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p-Harmonic l-forms on Complete Noncompact Submanifolds in Sphere with Flat Normal Bundle

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Abstract In this paper, we investigate a complete noncompact submanifold M^m in a sphere S^{m+t} with flat normal bundle. We prove that the dimension of the space of L^p p-harmonic l-forms (when $m \geq 4$, $2 \leq l \leq m-2$ and when $m = 3$, $l = 2$) on M is finite if the total curvature of M is finite and $m \geq 3$. We also obtain that there are no nontrivial L^p p-harmonic l-forms on M if the total curvature is bounded from above by a constant depending only on m, p, l.

Keywords p-Harmonic l-form · Submanifolds

Mathematics Subject Classification Primary 53C21 · 53C25

1 Introduction

Let M^m be a submanifold in a Riemannian manifold N^{m+t}. Fix a point $x \in M$ and a local orthonormal frame $\{e_1, \ldots, e_{m+t}\}$ of N^{m+t} such that $\{e_1, \ldots, e_m\}$ are tangent fields of M^m at x. In the following we shall use the following convention on the ranges of indices: $1 \leq i, j, k, \ldots \leq m$ and $m + 1 \leq \alpha \leq m + t$. The second fundamental form A is defined by

$$A(X, Y) = \sum_{\alpha} \langle \overline{\nabla}_X Y, e_\alpha \rangle e_\alpha$$

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for any vector fields X, Y on M^m, where $\overline{\nabla}$ is the Riemannian connection of N^{m+t}. Denote $h_{ij}^{\alpha} = \langle \overline{\nabla}_{e_i} e_j, e_\alpha \rangle$, then $|A|^2 = \sum_\alpha \sum_{ij} (h_{ij}^\alpha)^2$, and the mean curvature vector field H is defined by

$$H = \frac{1}{m} \sum_\alpha H^\alpha e_\alpha = \frac{1}{m} \sum_\alpha \sum_i h_{ii}^\alpha e_\alpha.$$

The traceless second fundamental form Φ is defined by

$$\Phi(X, Y) = A(X, Y) - \langle X, Y \rangle H$$

for any vector fields X, Y on M. It is easy to see that

$$|\Phi|^2 = |A|^2 - m|H|^2$$

which measures how much the immersion deviates from being totally umbilical. We say M has finite total curvature if

$$||\Phi||_{L^m(M)} = \left(\int_M |\Phi|^m \right)^{\frac{1}{m}} < \infty.$$

In Cao et al. (1997), Cao, Shen and Zhu showed that a complete connected stable minimal hypersurface in Euclidean space must have exactly one end. Their strategy was to utilize a result of Schoen-Yau asserting that a complete stable minimal hypersurface in Euclidean space can not admit a non-constant harmonic function with finite integral Schoen and Yau (1976). According to the work of Li and Tam (1992), Li and Wang modified the proof to show that each end of a complete immersed minimal submanifold must be parabolic in Li and Wang (2002). Due to this connection with harmonic functions, this allows one to estimate the number of ends of the above hypersurface by estimating the dimension of the space of bounded harmonic functions with finite Dirichlet integral. They prove that if M has finite index, then the dimension of space of L^2 harmonic 1-forms on M is finite, and M must have finitely many ends in Li and Wang (2002). In Fu and Xu (2010), Fu and Xu proved that a complete submanifold M^m with finite total curvature and some conditions on mean curvaute in an $(n + p)$-dimensional simply connected space form $M^{m+p}(c)$ must have finitely many ends. In Cavalcante et al. (2014), Cavalcante, Mirandola and Vitório proved that a complete submanifold M^m with finite total curvature and some conditions on the first eigenvalue of the Laplace–Beltrami operator of M in an Hadamard manifold must have finitely many ends. In Lin (2015c), Lin proved vanishing and finiteness theorems for L^2 harmonic forms under the assumptions on Schrödinger operators involving the squared norm of the traceless second fundamental form. In Zhu and Fang (2014a, b), Zhu and Fang obtained some vanishing and finiteness theorems for L^2 harmonic 1-forms on submanifold in sphere. In Zhu (2016), Zhu obtained that the space of all L^2 harmonic 2-forms on submanifolds with finite total curvture in spheres had finite dimension. And in the same paper, Zhu also gave the following conjecture.

Conjecture Zhu (2016) *Let M^m $m \geq 3$ be an m-dimensional complete noncompact oriented manifold isometrically immersed in S^{m+1}. If the total curvature is finite, then the space of all L^2 harmonic l-forms $3 \leq l \leq m-3$ has finite dimension.*

For p-harmonic 1-forms, Zhang Zhang (2001) obtained vanishing results for p-harmonic 1-form. Chang Chang et al. (2010) obtained the compactness for any bounded set of p-harmonic 1-forms. The author and Pan in Han and Pan (2016) investigated L^p p-harmonic 1-forms on complete noncompact submanifolds in a Hadamard manifolds, and obtained some vanishing and finiteness theorems under finite total curvature and the first eigenvalues of Laplace–Beltrami operator. The author Zhang and Liang in Han et al. (0000) obtained some vanishing and finiteness theorems under the conditions of the scalar curvature and Ricci curvature. The author and Zhang in Han and Zhang (0000) obtained that if the total curvature of complete noncompact submanifold in S^{m+t} is finite, then the space of L^p p-harmonic 1-form is finite. In Dung and Seo (2016) Dung and Seo obtained some vanishing results for p-harmonic forms. In Dung (2017) Dung obtained some vanishing results for p-harmonic l-forms, for $2 \leq l \leq n-2$ on Riemannian manifolds with a weighted Poincaré inequality.

