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Properties of Shadowable Points: Chaos and Equicontinuity

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Abstract We extend the study on shadowable points recently introduced by Morales in relation to chaotic or non-chaotic properties. Firstly, some sufficient conditions for a quantitative shadowable point to be approximated by an entropy point are given. As a corollary, we get different three chaotic conditions from which a shadowable point becomes an entropy point. Secondly, we provide a dichotomy on the interior of the set of shadowable chain recurrent points by two canonical chaotic and non-chaotic dynamics, the full shift and odometers.

Keywords Shadowable points · Entropy points · Li–Yorke chaos · Odometer · Chain decomposition

Mathematics Subject Classification 37C50 · 37B40 · 74H65 · 37B20

1 Introduction

Shadowing property has been the subject of numerous studies in the qualitative theory of dynamical systems (Aoki and Hiraide 1994; Pilyugin 1999). Recently, Morales (2016) introduced the notion of shadowable points by individualizing the shadowing property into pointwise shadowings. A shadowable point of a continuous map is defined to be a point such that the shadowing lemma holds for pseudo orbits beginning at the point. It prompts us to reconsider the theory of shadowing from a local point of view. On the other hand, the notion of quantitative shadowable points was defined in

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Kawaguchi (2017), which is a quantitative version of shadowable points. Some basic properties and several results of (quantitative) shadowable points have been obtained in Kawaguchi (2017), Morales (2016). In this paper, we extend the study on (quantitative) shadowable points by the idea of localizing and quantifying the arguments on the shadowing property in connection with chaos and equicontinuity. The chaos includes the positive entropy, sensitivity, and Li–Yorke chaos, and corresponds to the full-shift, while the equicontinuity corresponds to the odometers (or adding machines).

Our first main result gives sufficient conditions for a quantitative shadowable point to be approximated by an entropy point, which concern the notions of sensitivity and Li–Yorke pairs (Theorem 1.1). As a corollary, we obtain relatively simple sufficient conditions for a shadowable point to be an entropy point (Corollary 1.1). By this corollary, owing to the notion of shadowable points, we can concisely specify entropy points in connection with other chaotic properties of dynamical systems. As a consequence, we establish the equivalence of two definitions of chaos, Li–Yorke chaos and the positive topological entropy, under the shadowing property (Corollary 1.2). Moreover, we give a lower estimate of the positive topological entropy under the presence of a Li–Yorke pair and a quantitative shadowing property (Theorem 1.2).

Our second main result provides a dichotomy on interior points in the set of shadowable points under the assumption of chain recurrence (Theorem 1.3). It tells us that being an interior point in the set of shadowable points (with chain recurrence) enables us to characterize the point as a chaotic point or a non-chaotic point in comparison with two canonical dynamics, i.e., the full shift and odometers (see properties (S2) and (E2) in Theorem 1.3). It also depicts how chaotic points and non-chaotic points, or full shift extensions and odometers are mixed in the chain recurrent set. According to Akin et al. (2003), the mixture of full-shift extensions and odometers is a C^0-generic property of homeomorphisms on a smooth closed manifold, so our results complement such a picture. In the classical topological theory of hyperbolic dynamics, some kind of expansiveness, which is also a topological expression of hyperbolicity, is often assumed with the shadowing property, where the possibility of the presence of non-trivial equicontinuous subsystems is excluded. Therefore, our results seem to give an insight into a certain non-hyperbolic behavior.

We begin by defining (quantitative) shadowable points. Throughout this paper, we deal with a continuous map $f : X \to X$ on a compact metric space (X, d). An infinite sequence of points $(x_i)_{i=0}^{\infty}$ in X is called a δ*-pseudo orbit* of f if $d(f(x_i), x_{i+1}) \le \delta$ for all $i \ge 0$. For $\epsilon > 0$, a δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f is said to be ϵ*-shadowed* by $x \in X$ if $d(x_i, f^i(x)) \le \epsilon$ for all $i \ge 0$. For $b > 0$, we say that f has a b*-shadowing property* if there exists $\delta > 0$ such that every δ-pseudo orbit of f is b-shadowed by some $x \in X$. Then, for $c \ge 0$, we say that f has a $c+$*-shadowing property* if f has the b-shadowing property for every $b > c$. Note that the $0+$-shadowing property corresponds with the usual shadowing property. Now let us define quantitative shadowable points. For $b > 0$, a b*-shadowable point* of f is a point $x \in X$ such that there exists $\delta > 0$ for which every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is b-shadowed by some point of X. We denote by $Sh_b^+(f)$ the set of b-shadowable points of f. Then, for $c \ge 0$, we define

$$Sh^+_{c+}(f) = \bigcap_{b>c} Sh^+_b(f),$$

and a point of $Sh^+_{c+}(f)$ is called a *c+-shadowable point* of f. A 0+-shadowable point coincides with a *shadowable point* introduced by Morales (2016), and $Sh^+_{0+}(f)$ is also denoted by $Sh^+(f)$. It was proved in Kawaguchi (2017) that for any $c \geq 0$, $Sh^+_{c+}(f)$ is an f-invariant (i.e. $f(Sh^+_{c+}(f)) \subset Sh^+_{c+}(f)$) Borel set in X, and f has the $c+$-shadowing property iff $Sh^+_{c+}(f) = X$.

Here, we give some basic definitions and notations. A *dynamical system* is a pair (X, f) of a compact metric space X with a metric d and a continuous map f from X to itself.

A subset $S \subset X$ is *f-invariant* if $f(S) \subset S$. A *subsystem* of (X, f) is a pair of a closed f-invariant subset $S \subset X$ and $f|_S$. We say that (X, f) (or f) is *minimal* if X does not contain any non-empty, proper, and closed f-invariant subset. For dynamical systems (X, f) and (Y, g), a *factor map* is a continuous surjection $\pi : X \to Y$ with $\pi \circ f = g \circ \pi$. When there is a factor map $\pi : X \to Y$, then we say that (Y, g) is a *factor* of (X, f), and (X, f) is an *extension* of (Y, g). When a factor map $\pi : X \to Y$ is 1-1, π is said to be a *conjugacy*, and we say that (X, f) is *conjugate* to (Y, g). For $e > 0$, a point $x \in X$ is said to be an *e-sensitive point* of f if for any neighborhood U of x, there exist $y, z \in U$ and $n \in \mathbb{N}$ such that $d(f^n(y), f^n(z)) > e$. We denote by $Sen_e(f)$ the set of e-sensitive points of f and define $Sen(f) = \bigcup_{e>0} Sen_e(f)$. Define also $EC(f) = X \setminus Sen(f)$. Then, a point of $Sen(f)$ (resp. $EC(f)$) is called a *sensitive* (resp. *equicontinuous*) point of f. If $X = Sen_e(f)$ for some $e > 0$, then f is said to be *sensitive*. We say that f is *equicontinuous* if for every $\epsilon > 0$, there is $\delta > 0$ such that $d(x, y) \leq \delta$ implies $d(f^n(x), f^n(y)) \leq \epsilon$ for all $x, y \in X$ and $n \geq 0$. It is easy to see that f is equicontinuous iff $X = EC(f)$. A minimal system is either sensitive or equicontinuous. A finite sequence of points $(x_i)_{i=0}^k$ in X (where k is a positive integer) is called a *δ-chain of f* if $d(f(x_i), x_{i+1}) \leq \delta$ for every $0 \leq i \leq k-1$. A δ-chain $(x_i)_{i=0}^k$ of f is a *δ-cycle of f* if $x_0 = x_k$. A point $x \in X$ is said to be *chain recurrent* if for every $\delta > 0$, there is a δ-cycle $(x_i)_{i=0}^k$ of f with $x_0 = x_k = x$. The set of chain recurrent points of f is denoted by $CR(f)$. Note that $\Omega(f) \subset CR(f)$, where $\Omega(f)$ denotes the non-wandering set of f. We say that f is *chain transitive* if for every $x, y \in X$ and every $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ of f such that $x_0 = x$ and $x_k = y$.

Before stating our first result, we need to define *entropy points*, *Li–Yorke pairs*, and *Li–Yorke chaos*. Given a continuous map $f : X \to X$ and $n \geq 1$, define a metric d_n on X by $d_n(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y))$. For $n \geq 1$ and $\epsilon > 0$, a subset $E \subset X$ is called *(n, ϵ)-separated* if $x \neq y$ $(x, y \in E)$ implies $d_n(x, y) > \epsilon$. For $A \subset X$, let $S(A, n, \epsilon)$ denote the maximal cardinality of an (n, ϵ)-separated set contained in A and consider

$$h(f, A, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log S(A, n, \epsilon).$$

Note that $\epsilon_2 < \epsilon_1$ implies $h(f, A, \epsilon_1) \leq h(f, A, \epsilon_2)$, which guarantees the existence of $\lim_{\epsilon \to 0} h(f, A, \epsilon) \in [0, \infty]$. The *topological entropy of f on A*, denoted by $h(f, A)$, is $h(f, A) = \lim_{\epsilon \to 0} h(f, A, \epsilon)$. Then, the *topological entropy* of f, denoted by $h_{top}(f)$, is defined by $h_{top}(f) = h(f, X)$. Ye and Zhang (2007) introduced the notion of entropy points. A point $x \in X$ is said to be an *entropy point* of f if $h(f, \overline{U}) > 0$ for any neighborhood U of x. Let $Ent(f)$ denote the set of entropy points of f. It is known that $Ent(f)$ is a closed f-invariant subset of X, and $h_{top}(f) > 0$ iff $Ent(f) \neq \emptyset$ (see Walters 1982; Ye and Zhang 2007).

For a dynamical system (X, f), a pair of points $\{x, y\} \subset X$ is said to be a *Li–Yorke pair* (with modulus $e > 0$) if one has simultaneously,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > e > 0.$$

A subset $S \subset X$ is called *scrambled* if any pair of distinct points $x, y \in S$ is a Li–Yorke pair. Then, a system (X, f) is called *Li–Yorke chaotic* if X contains an uncountable scrambled set.

Now let us state our first result.

Theorem 1.1 *Let $f : X \to X$ be a continuous map. Given $x \in Sh^+_{c+}(f)$ with $c \geq 0$, if $e > 2c$ and one of the following conditions is satisfied, then there exists $w \in Ent(f)$ such that $d(x, w) \leq c$.*

(1) *There is a closed f-invariant subset $S \subset X$ such that $CR(f|_S) = S$ and $\omega(x, f) \cap Sen_e(f|_S) \neq \emptyset$.*
(2) *There is $y \in X$ such that $\{x, y\} \subset X$ is a Li–Yorke pair with modulus e.*
(3) *There is a closed f-invariant subset $S \subset \omega(x, f)$ such that $\omega(x, f)\backslash B_e(S) \neq \emptyset$, where $B_e(S) = \{y \in X : d(y, S) \leq e\}$.*

Theorem 1.1 gives three sufficient conditions for a quantitative shadowable point to be approximated by an entropy point. Roughly speaking, our proof of Theorem 1.1 is based on the observation that if one of the conditions (1)–(3) is satisfied, then x limits to a point such that there are sufficiently "separated" pairs of two cycles through the point. By constructing pseudo orbits beginning at x and eventually turning around the cycles, and then by shadowing them, we prove that x is approximated by an entropy point (Lemma 2.3). Indeed, it has been observed so far that the existence of such a "separated" pair of two cycles near a point together with the shadowing property enables us to obtain a factor map onto the full shift from a subsystem of some power of the map (see, for example, Kocielniak and Mazur 2007; Li and Oprocha 2013; Moothathu and Oprocha 2013). As far as the author knows, such an idea goes back to 80s (Kirchgraber and Stoffer 1989). In this paper, we explicitly define "e-separated pairs of two δ-cycles at a point" (Definition 2.1) and provide three sufficient conditions for the existence of such objects, each of which corresponds to one of the conditions in Theorem 1.1 (Lemma 2.1). By using them, we prove Theorem 1.1. The method to obtain a factor map onto the full shift is described in Lemma 2.4.

