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Pseudo-parallel surfaces of $\mathbb{S}_c^n \times \mathbb{R}$ and $\mathbb{H}_c^n \times \mathbb{R}$

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Abstract
In this work we give a characterization of pseudo-parallel surfaces in $\mathbb{S}_c^n \times \mathbb{R}$ and $\mathbb{H}_c^n \times \mathbb{R}$, extending an analogous result by Asperti-Lobos-Mercuri for the pseudo-parallel case in space forms. Moreover, when $n = 3$, we prove that any pseudo-parallel surface has flat normal bundle. We also give examples of pseudo-parallel surfaces which are neither semi-parallel nor pseudo-parallel surfaces in a slice. Finally, when $n \geq 4$ we give examples of pseudo-parallel surfaces with non vanishing normal curvature.

Keywords Surface · Parallel · Semi-parallel · Pseudo-parallel · λ-Isotropic · Minimal

Mathematics Subject Classification 53B25 · 53C42

1 Introduction

In the theory of submanifolds of a space form, Asperti-Lobos-Mercuri introduced in Asperti et al. (1999) pseudo-parallel submanifolds as a direct generalization of semi-parallel submanifolds in the sense of Deprez (1985), which in turn, are a generalization of parallel submanifolds (extrinsically symmetric in Ferus' terminology) Ferus (1980) (in particular, of umbilical and totally geodesic submanifolds), and as extrinsic analogues of pseudo-symmetric spaces in the sense of Deszcz (1992). They

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studied pseudo-parallel surfaces of a space form in Asperti et al. (2002), Lobos (2002), and proved that they are surfaces with flat normal bundle or λ-isotropic surfaces in the sense of O'Neill (1965) (i.e. surfaces whose ellipse of curvature in any point is a circle). In particular, they proved that pseudo-parallel surfaces of space forms with non vanishing normal curvature in codimension 2 are superminimal surfaces in the sense of Bryant (1982) (i.e. surfaces which are minimal and λ-isotropic).

An isometric immersion $f : M^m \to \tilde{M}^n$ is said to be *pseudo-parallel* if its second fundamental form α satisfies the following condition:

$$\tilde{R}(X, Y) \cdot \alpha = \phi(X \wedge Y) \cdot \alpha,$$

for some smooth real-valued function ϕ on M^m, where $\tilde{R}$ is the curvature tensor corresponding to the Van der Waerden-Bortolotti connection $\tilde{\nabla}$ of f and $X \wedge Y$ denotes the endomorphism defined by

$$(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y.$$

Considering the product space $\mathbb{Q}_c^n \times \mathbb{R}$ as the ambient space, the first studies of pseudo-parallel submanifolds were started in Lin and Yang (2014) and Lobos and Tassi (2019), where a classification of its hypersurfaces was given, generalizing the classification of parallel and semi-parallel hypersurfaces in Calvaruso et al. (2010) and Van der Veken and Vrancken (2008).

In this work we started the study of pseudo-parallel surfaces in $\mathbb{Q}_c^n \times \mathbb{R}$ (with $c \neq 0$). We begin by observing that any isometric immersion $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ with flat normal bundle is pseudo-parallel (see Proposition 2.2). So, we state the main result of this work:

Theorem 1.1 *Let $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ be a pseudo-parallel surface which does not have flat normal bundle on any open subset of M^2. Then $n \geq 4$, f is λ-isotropic and*

$$K > \phi, \tag{1.1}$$

$$\lambda^2 = 4K - 3\phi + c(\|T\|^2 - 1) > 0, \tag{1.2}$$

$$\|H\|^2 = 3K - 2\phi + c(\|T\|^2 - 1) \geq 0, \tag{1.3}$$

where K is the Gaussian curvature, λ is a smooth real-valued function on M^2, H is the mean curvature vector field of f and T is the tangent part of $\frac{\partial}{\partial t}$, the canonical unit vector field tangent to the second factor of $\mathbb{Q}_c^n \times \mathbb{R}$.

Conversely, if f is λ-isotropic then f is pseudo-parallel.