Let (M^m, g) be a Riemannian manifold, and let u be a real C^∞ function on M^m. Fix $p \in R$, $p \geq 2$ and consider a compact domain $\Omega \subset M^m$. The p-energy of u on Ω, is defined to be

$$E_p(\Omega, u) = \frac{1}{p}\int_\Omega |\nabla u|^p.$$

The function u is said to be p-harmonic on M^m if u is a critical point of $E_p(\Omega, *)$ for every compact domain $\Omega \subset M^m$. Equivalently, u satisfies the Euler-Lagrange equation.

$$div(|\nabla u|^{p-2}\nabla u) = 0.$$

Thus, the concept of p-harmonic function is a natural generalization of that of harmonic function, that is, of a critical point of the 2-energy functional.

Definition 1.1 A p-harmonic l-form is a differentiable l-form on M^m satisfying the following properties:

$$\begin{cases} d\omega = 0, \\ \delta(|\omega|^{p-2}\omega) = 0, \end{cases}$$

where δ is the codifferential operator. It is easy to see that the differential of a p-harmonic function is a p-harmonic 1-form.

In this paper, we investigate the properties for p-harmonic l-form (when $m \geq 4$, $2 \leq l \leq m-2$ and when $m = 3$, $l = 2$) on complete noncompact submanifolds in space forms. We assume that M^m is a complete noncompact manifold and define the space of the L^p p-harmonic l-forms on M by

$$H^{l,p}(M) = \{\omega | \int_M |\omega|^p < \infty, \quad d\omega = 0 \quad \text{and} \quad \delta(|\omega|^{p-2}\omega) = 0\}$$

where $p \geq 2$ and when $m \geq 4, 2 \leq l \leq m-2$ and when $m = 3, l = 2$. We obtain the following results:

Theorem 1.2 (cf. Theorem 3.1) *Let M^m, $m \geq 3$ be an m-dimensional complete noncompact oriented manifold isometrically immersed in an $(m+t)$-dimensional sphere S^{m+t} with flat normal bundle. If the total curvature is finite, then we have $dim H^{l,p}(M) < \infty$ for $p \geq 2$ and when $m \geq 4, 2 \leq l \leq m-2$ and when $m = 3, l = 2$.*

Remark 1.3 When $p = 2, l = 2$ and $t = 1$, we can obtain the Theorem 1 in Zhu (2016). When $p = 2, 3 \leq l \leq m-3$ and $t = 1$, we know that the Conjecture in Zhu (2016) is true.

Theorem 1.4 (cf. Theorem 3.2) *Let M^m, $m \geq 3$ be an m-dimensional complete noncompact oriented manifold isometrically immersed in an $(m+t)$-dimensional sphere S^{m+t}. There exists a positive constant Λ depending only on m, p, l, such that if $||\Phi||_{L^m(M)} < \Lambda$, then there is no nontrivial L^p p-harmonic l-forms on M, i.e. $H^{l,p}(M) = \{0\}$, for $p \geq 2$ and when $m \geq 4$, $2 \leq l \leq m-2$, when $m = 3$, $l = 2$. More precisely, Λ can be given explicitly by a constant $C(M)$ in (7) as follows:*

$$\Lambda < \min\left\{\sqrt{\frac{8(p-1)}{p^2 m C(M)}}, \sqrt{\frac{1}{m(m-1)C(M)}}, \sqrt{\frac{2l(m-l)}{m^2(m-1)C(M)}}\right\}.$$

2 Preliminaries

Let M^m be an m-dimensional complete noncompact submanifold in $F^{m+t}(c)$, and let $\triangle$ be the Hodge Laplace-Beltrami operator of M^m acting on the space of differential l-forms. Given two l-forms ω and θ, we define a pointwise inner product

$$\langle \omega, \theta \rangle = \sum_{i_1, \ldots, i_l = 1}^{m} \omega(e_{i_1}, \ldots, e_{i_l}) \theta(e_{i_1}, \ldots, e_{i_l})$$

Here we omit the normalizing factor $\frac{1}{l!}$. The Weitzenböck formula Wu (1988) gives

$$\triangle = \nabla^2 - W_l, \tag{1}$$

where ∇^2 is the Bochner Laplacian and W_l is an endomorphism depending upon the curvature tensor of M^m. Let $\{\theta^1, \ldots, \theta^m\}$ be an orthonormal basis dual to $\{e_1, \ldots, e_m\}$, then

$$\left\langle W_l(\omega), \omega \right\rangle = \langle \sum_{j,k=1}^{m} \theta^k \wedge i_{e_j} R(e_k, e_j)\omega, \omega \Big\rangle \tag{2}$$

for any l-form ω. For any $\omega \in H^{l,p}(M)$, by (1) and (2) we have

$$\frac{1}{2}\triangle|\omega|^{2(p-1)} = |\nabla(|\omega|^{p-2}\omega)|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega \rangle$$

$$
\begin{aligned}
&+|\omega|^{2(p-2)}\left\langle \sum_{j,k=1}^{m} \theta^k \wedge i_{e_j} R(e_k, e_j)\omega, \omega \right\rangle \\
&= |\nabla(|\omega|^{p-2}\omega)|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle \\
&+ l|\omega|^{2(p-2)} \left(\sum_{iji_2\cdots i_l} R_{ij}\omega^{ii_2\cdots i_l}\omega^{j}_{i_2\cdots i_l} \right. \\
&\left. -\frac{l-1}{2} \sum_{ijksi_3\cdots i_l} R_{ijks}\omega^{iji_3\cdots i_l}\omega^{ks}_{i_3\cdots i_l} \right)
\end{aligned}
$$

where we used ω is l-harmonic in the second equality. This can be read as

$$
\begin{aligned}
|\omega|^{p-1}\triangle|\omega|^{p-1} &= |\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle \\
&+ l|\omega|^{2(p-2)} \left(\sum_{iji_2\cdots i_l} R_{ij}\omega^{ii_2\cdots i_l}\omega^{j}_{i_2\cdots i_l} \right. \\
&\left. -\frac{l-1}{2} \sum_{ijksi_3\cdots i_l} R_{ijks}\omega^{iji_3\cdots i_l}\omega^{ks}_{i_3\cdots i_l} \right)
\end{aligned}
$$