Applying Theorem 1.1 with $c = 0$, it is immediate to obtain the following corollary, which provides sufficient conditions for a shadowable point to be an entropy point.

Corollary 1.1 *Let $f : X \to X$ be a continuous map. Given $x \in Sh^+(f)$, if one of the following conditions is satisfied, then $x \in Ent(f)$.*

(1) *There is a closed f-invariant subset $S \subset X$ such that $CR(f|_S) = S$ and $\omega(x, f) \cap Sen(f|_S) \neq \emptyset$.*
(2) *There is $y \in X$ such that $\{x, y\} \subset X$ is a Li–Yorke pair.*
(3) *$\omega(x, f)$ is non-minimal for f.*

By Theorem 1.1 and Lemma 2.4 together with the result of Blanchard et al. (2002), we obtain the following corollary.

Corollary 1.2 *Let $f : X \to X$ be a continuous map with the shadowing property. Then, the following properties are equivalent.*

(1) $h_{top}(f) > 0$.
(2) *(X, f) has a Li–Yorke pair.*
(3) *(X, f) is Li–Yorke chaotic.*
(4) *There exists $x \in X$ such that $\omega(x, f)$ is non-minimal for f.*

Positive topological entropy is a characteristic feature of chaos. It is well-known that positive topological entropy implies Li–Yorke chaos for any surjective continuous map on a compact metric space (Blanchard et al. 2002, Corollary 2.4). Corollary 1.2 claims that when the shadowing property is assumed, the presence of a Li–Yorke pair implies positive topological entropy, and so does Li–Yorke chaos by the fact above. As a consequence, two definitions of chaos coincide under the shadowing property. We remark here that for interval maps, the presence of a Li–Yorke pair implies Li–Yorke chaos, but there are Li–Yorke chaotic interval maps with zero topological entropy (Kuchta and Smital 1989; Smital 1986; Xiong 1986).

As the next step, we give a lower estimate of the topological entropy under the presence of a Li–Yorke pair and a quantitative shadowing property. Let d_2 denote the metric on $X^2 = X \times X$ defined by $d_2((a, b), (a', b')) = \max\{d(a, a'), d(b, b')\}$.

Theorem 1.2 *Let $f : X \to X$ be a continuous map and suppose that the following three conditions hold:*

(1) $e > 2b > 0$*;*
(2) *$x \in Sh_b^+(f)$ and every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is b-shadowed by some point of X; and*
(3) *There is $y \in X$ such that $\{x, y\} \subset X$ is a Li–Yorke pair with modulus e.*

Then, we have

$$h_{top}(f) \geq \frac{1}{2N_2(\delta)} \log 2,$$

where $N_2(\delta)$ denotes the minimum cardinality of an open cover of (X^2, d_2) whose mesh is $\leq \delta$.

Then, we proceed to a study on the presence of regularly recurrent points near a chain recurrent point in the interior of the set of shadowable points. The following proposition claims that there is a periodic point or a point whose orbit closure is conjugate to an odometer in any neighborhood of such a point. It is a slight extension of a recent result by Li and Oprocha (2016, Corollary 3.3), and we also give an alternative proof of it through the construction of a factor map (Lemma 4.1).

Let us briefly review the definition of odometers. Given a continuous map $f : X \to X$, a point $x \in X$ is said to be *regularly recurrent* if for every neighborhood U of x, there is $k \in \mathbb{N}$ such that $f^{kn}(x) \in U$ for all $n \geq 0$, and *minimal* (or *almost periodic*) if the restriction of f to the orbit closure $\overline{O_f(x)} = \overline{\{f^n(x) : n \geq 0\}}$ is minimal. We denote by $RR(f)$ (resp. $M(f)$) the set of regularly recurrent (resp. minimal) points of f. Note that $RR(f) \subset M(f)$. It holds that $M(f) = M(f^m)$ for every $m \in \mathbb{N}$ (see, for example, Moothathu 2011). An *odometer* (also called an *adding machine*) is defined as follows. Given a strictly increasing sequence $m = (m_k)_{k=1}^{\infty}$ of positive integers such that $m_1 \geq 2$ and m_k divides m_{k+1} for each $k = 1, 2, \dots,$ we define

- $X(k) = \{0, 1, \dots, m_k - 1\}$ (with the discrete topology);
- $X_m = \{x = (x_k)_{k=1}^{\infty} \in \prod_{k=1}^{\infty} X(k) : x_k \equiv x_{k+1} \pmod{m_k}\}$;
- $g(x)_k = x_k + 1 \pmod{m_k}$ for $x \in X_m$.

The resulting dynamical system (X_m, g) is called an odometer with the periodic structure m. An odometer is characterized as a minimal equicontinuous system on Cantor space (see Kurka 2003). Any infinite minimal system with the shadowing property is conjugate to an odometer. It is also known that for every continuous map $f : X \to X$ and $x \in RR(f) \setminus Per(f)$, a dynamical system $(\overline{O_f(x)}, f)$ is an almost 1–1 extension of an odometer. Moreover, if $\overline{O_f(x)} \subset RR(f)$, then $(\overline{O_f(x)}, f)$ is conjugate to an odometer (see Blokh and Keesling 2004; Downarowicz 2005).

Proposition 1.1 *Let $f : X \to X$ be a continuous map and let $p \in \mathrm{Int}\, Sh^+(f) \cap CR(f)$. Then, for every $\epsilon > 0$, there exists $q \in X$ with $d(p, q) \leq \epsilon$ such that $q \in Per(f)$ or $(\overline{O_f(q)}, f)$ is conjugate to an odometer.*

Remark 1.1 If a continuous map $f : X \to X$ satisfies the shadowing property, then $Sh^+(f) = X$. In this case, as seen from Proposition 1.1, $RR(f)$ is dense in the non-wandering set of f. Therefore, one may expect that if $f : X \to X$ has the b-shadowing property with $b > 0$, then for every $x \in \Omega(f)$, there exists $y \in RR(f)$ with $d(x, y) \leq b$, but this is not the case as shown in the following example. Let $\sigma : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ be the full shift and let $g_b : Y_b \to Y_b$ be a minimal rigid rotation on a circle Y_b with radius $b > 0$. Then, since σ has the shadowing property, $\sigma \times g_b : \{0, 1\}^{\mathbb{Z}} \times Y_b \to \{0, 1\}^{\mathbb{Z}} \times Y_b$ has the b-shadowing property. However, $RR(\sigma \times g_b) = RR(\sigma) \times RR(g_b) = \emptyset$ because $RR(g_b) = \emptyset$.

The next theorem describes local features of interior points in the set of shadowable points, under the assumption that $\mathrm{Int}\, Sh^+(f)$ is contained in a chain recurrent subset. A key idea of the proof is *Bowen type decomposition* of chain recurrent subsets.

It has been observed so far that if a continuous map $f : X \to X$ is chain recurrent, then X admits a canonical decomposition into finitely many chain components. Such an idea goes back to Smale's spectral decomposition theorem on Axiom A diffeomorphisms. It states that the non-wandering set of an Axiom A diffeomorphism is decomposed into finitely many clopen transitive components (Smale 1967). Then, Bowen decomposed each of the components into cyclically alternating clopen components for which the power of the diffeomorphism restricted to each component is topologically mixing, and used it to develop the ergodic theory of Axiom A diffeomorphisms (Bowen 1975). A topological version of Smale and Bowen decomposition is presented in Aoki and Hiraide (1994) for instance.

Relatively recently, such a type of decomposition is generalized for chain transitive maps. An idea leading to the generalization was already presented in Akin (1993). It was used in Richeson and Wiseman (2008) to give a structure theorem of chain transitive maps, and used in Brian et al. (2015) to prove a certain kind of shadowing property for chain transitive maps. We consider such a type of decomposition of chain recurrent subsets by chain equivalence relations without assuming the chain transitivity, and use it to prove Theorem 1.3.

Theorem 1.3 *Let $f : X \to X$ be a continuous map and let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the full shift. Suppose that there is a closed f-invariant subset $S \subset X$ such that $CR(f|_S) = S$ and $\mathrm{Int}\, Sh^+(f) \subset S$. Then, for any $x \in \mathrm{Int}\, Sh^+(f)$, each of the following two families of properties (S1)–(S5) and (E1)–(E4) consists of equivalent properties, and either (S1) or (E1) holds.*

- (S1) $x \in \overline{Sen(f)}$.
- (S2) *For every $\epsilon > 0$, there are $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset B_\epsilon(x)$ for which we have a factor map $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$, and there exists $y \in X$ with $d(x,y) \le \epsilon$ such that $y \in Per(f)$ or $(\overline{O_f(y)}, f)$ is conjugate to an odometer.*
- (S3) *For every $\epsilon > 0$, there exists $y \in X$ with $d(x,y) \le \epsilon$ such that $(\overline{O_f(y)}, f)$ is a minimal sensitive subsystem.*
- (S4) $x \in Ent(f)$.
- (S5) $x \notin \mathrm{Int}\, RR(f)$.

- (E1) $x \in \mathrm{Int}\, EC(f)$.
- (E2) *There is a neighborhood U of x such that for every $y \in U$, $y \in Per(f)$ or $(\overline{O_f(y)}, f)$ is conjugate to an odometer.*
- (E3) $x \notin Ent(f)$.
- (E4) $x \in \mathrm{Int}\, RR(f)$.

Moreover, if $x \in EC(f)$, then $x \in Per(f)$ or $(\overline{O_f(x)}, f)$ is conjugate to an odometer.

Remark 1.2 If $CR(f) = X$, then the hypothesis of Theorem 1.3 is satisfied for $S = X$. When $f : X \to X$ is a homeomorphism or an open map and $\mathrm{Int}\, Sh^+(f) \subset CR(f)$, putting $S = \overline{\mathrm{Int}\, Sh^+(f)}$, we have $f(S) \subset S$ and $\Omega(f|_S) = S$, implying $CR(f|_S) = S$. Then, the hypothesis of Theorem 1.3 is satisfied. Note that if a continuous map $f : X \to X$ satisfies the shadowing property and $S = \Omega(f)$, then we have $Sh^+(f|_S) = S$ and $CR(f|_S) = S$. Hence, Theorem 1.3 applies to any dynamical system with the shadowing property restricted to its non-wandering set.

This paper consists of six sections. In Sect. 2, we prove Theorem 1.1 and Corollary 1.2. Theorem 1.2 is proved in Sect. 3. We prove Proposition 1.1 in Sect. 4. In Sect. 5, we give Bowen type decomposition of chain recurrent subsets and some consequences. Finally, we prove Theorem 1.3 in Sect. 6.