We remark that Theorem 1.1 extends for $\mathbb{Q}_c^n \times \mathbb{R}$ a similar result for pseudo-parallel surfaces into space forms given by Asperti-Lobos-Mercuri in Asperti et al. (2002).

However, the class of pseudo-parallel surfaces in $\mathbb{Q}_c^3 \times \mathbb{R}$ is not empty. In the last section we give examples of semi-parallel surfaces which are not parallel as well as examples of pseudo-parallel surfaces in $\mathbb{S}_c^3 \times \mathbb{R}$ and $\mathbb{H}_c^3 \times \mathbb{R}$ which are neither semi-parallel nor pseudo-parallel surfaces in a slice.

Finally, we remark that pseudo-parallel surfaces in $\mathbb{Q}_c^n \times \mathbb{R}$ with $n \geq 4$ and non vanishing normal curvature do exist, as shown in Examples 4.3, 4.5 and 4.6 in the last section.

2 Preliminaries

First of all, we establish the notation that we use along this work. Let $f : M^m \to \tilde{M}^n$ be an isometric immersion. We decompose the tangent bundle $T\tilde{M}$ of $\tilde{M}^n$ in its tangent and normal parts, as a sum $T\tilde{M} = TM \oplus N_f M$, where TM and $N_f M$ are the tangent bundle of M^m and the normal bundle of f, respectively. Using this notation we consider $\tilde{\nabla} = \nabla \oplus \nabla^\perp$ the Van der Waerden-Bortolotti connection of f and $\tilde{R} = R \oplus R^\perp$ its curvature tensor. The second fundamental form of f is the symmetric 2-tensor denoted by $\alpha : TM \times TM \to N_f M$. For any $\xi \in N_f M$ the correspondent Weingarten operator in the ξ-direction is denoted by A_ξ, that is,

$$\langle \alpha(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle, \quad \text{for all} \quad X, Y \in TM, \quad \text{and} \quad \xi \in N_f M.$$

The mean curvature vector field of f is denoted by H. Finally, we say that f has flat normal bundle (or vanishing normal curvature) if $R^\perp = 0$.

An isometric immersion $f : M^m \to \tilde{M}^n$ is said to be:

1. *Totally geodesic* if
$$\alpha(X, Y) = 0; \tag{2.1}$$
2. *Umbilical* if the mean curvature vector field H of f satisfies
$$\alpha(X, Y) = \langle X, Y \rangle H; \tag{2.2}$$
3. *Locally parallel* if
$$(\tilde{\nabla}_X \alpha)(Y, Z) = 0; \tag{2.3}$$
4. *Semi-parallel* if
$$(\tilde{R}(X, Y) \cdot \alpha)(Z, W) = 0; \tag{2.4}$$
5. *Pseudo-parallel* if
$$(\tilde{R}(X, Y) \cdot \alpha)(Z, W) = \phi[(X \wedge Y) \cdot \alpha](Z, W), \tag{2.5}$$
for some smooth real-valued function ϕ on M^m and for any vector X,Y,Z and W tangents to M.

Here the notation means

$$\begin{aligned}
(\tilde{\nabla}_X \alpha)(Y, Z) &= \nabla_X^\perp \alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z),\\
(\tilde{R}(X, Y) \cdot \alpha)(Z, W) &= R^\perp(X, Y)[\alpha(Z, W)] - \alpha(R(X, Y)Z, W)\\
&\quad - \alpha(Z, R(X, Y)W),\\
[(X \wedge Y) \cdot \alpha](Z, W) &= -\alpha((X \wedge Y)Z, W) - \alpha(Z, (X \wedge Y)W).
\end{aligned}$$

