\int F(\alpha)d\alpha} = \frac{\sqrt{|8b^2 + 3\rho\sin^2\alpha||-2b^2 + c_3(8b^2 + 3\rho\sin^2\alpha)|}}{\sin\alpha}. \tag{3.7}$$

By (3.2) and (3.4), the imaginary part of $a$ is expressed in terms of $\sin\alpha$ as follows:

$$(\Im a)^2 = \frac{(1 - 4c_3)}{36b^2\rho^2\sin^4\alpha}(8b^2 + 3\rho\sin^2\alpha)^3(-2b^2 + c_3(8b^2 + 3\rho\sin^2\alpha)), \quad (\rho \neq 0).$$
(3.8)

$\square$

*Remark 3.3* We may assume $\Im a \geq 0$, by Remark 2.1 (2).

If $\rho \neq 0$, Eq. (3.8) implies

$$(1 - 4c_3)(8b^2 + 3\rho\sin^2\alpha)(-2b^2 + c_3(8b^2 + 3\rho\sin^2\alpha)) \geq 0. \tag{3.9}$$

The next lemma is the main result of this section.

**Lemma 3.4** *Let $x : M \longrightarrow \overline{M}[4\rho]$ be a parallel mean curvature surface of a general type. If $\rho \neq 0$, then $\rho = -3b^2 < 0$ and $k = 0$ or $c_3 = c \equiv 0$. Moreover, in the latter case the Gaussian curvature of $M$ is constant equal to $= -2b^2$.*

*Proof* By (2.16) of Kenmotsu (2018) and (2.11) of this article, we have:

$$2k|a + b|^2|c|^2 = \rho e^{\int F(\alpha)d\alpha}(8|a|^2 + 9b(a + \bar{a})\sin^2\alpha - 8b^2 + 18b^2)\cot\alpha, \tag{3.10}$$

where $k_1$ in Kenmotsu (2018) is replaced here by $k$. It follows from (2.5), (3.1) and (3.4) that

$$\begin{cases} 8|a|^2 + 9b(a + \bar{a})\sin^2\alpha - 8b^2 + 18b^2\sin^2\alpha = \frac{6(\rho + 3b^2)}{b}(\Re a + b)\sin^2\alpha, \\ |c|^2 = -(\rho + 3b^2) + \frac{c_3}{4b^2}(8b^2 + 3\rho\sin^2\alpha)^2. \end{cases} \tag{3.11}$$

By (3.5), (3.7), (3.10), and the above formulas, we obtain

$$\rho^2(\rho + 3b^2)^2(-2b^2 + c_3(8b^2 + 3\rho\sin^2\alpha))(1 - \sin^2\alpha)$$
$$- k^2(8b^2 + 3\rho\sin^2\alpha)\left(-(\rho + 3b^2) + \frac{c_3}{4b^2}(8b^2 + 3\rho\sin^2\alpha)^2\right)^2 = 0. \tag{3.12}$$

This is a polynomial equation of degree 5 in $\sin^2\alpha$ with constant coefficients. It follows that if $\rho \neq 0$ and $\alpha$ is not constant, then all the coefficients of the polynomial vanish, in particular, $k = 0$ or $c_3 = 0$ which comes from the term of the highest degree. Then we have $\rho = -3b^2$ by (3.12). In the case when $c_3 = 0$, we get $c \equiv 0$ by (3.11) and $K = \text{constant} = -2b^2$ by (2.2) and (2.5). Consequently, this is Hirakawa surface (Hirakawa 2006), which concludes the proof. □

It shall be noted that Lemma 3.4 does not need any global assumption of $M$. As a direct application of Lemma 3.4, we have: *Even locally, there is no parallel mean curvature surface of a general type in $\mathbb{C}P^2$.* So, a parallel mean curvature surface in $\mathbb{C}P^2$ has either constant Kaehler angle or non-constant Kaehler angle satisfying $a = \bar{a}$ on an open subset of $M$. However, the latter does not occur by Theorem 4.1 of Kenmotsu and Zhou (2000) (we note that all results in Kenmotsu and Zhou 2000 hold under the additional condition $a = \bar{a}$ by Kenmotsu and Ogata 2015). Therefore, a parallel mean curvature surface in $\mathbb{C}P^2$ must have a constant Kaehler angle, in particular, a constant Gaussian curvature. Then, such a surface is flat and totally real (Hirakawa 2006). Thus, we proved