2 Proof of Theorem 1.1 and Corollary 1.2

In this section, we prove Theorem 1.1 and Corollary 1.2. We first give the definition of "e-separated pairs of two δ-cycles at a point" mentioned in Sect. 1. Let $f : X \to X$ be a continuous map on a compact metric space (X, d).

Definition 2.1 For $x \in X$, a δ-chain $(x_i)_{i=0}^k$ of f is said to be a δ-cycle of f at x if $x_0 = x_k = x$. For $e > 0$, we say that a pair $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ of two δ-cycles of f at x is e-separated if $d(z_i^{(0)}, z_i^{(1)}) > e$ for some $0 < i < m$.

Note that when we say that a pair of δ-cycles of f is e-separated, the two δ-cycles have the same length, which will be called the *period* of the pair. In what follows, δ-cycles mean δ-cycles of f unless otherwise specified.

Remark 2.1 Let $((z_i^{(0)})_{i=0}^k, (z_i^{(1)})_{i=0}^l)$ be a pair of δ-cycles at x with $d(z_j^{(0)}, z_j^{(1)}) > e$ for some $0 < j < \min\{k, l\}$. Then, the pair of the following δ-cycles:

$$(z_0^{(0)}, z_1^{(0)}, \dots, z_j^{(0)}, \dots, z_{k-1}^{(0)}, z_0^{(1)}, z_1^{(1)}, \dots, z_{l-1}^{(1)}, z_0^{(1)}),$$
$$(z_0^{(1)}, z_1^{(1)}, \dots, z_j^{(1)}, \dots, z_{l-1}^{(1)}, z_0^{(0)}, z_1^{(0)}, \dots, z_{k-1}^{(0)}, z_0^{(0)})$$

is an e-separated pair of δ-cycles at x with the period $k + l$.

Lemma 2.1 *Let $f : X \to X$ be a continuous map. Given $e > 0$ and $z \in X$, if one of the following conditions is satisfied, then for any $\delta > 0$, X contains an e-separated pair of two δ-cycles of f at z.*

1. *There is a closed f-invariant subset $S \subset X$ such that $CR(f|_S) = S$ and $z \in Sen_e(f|_S)$.*
2. *There are a Li–Yorke pair $\{x, y\} \subset X$ with modulus e and a sequence of integers $0 < n_1 < n_2 < \cdots$ such that*

$$\lim_{j\to\infty} d(f^{n_j}(x), f^{n_j}(y)) = 0 \quad \textit{and} \quad \lim_{j\to\infty} f^{n_j}(x) = z.$$

3. *There are $x \in X$ and a closed f-invariant subset $S \subset X$ such that $z \in S \subset \omega(x, f)$ and $\omega(x, f) \setminus B_e(S) \neq \emptyset$.*

□

Proof (1) This proof is a modification of that of Kocielniak and Mazur (2007), Theorem 2. Given $\delta > 0$, fix $0 < \delta_0 < \delta/2$ and take $0 < \delta_1 < \delta/2$ so that $d(a, b) < \delta_1$ implies $d(f(a), f(b)) < \delta_0$ for all $a, b \in X$. Then, since $z \in Sen_e(f|_S)$, there are $z_0^{(0)}, z_0^{(1)} \in S$ and $N \in \mathbb{N}$ such that $\max\{d(z, z_0^{(0)}), d(z, z_0^{(1)})\} < \delta_1$ and $d(f^N(z_0^{(0)}), f^N(z_0^{(1)})) > e$. Choose $\epsilon > 0$ with $d(f^N(z_0^{(0)}), f^N(z_0^{(1)})) > e + 2\epsilon$ and take $0 < \delta_2 < \delta/2$ such that for every δ_2-chain $(x_0, x_1, \dots, x_N)$ of f, we have $d(f^N(x_0), x_N) < \epsilon$. Since $z_0^{(0)}, z_0^{(1)} \in S = CR(f|_S)$, there exists a pair of δ_2-cycles in S

$$((z_0^{(0)}, z_1^{(0)}, \dots, z_{k-1}^{(0)}, z_0^{(0)}), (z_0^{(1)}, z_1^{(1)}, \dots, z_{l-1}^{(1)}, z_0^{(1)}))$$

with $\min\{k, l\} > N$. By the choice of δ_2, we have

$$d(z_N^{(0)}, z_N^{(1)}) \geq d(f^N(z_0^{(0)}), f^N(z_0^{(1)})) - d(f^N(z_0^{(0)}), z_N^{(0)}) - d(f^N(z_0^{(1)}), z_N^{(1)})$$
$$> e + 2\epsilon - 2\epsilon = e.$$

From

$$d(f(z), z_1^{(0)}) \leq d(f(z), f(z_0^{(0)})) + d(f(z_0^{(0)}), z_1^{(0)}) < \delta_0 + \delta_2 < \delta$$

and

$$d(f(z_{k-1}^{(0)}), z) \leq d(f(z_{k-1}^{(0)}), z_0^{(0)}) + d(z_0^{(0)}, z) < \delta_2 + \delta_1 < \delta,$$

it follows that $(z, z_1^{(0)}, \ldots, z_{k-1}^{(0)}, z)$ is a δ-cycle at z. Similarly, $(z, z_1^{(1)}, \ldots, z_{l-1}^{(1)}, z)$ is also a δ-cycle at z. Hence, as in Remark 2.1, S contains an e-separated pair of δ-cycles at z with period $k + l$.

(2) Given $\delta > 0$, take $0 < \eta = \eta(\delta) < \delta$ such that $d(a, b) \leq \eta$ implies $d(f(a), f(b)) \leq \delta$ for all $a, b \in X$. Then, there are $1 \leq N_1 < N_2 < N_3$ with $N_2 - N_1$ and $N_3 - N_2$ arbitrarily large such that

$$\{f^{N_1}(x), f^{N_1}(y), f^{N_3}(x), f^{N_3}(y)\} \subset B_\eta(z) = \{u \in X : d(z, u) \leq \eta\}.$$

and $d(f^{N_2}(x), f^{N_2}(y)) > e$. Then, the pair of the following

$$(z, f^{N_1+1}(x), \ldots, f^{N_2-1}(x), f^{N_2}(x), f^{N_2+1}(x), \ldots, f^{N_3-1}(x), z),$$
$$(z, f^{N_1+1}(y), \ldots, f^{N_2-1}(y), f^{N_2}(y), f^{N_2+1}(y), \ldots, f^{N_3-1}(y), z)$$

is an e-separated pair of δ-cycles at z.

(3) Fix $p \in \omega(x, f)$ with $d(p, S) > e$. Given $\delta > 0$, since $f|_{\omega(x,f)}$ is chain transitive, there is a δ-chain $(x_i^{(1)})_{i=0}^a$ of $f|_{\omega(x,f)}$ such that $x_0^{(1)} = z$ and $x_a^{(1)} = p$. Note that $f^a(z) \in S$, and hence $d(f^a(z), p) \geq d(p, S) > e$. By the chain transitivity of $f|_{\omega(x,f)}$ again, there is a pair $((y_i^{(0)})_{i=0}^b, (y_i^{(1)})_{i=0}^c)$ of δ-chains of $f|_{\omega(x,f)}$ such that $(y_0^{(0)}, y_0^{(1)}) = (f^a(z), p)$ and $(y_b^{(0)}, y_c^{(1)}) = (z, z)$. Consider the following pair of δ-cycles of f:

$$((z, f(z), \ldots, f^{a-1}(z), y_0^{(0)}, y_1^{(0)}, \ldots, y_{b-1}^{(0)}, z), (z, x_1^{(1)}, \ldots, x_{a-1}^{(1)}, y_0^{(1)}, y_1^{(1)}, \ldots, y_{c-1}^{(1)}, z)).$$

Since $d(y_0^{(0)}, y_0^{(1)}) = d(f^a(z), p) > e$, as in Remark 2.1, there is an e-separated pair of δ-cycles at z with period $2a + b + c$ contained in $\omega(x, f)$.

□

Remark 2.2 Under the assumption of (1) (resp. (3)), the e-separated pairs of δ-cycles of f at z can be taken in S (resp. $\omega(x, f)$).

We need the following lemma given by Ye and Zhang (2007).

Lemma 2.2 (Ye and Zhang 2007, Proposition 2.5) *If $h(f, A) > 0$ for a closed subset $A \subset X$, then $A \cap Ent(f) \neq \emptyset$.*

This lemma is obtained by the fact that for any choice of compact subsets $K_1, \ldots, K_m \subset X$, we have $h(f, \bigcup_{i=1}^m K_i) = \max\{h(f, K_i) : 1 \leq i \leq m\}$ and a simple concentration argument.

The next lemma is essential in the proof of Theorem 1.1.

Lemma 2.3 *Let $f : X \to X$ be a continuous map and let $x \in Sh_b^+(f)$ with $b > 0$. Given $e, \delta > 0$, and $z \in \omega(x, f)$, suppose that the following conditions are satisfied.*

- *$e > 2b$.*
- *Every δ-pseudo orbit $(x_i)_{i=0}^\infty$ of f with $x_0 = x$ is b-shadowed by some point of X.*
- *There is an e-separated pair $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ of δ-cycles of f at z with period m.*

Then, $h_{top}(f) \geq (\log 2)/m$, and there exists $w \in Ent(f)$ such that $d(x, w) \leq b$.

Proof Fix $0 < j < m$ with $d(z_j^{(0)}, z_j^{(1)}) > e$ and take $k > 0$ with $d(f^k(x), z) \leq \delta$. By the hypothesis, given $n \in \mathbb{N}$, for each $s = (s_1, \ldots, s_n) \in \{0, 1\}^n$, we can consider the following δ-chain of f:

$$(x, f(x), \ldots, f^{k-1}(x), z_0^{(s_1)}, z_1^{(s_1)}, \ldots, z_{m-1}^{(s_1)}, \ldots, z_0^{(s_n)}, z_1^{(s_n)}, \ldots, z_{m-1}^{(s_n)}),$$

which is b-shadowed by $y(s) \in B_b(x)$. Put $E_n = \{y(s) \in X : s \in \{0, 1\}^n\}$ and let us claim that E_n is a $(k + mn, e - 2b)$-separated set. In fact, for any $s, t \in \{0, 1\}^n$, if $s \neq t$, then $s_a \neq t_a$ for some $1 \leq a \leq n$, and letting $K = k + (a - 1)m + j$, we have $K < k + mn$, and

$$d\big(f^K(y(s)), f^K(y(t))\big) \geq d(z_j^{(s_a)}, z_j^{(t_a)}) - d(f^K(y(s)), z_j^{(s_a)}) - d(f^K(y(t)), z_j^{(t_a)}) > e - 2b.$$

Note that $E_n \subset B_b(x)$ and the cardinality of E_n is 2^n. Hence, we have $S(B_b(x), k + mn, e - 2b) \geq 2^n$ for every $n \in \mathbb{N}$, and then

$$\begin{aligned} h(f, B_b(x)) \geq h(f, B_b(x), e - 2b) &= \limsup_{n\to\infty} \frac{1}{n} \log S(B_b(x), n, e - 2b) \\ &\geq \limsup_{n\to\infty} \frac{1}{k + mn} \log S(B_b(x), k + mn, e - 2b) \\ &\geq \limsup_{n\to\infty} \frac{1}{k + mn} \log 2^n \\ &= \frac{1}{m} \log 2 > 0. \end{aligned}$$

Thus, we obtain $h_{top}(f) \geq h(f, B_b(x)) \geq (\log 2)/m$, and from Lemma 2.2, it follows that $B_b(x) \cap Ent(f) \neq \emptyset$. □

Now let us prove Theorem 1.1.