A space form $\mathbb{Q}_c^n$ is a simply connected, complete, n-dimensional Riemannian manifold with constant sectional curvature c. Namely, $\mathbb{Q}_c^n$ is the n-dimensional sphere $\mathbb{S}_c^n$ or the n-dimensional hyperbolic space $\mathbb{H}_c^n$, respectively given by

$$\mathbb{S}_c^n = \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; \sum_{i=1}^{n+1} x_i^2 = \frac{1}{c}\right\}, \quad \text{if } \ c > 0,$$

$$\mathbb{H}_c^n = \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{L}^{n+1}; -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}, x_1 > 0\right\}, \quad \text{if } c < 0,$$

where $\mathbb{L}^{n+1}$ is the $(n+1)$-dimensional Minkowski space, that is, the $(n+1)$-dimensional euclidean space $\mathbb{R}^{n+1}$ endowed with the inner product

$$\langle (x_1, x_2, \ldots, x_{n+1}), (y_1, y_2, \ldots, y_{n+1})\rangle = -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i.$$

This work is devoted to the study of these classes of surfaces with $\mathbb{Q}_c^n \times \mathbb{R}$ as the ambient space and we always assume $c \neq 0$. Thus, let $\frac{\partial}{\partial t}$ be the canonical unit vector field tangent to the second factor of $\mathbb{Q}_c^n \times \mathbb{R}$. For a given isometric immersion $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$, it is convenient to consider the following decomposition of $\frac{\partial}{\partial t}$ in its tangent and normal parts:

$$\frac{\partial}{\partial t} = f_* T + \eta, \tag{2.6}$$

for some $T \in TM$ and some $\eta \in N_f M$.

Another tools we make use are the Fundamental Equations for a surface $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ and now we recall them. Let $\{e_1, e_2\}$ be an orthonormal local frame for M^2 and set $\alpha_{ij} = \alpha(e_i, e_j)$. By δ_{ij} we mean the Kronecker's Delta. From Mendonça and Tojeiro (2013) we have the following equations:

GAUSS:

$$\begin{aligned} R(e_1, e_2)e_k = c(&\delta_{2k} e_1 - \delta_{1k} e_2 - \langle e_2, T\rangle\langle e_k, T\rangle e_1 + \delta_{1k}\langle e_2, T\rangle T \\ &- \delta_{2k}\langle e_1, T\rangle T + \langle e_1, T\rangle\langle e_k, T\rangle e_2) + A_{\alpha_{2k}} e_1 - A_{\alpha_{1k}} e_2. \end{aligned} \tag{2.7}$$

CODAZZI:

$$(\tilde{\nabla}_{e_1}\alpha)(e_2, e_k) - (\tilde{\nabla}_{e_2}\alpha)(e_1, e_k) = c(\delta_{1k}\langle e_2, T\rangle - \delta_{2k}\langle e_1, T\rangle)\eta \tag{2.8}$$

RICCI:

$$R^{\perp}(e_1, e_2)\xi = \alpha(e_1, A_\xi e_2) - \alpha(A_\xi e_1, e_2). \tag{2.9}$$

It follows from the Ricci equation that

$$R^{\perp}(e_1, e_2)\xi \in \operatorname{span}\{\alpha(X, Y); X, Y \in TM\}, \quad \text{for all } \ \xi \in N_f M(x).$$

Thus, the equation (2.9) is equivalent to the following equation:

$$R^{\perp}(e_1, e_2)\alpha_{ij} = \langle \alpha_{12}, \alpha_{ij} \rangle (\alpha_{11} - \alpha_{22}) + \langle \alpha_{22} - \alpha_{11}, \alpha_{ij} \rangle \alpha_{12}. \quad (2.10)$$

On the other hand, the pseudo-parallelism condition is equivalent to the following two equations:

$$R^{\perp}(e_1, e_2)\alpha_{ii} = (-1)^i 2(K - \phi)\alpha_{12}, \quad (2.11)$$

$$R^{\perp}(e_1, e_2)\alpha_{12} = (K - \phi)(\alpha_{11} - \alpha_{22}), \quad (2.12)$$

where

$$K = c(1 - \|T\|^2) + \langle \alpha_{11}, \alpha_{22} \rangle - \|\alpha_{12}\|^2 \quad (2.13)$$

is the Gaussian curvature of M^2. As a consequence, we have the next lemma.