By Kato type inequality $|\nabla\left(|\omega|^{p-2}\omega\right)|^2 \geq |\nabla|\omega|^{p-1}|^2$, we have

$$
\begin{aligned}
&|\omega|^{p-1}\triangle|\omega|^{p-1} \geq -\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle \\
&+ l|\omega|^{2(p-2)} \left(\sum_{iji_2\cdots i_l} R_{ij}\omega^{ii_2\cdots i_l}\omega^{j}_{i_2\cdots i_l} - \frac{l-1}{2} \sum_{ijksi_3\cdots i_l} R_{ijks}\omega^{iji_3\cdots i_l}\omega^{ks}_{i_3\cdots i_l} \right)
\end{aligned}
\tag{3}
$$

Here when $m \geq 4$, $2 \leq l \leq m-2$ and when $m = 3$, $l = 2$. By the Gauss equation, we have

$$
R_{ijks} = c(\delta_{ik}\delta_{js} - \delta_{is}\delta_{jk}) + \sum_{\alpha=m+1}^{m+t} [h^\alpha_{ik}h^\alpha_{js} - h^\alpha_{is}h^\alpha_{jk}],
$$

and

$$
R_{ij} = (m-1)c\delta_{ij} + \sum_{\alpha=m+1}^{m+t} [mH^\alpha h^\alpha_{ij} - h^\alpha_{ik}h^\alpha_{jk}].
$$

Thus,

$$
\sum_{iji_2\cdots i_l} R_{ij}\omega^{ii_2\cdots i_l}\omega^{j}_{i_2\cdots i_l} - \frac{l-1}{2} \sum_{ijksi_3\cdots i_l} R_{ijks}\omega^{iji_3\cdots i_l}\omega^{ks}_{i_3\cdots i_l} = F_1(\omega) + F_2(\omega)
$$

where

$$F_1(\omega) = c(m-1)\sum_{iji_2\cdots i_l}\delta_{ij}\omega^{ii_2\cdots i_l}\omega^{j}_{i_2\cdots i_l} - c\frac{l-1}{2}\sum_{ijkli_3\cdots i_l}(\delta_{ik}\delta_{jl} -\delta_{il}\delta_{jk})\omega^{iji_3\cdots i_l}\omega^{kl}_{i_3\cdots i_l} = (m-l)c|\omega|^2 \tag{4}$$

and

$$F_2(\omega) = \sum_{iji_2\cdots i_l}\left(\sum_{\alpha=m+1}^{m+t}\left[mH^\alpha h^\alpha_{ij} - \sum_{k=1}^{m} h^\alpha_{ik}h^\alpha_{jk}\right]\right)\omega^{ii_2\cdots i_l}\omega^{j}_{i_2\cdots i_l} - \frac{l-1}{2}\sum_{ijksi_3\cdots i_l}\left(\sum_{\alpha=m+1}^{m+t}\left[h^\alpha_{ik}h^\alpha_{js} - h^\alpha_{is}h^\alpha_{jk}\right]\right)\omega^{iji_3\cdots i_l}\omega^{ks}_{i_3\cdots i_l}$$

From the computation in Lin (2015a, b), it follows that

$$F_2(\omega) \geq \frac{1}{2l}(m^2H^2 - \max\{l, m-l\}|A|^2)|\omega|^2, \tag{5}$$

where the assumption of flat normal bundle is used. Substituting (4), (5) into (3), we have

$$\begin{aligned} |\omega|^{p-1}\triangle|\omega|^{p-1} &\geq -\langle\delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + l(m-l)c|\omega|^{2p-2} \\ &\quad + \frac{1}{2}(m^2H^2 - \max\{l, m-l\}|A|^2)|\omega|^{2p-2} \\ &\geq -\langle\delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + l(m-l)c|\omega|^{2p-2} \\ &\quad - \frac{m-1}{2}|\Phi|^2|\omega|^{2p-2} + \frac{m}{2}|H|^2|\omega|^{2p-2} \end{aligned}$$

that is,

$$\begin{aligned} |\omega|\triangle|\omega|^{p-1} &\geq -\langle\delta d(|\omega|^{p-2}\omega), \omega\rangle + l(m-l)c|\omega|^p \\ &\quad - \frac{m-1}{2}|\Phi|^2|\omega|^p + \frac{m}{2}|H|^2|\omega|^p \end{aligned} \tag{6}$$

In order to prove our main result, we need the following results:

Lemma 2.1 Li (1980) *Let E be a finite dimensional subspace of the space L^2 q-forms on a compact Riemannian manifold $\widetilde{M}^m$. Then there exists $\omega \in E$ such that*

$$\frac{dim E}{Vol(\widetilde{M})}\int_{\widetilde{M}}|\omega|^2 dv \leq \min\{\tbinom{m}{q}, dim E\}\sup_{\widetilde{M}}|\omega|^2.$$

From Lemma 2.1, the author and H. Pan in Han and Pan (2016) obtained the following result.

Lemma 2.2 Han and Pan (2016) *Let E be a finite dimensional subspace of the space L^p q-forms on a compact Riemannian manifold $\widetilde{M}^m$. Then there exists $\omega \in E$ such that*

$$\frac{dimE}{Vol(\widetilde{M})}\int_{\widetilde{M}}|\omega|^p dv \le \min\{C_p\tbinom{m}{q}, dimE\}\sup_{\widetilde{M}}|\omega|^p,$$

where C_p is a positive constant depending only p and $p \ge 2$.

Lemma 2.3 Hoffman and Spruck (1974); Michael and Simon (1973); Zhu and Fang (2014a) *Let M^m be a complete noncompact oriented manifold isometrically immersed in a sphere S^{m+t}. Then we have*

$$\left(\int_M |f|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \le C(m)\left(\int_M |\nabla f|^2 + m^2\int_M (1+|H|^2)f^2\right) \tag{7}$$

for all $f \in C_0^\infty(M)$, where $C(m)$ depends only on m and H is the mean curvature vector of M in S^{m+t}.