**Theorem 3.5** *Any complete parallel mean curvature surface with $|H| = 2b$ $(> 0)$ in $\mathbb{C}P^2$ is the flat torus up to rigid motions of $\mathbb{C}P^2$.*

Theorem 3.5 is in contrast to the fruitful theory of minimal surfaces in $\mathbb{C}P^2$ (see Introduction of Inoguchi et al. 2016 for references).

## 4 Jacobi Elliptic Functions

In this section, non-zero parallel mean curvature surfaces in $\mathbb{C}H^2$ are studied by applying the results of Sect. 3, and we will prove that such a surface is determined by a Jacobi elliptic function. Let $x : M \longrightarrow \tilde{M}[4\rho]$ be a parallel mean curvature surface of a general type with $|H| = 2b\ (> 0)$. We can assume $\rho = -3b^2$ by Lemma 3.4 and $a \neq \bar{a}$ by Theorem 2.2. Lemma 3.4 and the second formula of (3.11) yield $c_3 \geq 0$, and $a$ is explicitly expressed in terms of $\sin \alpha$ by (3.4), (3.8), and Remark 3.3:

$$
\begin{aligned}
a = \frac{b}{9 \sin^2 \alpha} \{ & 16 - 27 \sin^2 \alpha - c_3(8 - 9 \sin^2 \alpha)^2 \\
& + \frac{i}{2} |8 - 9 \sin^2 \alpha| \sqrt{(1 - 4c_3)(8 - 9 \sin^2 \alpha)(-2 + c_3(8 - 9 \sin^2 \alpha))} \}.
\end{aligned}
\tag{4.1}
$$

From now on, let $p = c_3$. By (3.6) with $\rho + 3b^2 = 0$ and (3.9), since $2 - p(8 - 9 \sin^2 \alpha) > 0$, two cases to be studied arise: (i) $0 < p < 1/4$ and $8 - 9 \sin^2 \alpha < 0$, and (ii) $1/4 < p$ and $8 - 9 \sin^2 \alpha > 0$.

First, we study case (i). By (2.10) and (3.7), $\alpha = \alpha(u)$ satisfies the ODE

$$
\frac{d\alpha}{du} = 2b^2 \frac{\sqrt{(-8 + 9 \sin^2 \alpha)(2 - p(8 - 9 \sin^2 \alpha))}}{\sin \alpha}.
\tag{4.2}
$$

Consider the function $\gamma = \sin^{-1}(-3 \cos \alpha)$. By (4.2), this satisfies the ODE $d\gamma/du = 6b^2 \sqrt{2 + p - p \sin^2 \gamma}$. Hence, $\gamma$ is the amplitude of a Jacobi elliptic function with the modulus $\sqrt{p/(2 + p)}$, i.e., $\gamma = \gamma(u)$ is written as

$$
\gamma = am(6b^2 \sqrt{2 + p}u, \sqrt{p/(2 + p)}), \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2}.
\tag{4.3}
$$

For case (ii), $\alpha = \alpha(u)$ satisfies the ODE

$$
\frac{d\alpha}{du} = 2b^2 \frac{\sqrt{(8 - 9 \sin^2 \alpha)(2 - p(8 - 9 \sin^2 \alpha))}}{\sin \alpha}.
\tag{4.4}
$$

Consider the function $\tilde{\gamma} = \cos^{-1} \sqrt{p(8 - 9 \sin^2 \alpha)/2}$. This satisfies the ODE $d\tilde{\gamma}/du = 6b^2 \sqrt{2 + p - 2 \sin^2 \tilde{\gamma}}$. Hence, $\tilde{\gamma}$ is the amplitude of a Jacobi elliptic function with the modulus $\sqrt{2/(2 + p)}$, i.e., $\tilde{\gamma} = \tilde{\gamma}(u)$ is written as

$$
\tilde{\gamma} = am(6b^2 \sqrt{2 + p}u, \sqrt{2/(2 + p)}), \quad -\frac{\pi}{2} < \tilde{\gamma} < \frac{\pi}{2}.
\tag{4.5}
$$