Lemma 2.1 *Let $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ be a pseudo-parallel surface. Then $R^{\perp}(X, Y)H = 0$, for all $X, Y \in TM$.*

Proof Immediate by equation (2.11), since $H = \frac{1}{2}(\alpha_{11} + \alpha_{22})$. □

Proposition 2.2 *Let $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ be a surface with flat normal bundle. Then f is a pseudo-parallel immersion.*

Proof Since f has flat normal bundle, by equations (2.11) and (2.12) we conclude that f is ϕ-pseudo-parallel by taking $\phi = K$, where K is the Gaussian curvature of M^2. □

In the following, we have two propositions that is useful to construct examples of pseudo-parallel surfaces.

Proposition 2.3 *Let $f : M^m \to \mathbb{Q}_c^n$ be an isometric immersion and let $j : \mathbb{Q}_c^n \to \mathbb{Q}_c^n \times \mathbb{R}$ be a totally geodesic immersion. If f is ϕ-pseudo-parallel, then $j \circ f$ is ϕ-pseudo-parallel.*

Proof In this proof, we denote the second fundamental form of f and $j \circ f$ respectively by α^f and $\alpha^{j \circ f}$. In the same way, we denote the normal curvature tensors of f and $j \circ f$ respectively by $R_f^{\perp}$ and $R_{j \circ f}^{\perp}$. Since j is a totally geodesic immersion, we have the following relations:

$$\alpha^{j \circ f}(Z, W) = j_* \alpha^f(Z, W),$$
$$R_{j \circ f}^{\perp}(X, Y)\alpha^{j \circ f}(Z, W) = j_* R_f^{\perp}(X, Y)\alpha^f(Z, W),$$

Therefore, applying Definition 2.5 we obtain

$$\begin{aligned}(\tilde{R}(X, Y) \cdot \alpha^{j \circ f})(Z, W) &= R_{j \circ f}^{\perp}(X, Y)\alpha^{j \circ f}(Z, W) - \alpha^{j \circ f}(R(X, Y)Z, W) \\ &\quad - \alpha^{j \circ f}(Z, R(X, Y)W) \\ &= j_* R_f^{\perp}(X, Y)\alpha^f(Z, W) - j_* \alpha^f(R(X, Y)Z, W)\end{aligned}$$

$$
\begin{aligned}
&- j_*\alpha^f(Z, R(X,Y)W) \\
&= \phi\{-j_*\alpha^f((X\wedge Y)Z, W) - j_*\alpha^f(Z, (X\wedge Y)W)\} \\
&= \phi\{-\alpha^{j\circ f}((X\wedge Y)Z, W) - \alpha^{j\circ f}(Z, (X\wedge Y)W)\} \\
&= \phi[(X\wedge Y)\cdot \alpha^{j\circ f}](Z, W).
\end{aligned}
$$

□

Proposition 2.4 *Let $f : M^m \to \mathbb{Q}_c^n \times \mathbb{R}$ be an isometric immersion and let $j : \mathbb{Q}_c^n \times \mathbb{R} \to \mathbb{Q}_c^{n+l} \times \mathbb{R}$ be a totally geodesic immersion. If f is ϕ-pseudo-parallel, then $j \circ f$ is ϕ-pseudo-parallel.*

Proof Is analogous to that of Proposition 2.3. □

3 Proof of the main theorem

Before we give a proof of Theorem 1.1 we recall that $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ is a λ-isotropic surface if, for each $x \in M$, the ellipse of curvature $\{\alpha(X,X) \in N_f M(x); X \in T_x M \text{ with } \|X\| = 1\}$ is a sphere with radius $\lambda(x)$, where $\lambda : M^2 \to \mathbb{R}$ is a smooth function. The following result, due to Sakaki in Sakaki (2015) plays a vital role in the proof of Theorem 1.1. Its statement is:

