BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 47(2), 799-809

ISSN: 1678-7544 (Print) / 1678-7714 (Online)

On Cauchy problems of thermal non-equilibrium flows with small data*

Yanni Zeng

Abstract. We study the equations describing the motion of a thermal non-equilibrium gas with one non-equilibrium mode. In three space dimensions it is a hyperbolic system of six equations with a relaxation term. The dissipation mechanism induced by the relaxation is weak in the sense that Shizuta-Kawashima criterion is violated. However, there is a significant difference between one dimensional and three dimensional flows in how the criterion is violated. As a consequence, the velocity components in their solutions behave differently while thermal dynamic variables share common properties.

Keywords: hyperbolic system, relaxation, partially decaying solution, Green's function, thermal non-equilibrium flow, multi space dimensions.

Mathematical subject classification: 35L65, 35Q35, 35L45, 35C99.

1 Introduction

In this paper we consider initial value problems of the dynamics of real gasses. A real gas has at least one non-equilibrium molecular process, such as vibration, rotation, chemical composition, etc. We focus our discussion on thermal non-equilibrium flows. Thus the motion of a non-equilibrium mode is a relaxation towards its local equilibrium value in a time scale called relaxation time. To simplify our notation we further assume that the gas has only one non-equilibrium mode. At least in the case of one space dimension, however, the results on global existence and large time behavior to be discussed below apply to flows with several non-equilibrium modes as well. The purpose of this paper is to compare multi-dimensional flows with one-dimensional flows, and discuss the difference

Received 2 March 2015.

*This work was partially supported by a grant from the Simons Foundation (#244905 to Yanni Zeng).

in the structure of the partial differential equations describing these flows, and as a consequence, the difference in the solution behavior.

Without loss of generality we assume that the non-equilibrium mode is the vibrational mode. Then in three space dimensions, the motion of the gas is described by the following equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u u^t) + \nabla p = 0, \\ (\rho \mathscr{E})_t + \operatorname{div}(\rho \mathscr{E} u + p u) = 0, \\ (\rho q)_t + \operatorname{div}(\rho q u) = \rho \dfrac{Q - q}{\tau}, \end{cases} \tag{1.1}$$

where ρ, $u = (u_1, u_2, u_3)^t$, p, $\mathscr{E}$, q, Q and τ are the gas density, velocity, pressure, specific total energy, specific vibrational energy, local equilibrium value of specific vibrational energy, and local relaxation time, respectively. All these are functions of the space variable $x \in \mathbb{R}^3$ and time variable $t \in \mathbb{R}^+$. We note that the first three equations are the conservation of mass, momentum and energy, while the last one is the relaxation equation for the vibrational energy. The total energy $\mathscr{E}$ consists of internal energy and kinetic energy:

$$\mathscr{E} = e + \frac{1}{2}|u|^2, \qquad |u|^2 = \sum_{j=1}^{3} u_j^2, \tag{1.2}$$

and the internal energy e is further divided into the equilibrium energy e_1 and the vibrational energy q:

$$e = e_1 + q. \tag{1.3}$$

We use subscript "1" to denote thermodynamic variables related to equilibrium modes, and subscript "2" to denote those related to the non-equilibrium mode. Thus s_1 and T_1 are equilibrium entropy and temperature, respectively, while s_2 and T_2 are non-equilibrium entropy and temperature. Under these notations the thermodynamic equations read:

$$de_1 = T_1 ds_1 - p dv, \qquad v = 1/\rho, \qquad dq = T_2 ds_2. \tag{1.4}$$

From (1.4) we see that for the equilibrium modes only two thermodynamic variables are independent, and others can be regarded as known functions of them. In particular, the local equilibrium value of vibrational energy and the local relaxation time are given functions of, say, v and e_1:

$$Q = Q(v, e_1), \qquad \tau = \tau(v, e_1). \tag{1.5}$$

Similarly, for the non-equilibrium mode only one of the variables is independent. Therefore, we have six unknowns: three thermodynamic variables and three components of the velocity. Equation (1.1) is a system of six equations for these unknowns.

To state our basic assumptions we introduce the following notations:

$$p = p(v, e_1) = \breve{p}(v, s_1) = \tilde{p}(v, T_1), \quad T_1 = T_1(v, e_1), \quad q = \omega(T_2), \tag{1.6}$$

for some known function ω. Recall that Q is the local equilibrium value of q, and a state is an equilibrium state if and only if $T_2 = T_1$. Therefore, (1.6) implies that

$$Q = Q(v, e_1) = \omega(T_1). \tag{1.7}$$

The basic physical assumptions for this paper are:

$$\begin{gathered} \tilde{p}_v = \frac{\partial}{\partial v}\tilde{p}(v, T_1) < 0, \quad T_{1e_1} = \frac{\partial}{\partial e_1}T_1(v, e_1) > 0, \\ \omega'(T_1) > 0, \quad p_{e_1} = \frac{\partial}{\partial e_1}p(v, e_1) \neq 0. \end{gathered} \tag{1.8}$$