Lemma 2.4 *Let $f : M^m \to R$ be a smooth function on Riemannian manifold M, and ω be an l-form on M, $l \ge 2$. Then we have*

$$|df \wedge \omega| \le |df||\omega|.$$

Proof We can choose a local orthonormal basis $e_1, \ldots, e_m$ with the dual basis $\theta^1, \ldots, \theta^m$ on M. df and ω are denoted by the following: $df = \sum_{i=1}^m f_i\theta^i$ and

$$\omega = \sum_{i_1<\cdots<i_l} \omega_{i_1,\cdots,i_l}\theta^{i_1}\wedge\cdots\wedge\theta^{i_l}$$

so we have

$$df \wedge \omega = \sum_{i_1<\cdots<i_{l+1}}\left[\sum_{k=1}^{l+1}(-1)^{k-1}f_{i_k}\omega_{i_1\cdots\hat{i_k}\cdots i_{l+1}}\right]\theta^{i_1}\wedge\cdots\wedge\theta^{i_{l+1}}.$$

Now we compute

$$|df|^2|\omega|^2 - |df\wedge\omega|^2$$
$$= \left(\sum_{i=1}^m f_i^2\right)\left(\sum_{i_1<\cdots<i_l}\omega^2_{i_1\cdots i_l}\right) - \sum_{i_1<\cdots<i_{l+1}}\left(\sum_{k=1}^{l+1}(-1)^{k-1}f_{i_k}\omega_{i_1\cdots\hat{i_k}\cdots i_{l+1}}\right)^2$$

$$= \sum_{i_1<\cdots<i_{l+1}} \left[\sum_{k\neq t} f_{i_k}^2 \omega^2_{i_1\cdots i_k\cdots \hat{i}_t \cdots i_{l+1}} \right.$$

$$\left. + \sum_{k\neq t} (-1)^{k+t} (f_{i_k}\omega_{i_1\cdots i_k\cdots \hat{i}_t\cdots i_{l+1}})(f_{i_t}\omega_{i_1\cdots \hat{i}_k\cdots i_t\cdots i_{l+1}}) \right] \geq 0.$$

Here $\hat{i_k}$ means that i_k does not appear. This proves the lemma. □

3 Proof of the main results

In this section, we obtain the following results.

Theorem 3.1 *Let M^m, $m \geq 3$ be an m-dimensional complete noncompact oriented manifold isometrically immersed in an $(m+t)$-dimensional sphere S^{m+t} with flat normal bundle. If the total curvature is finite, then we have $dim H^{l,p}(M) < \infty$ for $p \geq 2$ and when $m \geq 4$, $2 \leq l \leq m-2$, when $m = 3$, $l = 2$.*

Proof Assume that ω is a p-harmonic l-form on M^m, i.e. $\omega \in H^{l,p}(M)$. From (6), we have

$$|\omega|\triangle|\omega|^{p-1} \geq -\langle \delta d(|\omega|^{p-2}\omega), \omega\rangle + l(m-l)|\omega|^p \\ -\frac{m-1}{2}|\Phi|^2|\omega|^p + \frac{m}{2}|H|^2|\omega|^p \tag{8}$$

Fixed a point $x_0 \in M$ and denote by $\rho(x)$ the geodesic distance on M from x_0 to x. Let us choose $\eta \in C_0^\infty(M)$ satisfying

$$\eta = \begin{cases} 0 & \text{on } B_{x_0}(r_0) \cup \left(M \backslash B_{x_0}(2r)\right), \\ \rho(x_0, x) - r_0 & \text{on } B_{x_0}(r_0+1) \backslash B_{x_0}(r_0), \\ 1 & \text{on } B_{x_0}(r) \backslash B_{x_0}(r_0+1), \\ \frac{2r-\rho(x_0,x)}{r} & \text{on } B_{x_0}(2r) \backslash B_{x_0}(r), \end{cases}$$

where $r > r_0 + 1$ and r_0 will be determined later. Multiplying (8) by η^2 and integrating over M, we have

$$-\int_M \eta^2 \langle \nabla|\omega|, \nabla|\omega|^{p-1}\rangle - 2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|^{p-1}\rangle + \int_M \langle \delta d(|\omega|^{p-2}\omega), \eta^2\omega\rangle \\ \geq l(m-l)\int_M \eta^2|\omega|^p - \frac{m-1}{2}\int_M |\Phi|^2|\omega|^p\eta^2 + \frac{m}{2}\int_M |H|^2|\omega|^p\eta^2 \tag{9}$$

Now we first estimate the third term of the left side of (9)

$$\left|\int_M \langle \delta d(|\omega|^{p-2}\omega), \eta^2\omega\rangle\right| = \left|\int_M \langle d(|\omega|^{p-2}\omega), d(\eta^2\omega)\rangle\right|$$

$$\le \int_M |d(|\omega|^{p-2}\omega)||d(\eta^2\omega)| \le 2\int_M \eta|d\eta||\omega|^2|d|\omega|^{p-2}| \tag{10}$$
$$= \frac{4(p-2)}{p}\int_M \eta|\nabla\eta||\omega|^{\frac{p}{2}}|\nabla|\omega|^{\frac{p}{2}}|.$$