Proof of Theorem 1.1 Take $b > c$ with $e > 2b > 2c$ and choose $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is b-shadowed by some point of X. For such δ, if one of the conditions (1)-(3) in Theorem 1.1 (corresponding to those in Lemma 2.1) is satisfied, then there exist $z \in \omega(x, f)$ and an e-separated pair of δ-cycles of f at z by Lemma 2.1. Hence, using Lemma 2.3, we see that there exists $w \in Ent(f)$ such that $d(x, w) \leq b$. Since $b > c$ can be taken arbitrarily close to c, and $Ent(f)$ is a closed subset of X, there exists $w \in Ent(f)$ such that $d(x, w) \leq c$, proving the theorem. □

Let $\sigma : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ be the full shift. The following lemma is a restatement (with modification) of Proposition 2 in Sect. 2 of Kirchgraber and Stoffer (1989). It describes how we obtain from an e-separated pair of δ-cycles at x together with the shadowing property, a subsystem of some power of f which is an extension of the full shift.

Lemma 2.4 *Let $e \geq 2b > 0$ and let $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ be an e-separated pair of δ-cycles at $x \in X$. For each $s = (s_1, s_2, \dots) \in \{0, 1\}^{\mathbb{N}}$, define a δ-pseudo orbit $\gamma(s)$ as follows:*

$$\gamma(s) = (z_0^{(s_1)}, z_1^{(s_1)}, \dots, z_{m-1}^{(s_1)}, z_0^{(s_2)}, z_1^{(s_2)}, \dots, z_{m-1}^{(s_2)}, z_0^{(s_3)}, z_1^{(s_3)}, \dots, z_{m-1}^{(s_3)}, \dots).$$

If every $\gamma(s)$, $s \in \{0, 1\}^{\mathbb{N}}$, is b-shadowed by some point of X, then there exist a closed f^m-invariant subset $Y \subset B_b(x)$ and a factor map $\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma)$.

Proof Let

$$Y = \{y \in X : y \text{ is a } b\text{-shadowing point of } \gamma(s) \text{ for some } s \in \{0, 1\}^{\mathbb{N}}\},$$

and define a map $\pi : Y \to \{0, 1\}^{\mathbb{N}}$ so that y is a b-shadowing point of $\gamma(\pi(y))$. Then, it is easy to see that the following properties hold.

(1) Y is a closed subset of X;
(2) $f^m(Y) \subset Y$;
(3) π is well-defined;
(4) π is surjective;
(5) π is continuous; and
(6) $\pi \circ f^m = \sigma \circ \pi$.

Hence, $\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma)$ is a factor map, and $Y \subset B_b(x)$ is obvious. □

Remark 2.3 Let $Id : [0, 1] \to [0, 1]$ be the identity map on the unit interval. Then, for any $\delta > 0$, if $m \geq 1$ is large enough, we can take a δ-cycle $(z_i^{(0)})_{i=0}^m$ of Id at 0 with $z_j^{(0)} = 1$ for some $0 < j < m$. Consider the δ-cycle $(z_i^{(1)})_{i=0}^m$ of Id at 0 defined by $z_i^{(1)} = 0$ for all $0 \leq i \leq m$. Then, we have $d(z_j^{(0)}, z_j^{(1)}) = 1$, and every δ-pseudo orbit $\gamma(s)$, $s \in \{0, 1\}^{\mathbb{N}}$, defined as in Lemma 2.4 is 1/2-shadowed by 1/2. But, it is obvious that there is no subsystem of (powers of) Id admitting a factor map to the full shift. Note that $((z_i^{(0)})_{i=0}^m, (z_i^{(1)})_{i=0}^m)$ is not a 1-separated pair of δ-cycles at 0 by the definition. This example shows that the assumption of the separation $> e$ cannot be replaced by $\geq e$ in order that Lemma 2.4 holds.

Remark 2.4 There is a sensitive continuous map $f : X \to X$ with the shadowing property such that (X, f) admits $(\{0,1\}^{\mathbb{N}}, \sigma)$ as a factor, but any subsystem of powers of f is not conjugate to $(\{0,1\}^{\mathbb{N}}, \sigma)$. In fact, $(\{0,1\}^{\mathbb{N}} \times X_m, \sigma \times g)$ with an odometer (X_m, g) gives such an example. The natural projection onto $(\{0,1\}^{\mathbb{N}}, \sigma)$ is a factor map. Note that $Per(\sigma \times g) = \emptyset$ because $Per(g) = \emptyset$, but if some subsystem of some power of $\sigma \times g$ were conjugate to $(\{0,1\}^{\mathbb{N}}, \sigma)$, then $Per(\sigma \times g)$ would be non-empty.

As a corollary of Lemmas 2.1 and 2.4, we obtain the following lemma.

Lemma 2.5 *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset such that $CR(f|_S) = S$. If $x \in Sen(f|_S) \cap Sh^+(f)$, then for every $\epsilon > 0$, there are $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset B_\epsilon(x)$ for which we have a factor map $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$.*

Proof Take positive constants e, ϵ, and $\delta > 0$ with the following properties.

- $x \in Sen_e(f|_S)$.
- $e > 2\epsilon$.
- Every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is ϵ-shadowed by some point of X.

Then, the condition (1) of Lemma 2.1 is satisfied, and therefore S contains an e-separated pair of δ-cycles of f at x by Lemma 2.1. Hence, we can use Lemma 2.4 to obtain the conclusion. □

For the proof of Corollary 1.2, we need the following lemma, which is also used in the proof of Theorem 1.3.

Lemma 2.6 *Let $f : X \to X$ be a continuous map. If $Y \subset X$ is a closed f^m-invariant subset with $m \in \mathbb{N}$, and $\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$ is a factor map, then we have the following properties.*

(1) *There is $y \in Y$ such that $\omega(y, f)$ is non-minimal for f.*
(2) *There is $y \in Y$ such that $(\overline{O_f(y)}, f)$ is a minimal sensitive subsystem.*

Proof (1) Take $s \in \{0,1\}^{\mathbb{N}}$ with $\omega(s, \sigma) = \{0,1\}^{\mathbb{N}}$ and $y \in \pi^{-1}(s)$. Putting $Z = \omega(y, f^m)$, we have $f^m(Z) \subset Z$ and $\pi(Z) = \{0,1\}^{\mathbb{N}}$, which implies that $\pi : (Z, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$ is a factor map. Then, defining $W = Z \cup f(Z) \cup \cdots \cup f^{m-1}(Z)$, we have $W = \omega(y, f)$. To show that W is non-minimal for f by contradiction, assume that W is minimal for f. Then, we have $Z \subset W \subset M(f) = M(f^m)$, which contradicts that $M(\sigma) \neq \{0,1\}^{\mathbb{N}}$, because the π-image of any minimal point for f^m is also a minimal point for σ. Thus, W is non-minimal for f.

(2) It suffices to show that there exists $y \in Y$ such that $(\overline{O_{f^m}(y)}, f^m)$ is a minimal sensitive subsystem of (Y, f^m). Put $g = f^m$ and take an infinite minimal subshift $\Sigma \subset \{0,1\}^{\mathbb{N}}$. Since $Z = \pi^{-1}(\Sigma)$ is g-invariant, there is a minimal g-invariant subset $W \subset Z$. Then, since $\pi(W) \subset \Sigma$ is σ-invariant and Σ is minimal, we have $\pi(W) = \Sigma$. Let us claim that $g|_W$ is sensitive. Fix any $w \in W$. Note that $\sigma|_\Sigma$ is

positively expansive, and hence if $\pi(U) \neq \{\pi(w)\}$ for every neighborhood U of w in W, then w is a sensitive point of $g|_W$. Assume the contrary, i.e., there is a neighborhood U of w in W such that $\pi(U) = \{\pi(w)\}$ to exhibit a contradiction. Since W is minimal for g, there is $n > 0$ such that $g^n(w) \in U$. Then, $\pi(w) \in \Sigma$ and $\sigma^n(\pi(w)) = \pi(g^n(w)) = \pi(w)$, which contradicts that Σ is infinite and minimal. Thus, for every $w \in W$, $(\overline{O_g(w)}, g)$=(W, g) is a minimal sensitive subsystem of (Y, g), proving the lemma.

□

As the final proof of this section, we give a proof of Corollary 1.2.

Proof of Corollary 1.2 (1) ⇒ (3): $h_{top}(f) > 0$ implies that $h_{top}(f|_{\Omega(f)}) = h_{top}(f) > 0$. By the shadowing property of f, we see that $f|_{\Omega(f)}$ is surjective. Hence, from Blanchard et al. (2002), Corollary 2.4, it follows that $f|_{\Omega(f)}$ is Li-Yorke chaotic, and so is f.

(3) ⇒ (2): This is obvious by the definition.

(2) ⇒ (1): Let $\{x, y\} \subset X$ be a Li–Yorke pair with modulus e. Note that $x \in Sh^+(f)$ since $Sh^+(f) = X$. Applying Theorem 1.1 (2) with $c = 0$, we have $x \in Ent(f)$, implying that $Ent(f) \neq \emptyset$, and thus $h_{top}(f) > 0$.

(1) ⇒ (4): If $Sen(f|_{\Omega(f)}) = \emptyset$, then $h_{top}(f) = h_{top}(f|_{\Omega(f)}) = 0$. Therefore, when $h_{top}(f) > 0$, we have $Sen(f|_{\Omega(f)}) \neq \emptyset$. The shadowing property of f implies that $\Omega(f) = \Omega(f|_{\Omega(f)}) \subset CR(f|_{\Omega(f)}) \subset \Omega(f)$, so $CR(f|_{\Omega(f)}) = \Omega(f)$. Since $Sh^+(f) = X$, we can apply Lemma 2.5 with $S = \Omega(f)$ to have $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset X$ for which we have a factor map $\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma)$. Thus, by Lemma 2.6 (1), there is $x \in X$ such that $\omega(x, f)$ is non-minimal for f.

(4) ⇒ (1): Let $x \in X$ be a point such that $\omega(x, f)$ is non-minimal for f. Then, applying Theorem 1.1 (3) with $c = 0$, we have $x \in Ent(f)$, which implies $h_{top}(f) > 0$. □

3 Proof of Theorem 1.2

To prove Theorem 1.2, we need the following technical lemma, which is a version of the shortcut lemma proved in Kawaguchi (2016), Lemma 2.2. Intuitively, the open cover $\mathcal{U}$ of (X^2, d_2) in the lemma works as a "scale", and any pair of chains of f with sufficiently small gaps and an arbitrary length can be replaced by a pair of chains with the same beginning and end points, whose gaps and length are bounded by the mesh and the cardinality of $\mathcal{U}$, respectively.