By direct calculation one can verify that assumption (1.8) implies

$$\begin{gathered} c_f^2 \equiv -\breve{p}_v/\rho^2 = (pp_{e_1} - p_v)v^2 = [-\tilde{p}_v + T_1(\tilde{p}_{T_1})^2 T_{1e_1}]v^2 > 0, \\ c^2 \equiv \Big[\frac{p_{e_1}(p + Q_v)}{1 + Q_{e_1}} - p_v\Big]v^2 > 0. \end{gathered} \tag{1.9}$$

Here c_f is the frozen speed of sound, i.e., the sound speed of the frozen flow, which is the limit of the non-equilibrium flow as $\tau \to \infty$. On the other hand, c is the equilibrium speed of sound, the sound speed of the equilibrium flow, which is the limit of the non-equilibrium flow as $\tau \to 0$.

If we consider the special case of plane wave solutions to (1.1) we have one-dimensional flows. The equations may be further simplified by using Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \mathscr{E}_t + (pu)_x = 0, \\ q_t = \dfrac{Q - q}{\tau}. \end{cases} \tag{1.10}$$

Noting u is a scalar, (1.10) is a system of four equations for four unknowns.

2 Structural conditions for well-posedness and dissipation

Consider a general system of hyperbolic balance laws

$$w_t + \sum_{i=1}^{m} f_i(w)_{x_i} = r(w), \quad m \geq 1, \tag{2.1}$$

where $w \in \mathbb{R}^n$ is the unknown density function (mass density, momentum density, etc), $f_i \in \mathbb{R}^n$, $1 \leq i \leq m$, are flux functions, and $r \in \mathbb{R}^n$ represents external force, relaxation, chemical reactions and so on. We assume f_i and r are smooth functions of w, which depends on the space variable $x = (x_1, \ldots, x_m)^t \in \mathbb{R}^m$ and time variable $t \in \mathbb{R}^+$. A constant state $\bar{w}$ is an equilibrium state if $r(\bar{w}) = 0$. We consider (2.1) in a small neighborhood O of $\bar{w}$. We also define the equilibrium manifold as

$$E = \{w \in O | r(w) = 0\}.$$

Physical examples usually come with a certain number of conservation laws. Thus we assume (2.1) can be written as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_t + \sum_{i=1}^{m} \begin{pmatrix} f_{i1} \\ f_{i2} \end{pmatrix} (w)_{x_i} = \begin{pmatrix} 0 \\ r_2 \end{pmatrix} (w), \tag{2.2}$$

where $w_1,\ f_{i1} \in \mathbb{R}^{n_1}$, $w_2,\ f_{i2},\ r_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, and $n_1, n_2 > 0$. Here $n_1 > 0$ is demanded by physics as in (1.1) and (1.10), while $n_2 > 0$ is to set (2.1) apart from hyperbolic conservation laws, which have different solution behavior.

For (2.2) we impose the following assumptions:

(i) The matrix $(r_2)_{w_2} \in \mathbb{R}^{n_2 \times n_2}$ is non-singular.

(ii) There exists a strictly convex entropy U such that U'' symmetries (2.1) (hence (2.2)) in the following sense: $U'' f_i'$,$1 \leq i \leq m$, are symmetric in O; and $U'' r'$ is symmetric, semi-negative definite on E. Here U'' is the Hessian of U with respect to w, and f_i' is the Jacobian matrix of f_i with respect to w, etc.

Clearly, (1.1) and (1.10) are examples of (2.2). Under the physical assumptions (1.8), by direct calculation we can verify that they both satisfy Assumptions (i) and (ii), see [12, 13, 14].

The first part of Assumption (ii), i.e., the positive definite U'' symmetrizing f_i' in O, is in fact a condition for well-posedness. It is a classical result that it implies local existence for Cauchy problems with smooth and small data, see

[3, 6], etc. Such a condition, however, does not imply the decay of solution in time, hence may not be sufficient for global existence. There has been an extensive literature studying (2.1) or (2.2) under a variety of assumptions, [9, 8, 5, 2, 4, 11, 1, 7] and references therein. The key ingredients of those assumptions are a well-posedness condition and a dissipation criterion in some form. The dissipation criterion gives the full decay of solution in conjunction with the well-posedness condition. Therefore, energy estimates can be obtained, global existence can be established, and large time behavior can be studied, for one space dimension and multi-space dimensions.

A typical version of the dissipation criterion is through the strong coupling of the flux functions and the inhomogeneous term, known as the Shizuta-Kawashima condition, originally introduced for hyperbolic-parabolic systems, [10]. For (2.1) (or (2.2)) it reads:

(iii) The null space of $r'(\bar{w})$ contains no eigenvectors of $\sum_{i=1}^{m} \nu_i f_i'(\bar{w})$ for all unit vectors $\nu = (\nu_1, \ldots, \nu_m)^t \in \mathbb{R}^m$.