Here we use the inequality $|df \wedge \omega| \le |df||\omega|$, for any $f \in C^\infty(M)$. By direct computation, we get

$$\begin{aligned} &-\int_M \eta^2\langle\nabla|\omega|, \nabla|\omega|^{p-1}\rangle - 2\int_M \eta|\omega|\langle\nabla\eta, \nabla|\omega|^{p-1}\rangle \\ &= -\frac{4(p-1)}{p^2}\int_M \eta^2|\nabla|\omega|^{\frac{p}{2}}|^2 - \frac{4(p-1)}{p}\int_M \eta\langle\nabla\eta, \nabla|\omega|^{\frac{p}{2}}\rangle|\omega|^{\frac{p}{2}} \\ &\le -\frac{4(p-1)}{p^2}\int_M \eta^2|\nabla|\omega|^{\frac{p}{2}}|^2 + \frac{4(p-1)}{p}\int_M \eta|\nabla\eta||\omega|^{\frac{p}{2}}|\nabla|\omega|^{\frac{p}{2}}|. \end{aligned} \tag{11}$$

From (9), (10) and (11), we have

$$\begin{aligned} 0 \le &-\frac{4(p-1)}{p^2}\int_M \eta^2|\nabla|\omega|^{\frac{p}{2}}|^2 + \frac{4(2p-3)}{p}\int_M \eta|\nabla\eta||\omega|^{\frac{p}{2}}|\nabla|\omega|^{\frac{p}{2}}| \\ &-l(m-l)\int_M \eta^2|\omega|^p + \frac{m-1}{2}\int_M |\Phi|^2|\omega|^p\eta^2 - \frac{m}{2}\int_M |H|^2|\omega|^p\eta^2 \end{aligned}$$

For $\varepsilon_1 > 0$, we apply the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left[\frac{4(p-1)}{p^2} - \frac{4(2p-3)}{p}\varepsilon_1\right]\int_M \eta^2|\nabla|\omega|^{\frac{p}{2}}|^2 \le \frac{(2p-3)}{p}\frac{1}{\varepsilon_1}\int_M |\omega|^p|\nabla\eta|^2 \\ &-l(m-l)\int_M \eta^2|\omega|^p + \frac{m-1}{2}\int_M |\Phi|^2|\omega|^p\eta^2 - \frac{m}{2}\int_M |H|^2|\omega|^p\eta^2 \end{aligned} \tag{12}$$

On the other hand, since $m \ge 3$, we use Hölder, Sobolev inequality (7), and Cauchy-Schwartz inequalities to obtain

$$\begin{aligned} &\int_M |\Phi|^2|\omega|^p\eta^2 \le \left(\int_{supp(\eta)} |\Phi|^m\right)^{\frac{2}{m}} \left(\int_M (\eta|\omega|^{\frac{p}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \\ &\le C(m)\left(\int_{supp(\eta)} |\Phi|^m\right)^{\frac{2}{m}} \int_M (|\nabla(\eta|\omega|^{\frac{p}{2}})|^2 + m^2(1+|H|^2)\eta^2|\omega|^p) \\ &\le C(m)\phi^2(\eta)[(1+\varepsilon_2)\int_M \eta^2|\nabla|\omega|^{\frac{p}{2}}| + \left(1+\frac{1}{\varepsilon_2}\right)\int_M |\omega|^p|\nabla\eta|^2] \\ &\quad +C(m)\phi^2(\eta)m^2\int_M \eta^2|\omega|^p(1+|H|^2) \end{aligned} \tag{13}$$

for $\varepsilon_2 > 0$, where $\phi(\eta) = (\int_{supp\eta} |\Phi|^m)^{\frac{1}{m}}$. From (12) and (13), we have

$$A \int_M \eta^2 |\nabla|\omega|^{\frac{p}{2}}|^2 + B \int_M |H|^2 |\omega|^p \eta^2 + C \int_M |\omega|^p \eta^2 \leq D \int_M |\omega|^p |\nabla\eta|^2 \quad (14)$$

where

$$A = \frac{4(p-1)}{p^2} - \frac{4(2p-3)}{p}\varepsilon_1 - \frac{m-1}{2} C(m)\phi^2(\eta)(1+\varepsilon_2)$$
$$B = \frac{m}{2} - \frac{m-1}{2} C(m)\phi^2(\eta) m^2$$
$$C = l(m-l) - \frac{m-1}{2} C(m)\phi^2(\eta) m^2$$
$$D = \frac{(2p-3)}{p}\frac{1}{\varepsilon_1} + \frac{m-1}{2} C(m)\phi^2(\eta)(1+\frac{1}{\varepsilon_2})$$

We choose $0 < \varepsilon < \min\{\frac{-(mp+7p-12)+\sqrt{(mp+7p-12)^2+16(m-1)(p-1)}}{2(m-1)p}, \frac{1}{2(m-1)m}, \frac{l(m-l)}{(m-1)m^2}\}$ and a positive constant $\Lambda(\varepsilon) > 0$ satisfying:

$$\frac{4(2p-3)}{p}\varepsilon + (m-1)\varepsilon(1+\varepsilon) \leq \frac{4(p-1)}{p^2}$$
$$\frac{m-1}{2} C(m)\Lambda^2(\varepsilon) < (m-1)\varepsilon$$

Since M has finite total curvature, we can fix r_1 large enough such that

$$\left(\int_{M\setminus B_{x_0}(r_1)} |\Phi|^m\right)^{\frac{1}{m}} \leq \Lambda \quad (15)$$

Take $r_0 > r_1$, thus $supp(\eta) \subseteq M\setminus B_{x_0}(r_1)$ and $\phi(\eta) \leq \Lambda$. Choose $0 < \varepsilon_i < \varepsilon$, for $i = 1, 2$, we have

$$A \geq \widetilde{A} = \frac{4(p-1)}{p^2} - \frac{4(2p-3)}{p}\varepsilon - (m-1)\varepsilon(1+\varepsilon) > 0,$$
$$B \geq \widetilde{(B)} = \frac{m}{2} - m^2(m-1)\varepsilon > 0$$
$$C \geq \widetilde{C} = l(m-l) - m^2(m-1)\varepsilon > 0$$
$$0 < D \leq \widetilde{D} = \frac{(2p-3)}{p}\frac{1}{\varepsilon_1} + \frac{m-1}{2} C(m)\Lambda^2 \left(1 + \frac{1}{\varepsilon_2}\right)$$