Lemma 3.1 *Let $\delta > 0$ and let $\mathcal{U} = \{U_1, \ldots, U_K\}$ be an open cover of (X^2, d_2) with $mesh\,\mathcal{U} = \max_{1 \le i \le K} diam\, U_i \le \delta$. Suppose that $\beta > 0$ is a Lebesgue number of $\mathcal{U}$. Then, for every pair $((x_i^{(0)})_{i=0}^k, (x_i^{(1)})_{i=0}^k)$ of β-chains of f, there is a pair $((y_i^{(0)})_{i=0}^l, (y_i^{(1)})_{i=0}^l)$ of δ-chains of f such that $(y_0^{(0)}, y_0^{(1)}) = (x_0^{(0)}, x_0^{(1)})$ and $(y_l^{(0)}, y_l^{(1)}) = (x_k^{(0)}, x_k^{(1)})$ with $1 \le l \le K$.*

Proof Put $g = f \times f$ and $z_i = (x_i^{(0)}, x_i^{(1)}) \in X^2$ for each $0 \le i \le k$. Then, we have $d_2(g(z_i), z_{i+1}) \le \beta$ for every $0 \le i < k$. Since $d_2(g(z_0), z_1)) \le \beta$, there is

$1 \le i_0 \le K$ such that $\{g(z_0), z_1\} \subset U_{i_0}$. Put $j_1 = \max\{1 \le j \le k : z_j \in U_{i_0}\}$. Since $\{g(z_0), z_{j_1}\} \subset U_{i_0}$ and $diam\, U_{i_0} \le \delta$, we have $d_2(g(z_0), z_{j_1}) \le \delta$. If $j_1 < k$, then since $d_2(g(z_{j_1}), z_{j_1+1}) \le \beta$, there is $1 \le i_1 \le K$ such that $\{g(z_{j_1}), z_{j_1+1}\} \subset U_{i_1}$. Put $j_2 = \max\{j_1 + 1 \le j \le k : z_j \in U_{i_1}\}$. Since $\{g(z_{j_1}), z_{j_2}\} \subset U_{i_1}$ and $diam\, U_{i_1} \le \delta$, we have $d_2(g(z_{j_1}), z_{j_2}) \le \delta$. Note that $z_{j_1+1} \in U_{i_1} \backslash U_{i_0}$, and so $U_{i_1} \ne U_{i_0}$. If $j_2 < k$, we repeat the process, and so on. Inductively, we obtain a sequence of integers $0 = j_0 < j_1 < j_2 < \cdots$. If $j_K < k$, then $U_{i_0}, U_{i_1}, \dots, U_{i_K}$ would be $K + 1$ distinct elements of $\mathcal{U}$, which is absurd. Therefore, we have $j_l = k$ for some $1 \le l \le K$ and $d_2(g(z_{j_\alpha}), z_{j_{\alpha+1}}) \le \delta$ for every $0 \le \alpha < l$. Put $z_{j_\alpha} = (x^{(0)}_{j_\alpha}, x^{(1)}_{j_\alpha}) = (y^{(0)}_\alpha, y^{(1)}_\alpha)$ for each $0 \le \alpha \le l$. Then, we have $(y^{(0)}_0, y^{(1)}_0) = (x^{(0)}_0, x^{(1)}_0)$, $(y^{(0)}_l, y^{(1)}_l) = (x^{(0)}_k, x^{(1)}_k)$, and

$$\begin{aligned}\max\{d(f(y^{(0)}_\alpha), y^{(0)}_{\alpha+1}), d(f(y^{(1)}_\alpha), y^{(1)}_{\alpha+1})\} &= d_2\big((f(y^{(0)}_\alpha), f(y^{(1)}_\alpha)), (y^{(0)}_{\alpha+1}, y^{(1)}_{\alpha+1})\big)\\ &= d_2(g(z_{j_\alpha}), z_{j_{\alpha+1}}) \le \delta\end{aligned}$$

for all $0 \le \alpha < l$. Hence, $((y^{(0)}_\alpha)^l_{\alpha=0}, (y^{(1)}_\alpha)^l_{\alpha=0})$ is a pair of δ-chains of f satisfying the required property. □

By the virtue of Lemma 3.1, we can reduce the period of a separated pair of cycles at a point in some case.

Lemma 3.2 *Under the same hypothesis as in Lemma 3.1, for every e-separated pair* $((z^{(0)}_i)^m_{i=0}, (z^{(1)}_i)^m_{i=0})$ *of* β*-cycles at* $x \in X$*, there is an e-separated pair* $((w^{(0)}_i)^n_{i=0}, (w^{(1)}_i)^n_{i=0})$ *of* δ*-cycles at* x *with period* $n \le 2K$.

Proof Fix $0 < j < m$ with $d(z^{(0)}_j, z^{(1)}_j) > e$. We split the pair $((z^{(0)}_i)^m_{i=0}, (z^{(1)}_i)^m_{i=0})$ into two parts corresponding to $0 \le i \le j$ and $j \le i \le m$ respectively, and apply the shortcut lemma (Lemma 3.1) to each part. Then, by joining them, we obtain a separated pair with a shortened period. Precisely, by Lemma 3.1, there exist two pairs $((x^{(0)}_i)^k_{i=0}, (x^{(1)}_i)^k_{i=0})$ and $((y^{(0)}_i)^l_{i=0}, (y^{(1)}_i)^l_{i=0})$ of δ-chains of f such that

- $(x^{(0)}_0, x^{(1)}_0) = (z^{(0)}_0, z^{(1)}_0) = (x, x)$ and $(x^{(0)}_k, x^{(1)}_k) = (z^{(0)}_j, z^{(1)}_j)$;
- $(y^{(0)}_0, y^{(1)}_0) = (z^{(0)}_j, z^{(1)}_j)$ and $(y^{(0)}_l, y^{(1)}_l) = (z^{(0)}_m, z^{(1)}_m) = (x, x)$; and
- $\max\{k, l\} \le K$.

Then, the pair of the following δ-cycles:

$$\begin{gathered}(x^{(0)}_0, x^{(0)}_1, \dots, x^{(0)}_{k-1}, y^{(0)}_0, y^{(0)}_1, \dots, y^{(0)}_{l-1}, y^{(0)}_l),\\ (x^{(1)}_0, x^{(1)}_1, \dots, x^{(1)}_{k-1}, y^{(1)}_0, y^{(1)}_1, \dots, y^{(1)}_{l-1}, y^{(1)}_l)\end{gathered}$$

is an e-separated pair of δ-cycles at x with period $n = k + l \le 2K$. □

Finally, using Lemma 3.2, we prove Theorem 1.2.

Proof of Theorem 1.2 Take an open cover $\mathcal{U}$ of (X^2, d_2) such that $mesh\, \mathcal{U} \le \delta$ and $card\, \mathcal{U} = N_2(\delta)$. Let $\beta > 0$ be a Lebesgue number of $\mathcal{U}$. Then, by assumption (3) and

Lemma 2.1, there is $z \in \omega(x, f)$ such that X contains an e-separated pair of β-cycles at z. Applying Lemma 3.2 to this pair, we obtain an e-separated pair of δ-cycles at z whose period n is $\leq 2N_2(\delta)$. Then, by assumptions (1) and (2), we can use Lemma 2.3 to conclude that

$$h_{top}(f) \geq \frac{1}{n} \log 2 \geq \frac{1}{2N_2(\delta)} \log 2.$$

□

4 Proof of Proposition 1.1

In this section, we prove Proposition 1.1. We first prove the following lemma.

Lemma 4.1 *Let $f : X \to X$ be a continuous map. If $p \in \mathrm{Int}\, Sh^+(f)$ is a chain recurrent point of f, then for every $\epsilon > 0$, there exist a closed f-invariant subset $Y \subset X$ with $Y \cap B_\epsilon(p) \neq \emptyset$ and a factor map $\pi : (X_m, g) \to (Y, f)$, where $B_\epsilon(p) = \{x \in X : d(p, x) \leq \epsilon\}$ and (X_m, g) is an odometer.*

Given $p \in \mathrm{Int}\, Sh^+(f) \cap CR(f)$ and $\epsilon > 0$, take $\epsilon_k > 0, k \in \mathbb{N}$, with $\sum_{k\in\mathbb{N}} \epsilon_k \leq \epsilon$. For any subset $S \subset X$ and any $\delta > 0$, let $B_\delta(S) = \{x \in X : d(x, S) \leq \delta\}$. We may suppose that $B_\epsilon(p) \subset Sh^+(f)$.

The next two lemmas are needed to prove Lemma 4.1. The first lemma is similar to Moothathu and Oprocha (2013), Lemma 3.1, but we extend it to a sequence of finite collections of subsets of X.

Lemma 4.2 *There exist a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ of positive integers and a sequence $(\{A_j^{(k)} : 0 \leq j < m_k\})_{k\in\mathbb{N}}$ of finite collections of compact subsets of X such that the following properties are satisfied for each $k \in \mathbb{N}$.*

(1) $A_0^{(k)} \subset B_{\sum_{i=1}^k \epsilon_i}(p)$.
(2) *m_k divides m_{k+1}.*
(3) *$f^{m_k}(A_0^{(k)}) = A_0^{(k)}$ and $A_0^{(k)}$ is minimal for f^{m_k}.*
(4) *$f^j(A_0^{(k)}) = A_j^{(k)}$ for all $0 \leq j < m_k$.*
(5) *$diam\, A_j^{(k)} \leq 2\epsilon_k$ for all $0 \leq j < m_k$.*
(6) *For any $0 \leq j < m_{k+1}$, if $j = qm_k + r$ with $0 \leq r < m_k$, then $A_j^{(k+1)} \subset B_{\epsilon_{k+1}}(A_r^{(k)})$.*

Proof Let us prove the claim by induction on k. When $k = 1$, since $p \in Sh^+(f)$, there is $\delta_1 > 0$ such that every δ_1-pseudo orbit $(z_i)_{i=0}^\infty$ with $z_0 = p$ is ϵ_1-shadowed by some point of X. Then, since $p \in CR(f)$, there is a δ_1-cycle $(x_i^{(1)})_{i=0}^{m_1}$ with $x_0^{(1)} = x_{m_1}^{(1)} = p$. Consider the following m_1-periodic δ_1-pseudo orbit

$$(x_0^{(1)}, x_1^{(1)}, \ldots, x_{m_1-1}^{(1)}, x_0^{(1)}, x_1^{(1)}, \ldots, x_{m_1-1}^{(1)}, \ldots),$$

which is ϵ_1-shadowed by $y_1 \in X$. Then, for every $n \geq 0$, $f^{m_1 n}(y_1)$ is also an ϵ_1-shadowing point, and hence every $y \in \omega(y_1, f^{m_1})$ is an ϵ_1-shadowing point of the above pseudo orbit. Since $\omega(y_1, f^{m_1})$ is f^{m_1}-invariant, there is a minimal subset $Y_1 \subset \omega(y_1, f^{m_1})$ for f^{m_1}. Given $0 \le j < m_1$, we have $d(f^j(y), x_j^{(1)}) \le \epsilon_1$ for every $y \in Y_1$, and therefore $f^j(Y_1) \subset B_{\epsilon_1}(x_j^{(1)})$. For each $0 \le j < m_1$, put $A_j^{(1)} = f^j(Y_1)$. Then, $A_0^{(1)} = Y_1 \subset B_{\epsilon_1}(x_0^{(1)}) = B_{\epsilon_1}(p)$, and properties (3), (4), and (5) are satisfied for $k = 1$.