Theorem 3.1 (see Sakaki (2015)) *Let $f : M^2 \to \mathbb{Q}_c^3 \times \mathbb{R}$ be a minimal surface with $c \neq 0$. If f is λ-isotropic, then f is totally geodesic.*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let us suppose that $f : M^2 \to \mathbb{Q}_c^n \times \mathbb{R}$ is pseudo-parallel with non vanishing normal curvature. Combining equations (2.10) to (2.13) we get

$$\langle \alpha_{12}, \alpha_{ii}\rangle(\alpha_{11} - \alpha_{22}) + \{2(-1)^{i+1}(K - \phi) + \langle \alpha_{ii}, \alpha_{22} - \alpha_{11}\rangle\}\alpha_{12} = 0, \tag{3.1}$$

$$\{\|\alpha_{12}\|^2 + (\phi - K)\}(\alpha_{11} - \alpha_{22}) + \langle \alpha_{22} - \alpha_{11}, \alpha_{12}\rangle\alpha_{12} = 0. \tag{3.2}$$

Next, we prove that $\{\alpha_{12}, \alpha_{11} - \alpha_{22}\}$ is linearly independent. We can suppose $\phi \neq K$. Otherwise, since $\langle R^\perp(e_1,e_2)\xi, \zeta\rangle = -\langle R^\perp(e_1,e_2)\zeta, \xi\rangle$, by the equations (2.11) and (2.12) we would have $R^\perp = 0$, which is a contradiction. Notice that $\alpha_{12} \neq 0$ and $\alpha_{11} \neq \alpha_{22}$. In fact, if $\alpha_{12} = 0$ then $R^\perp(e_1,e_2)\alpha_{12} = 0$ which implies by equation (2.12) that $\alpha_{11} = \alpha_{22}$, and in this case f is umbilical and has flat normal bundle, a contradiction. If $\alpha_{11} - \alpha_{22} = 0$, then $R^\perp(e_1,e_2)(\alpha_{11} - \alpha_{22}) = 0$, which implies by equation (2.11) that $\alpha_{12} = 0$.

Assume that there exist $\lambda, \mu \in \mathbb{R}$ such that $\lambda\alpha_{12} + \mu(\alpha_{11} - \alpha_{22}) = 0$. Then, by equations (2.11) and (2.12) we get $\lambda(\alpha_{11} - \alpha_{22}) - 4\mu\alpha_{12} = 0$. If $\mu \neq 0$ then $(\alpha_{11} - \alpha_{22}) = \frac{-\lambda}{\mu}\alpha_{12}$ and thus $\left(\frac{-\lambda^2}{\mu} - 4\mu\right)\alpha_{12} = 0$, which lead us to $\lambda^2 = -4\mu^2 < 0$, a contradiction. So $\mu = 0$, and therefore $\lambda = 0$.

Using this and equations (3.1) and (3.2) we obtain

$$\langle \alpha_{12}, \alpha_{11}\rangle = \langle \alpha_{12}, \alpha_{22}\rangle = 0, \tag{3.3}$$

$$\langle \alpha_{22} - \alpha_{11}, \alpha_{ii} \rangle = (-1)^i 2(K - \phi), \tag{3.4}$$
$$\|\alpha_{12}\|^2 = K - \phi > 0. \tag{3.5}$$

From the equation (2.13) we get

$$\langle \alpha_{11}, \alpha_{22} \rangle = 2K - \phi + c(\|T\|^2 - 1), \tag{3.6}$$
$$\|\alpha_{11}\|^2 = \|\alpha_{22}\|^2 = 4K - 3\phi + c(\|T\|^2 - 1) > 0, \tag{3.7}$$
$$\|\alpha_{11} - \alpha_{22}\|^2 = 4(K - \phi) > 0, \tag{3.8}$$
$$\|H\|^2 = 3K - 2\phi + c(\|T\|^2 - 1). \tag{3.9}$$

In particular, f is λ-isotropic with $\lambda^2 = 4K - 3\phi + c(\|T\|^2 - 1)$.

Now, we prove that $n \geq 4$. Suppose that $n = 3$. Since f has non flat normal bundle, for any $x \in M^2$ we have that $R^\perp(x)(e_1, e_2) : N_f M(x) \to N_f M(x)$ is a non zero antisymmetric linear operator, defined in a two-dimensional vector space. Thus, by Lemma 2.1 we conclude that $H(x) = 0$. But from Theorem 3.1, we conclude that f is totally geodesic and in particular, $R^\perp(x) = 0$, which is a contradiction.