From (14), we have

$$\widetilde{A} \int_M \eta^2 |\nabla|\omega|^{\frac{p}{2}}|^2 + \widetilde{B} \int_M |H|^2 |\omega|^p \eta^2 + \widetilde{C} \int_M |\omega|^p \eta^2 \leq \widetilde{D} \int_M |\omega|^p |\nabla\eta|^2 \quad (16)$$

From (7) and the Cauchy-Schwarz inequality, we have

$$C^{-1}(m)\left(\int_M (\eta|\omega|^{\frac{p}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \int_M |\nabla(\eta|\omega|^{\frac{p}{2}})|^2 + m^2\int_M (1+|H|^2)\eta^2|\omega|^2$$
$$\leq (1+s)\int_M \eta^2|\nabla|\omega|^{\frac{p}{2}}|^2 + \left(1+\frac{1}{s}\right)\int_M |\omega|^p|\nabla\eta|^2 + m^2\int_M (1+|H|^2)\eta^2|\omega|^2 \tag{17}$$

From (16) and (17), we have

$$C^{-1}(m)\left(\int_M (\eta|\omega|^{\frac{p}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \left(m^2-(1+s)\frac{\widetilde{B}}{\widetilde{A}}\right)\int_M |H|^2|\omega|^p\eta^2$$
$$+\left(m^2-(1+s)\frac{\widetilde{C}}{\widetilde{A}}\right)\int_M |\omega|^p\eta^2 + (1+\frac{1}{s}+(1+s)\frac{\widetilde{D}}{\widetilde{A}})\int_M |\omega|^p|\nabla\eta|^2$$

Choose a sufficiently large s such that $m^2-(1+s)\frac{\widetilde{B}}{\widetilde{A}}<0$ and $m^2-(1+s)\frac{\widetilde{C}}{\widetilde{A}}<0$. Then we have

$$C^{-1}(m)\left(\int_M (\eta|\omega|^{\frac{p}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \left(1+\frac{1}{s}+(1+s)\frac{\widetilde{D}}{\widetilde{A}}\right)\int_M |\omega|^p|\nabla\eta|^2$$

That is,

$$\left(\int_M (\eta|\omega|^{\frac{p}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq E\int_M |\omega|^p|\nabla\eta|^2 \tag{18}$$

where E is a positive constant depending only on m, p. It follows from the definition of η and (18), we have

$$\left(\int_{B_{x_0}(r)\setminus B_{x_0}(r_0+1)} (|\omega|^{\frac{p}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq E\int_{B_{x_0}(r_0+1)\setminus B_{x_0}(r_0)} |\omega|^p + \frac{E}{r^2}\int_{B_{x_0}(2r)\setminus B_{x_0}(r)} |\omega|^p$$

Since $|\omega|\in L^p(M)$, taking $r\to\infty$, we have

$$\left(\int_{M\setminus B_{x_0}(r_0+1)} (|\omega|^{\frac{p}{2}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq E\int_{B_{x_0}(r_0+1)\setminus B_{x_0}(r_0)} |\omega|^p \tag{19}$$

It follows from the Hölder inequality that

$$\int_{B_{x_0}(r_0+2)\setminus B_{x_0}(r_0+1)} |\omega|^p \leq [\mathrm{Vol}(B_{x_0}(r_0+2))]^{\frac{2}{m}}(\int_{B_{x_0}(r_0+2)\setminus B_{x_0}(r_0+1)} |\omega|^{\frac{pm}{m-2}})^{\frac{m-2}{m}}. \tag{20}$$

From (19) and (20), we have

$$\int_{B_{x_0}(r_0+2)} |\omega|^p \leq C_1 \int_{B_{x_0}(r_0+1)} |\omega|^p, \tag{21}$$

where C_1 depends on $\mathrm{Vol}(B_{x_0}(r_0+2))$, m and p. From (8), we have

$$|\omega|\triangle|\omega|^{p-1} \geq -\langle \delta d(|\omega|^{p-2}\omega), \omega\rangle - F|\omega|^p, \tag{22}$$

where $F : M \to [0, \infty)$ is a function given by

$$F = |l(m-l) - \frac{m-1}{2}|\Phi|^2 + \frac{m}{2}|H|^2|.$$

Fix $x \in M$ and take $\eta \in C_0^\infty(B_x(1))$. Multiply both sides of (22) by $\eta^2|\omega|^{\frac{pq}{2}-p}$, with $q \geq 2$, and integrating by parts we obtain

$$\begin{aligned} &-\frac{4(p-1)}{p}\int_{B_x(1)} \eta|\omega|^{\frac{pq}{2}-\frac{p}{2}}\langle \nabla\eta, \nabla|\omega|^{\frac{p}{2}}\rangle \\ &\quad\geq \frac{2(p-1)(pq-2p+2)}{p^2}\int_{B_x(1)} |\omega|^{\frac{pq}{2}-p}|\nabla|\omega|^{\frac{p}{2}}|^2\eta^2 \\ &\quad\quad -F\int_{B_x(1)} \eta^2|\omega|^{\frac{pq}{2}} - \int_{B_x(1)} \langle d(|\omega|^{p-2}\omega), d(\eta^2|\omega|^{\frac{pq}{2}-p}\omega)\rangle. \end{aligned} \tag{23}$$