Now given $k \in \mathbb{N}$, assume that m_k and $\{A_j^{(k)} : 0 \le j < m_k\}$ satisfying (1), (3), (4), and (5) are chosen. Fix $x_0^{(k+1)} \in A_0^{(k)}$. Then, since $A_0^{(k)} \subset B_{\sum_{i=1}^k \epsilon_i}(p) \subset B_\epsilon(p) \subset Sh^+(f)$, there is $\delta_{k+1} > 0$ such that every δ_{k+1}-pseudo orbit $(z_i)_{i=0}^\infty$ with $z_0 = x_0^{(k+1)}$ is ϵ_{k+1}-shadowed by some point of X. Since $A_0^{(k)}$ is minimal for f^{m_k}, there is $a_k \ge 2$ such that $d(x_0^{(k+1)}, f^{a_k m_k}(x_0^{(k+1)})) \le \delta_{k+1}$. Put $m_{k+1} = a_k m_k$ and consider the following m_{k+1}-periodic δ_{k+1}-pseudo orbit

$$(x_0^{(k+1)}, f(x_0^{(k+1)}), \dots, f^{m_{k+1}-1}(x_0^{(k+1)}), x_0^{(k+1)}, f(x_0^{(k+1)}), \dots, f^{m_{k+1}-1}(x_0^{(k+1)}), \dots),$$

which is ϵ_{k+1}-shadowed by some $y_{k+1} \in X$. Similarly to the above, we take a minimal subset $Y_{k+1} \subset \omega(y_{k+1}, f^{m_{k+1}})$ for $f^{m_{k+1}}$. Note that every $y \in Y_{k+1}$ is an ϵ_{k+1}-shadowing point of the pseudo orbit above. Given $0 \le j < m_{k+1}$, we have $d(f^j(y), f^j(x_0^{(k+1)})) \le \epsilon_{k+1}$ for every $y \in Y_{k+1}$, and therefore $f^j(Y_{k+1}) \subset B_{\epsilon_{k+1}}(f^j(x_0^{(k+1)}))$. Put $A_j^{(k+1)} = f^j(Y_{k+1})$ for every $0 \le j < m_{k+1}$. Then,

$$A_0^{(k+1)} = Y_{k+1} \subset B_{\epsilon_{k+1}}(x_0^{(k+1)}) \subset B_{\epsilon_{k+1}}(A_0^{(k)}) \subset B_{\epsilon_{k+1}}(B_{\sum_{i=1}^k \epsilon_i}(p)) \subset B_{\sum_{i=1}^{k+1} \epsilon_i}(p),$$

(2) is satisfied for k, and (3), (4), and (5) are satisfied for $k+1$. Suppose that $0 \le j < m_{k+1}$ is written as $j = qm_k + r$ with $0 \le r < m_k$. Then, $f^j(x_0^{(k+1)}) \in f^j(A_0^{(k)}) = A_r^{(k)}$, which implies $A_j^{(k+1)} = f^j(Y_{k+1}) \subset B_{\epsilon_{k+1}}(A_r^{(k)})$. Hence, (6) is also satisfied for k, and thus the lemma has been proved. □

Recall that the definition of the odometer (X_m, g) with the periodic structure $m = (m_k)_{k\in\mathbb{N}}$ was given in Sect. 1. For $m = (m_k)_{k\in\mathbb{N}}$ and $(\{A_j^{(k)} : 0 \le j < m_k\})_{k\in\mathbb{N}}$ constructed in Lemma 4.2, we have the following property.

Lemma 4.3 *Let $r = (r_l)_{l\in\mathbb{N}} \in X_m$ and $k \in \mathbb{N}$. Then, we have*

$$A_{r_{k+N}}^{(k+N)} \subset B_{\epsilon_{k+1}+\dots+\epsilon_{k+N}}(A_{r_k}^{(k)})$$

for every $N \in \mathbb{N}$.

Proof We prove this lemma by induction on N. When $N = 1$, since $r_{k+1} \equiv r_k \pmod{m_k}$, by substituting r_{k+1} and r_k for j and r in Lemma 4.2 (6), we have $A_{r_{k+1}}^{(k+1)} \subset B_{\epsilon_{k+1}}(A_{r_k}^{(k)})$. Let us assume that the claim holds for some $N \in \mathbb{N}$ and prove it for $N+1$. Since $r_{k+N+1} \equiv r_{k+N} \pmod{m_{k+N}}$, we have $A_{r_{k+N+1}}^{(k+N+1)} \subset B_{\epsilon_{k+N+1}}(A_{r_{k+N}}^{(k+N)})$ by Lemma 4.2 (6). On the other hand, we have $A_{r_{k+N}}^{(k+N)} \subset B_{\epsilon_{k+1}+\dots+\epsilon_{k+N}}(A_{r_k}^{(k)})$ by the induction hypothesis. Hence,

$$A^{(k+N+1)}_{r_{k+N+1}} \subset B_{\epsilon_{k+N+1}}(B_{\epsilon_{k+1}+\dots+\epsilon_{k+N}}(A^{(k)}_{r_k})) \subset B_{\epsilon_{k+1}+\dots+\epsilon_{k+N}+\epsilon_{k+N+1}}(A^{(k)}_{r_k}),$$

which completes the induction. □

Now let us prove Lemma 4.1.

Proof of Lemma 4.1 Given $r = (r_l)_{l\in\mathbb{N}} \in X_m$, using Lemma 4.3, for every $k \in \mathbb{N}$ and every $N \in \mathbb{N}$, we have

$$A^{(k+N)}_{r_{k+N}} \subset B_{\epsilon_{k+1}+\dots+\epsilon_{k+N}}(A^{(k)}_{r_k}) \subset B_{\sum_{i=k+1}^{\infty}\epsilon_i}(A^{(k)}_{r_k}).$$

Since $diam\, A^{(k)}_{r_k} \le 2\epsilon_k$ by Lemma 4.2 (5), we see that $d_H(A^{(k)}_{r_k}, A^{(k+N)}_{r_{k+N}}) \le \sum_{i=k}^{\infty} 2\epsilon_i \to 0$ as $k \to \infty$, where d_H denotes the Hausdorff distance. In other words, the sequence $(A^{(l)}_{r_l})_{l\in\mathbb{N}}$ is a Cauchy sequence with respect to d_H, and so $\lim_{l\to\infty} d_H(A^{(l)}_{r_l}, C) = 0$ for some closed subset $C \subset X$. Since $diam\, A^{(l)}_{r_l} \le 2\epsilon_l \to 0$ as $l \to \infty$ by Lemma 4.2 (5) again, we have $C = \{x\}$ for some $x \in X$. Then, define a map $\pi : X_m \to X$ by putting $\pi(r) = x$, which implies

$$\lim_{l\to\infty} d_H(A^{(l)}_{r_l}, \{\pi(r)\}) = 0$$

for every $r = (r_l)_{l\in\mathbb{N}} \in X_m$. We need two claims concerning the map π.

Claim 1: $\pi : X_m \to X$ is continuous.

Given $r = (r_l)_{l\in\mathbb{N}}$ and $s = (s_l)_{l\in\mathbb{N}} \in X_m$, suppose $r_l = s_l$ for every $1 \le l \le K$. Then,

$$A^{(K+N)}_{r_{K+N}} \subset B_{\sum_{i=K+1}^{\infty}\epsilon_i}(A^{(K)}_{r_K})$$

for all $N \in \mathbb{N}$ as above. Taking the limit as $N \to \infty$, we obtain

$$\pi(r) \in B_{\sum_{i=K+1}^{\infty}\epsilon_i}(A^{(K)}_{r_K}).$$

Similarly,

$$\pi(s) \in B_{\sum_{i=K+1}^{\infty}\epsilon_i}(A^{(K)}_{s_K}) = B_{\sum_{i=K+1}^{\infty}\epsilon_i}(A^{(K)}_{r_K}).$$

By Lemma 4.2 (5), we have $diam\, A^{(K)}_{r_K} \le 2\epsilon_K$, and therefore $d(\pi(r), \pi(s)) \le \sum_{i=K}^{\infty} 2\epsilon_i \to 0$ as $K \to \infty$. Thus, $\pi : X_m \to X$ is continuous.

Claim 2: $\pi \circ g = f \circ \pi$.

Given $r \in X_m$, put $g(r) = s$. Then, for each $l \in \mathbb{N}$, we have $s_l = r_l + 1 \pmod{m_l}$ by the definition of g, and hence $A^{(l)}_{s_l} = f(A^{(l)}_{r_l})$ by Lemma 4.2 (4). Taking the limit as $l \to \infty$, we obtain $\pi(s) = f(\pi(r))$, that is, $\pi(g(r)) = f(\pi(r))$. Since $r \in X_m$ is arbitrary, this claim has been proved.

Putting $\pi(X_m) = Y$, from Claims 1 and 2, we see that $Y \subset X$ is a closed f-invariant subset, and $\pi : (X_m, g) \to (Y, f)$ is a factor map. Hence, it only remains to prove that there exists $q \in Y$ such that $q \in B_\epsilon(p)$. Put $q = \pi(\mathbf{0}) \in Y$, where $\mathbf{0} = (0, 0, 0, \dots) \in X_m$. By Lemma 4.2 (1), we have

$$A_0^{(k)} \subset B_{\sum_{i=1}^{k} \epsilon_i}(p) \subset B_{\sum_{i=1}^{\infty} \epsilon_i}(p) \subset B_\epsilon(p)$$

for every $k \in \mathbb{N}$. Taking the limit as $k \to \infty$, we obtain $q \in B_\epsilon(p)$, proving the theorem. □

Using Lemma 4.1, we prove Proposition 1.1.

Proof of Proposition 1.1 By Lemma 4.1, for any given $\epsilon > 0$, there are a closed f-invariant subset $Y \subset X$ with $Y \cap B_\epsilon(p) \neq \emptyset$ and a factor map $\pi : (X_m, g) \to (Y, f)$, where (X_m, g) is an odometer. Then, (Y, f) is minimal, and it holds that $Y \subset RR(f)$ because $X_m = RR(g)$. By Blokh and Keesling (2004), Corollary 2.5, we see that Y is a periodic orbit or (Y, f) is conjugate to an odometer. Thus, taking $q \in Y \cap B_\epsilon(p)$, we have $q \in Per(f)$ or $(\overline{O_f(q)}, f)$ is conjugate to an odometer. □

5 Bowen Type Decomposition of Chain Recurrent Subsets

In this section, we give Bowen type decomposition of chain recurrent subsets and present some consequences.