We note that $\tilde{\gamma}$ is the conjugate of $\gamma$. We showed that in these two cases, *Kaehler angles are determined by a Jacobi elliptic function or its conjugate.*

Let us put

$$\mathbb{D} = \left\{ (u, v) \in R^2 \mid -\frac{K(q)}{6b^2\sqrt{2+p}} < u < \frac{K(q)}{6b^2\sqrt{2+p}}, \ -\infty < v < \infty \right\}, \quad (4.6)$$

where $K(q)$ denotes the complete elliptic integral of the first kind with the modulus $q = \sqrt{p/(2+p)}$ for $0 < p < 1/4$ and $q = \sqrt{2/(2+p)}$ for $p > 1/4$. We note that $\mathbb{D}$ is transformed to the slab $\{(\tilde{u}, v) \in \mathbb{R}^2 \mid -\pi/2 < \tilde{u} < \pi/2, \ v \in \mathbb{R}\}$, where $\tilde{u} = \gamma$ for $0 < p < 1/4$ and $\tilde{u} = \tilde{\gamma}$ for $p > 1/4$.

Given $p \in (0, \infty)$, let us define a real-valued function of $\alpha$, $\theta(\alpha)$, by

$$e^{i\theta(\alpha)} = \frac{1}{3\sin\alpha}\left(2\sqrt{2 - p(8 - 9\sin^2\alpha)} \pm i\sqrt{|(1 - 4p)(8 - 9\sin^2\alpha)|}\right), \quad (4.7)$$

where the plus sign is taken for $0 < p \le 1/4$ and the minus one for $1/4 < p$. By (2.11), (2.12), (3.7) and (4.1), $\mu$, $a$, and $c$ are expressed as follows:

$$\mu = \frac{6be^{i\theta(\alpha)}}{\sqrt{|8 - 9\sin^2\alpha|}}, \quad (8 - 9\sin^2\alpha \neq 0),$$

$$a = -b + \frac{b(8 - 9\sin^2\alpha)\sqrt{2 - p(8 - 9\sin^2\alpha)}}{6\sin\alpha}e^{-i\theta(\alpha)}$$

$$c = \frac{b}{2}\sqrt{p}|8 - 9\sin^2\alpha|e^{i(2\theta(\alpha)+t)}, \quad (t \in [0, \pi]). \quad (4.8)$$

The last main result of this article is the following theorem.

**Theorem 4.1** *Let $x : M \longrightarrow \overline{M}[-12b^2]$ be a parallel mean curvature surface with $|H| = 2b \ (> 0)$. If $x$ is of a general type, then there exists a positive number $p \ (\neq 1/4)$ such that the Kaehler angle is determined by a Jacobi elliptic function (4.3) if $0 < p < 1/4$ and (4.5) if $1/4 < p < \infty$. Moreover, the first and second fundamental forms of $x$ are given by (4.8). Conversely, given three real numbers $b \ (> 0), t \in [0, \pi], \ p \ (> 0, \neq 1/4)$, there exist a domain $\mathbb{D} \subset \mathbb{R}^2$ and a parallel mean curvature immersion of a general type from $\mathbb{D}$ into $\overline{M}[-12b^2]$ such that the first and second fundamental forms are determined by $t$ and a Jacobi elliptic function whose modulus depends only on $p$, and the length of the mean curvature vector is equal to $2b$.*

*Proof* We only have to prove the last part of Theorem 4.1. Consider three numbers $b, t, p$ presented in Theorem 4.1 and define the domain $\mathbb{D} \subset \mathbb{R}^2$ by (4.6). Suppose, first, that $0 < p < 1/4$. Take the amplitude $\gamma$ of a Jacobi elliptic function as in (4.3). Define a function $\alpha$ by $\alpha = \alpha(u; p) = \cos^{-1}(-\sin\gamma/3)$. Then, $8 - 9\sin^2\alpha < 0$, $2 - p(8 - 9\sin^2\alpha) > 0$, and $\alpha$ satisfies the ODE (4.2). If $p > 1/4$, for a given amplitude $\tilde{\gamma}$ as in (4.5), we have a function $\alpha$ which satisfies the ODE (4.4). Let us define a real-valued function $\theta(\alpha)$ by (4.7), and $\mu$, $a$, $c$ by (4.8). Now, we prove that the data $\{\mathbb{D}, \phi = \mu(du + idv), a, c\}$ satisfy the structure equations (2.1)–(2.5) under the condition $\rho = -3b^2$. In fact, by the first and second formulas of (4.8), we see that $(a + b)\mu = (\bar{a} + b)\bar{\mu}$, which proves the first equation of (2.1). To show the