From the inequality $|df \wedge \omega| \leq |df||\omega|$ and Cauchy-Schwatz inequality, we have

$$\begin{aligned} &\int_{B_x(1)} |\langle d(|\omega|^{p-2}\omega), d(\eta^2|\omega|^{\frac{pq}{2}-p}\omega)\rangle| \leq \int_{B_x(1)} |d(|\omega|^{p-2}\omega)||d(\eta^2|\omega|^{\frac{pq}{2}-p}\omega)| \\ &\quad\leq \int_{B_x(1)} |\nabla|\omega|^{p-2}||\omega|^2|[d(\eta^2)|\omega|^{\frac{pq}{2}-p} + \eta^2 d|\omega|^{\frac{pq}{2}-p}]| \\ &\qquad\qquad \leq \frac{4(p-2)}{p}\int_{B_x(1)} \eta|\omega|^{\frac{pq}{2}-\frac{p}{2}}|\nabla\eta||\nabla|\omega|^{\frac{p}{2}}| \\ &\qquad\qquad\quad +\frac{2(p-2)(q-2)}{p}\int_{B_x(1)} \eta^2|\omega|^{\frac{pq}{2}-p}|\nabla|\omega|^{\frac{p}{2}}|^2 \\ &\qquad\qquad \leq \frac{2\varepsilon_3}{p^2(m-1)}\int_{B_x(1)} |\omega|^{\frac{pq}{2}-p}|\nabla|\omega|^{\frac{p}{2}}|^2\eta^2 \\ &\qquad\qquad\quad +2(p-2)^2(m-1)\frac{1}{\varepsilon_3}\int_{B_x(1)} |\nabla\eta|^2|\omega|^{\frac{pq}{2}} \\ &\qquad\qquad\quad +\frac{2(p-2)(q-2)}{p}\int_{B_x(1)} \eta^2|\omega|^{\frac{pq}{2}-p}|\nabla|\omega|^{\frac{p}{2}}|^2 \end{aligned} \tag{24}$$

where $\varepsilon_3 > 0$ is a positive constant. And

$$\begin{aligned}
&-\frac{4(p-1)}{p}\int_{B_x(1)}\eta|\omega|^{\frac{pq}{2}-\frac{p}{2}}\langle\nabla\eta,\nabla|\omega|^{\frac{p}{2}}\rangle \\
&\le \frac{2\varepsilon_3}{p^2(m-1)}\int_{B_x(1)}|\omega|^{\frac{pq}{2}-p}|\nabla|\omega|^{\frac{p}{2}}|^2\eta^2 \\
&\quad +2(p-1)^2(m-1)\frac{1}{\varepsilon_3}\int_{B_x(1)}|\nabla\eta|^2|\omega|^{\frac{pq}{2}},
\end{aligned} \tag{25}$$

From (23), (24) and (25), we have

$$\begin{aligned}
&\Big[\frac{2(p-1)(pq-2p+2)}{p^2}-\frac{2(p-2)(q-2)}{p} \\
&\quad -\frac{4\varepsilon_3}{(m-1)p^2}\Big]\int_{B_x(1)}|\omega|^{\frac{pq}{2}-p}|\nabla|\omega|^{\frac{p}{2}}|^2\eta^2 \\
&\quad \le F\int_{B_x(1)}\eta^2|\omega|^{\frac{pq}{2}}+[2(p-1)^2 \\
&\quad +2(p-2)^2](m-1)\frac{1}{\varepsilon_3}\int_{B_x(1)}|\nabla\eta|^2|\omega|^{\frac{pq}{2}}.
\end{aligned} \tag{26}$$

We can choose ε_3 small enough such that $[\frac{2(p-1)(pq-2p+2)}{p^2}-\frac{2(p-2)(q-2)}{p}-\frac{4\varepsilon_3}{(m-1)p^2}] > 0$. By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\int_{B_x(1)}|\nabla(\eta|\omega|^{\frac{pq}{4}})|^2 &= \int_{B_x(1)}|\omega|^{\frac{pq}{2}}|\nabla\eta|^2+\frac{q^2}{4}\int_{B_x(1)}\eta^2|\omega|^{\frac{pq}{2}-p}|\nabla|\omega|^{\frac{p}{2}}|^2 \\
&\quad +q\int_{B_x(1)}\eta|\omega|^{\frac{pq}{2}-\frac{p}{2}}\langle\nabla\eta,\nabla|\omega|^{\frac{p}{2}}\rangle \\
&\le (1+q)\int_{B_x(1)}|\omega|^{\frac{pq}{2}}|\nabla\eta|^2+\frac{q}{4}(q+1) \\
&\quad \times\int_{B_x(1)}\eta^2|\omega|^{\frac{pq}{2}-p}|\nabla|\omega|^{\frac{p}{2}}|^2.
\end{aligned} \tag{27}$$

From (26) and (27), we have

$$\int_{B_x(1)}|\nabla(\eta|\omega|^{\frac{pq}{4}})|^2 \le C_1\int_{B_x(1)}|\omega|^{\frac{pq}{2}}|\nabla\eta|^2+C_2\int_{B_x(1)}F\eta^2|\omega|^{\frac{pq}{2}}, \tag{28}$$

where

$$\begin{aligned}
&C_1 = 1+q+\frac{q}{4}(q+1)[2(p-1)^2+2(p-2)^2](m-1)\frac{1}{\varepsilon_3} \\
&\Big[\frac{2(p-1)(pq-2p+2)}{p^2}-\frac{2(p-2)(q-2)}{p}-\frac{4\varepsilon_3}{(m-1)p^2}\Big]^{-1} \le C(p,\varepsilon_3)mq,
\end{aligned}$$

$$C_2=\frac{q}{4}(q+1)\left[\frac{2(p-1)(pq-2p+2)}{p}-\frac{2(p-2)(q-2)}{p}\right.$$
$$\left.-\frac{4\varepsilon_3}{(m-1)p^2}\right]^{-1}\le C(p,\varepsilon_3)q,$$

where $C(p,\varepsilon_3)$ is a positive constant depending only on p,ε_3. Applying (7) to $\eta|\omega|^{\frac{pq}{4}}$ and using (28), we have