Let $g : S \to S$ be a chain recurrent continuous map on a compact metric space S. For $\delta > 0$, we define a relation $\sim_\delta$ on S as follows. For $x, y \in S$, $x \sim_\delta y$ iff there are a δ-chain $(x_i)_{i=0}^{k}$ of g with $x_0 = x$ and $x_k = y$, and a δ-chain $(y_i)_{i=0}^{l}$ of g with $y_0 = y$ and $y_l = x$. By the chain recurrence of g, we can show that $x \sim_\delta g(x)$ for every $x \in S$, and $x \sim_\delta y$ for all $x, y \in S$ with $d(x, y) < \delta$. Hence, every equivalence class C with respect to $\sim_\delta$ is clopen in S and g-invariant, i.e., $g(C) \subset C$. Then, each equivalence class is called a *δ-chain component* of S (with respect to g), and so S is decomposed into finitely many δ-chain components. Such a decomposition is called a *δ-chain decomposition* of S (with respect to g). Now, fix a δ-chain component C. Note that for any δ-cycle $c = (x_i)_{i=0}^{n}$ of g, if $x_i \in C$ for some $0 \le i \le n$, then $x_i \in C$ for all $0 \le i \le n$. In such a case, we write $c \subset C$. Set $l(c) = n$ for any δ-cycle $c = (x_i)_{i=0}^{n}$. Define

$$\mathcal{N} = \{n \in \mathbb{N} : \exists\ \delta\text{-cycle } c \text{ of } g \text{ with } c \subset C \text{ and } l(c) = n\},$$

and put

$$m = \gcd \mathcal{N} = \max\{j \in \mathbb{N} : j | n \text{ for every } n \in \mathcal{N}\}.$$

Then, we define a relation $\sim_{\delta,m}$ on C as follows. For any $x, y \in C$, $x \sim_{\delta,m} y$ iff there is a δ-chain $(x_i)_{i=0}^{k}$ of g with $x_0 = x$, $x_k = y$ and $m|k$. By the definition of m, we see that $\sim_{\delta,m}$ is an equivalence relation on C, and by the chain recurrence of g, for all $x, y \in C$ with $d(x, y) < \delta$, we have $x \sim_{\delta,m} y$. Hence, every equivalence class D with respect to $\sim_{\delta,m}$ is clopen in S. Take $p \in C$ and consider m points $p, g(p), \dots, g^{m-1}(p)$. Then, it is easy to see that $C = \bigsqcup_{i=0}^{m-1} [g^i(p)]$ is the partition of C into equivalence classes with respect to $\sim_{\delta,m}$, where $[g^i(p)]$ denotes the equivalence class containing $g^i(p)$. Put $D_i = [g^i(p)]$ for $0 \le i \le m-1$ and $D_m = D_0$. Then, we have

(D1) $C = \bigsqcup_{i=0}^{m-1} D_i$ and every D_i, $0 \le i \le m-1$, is clopen in S;
(D2) $g(D_i) \subset D_{i+1}$ for every $0 \le i \le m-1$ (Lemma 5.1);
(D3) Given $x, y \in D_i$ with $0 \le i \le m-1$, there exists $M \in \mathbb{N}$ such that for any integer $N \ge M$, there is a δ-chain $c = (x_i)_{i=0}^k$ of g in C with $x_0 = x$, $x_k = y$, and $l(c) = k = mN$.

(D3) is proved in Brian et al. (2015), Lemma 2.3. The proof is based on the fact that for every positive integers $n_1, n_2, \ldots, n_l \in \mathbb{N}$ with $\gcd\{n_1, n_2, \ldots, n_l\} = m$, there exists $L \in \mathbb{N}$ such that for every integer $N \ge L$, we have $n_1a_1 + n_2a_2 + \cdots + n_la_l = mN$ for some integers $a_1, a_2, \ldots, a_l \ge 0$. We call each D_i, $0 \le i \le m-1$, a *δ-cyclic component* of C, and $C = \bigsqcup_{i=0}^{m-1} D_i$ is called a *δ-cyclic decomposition* of C.

Proof of Lemma 5.1 It is obvious from the definition that $x \sim_{\delta,m} g^m(x)$ for every $x \in C$, and hence $g^{3m+i}(p) \in D_i$ for every $0 \le i \le m-1$. Fix $0 \le i \le m-1$ and $x \in D_i$. Since both $g^{3m+i}(p)$ and x are in D_i, there are $N \in \mathbb{N}$ and a δ-chain $(x_i)_{i=0}^{mN}$ of g such that $x_0 = g^{3m+i}(p)$ and $x_{mN} = x$. Then, the following

$$(g^{i+1}(p), g^{i+2}(p), \ldots, g^{i+3m}(p), x_1, \ldots, x_{mN-1}, x, g(x))$$

is a δ-chain of g of length $m(N+3)$, which implies $g^{i+1}(p) \sim_{\delta,m} g(x)$. Thus, we have $g(x) \in D_{i+1}$, and since $x \in D_i$ is arbitrary, $g(D_i) \subset D_{i+1}$ has been proved. □

In what follows, for $x \in S$, we denote by $C(x, \delta, g)$ the δ-chain component containing x. For the given δ-cyclic decomposition $C(x, \delta, g) = \bigsqcup_{i=0}^{m-1} D_i$ with $x \in D_0$, we define

- $D(x, \delta, g) = D_0$;
- $r(x, \delta, g) = \max\{diam\ D_i : 0 \le i \le m-1\}$; and
- $m(x, \delta, g) = m$.

Note that for any $0 < \delta_2 < \delta_1$, we have

- $C(x, \delta_2, g) \subset C(x, \delta_1, g)$;
- $D(x, \delta_2, g) \subset D(x, \delta_1, g)$;
- $r(x, \delta_2, g) \le r(x, \delta_1, g)$; and
- $m(x, \delta_1, g) | m(x, \delta_2, g)$.

Then, we present some consequences of the Bowen type decomposition. The following lemma characterizes the dynamics of a point $x \in S$ satisfying $\lim_{\delta \to 0} r(x, \delta, g) = 0$.

Lemma 5.2 *Let $g : S \to S$ be a chain recurrent continuous map. Suppose that* $\lim_{\delta \to 0} r(x, \delta, g) = 0$. *Then, we have $\overline{O_g(x)} \subset EC(g) \cap RR(g)$ and $dim\ \overline{O_g(x)} = 0$. Moreover,*

(1) *If* $\lim_{\delta \to 0} m(x, \delta, g) = \infty$, *then $(\overline{O_g(x)}, g)$ is conjugate to an odometer.*
(2) *If* $\lim_{\delta \to 0} m(x, \delta, g) < \infty$, *then $x \in Per(g)$.*

Proof It is obvious from the definition of the δ-cyclic decomposition that $\overline{O_g(x)} \subset EC(g) \cap RR(g)$ and $dim\, \overline{O_g(x)} = 0$. Note that $\overline{O_g(x)}$ is minimal and $\overline{O_g(x)} \subset RR(f)$. If $\overline{O_g(x)}$ is a finite set, then $g^n(x) = x$ for some $n \in \mathbb{N}$, and hence $m(x, \delta, g) \leq n$ for every $\delta > 0$. Therefore, if $\lim_{\delta \to 0} m(x, \delta, g) = \infty$, then $\overline{O_g(x)}$ is infinite, and thus by Blokh and Keesling (2004), Corollary 2.5, $(\overline{O_g(x)}, g)$ is conjugate to an odometer. If $\lim_{\delta \to 0} m(x, \delta, g) = n < \infty$, then it is easy to see that $g^n(x) = x$. □

The next lemma gives a quantitative relation between the Bowen type decomposition and the presence of separated pairs of cycles at a point, whose definition was given in Definition 2.1.

Lemma 5.3 *Let $g : S \to S$ be a chain recurrent continuous map. For every $x \in S$ and every $e, \delta > 0$, S contains an e-separated pair of δ-cycles of g at x iff $r(x, \delta, g) > e$.*

Proof Put $m = m(x, \delta, g)$ and let $C(x, \delta, g) = \bigsqcup_{i=0}^{m-1} D_i$ be the δ-cyclic decomposition of $C(x, \delta, g)$ with $x \in D_0$.

Assume that $r(x, \delta, g) \leq e$. Then, by the definition of $r(x, \delta, g)$, we have $diam\, D_i \leq e$ for every $0 \leq i \leq m-1$. Let $((z_j^{(0)})_{j=0}^n, (z_j^{(1)})_{j=0}^n)$ be a pair of δ-cycles of g such that $z_0^{(0)} = z_0^{(1)} = z_n^{(0)} = z_n^{(1)} = x$. Then, both $(z_j^{(0)})_{j=0}^n$ and $(z_j^{(1)})_{j=0}^n$ are contained in $C(x, \delta, g)$, and $m|n$. Moreover, for given $0 \leq j \leq n$ and $0 \leq i \leq m-1$, if $j \equiv i \pmod m$, then we have $\{z_j^{(0)}, z_j^{(1)}\} \subset D_i$. Hence, $d(z_j^{(0)}, z_j^{(1)}) \leq e$ for all $0 \leq j \leq n$, and thus $((z_j^{(0)})_{j=0}^n, (z_j^{(1)})_{j=0}^n)$ is not e-separated.

Conversely, assume that $r(x, \delta, g) > e$ and take $0 \leq i \leq m-1$ such that $diam\, D_i > e$. Choose $y_0, y_1 \in D_i$ with $d(y_0, y_1) > e$. Then, by (D3), there are $N_1 \in \mathbb{N}$ and a pair of δ-chains $((x_j^{(0)})_{j=0}^{mN_1}, (x_j^{(1)})_{j=0}^{mN_1})$ of g with $x_0^{(0)} = x_0^{(1)} = f^i(x)$ and $(x_{mN_1}^{(0)}, x_{mN_1}^{(1)}) = (y_0, y_1)$. Since $x \in D_0$ and $f^{m-i}(y_0), f^{m-i}(y_1) \in D_0$, using (D3) again, we have $N_2 \in \mathbb{N}$ and a pair of δ-chains $((y_j^{(0)})_{j=0}^{mN_2}, (y_j^{(1)})_{j=0}^{mN_2})$ of g with $(y_0^{(0)}, y_0^{(1)}) = (f^{m-i}(y_0), f^{m-i}(y_1))$ and $y_{mN_2}^{(0)} = y_{mN_2}^{(1)} = x$. Then, the pair of the following δ-cycles

$$(x, \ldots, f^{i-1}(x), x_0^{(0)}, x_1^{(0)}, \ldots, x_{mN_1-1}^{(0)}, y_0, \ldots, f^{m-i-1}(y_0), y_0^{(0)}, y_1^{(0)}, \ldots, y_{mN_2-1}^{(0)}, x),$$
$$(x, \ldots, f^{i-1}(x), x_0^{(1)}, x_1^{(1)}, \ldots, x_{mN_1-1}^{(1)}, y_1, \ldots, f^{m-i-1}(y_1), y_0^{(1)}, y_1^{(1)}, \ldots, y_{mN_2-1}^{(1)}, x)$$

is an e-separated pair of δ-cycles of g at x with period $m(N_1 + N_2 + 1)$, proving the lemma. □

By Lemmas 5.3 and 2.4, we obtain the following lemma.

Lemma 5.4 *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset such that $CR(f|_S) = S$. Given $x \in S$, suppose that the following conditions are satisfied.*

- *Every δ-pseudo orbit $(x_i)_{i=0}^\infty$ contained in S with $x_0 = x$ is b-shadowed by some point of X.*

- $r(x, \delta, f|_S) > 2b$.