second equation of (2.1), we use that an equation $d\phi = -\log \mu'(\alpha)(\bar{a} + b)\phi \wedge \bar{\phi}$ which is derived by virtue of the first equation of (2.1). Thus, the second equation of (2.1) is equivalent to $\mu'(\alpha)(\bar{a} + b) + \mu(\bar{a} - b)\cot\alpha = 0$. This follows from (4.8) by direct computation. To prove (2.2), we recall the formula of Gauss curvature $K = -e^{-2\sigma}(\sigma_{uu} + \sigma_{vv})$ in terms of isothermal coordinates $(u, v)$ given by $ds^2 = e^{2\sigma}(du^2 + dv^2)$. Now, $e^{2\sigma} = 36b^2/(-8 + 9\sin^2\alpha(u; p))$, and the direct computation using (4.2) implies $\sigma_{uu} = 36b^4(2 + p(-8 + 9\sin^2\alpha)^2)/(-8 + 9\sin^2\alpha)$, which implies (2.2). To prove (2.3), we first note that $\mu^2(8a + 9b\sin^2\alpha) = $ constant, which is observed by (4.8). Taking the exterior derivative of this formula, (2.3) with $\rho = -3b^2$ follows from (2.1). To prove (2.4), we use $\mu^2\bar{c} = $ constant which can be seen from (4.8). Taking the exterior derivative of this formula and using (2.1), (2.4) is proved. (2.5) follows from the formula $|a|^2 = |a + b|^2 - 2b\Re(a + b) + b^2$ and (4.8). For the case when $p > 1/4$, the proof is done by the similar way as before using (4.4) and (4.5), so we omit it, proving Theorem 4.1. $\qquad\square$

Let $\mathbb{C}H^2[4\rho]$ be the complex hyperbolic plane of constant holomorphic sectional curvature $4\rho$, $(\rho < 0)$. By $x_{t,p}$, $(0 \leq t \leq \pi, 0 < p < \infty, p \neq 1/4)$, we denote the immersion from $\mathbb{D} \subset \mathbb{R}^2$ into $\mathbb{C}H^2[-12b^2]$ obtained by Theorem 4.1. Note that these surfaces are the simplified version of the ones given in Sect. 4 of Kenmotsu (2016), because $\xi(t)$ in Kenmotsu (2016) is explicitly integrated here by using the special isothermal coordinates.

Now, we list the properties of $x_{t,p}$:

1. The mean curvature vector $H$ is parallel, $|H| = 2b$, and the Kaehler angle is not constant.
   It follows from the structure equations (2.1)–(2.5).
2. $x_{t,p} : \mathbb{D} \longrightarrow \mathbb{C}H^2[-12b^2]$ is complete.
   We do not give the proof here, as it can be performed almost in the same way as the proof of Proposition 2.4.
3. For any $p, q$ with $0 < p < q < \infty$, $x_{t,p}$ is not isometric to $x_{t,q}$, and hence they are not congruent each other, because of $\sin^2\alpha(u; p) \neq \sin^2\alpha(u; q)$.
4. By letting $p \to 0$, $x_{t,p}$ has a limit, say $x_{t,0}$, that is Hirakawa surface (Hirakawa 2006).
   In fact, as $p \to 0$ in (4.2), (4.7) and (4.8), $\alpha$ satisfies an ODE, $\phi$ and $a$ are well defined and $c \equiv 0$. It is verified in the same way as in the case when $0 < p < 1/4$ in Theorem 4.1 that these data satisfy the structure equations (2.1)–(2.5) under the condition $\rho = -3b^2$. Therefore, there exists an immersion $x_{t,0} : \mathbb{D} \longrightarrow \mathbb{C}H^2[-12b^2]$. It can be easily shown that the immersion $x_{t,0}$ has a constant Gaussian curvature $K = -2b^2$ by (2.2), but is not of a constant Kaehler angle. So, $x_{t,0}$ is congruent to Hirakawa surface by Theorem 1.1 (1) of Hirakawa (2006).