$$\begin{aligned}\left(\int_{B_x(1)}(\eta|\omega|^{\frac{pq}{4}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}&\le C(m)\left(\int_{B_x(1)}|\nabla(\eta|\omega|^{\frac{pq}{4}})|^2\right.\\&\quad\left.+m^2\int_{B_x(1)}(1+|H|^2)\eta^2|\omega|^{\frac{pq}{2}}\right)\\&\le\int_{B_x(1)}[C_2F+m^2(1+|H|^2)]\eta^2|\omega|^{\frac{pq}{2}}\\&\quad+C_1\int_{B_x(1)}|\omega|^{\frac{pq}{2}}|\nabla\eta|^2.\end{aligned}$$

so we have

$$\left(\int_{B_x(1)}(\eta|\omega|^{\frac{pq}{4}})^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}\le qC_3\int_{B_x(1)}[\eta^2+|\nabla\eta|^2]|\omega|^{\frac{pq}{2}},\tag{29}$$

for a constant $C_3>0$ depending m,p,ε_3, $\mathrm{Vol}(B_x(1))$, $\sup_{B_x(1)}F$ and $\sup_{B_x(1)}|H|$.

Given an integer $k\ge 0$, we set $q_k=\frac{2m^k}{(m-2)^k}$ and $\rho_k=\frac{1}{2}+\frac{1}{2^{k+1}}$. Take a function $\xi_k\in C_0^\infty(B_x(\rho_k))$ satisfying $\eta_k\ge 0$, $\eta_k=1$ on $B_x(\rho_{k+1})$ and $|\nabla\eta_k|\le 2^{k+3}$. Replacing q and η in (29) by q_k and η_k respectively, we have

$$\left(\int_{B_x(\rho_{k+1})}|\omega|^{\frac{pq_{k+1}}{2}}\right)^{\frac{1}{q_{k+1}}}\le(q_kC_34^{k+4})^{\frac{1}{q_k}}\left(\int_{B_x(\rho_k)}|\omega|^{\frac{pq_k}{2}}\right)^{\frac{1}{q_k}}.\tag{30}$$

Applying the Moser iteration to (30), we conclude that

$$|\omega|^p(x)\le||\omega||^p_{L^\infty(B_x(\frac{1}{2}))}\le C_4\int_{B_x(1)}|\omega|^p\tag{31}$$

for a constant $C_4>0$ depending only on m,p,ε_3, $\mathrm{Vol}(B_x(1))$, $\sup_{B_x(1)}F$ and $\sup_{B_x(1)}|H|$. Take $x\in B_{x_0}(r_0+1)$ such that

$$|\omega|^p(x)=\sup_{B_{x_0}(r_0+1)}|\omega|^p.\tag{32}$$

From (31) and (32), we have

$$\sup_{B_{x_0}(r_0+1)} |\omega|^p \leq C_4 \int_{B_{x_0}(r_0+2)} |\omega|^p. \tag{33}$$

From (21) and (33), we have

$$\sup_{B_{x_0}(r_0+1)} |\omega|^p \leq C_5 \int_{B_{x_0}(r_0+1)} |\omega|^p, \tag{34}$$

where $C_7 > 0$ is a constant depending on m, p, ε_3, $\mathrm{Vol}(B_x(r_0+2))$, $\sup_{B_x(r_0+2)} F$ and $\sup_{B_x(r_0+2)} |H|$.

Finally, let V be any finite-dimensional subspace of $H^{l,p}(M)$. From Lemma 2.2, there exists $\omega \in V$ such that

$$\frac{dimV}{Vol(B_{x_0}(r_0+1))} \int_{B_{x_0}(r_0+1)} |\omega|^p \leq \min\{C_p\tbinom{m}{l}, dimV\} \sup_{B_{x_0}(r_0+1)} |\omega|^p. \tag{35}$$

From (34) and (35), we have $dimV \leq C_6$, where $C_6 > 0$ depends only on on m, p, ε_3, $\mathrm{Vol}(B_x(r_0+2))$, $\sup_{B_x(r_0+2)} F$ and $\sup_{B_x(r_0+2)} |H|$. This implies that $H^{l,p}(M)$ has finite dimension.

Theorem 3.2 *Let M^m, $m \geq 3$ be an m-dimensional complete noncompact oriented manifold isometrically immersed in an $(m+l)$-dimensional sphere S^{m+l}. There exists a positive constant Λ depending only on m, p, l, such that if $||\Phi||_{L^m(M)} < \Lambda$, then there admit no nontrivial L^p p-harmonic l-forms on M, i.e. $H^{l,p}(M) = \{0\}$, for $p \geq 2$ and when $m \geq 4$, $2 \leq l \leq m-2$, when $m = 3$, $l = 2$. More precisely, Λ can be given explicitly by a constant $C(M)$ in (7) as follows:*

$$\Lambda < \min\left\{\sqrt{\frac{8(p-1)}{p^2mC(M)}}, \sqrt{\frac{1}{m(m-1)C(M)}}, \sqrt{\frac{2l(m-l)}{m^2(m-1)C(M)}}\right\}. \tag{36}$$

Proof From (36), we know that $A > 0$, $B > 0$ and $C > 0$ in (14). For a fixed point x_0 and take a cut-off function η such that

$$\begin{cases} 0 \leq \eta \leq 1, \\ \eta = 1 \quad on \quad B_{x_0}(r) \\ \eta = 0 \quad on \quad M \backslash B_{x_0}(2r) \\ |\nabla\eta| \leq \frac{c}{r}, \end{cases}$$

where c is a positive real number. From (14), we have

$$A \int_{B_{x_0}(r)} |\nabla|\omega|^{\frac{p}{2}}|^2 + B \int_{B_{x_0}(r)} |H|^2|\omega|^p + C \int_{B_{x_0}(r)} |\omega|^p \leq \frac{Dc^2}{r^2} \int_M |\omega|^p,$$

Let $r \to \infty$. We obtain that $|\omega| = 0$, that is, $\omega = 0$. □

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