Conversely, let us assume that f is λ-isotropic. Set $X = \cos\theta e_1 + \sin\theta e_2$. Then

$$\begin{aligned}\lambda^2 &= \|\alpha(X, X)\|^2 \\ &= (\cos^4\theta + \sin^4\theta)\lambda^2 + 2\sin^2\theta\cos^2\theta\langle \alpha_{11}, \alpha_{22}\rangle \\ &\quad + 4\sin^3\theta\cos\theta\langle \alpha_{22}, \alpha_{12}\rangle + 4\sin\theta\cos^3\theta\langle \alpha_{11}, \alpha_{12}\rangle \\ &\quad + 4\sin^2\theta\cos^2\theta\|\alpha_{12}\|^2.\end{aligned}$$

Since λ does not depend on θ, taking the derivative with respect to θ we get

$$\begin{aligned}0 &= \left.\frac{d\lambda^2}{d\theta}\right|_{\theta=0} = \frac{d}{d\theta}(\|\alpha(X, X)\|^2)|_{\theta=0} = 4\langle \alpha_{11}, \alpha_{12}\rangle, \\ 0 &= \left.\frac{d\lambda^2}{d\theta}\right|_{\theta=\frac{\pi}{2}} = \frac{d}{d\theta}(\|\alpha(X, X)\|^2)|_{\theta=\frac{\pi}{2}} = -4\langle \alpha_{22}, \alpha_{12}\rangle.\end{aligned}$$

On the other hand, with $Y = \frac{1}{\sqrt{2}}(e_1 + e_2)$ we get

$$\begin{aligned}\lambda^2 &= \|\alpha(Y, Y)\|^2 \\ &= \frac{1}{4}\{2\lambda^2 + 4\|\alpha_{12}\|^2 + 2\langle \alpha_{11}, \alpha_{22}\rangle\},\end{aligned}$$

that is,

$$\lambda^2 = 2\|\alpha_{12}\|^2 + \langle \alpha_{11}, \alpha_{22}\rangle.$$

Using this and the Gauss equation we get

$$\|\alpha_{12}\|^2 = \frac{1}{3}\{\lambda^2 - K + c(1 - \|T\|^2)\}.$$

From the Ricci equation $R^{\perp}(e_1, e_2)\alpha_{ii} = \langle \alpha_{22} - \alpha_{11}, \alpha_{ii} \rangle \alpha_{12}$, $i = 1, 2$, we obtain

$$\begin{aligned} R^{\perp}(e_1, e_2)\alpha_{ii} &= \langle \alpha_{22} - \alpha_{11}, \alpha_{ii} \rangle \alpha_{12} \\ &= (-1)^i 2\|\alpha_{12}\|^2 \alpha_{12} \\ &= (-1)^i \frac{2}{3}\{\lambda^2 - K + c(1 - \|T\|^2)\}\alpha_{12}, \end{aligned}$$

Using the Ricci equation once more we obtain

$$\begin{aligned} R^{\perp}(e_1, e_2)\alpha_{12} &= \|\alpha_{12}\|^2(\alpha_{11} - \alpha_{22}) \\ &= \frac{1}{3}\{\lambda^2 - K + c(1 - \|T\|^2)\}(\alpha_{11} - \alpha_{22}). \end{aligned}$$

Therefore, taking $\phi = \frac{4K - \lambda^2 + c(\|T\|^2 - 1)}{3}$, we conclude that f is pseudo-parallel according to equations (2.11) and (2.12). □

4 Some Examples

We now introduce the first examples of semi-parallel and pseudo-parallel surfaces of $\mathbb{Q}_c^3 \times \mathbb{R}$ which are not locally parallel and semi-parallel, respectively, and that are not just inclusions of surfaces of $\mathbb{Q}_c^3$ into $\mathbb{Q}_c^3 \times \mathbb{R}$.