*Remark 4.2* At page 230 of Hirakawa (2006), author describes two parallel mean curvature surfaces of constant Gaussian curvature $K = -2b^2$, but both surfaces are congruent to each other in $\mathbb{C}H^2[-12b^2]$ by Remark 2.1 (2).

5. As $p \to 1/4 - 0$, $x_{t,p}$ has a limit which is $x_{t,1/4-}$, and as $p \to 1/4 + 0$, $x_{t,p}$ has also a limit which is $x_{t,1/4+}$.

In fact, as $p \to 1/4 - 0$ in (4.2), (4.7), and (4.8), $\alpha$, $\phi$, $a$ and $c$ have limits which are the same ones in Theorem 2.2. In the case when $p > 1/4$, we have similar formulas.

6. $x_{t,p}$ is isometric to $x_{\tilde{t},p}$ for any $t, \tilde{t} \in [0, \pi]$ and both surfaces have the same length of the mean curvature vector. Hence, for any fixed $p$, $\{x_{t,p} \mid t \in [0, \pi]\}$ is the associated family of $x_{0,p}$.

In fact, the metric is determined only by the Kaehler angle function and therefore independent from $c$.

Finally, we describe the manifold structure of the set of the complete parallel mean curvature surfaces in $\mathbb{C}H^2[4\rho]$ as follows: By *the moduli space of parallel mean curvature surfaces*, we mean the quotient set of all parallel mean curvature surfaces, where the equivalence relation is given by rigid motions of the target space. By Theorem 3.5, the moduli space of complete parallel mean curvature surfaces in $\mathbb{C}P^2$ reduces to a point.

If there exists a parallel mean curvature surface of a constant Gaussian curvature in $\mathbb{C}H^2[4\rho]$, then $\rho$ satisfies the condition (i) $-2b^2 \le \rho < 0$ or (ii) $\rho = -3b^2$ (Hirakawa 2006). For case (i), by Theorem 2.2 and Lemma 3.4, a parallel mean curvature surface with $|H| = 2b$ in $\mathbb{C}H^2[4\rho]$, $(-2b^2 \le \rho < 0)$, is of a constant Kaehler angle, and hence has a constant Gaussian curvature. Then, the surface is flat (Hirakawa 2006) and the image is explicitly determined in Hirakawa (2004). For case (ii), if the Kaehler angle is constant, then the surface is uniquely determined as Chen surface (Chen 1998) and if the Kaehler angle is not constant, then we have Hirakawa surface (Hirakawa 2006). Applying Theorems 2.2 and 4.1, we get the following:

**Theorem 4.3** *The moduli space of complete parallel mean curvature surfaces with* $|H| = 2b$ *in* $\mathbb{C}H^2[-12b^2]$ *is the disjoint union of two cones in* $\mathbb{R}^3$.