Then, there exist $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset B_b(x)$ for which we have a factor map $\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma)$.

From Lemma 5.4, we obtain the following corollary, which is a quantitative localized version of Moothathu (2011), Corollary 6. For $b > 0$ and $S \subset X$, we say that a continuous map $f : X \to X$ has the *b-shadowing property around S* if there is $\delta > 0$ such that every δ-pseudo orbit of f contained in S is b-shadowed by some point of X.

Corollary 5.1 *Let $f : X \to X$ be a continuous map with the b-shadowing property around a closed f-invariant subset S. If $CR(f|_S) = S$ and $h_{top}(f) = 0$, then for every $x \in S$, there is a clopen subset D of S such that $x \in D$ and $diam\ f^n(D) \leq 2b$ for all $n \geq 0$.*

Proof Choose $\delta > 0$ such that every δ-pseudo orbit contained in S is b-shadowed by some point of X. Given $x \in S$, by Lemma 5.4, we have $r(x, \delta, f|_S) \leq 2b$. Put $D = D(x, \delta, f|_S)$. Then, D is a clopen subset of S containing x, and we have $diam\ f^n(D) \leq 2b$ for all $n \geq 0$ by (D2) and the definition of $r(x, \delta, f|_S)$. □

By Corollary 5.1, we can recover (Moothathu 2011, Corollary 6) as the continuous limit when $b \to 0$.

Corollary 5.2 (Moothathu 2011, Corollary 6) *Let $f : X \to X$ be a continuous map with the shadowing property. If $h_{top}(f) = 0$, then $dim\,\Omega(f) = 0$, and $f|_{\Omega(f)}$ is equicontinuous.*

Proof The shadowing property of f implies that $\Omega(f) = \Omega(f|_{\Omega(f)}) \subset CR(f|_{\Omega(f)}) \subset \Omega(f)$, so $CR(f|_{\Omega(f)}) = \Omega(f)$. Note that for every $b > 0$, f has the b-shadowing property around $\Omega(f)$, and hence Corollary 5.1 applies to $S = \Omega(f)$. By taking the limit as $b \to 0$, we obtain $dim\,\Omega(f) = 0$, and $f|_{\Omega(f)}$ is equicontinuous. □

The next lemma gives a quantitative relation between the Bowen type decomposition and the distribution of sensitive points under quantitative pointwise shadowability.

Lemma 5.5 *Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset such that $CR(f|_S) = S$. Given $x \in S$, suppose that every δ-pseudo orbit $(x_i)_{i=0}^{\infty}$ contained in S with $x_0 = x$ is b-shadowed by some point of X. Then, we have the following properties.*

(1) *If $r(x, \delta, f|_S) - 2b > e > 0$, then there exists $y \in Sen_e(f)$ such that $d(x, y) \leq b$.*
(2) *If $r(x, \delta, f|_S) \leq e$, then $x \notin Sen_e(f|_S)$.*

Proof We first prove (1). Since $r(x, \delta, f|_S) > e + 2b$, by Lemma 5.3, S contains an $(e + 2b)$-separated pair of δ-cycles at x. By Lemma 2.4, there exist $m \in \mathbb{N}$, a closed f^m-invariant subset $Y \subset B_b(x)$, and a factor map $\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma)$. Moreover, the construction of the factor map π in the proof of Lemma 2.4 implies that if $\pi(a) \neq \pi(b)$ for $a, b \in Y$, then $d(f^n(a), f^n(b)) > e$ for some $n \geq 0$. Now, since Y

is compact and π is surjective, there exists $y \in Y$ such that for every neighborhood U of y in Y, there is $z \in U$ with $\pi(z) \neq \pi(y)$. Then, we have $y \in Sen_e(f)$ and $d(x, y) \leq b$, proving (1). Suppose that $r(x, \delta, f|_S) \leq e$. Then, putting $D = D(x, \delta, f|_S)$, we have $diam\ f^n(D) \leq e$ for every $n \geq 0$ by (D2) and the definition of $r(x, \delta, f|_S)$. Hence, we have $x \notin Sen_e(f|_S)$, proving (2). □

6 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Let $f : X \to X$ be a continuous map and let $S \subset X$ be a closed f-invariant subset such that $CR(f|_S) = S$ and $\mathrm{Int}\, Sh^+(f) \subset S$. Then, S admits Bowen type decomposition with respect to $f|_S$.

The following lemma claims that for any $x \in \mathrm{Int}\, Sh^+(f)$, we have a dichotomy, $\lim_{\delta\to 0} r(x, \delta, f|_S) > 0$ with $x \in Sen(f)$ or $\lim_{\delta\to 0} r(x, \delta, f|_S) = 0$ with $x \in EC(f)$.

Lemma 6.1 *For any $x \in \mathrm{Int}\, Sh^+(f)$, we have the following properties.*

(1) *If $\lim_{\delta\to 0} r(x, \delta, f|_S) > 0$, then $x \in Sen(f)$.*
(2) *If $\lim_{\delta\to 0} r(x, \delta, f|_S) = 0$, then $x \in EC(f) \cap RR(f)$.*

Proof Let us suppose that $\lim_{\delta\to 0} r(x, \delta, f|_S) > e > 0$ and prove that $x \in Sen_e(f)$. Take $\epsilon > 0$ and $\delta_0 > 0$ such that $\lim_{\delta\to 0} r(x, \delta, f|_S) > e + 2\epsilon > e$ and every δ_0-pseudo orbit $(x_i)_{i=0}^{\infty}$ of f with $x_0 = x$ is ϵ-shadowed by some point of X. Then, since $r(x, \delta_0, f|_S) - 2\epsilon > e > 0$, by Lemma 5.5, there exists $y \in Sen_e(f)$ such that $d(x, y) \leq \epsilon$. Since $\epsilon > 0$ can be taken arbitrarily small, we obtain $x \in \overline{Sen_e(f)} = Sen_e(f)$. As for (2), if $\lim_{\delta\to 0} r(x, \delta, f|_S) = 0$, then from Lemma 5.2, it follows that $x \in EC(f|_S) \cap RR(f|_S)$. Since $x \in \mathrm{Int}\, Sh^+(f) \subset \mathrm{Int}\, S$, we have $x \in EC(f)$, and obviously $x \in RR(f)$. □

By Lemmas 5.2 and 6.1, we obtain the following corollary.

Corollary 6.1 *For every $x \in \mathrm{Int}\, Sh^+(f)$, if $x \in EC(f)$, then $x \in Per(f)$ or $(\overline{O_f(x)}, f)$ is conjugate to an odometer.*

Now let us prove Theorem 1.3.

Proof of Theorem 1.3 $(S1) \Rightarrow (S2)$: Let $x \in \overline{Sen(f)}$. By Proposition 1.1, for every $\epsilon > 0$, there exists $y \in X$ with $d(x, y) \leq \epsilon$ such that $y \in Per(f)$ or $(\overline{O_f(y)}, f)$ is conjugate to an odometer. On the other hand, given $\epsilon > 0$, take $y \in Sen(f) \cap \mathrm{Int}\, Sh^+(f)$ with $d(x, y) \leq \epsilon/2$. Note that $y \in \mathrm{Int}\, Sh^+(f) \subset \mathrm{Int}\, S$, and so $y \in Sen(f|_S) \cap Sh^+(f)$. By Lemma 2.5, there are $m \in \mathbb{N}$ and a closed f^m-invariant subset $Y \subset B_{\epsilon/2}(y)$ for which we have a factor map $\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma)$. Since $B_{\epsilon/2}(y) \subset B_\epsilon(x)$, $(S1) \Rightarrow (S2)$ has been proved.

$(S2) \Rightarrow (S3)$: This follows from Lemma 2.6 (2).

$(S2) \Rightarrow (S4)$: Let U be a neighborhood of x in X. Then, there are $m \in \mathbb{N}$, a closed f^m-invariant subset $Y \subset U$, and a factor map $\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma)$. We have

$$h(f, \overline{U}) \geq \frac{1}{m} h(f^m, \overline{U}) \geq \frac{1}{m} h(f^m, Y) \geq \frac{1}{m} \log 2 > 0.$$

Since U is arbitrary, we have $x \in Ent(f)$.

$(S2) \Rightarrow (S5)$: Take $s \notin RR(\sigma)$ and $y \in \pi^{-1}(s)$. Then, we have $y \notin RR(f^m) = RR(f)$. Hence, there exists $y \in B_\epsilon(x)$ with $y \notin RR(f)$ for every $\epsilon > 0$, which implies (S5).

$(S3) \Rightarrow (S1)$: This is obvious.

$(S4) \Rightarrow (S1)$: Suppose that $x \notin \overline{Sen(f)}$. Then, $x \in \operatorname{Int} EC(f)$. Take a neighborhood U of x such that $\overline{U} \subset EC(f)$. Then, we have $h(f, \overline{U}) = 0$, and hence $x \notin Ent(f)$.

$(S5) \Rightarrow (S1)$: Suppose that $x \notin \overline{Sen(f)}$. Then, $x \in \operatorname{Int} EC(f)$. Take a neighborhood U of x such that $U \subset EC(f) \cap \operatorname{Int} Sh^+(f)$. Then, using Lemma 6.1, we have $\lim_{\delta\to 0} r(y, \delta, f|_S) = 0$ for every $y \in U$, and hence $y \in RR(f)$ by Lemma 5.2. Thus, we have $x \in \operatorname{Int} RR(f)$.

$(E1) \iff (E3) \iff (E4)$ has been already proved.

$(E1) \Rightarrow (E2)$: Take a neighborhood U of x such that $U \subset EC(f) \cap \operatorname{Int} Sh^+(f)$. Then, using Lemma 6.1, we have $\lim_{\delta\to 0} r(y, \delta, f|_S) = 0$ for every $y \in U$, and hence by Lemma 5.2, $y \in Per(f)$ or $(\overline{O_f(y)}, f)$ is conjugate to an odometer.

$(E2) \Rightarrow (E4)$: This is obvious.

The last claim has been already proved as Corollary 6.1. □

Finally, we give an example in which Theorem 1.3 holds.

Example 6.1 Let $C \subset [0, 1]$ be the Cantor ternary set. Take a homeomorphism $g : C \to C$ which is conjugate to the full shift $\sigma : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$. Then, g has the shadowing property. Set $x_n = 1/n$, $C_n = \{y/n : y \in C\}$ for each $n \in \mathbb{N}$, and let

$$X = \{(0, 0)\} \cup \bigcup_{n\in\mathbb{N}} \{x_n\} \times C_n \subset \mathbb{R}^2.$$

Then, X is a compact subset of $\mathbb{R}^2$. Define a homeomorphism $f : X \to X$ by $f((0, 0)) = (0, 0)$, and $f(x_n, y/n) = (x_n, g(y)/n)$ for $y \in C_n$, $n \in \mathbb{N}$. Then, it is easy to see that $Sh^+(f) = X$, and so f has the shadowing property. It is also obvious that f is non-wandering. Note that $(0, 0) \in EC(f)$, but $(0, 0) \notin \operatorname{Int} EC(f)$. Then, we see that (S1)–(S5) are satisfied for $x = (0, 0)$.

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