Example 4.1 A general construction of submanifolds of $\mathbb{Q}_c^n \times \mathbb{R}$ with flat normal bundle and T as a principal direction can be found in Mendonça and Tojeiro (2014), by Mendonça-Tojeiro. For our purpose, based on this work, the construction becomes: let $g : J \to \mathbb{Q}_c^3$ be a regular curve and $\{\xi_1, \xi_2\}$ an orthonormal set of vector fields normal to g. Put

$$\begin{aligned} \tilde{g} &= i \circ j \circ g, \\ \tilde{\xi}_k &= i_* j_* \xi_k, \quad \text{for} \quad k \in \{1, 2\}, \\ \tilde{\xi}_0 &= \tilde{g}, \quad \tilde{\xi}_3 = i_* \frac{\partial}{\partial t}, \end{aligned}$$

where $j : \mathbb{Q}_c^3 \to \mathbb{Q}_c^3 \times \mathbb{R}$ and $i : \mathbb{Q}_c^3 \times \mathbb{R} \to \mathbb{E}^5$ are the canonical inclusions. If $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) : I \to \mathbb{Q}_c^2 \times \mathbb{R}$ is a smooth regular curve with $\alpha_3'(s) \neq 0, \quad \forall s \in I$, we have the following isometric immersion $f : M^2 = J \times I \to \mathbb{Q}_c^3 \times \mathbb{R}$ given by

$$\tilde{f}(x, s) = (i \circ f)(x, s) = \sum_{k=0}^{3} \alpha_k(s)\tilde{\xi}_k(x). \tag{4.1}$$

At regular points, f is an isometric immersion with flat normal bundle and T as a principal direction. Conversely, if $f : M^2 \to \mathbb{Q}_c^3 \times \mathbb{R}$ is an isometric immersion with flat normal bundle and T as a principal direction, then f is given by (4.1) for some isometric immersion $g : \mathbb{Q}^3_c \times \mathbb{R}$ and some smooth regular curve $\alpha : I \to \mathbb{Q}^2_c \times \mathbb{R}$ whose its last coordinate has non vanishing derivative.

Geometrically, f is obtained by parallel transporting a curve in a product submanifold $\mathbb{Q}^2_c \times \mathbb{R}$ of a fixed normal space of $\tilde{g}$ with respect to its normal connection.

In particular, when dealing with pseudo-parallel surfaces in $\mathbb{Q}^3_c \times \mathbb{R}$, at least those that have T as a principal direction are fully described by this method.

We now construct two simple examples. Let us define

$$C_c(s) = \begin{cases} \cos(s), & \text{if } c > 0 \\ \cosh(s), & \text{if } c < 0 \end{cases} \quad \text{and} \quad S_c(s) = \begin{cases} \sin(s), & \text{if } c > 0 \\ \sinh(s), & \text{if } c < 0. \end{cases}$$

By taking

$$\begin{aligned} &\tilde{g}(x) = (C_c(\theta(x)), S_c(\theta(x)), 0, 0, 0), \\ &\tilde{\xi}_1(x) = (0, 0, 1, 0, 0), \quad \tilde{\xi}_2(x) = (0, 0, 0, 1, 0), \\ &\alpha_0(s) = \sqrt{1 - \operatorname{sgn}(c)d^2}, \quad \alpha_1(s) = d\cos s, \quad \alpha_2(s) = d\sin s, \quad \alpha_3(s) = s. \end{aligned}$$

where $0 < d < 1$, if $c > 0$, or $d > 0$, if $c < 0$, and $\theta : \mathbb{R}^2 \to \mathbb{R}$ is the smooth function given by

$$\theta(u) = \frac{u}{\sqrt{1 - \operatorname{sgn}(c)d^2}},$$

we obtain a semi-parallel surface in $\mathbb{Q}^3_c \times \mathbb{R}$ that is not locally parallel.