*Proof* Let $x : M \longrightarrow \mathbb{C}H^2[-12b^2]$ be a parallel mean curvature surface with $|H| = 2b > 0$, where it is not assumed that $x$ is of a general type. First, we prove that if $a \ne \bar{a}$ *at a point of* $M_0$, *then this holds throughout* $M_0$. In fact, suppose that there is a point $\mathbf{u}_0 \in M_0$ with $a = \bar{a}$ at $\mathbf{u}_0$. Since the set of points in $M_0$ satisfying $a = \bar{a}$ is not open, as $n \to \infty$, we have a sequence of points $\mathbf{u}_n \to \mathbf{u}_0$ such that $a \ne \bar{a}$ at $\mathbf{u}_n$. Then, at $\mathbf{u}_n$, we have (4.1) for $a$ and, in particular, $1 - 4p \ne 0$. As $n$ tends to $\infty$, both functions $8 - 9\sin^2 \alpha$ and $2 - p(8 - 9\sin^2 \alpha)$ do not vanish at $\mathbf{u}_0$, because, otherwise, $a \to -b$ ($\mathbf{u}_n \to \mathbf{u}_0$) by the second formula of (4.8), which contradicts $|\mu| < \infty$ at $\mathbf{u}_0$ by (2.11). Consequently, since the imaginary part in (4.1) goes to zero, we have $1 - 4p = 0$, giving a contradiction. We proved that *the set of parallel mean curvature surfaces in* $\mathbb{C}H^2[-12b^2]$ *is divided by the surfaces with* $a = \bar{a}$ *on* $M_0$ *and those with* $a \ne \bar{a}$ *on* $M_0$. If a complete parallel mean curvature surface with $|H| = 2b$ in $\mathbb{C}H^2[-12b^2]$ is of a general type, then there are real numbers $p$ $(> 0)$ and $t \in [0, \pi]$ such that $x = x_{t,p}$, $p \ne 1/4$, up to rigid motions of $\mathbb{C}H^2[-12b^2]$ by Theorem 4.1, i.e., $(t, p)$ are the coordinates of the non-singular part of the moduli space we now considering. Since $x_{0,p}$ and $x_{\pi,p}$ are congruent because $c$ is defined up to sign, we identify $t = 0$ and $t = \pi$ in $[0, \pi]$, so we have a circle $\mathbb{S}^1$, and the moduli space of the surfaces satisfying $d\alpha \ne 0$ and $a \ne \bar{a}$ is the disjoint union of $\mathbb{S}^1 \times (0, 1/4) \cup \mathbb{S}^1 \times (1/4, \infty)$. When $p = 0$, up to rigid motions, there is only

one surface of $\mathbb{C}H^2[-12b^2]$ which is Hirakawa surface, and as $p \to 1/4-$, there is the one-parameter family $\{x_{t,1/4-} \mid t \in \mathbb{S}^1\}$. These make a $Cone_1$ in $R^3$, in which the vertex of the cone is Hirakawa surface and the base of the cone is the circle $\{x_{t,1/4-} \mid t \in [0, \pi]\}$. A $Cone_2$ is the one point compactification of infinite cylinder $\mathbb{S}^1 \times [1/4, \infty)$, where the base of the cone is the circle $\{x_{t,1/4+} \mid t \in \mathbb{S}^1\}$ and $\infty$ corresponds to Chen surface, proving Theorem 4.3. $\qquad\square$

Compact parallel mean curvature surfaces with the genus $\leq 1$ in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ are classified in Ogata (1995), Hirakawa (2006), Fetcu (2012) and Kenmotsu (2018). As a corollary of Theorem 4.3 of this article, we can see that the assumption of the genus in those classification results is not necessary.

**Corollary 4.4** *Let $M$ be a two-dimensional compact Riemannian manifold and let $x : M \longrightarrow \overline{M}[4\rho]$ be an isometric immersion with a non-zero parallel mean curvature vector. If $\rho \neq 0$, then both the Kaehler angle and the Gaussian curvature are constant.*

Parallel mean curvature surfaces of constant Gaussian curvature are even locally classified in Hirakawa (2006).

## 5 Added in Proof

We can state two open problems for parallel mean curvature surfaces which are related to this article.

1. A parallel mean curvature surface in a higher dimensional complex space-form is contained in a totally geodesic submanifold of the ambient space with codimension at most 8, if the surface is not totally real (Fetcu 2012, Ferreira and Tribuzy 2014). Can be the main results of this article, Theorems 3.5 and 4.1, generalized to $\mathbb{C}P^n$ and $\mathbb{C}H^n$ with $(3 \leq n \leq 5)$ ?
2. Parallel mean curvature surfaces in $\mathbb{S}^3 \times \mathbb{R}$, $\mathbb{H}^3 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^2$, and $\mathbb{H}^2 \times \mathbb{H}^2$ are studied in Fetcu and Rosenberg (2012) and Torralbo and Urbano (2011). What are the moduli spaces of the complete parallel mean curvature surfaces in these ambient spaces ?

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