Another example can be obtained by taking $0 < d < 1$ and

$$\begin{aligned} &\tilde{g}(x) = (0, \cos(x), \sin(x), 0, 0, 0), \\ &\tilde{\xi}_1(x) = (1, 0, 0, 0, 0), \quad \tilde{\xi}_2(x) = (0, 0, 0, 1, 0), \\ &\alpha_0(s) = dS_c(s), \quad \alpha_1(s) = dC_c(s), \quad \alpha_2(s) = \sqrt{\operatorname{sgn}(c)(1 - d^2)}, \quad \alpha_3(s) = s, \end{aligned}$$

where the surface obtained is pseudo-parallel in $\mathbb{Q}^3_c \times \mathbb{R}$ but not semi-parallel since its Gaussian curvature does not vanish. Also, notice that it is not contained in a totally geodesic slice of the form $\mathbb{Q}^3_c \times \{t\}$, for some $t \in \mathbb{R}$.

Question 4.2 Are there other examples, up to isometries, of pseudo-parallel surfaces in $\mathbb{Q}^3_c \times \mathbb{R}$ ($c \neq 0$), for which T is not a principal direction?

The next three examples show us that for $n > 3$ there exists pseudo-parallel surfaces with non vanishing normal curvature.

Example 4.3 Let $f : \mathbb{S}^2_{1/3} \to \mathbb{S}^4_1$ be the classical *Veronese surface*, given by

$$f(x, y, z) = \left(\frac{1}{\sqrt{3}}xy, \frac{1}{\sqrt{3}}xz, \frac{1}{\sqrt{3}}yz, \frac{1}{2\sqrt{3}}(x^2 - y^2), \frac{1}{6}(x^2 + y^2 - 2z^2)\right),$$

which is a locally parallel, minimal and λ-isotropic immersion (as we can see in Chern et al. (1970), Itoh and Ogiue (1973) and Sakamoto (1977)) in $\mathbb{S}_1^4$ with non vanishing normal curvature. If $i : \mathbb{S}_1^4 \to \mathbb{S}_1^4 \times \mathbb{R}$ is the totally geodesic inclusion given by $i(x) = (x, 0)$, then by Proposition 2.3 we have that $i \circ f$ is a pseudo-parallel immersion in $\mathbb{S}_1^4 \times \mathbb{R}$ with non vanishing normal curvature.

Conjecture 4.4 *The only minimal pseudo-parallel surface in $\mathbb{Q}_c^4 \times \mathbb{R}$ with non vanishing normal curvature and constant ϕ are these of Example 4.3.*

Example 4.5 It's known by Chern in Chern (1970) that: "Any minimal immersion of a topological 2-sphere $\mathbb{S}^2$ into $\mathbb{S}_c^4$ is a superminimal immersion". So, by Theorem 1.1, we have that any minimal immersion of a topological 2-sphere into a slice of $\mathbb{S}_c^4 \times \mathbb{R}$ whit non vanishing normal curvature is pseudo-parallel with $\phi = \frac{4K - c - \lambda^2}{3}$. Moreover, if the Gaussian curvature is not constant, the immersion is not semi-parallel.

Example 4.6 Let $f : \mathbb{R}^2 \to \mathbb{S}_c^5$ be the surface given by

$$f(x, y) = \frac{2}{\sqrt{6c}} \left(\cos u \cos v, \cos u \sin v, \frac{\sqrt{2}}{2} cos(2u), \sin u \cos v, \right.$$
$$\left. \sin u \sin v, \frac{\sqrt{2}}{2} \sin(2u) \right),$$

where $u = \sqrt{\frac{c}{2}} x$, $v = \frac{\sqrt{6c}}{2} y$.

This example, that appears in Sakamoto (1989), is a minimal λ-isotropic flat torus with $\lambda = \sqrt{\frac{c}{2}}$ and non vanishing normal curvature. In particular, f is a pseudo-parallel immersion in $\mathbb{S}_c^5$ with $\phi = \frac{-c}{2}$.

Thus, if $i : \mathbb{S}_c^5 \to \mathbb{S}_c^5 \times \mathbb{R}$ is the totally geodesic inclusion given by $i(x) = (x, 0)$, by Proposition 2.3 we have that $i \circ f$ is a pseudo-parallel immersion in $\mathbb{S}_c^5 \times \mathbb{R}$ with non vanishing normal curvature.

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