For the one dimensional thermal non-equilibrium flow, it has been shown in [12] that (1.10) violates Assumption (iii). Similarly, we can show that the three dimensional flow (1.1) also violates the assumption. The direct consequence is that if the initial data is a small perturbation of the constant equilibrium state $\bar{w}$, only a portion of the solution decays to $\bar{w}$. This can be seen by the following equilibrium solution to (1.10):

$$v = v(x), \quad u = 0, \quad p = \bar{p}, \quad e_1 = e_1(v, p) = e_1(x), \quad q = Q(v, p) = Q(x),$$

where $\bar{p} > 0$ is a constant. We see that this is a steady state solution, hence any initial perturbation of $v(x)$ to $\bar{v} > 0$ stays all the time. In fact, for a non-equilibrium solution, the entropy grows in time along the particle path: Let $s = s_1 + s_2$ be the total entropy. Equation (1.10) and assumptions (1.8) imply

$$s_t = \left(\frac{1}{T_2} - \frac{1}{T_1}\right) \frac{Q - q}{\tau} > 0, \tag{2.3}$$

see [12]. For three dimensional flows we have a similar equation for (1.1) as well, [14]. This is consistent with physics as the motion of a non-equilibrium flow is irreversible. The growth in entropy, however, is weak as it is easy to see from (2.3) that the linearized entropy equation is $s_t = 0$.

Although both one dimensional flow and three dimensional flow do not satisfy the Shizuta-Kawashima condition, there is a significant difference in their structure. To see this we use non-conserved variables $p, u, \chi = Q - q$ and s as unknowns. Equation (1.10) is equivalent to

$$\begin{cases} p_t + c_f^2 u_x = -p_{e_1}\chi/\tau, \\ u_t + p_x = 0, \\ \chi_t + a u_x = -(1 + Q_{e_1})\chi/\tau, \\ s_t = (1/T_2 - 1/T_1)\chi/\tau, \end{cases} \tag{2.4}$$

where $a = a(v, e_1)$ is a known function of equilibrium thermal dynamic variables.

Next we linearize (2.4) around a constant equilibrium state $(\bar{p}, 0, 0, \bar{s})^t$. Here we note that $\bar{\chi} = 0$ and without loss of generality we have set $\bar{u} = 0$. Using the bar accent to denote thermal dynamic variables taken at the constant equilibrium state, the linearized equation of (2.4) reads

$$\begin{cases} p_t + \bar{c}_f^2 u_x = -\bar{p}_{e_1}\chi/\bar{\tau}, \\ u_t + p_x = 0, \\ \chi_t + \bar{a} u_x = -(1 + \bar{Q}_{e_1})\chi/\bar{\tau}, \end{cases} \tag{2.5}$$

$$s_t = 0. \tag{2.6}$$

We observe that the linear entropy equation (2.6) is completely decoupled for the others. Although the 4×4 system (2.5), (2.6) violates assumption (iii), after the decoupling of (2.6), the 3×3 system (2.5) now satisfies the assumption, [12]. The significance of this fact is two-fold. Firstly, the non-decaying part of the solution is represented by the entropy, while the decaying part represented by the pressure, velocity and the departure of the vibrational energy from its local equilibrium value. Secondly, these two parts are weakly coupled (in the nonlinear level but not in the linear level). It is because of such a weak coupling, we are able to separate waves according to their decay rates. Consequently, we are able to obtain an energy estimate to establish the global existence of solution if the Cauchy datum is a small perturbation of the constant equilibrium state. We are even able to study the large time behavior of the solution in a space-time pointwise sense. These have been done in [12]. Parallel results are also obtained for one dimensional flows with a finite number of non-equilibrium modes, [13].

The structure of the three dimensional flow (1.1), however, is very different. To see this we use the same unknowns as in (2.4) and consider the following equations equivalent to (1.1):

$$\begin{cases} p_t + u^t \nabla p + \rho c_f^2 \operatorname{div} u = -p_{e_1} \chi/\tau, \\ u_{it} + u^t \nabla u_i + p_{x_i}/\rho = 0, \quad 1 \le i \le 3, \\ \chi_t + u^t \nabla \chi + a \operatorname{div} u = -(1 + Q_{e_1})\chi/\tau, \\ s_t + u^t \nabla S = (1/T_2 - 1/T_1)\chi/t. \end{cases} \tag{2.7}$$

Now we linearize (2.7) around a constant equilibrium state $(\bar{p}, 0, 0, 0, 0, \bar{s})^t$, where without loss of generality we have set $\bar{u} = (0, 0, 0)$. This gives us

$$\begin{cases} p_t + \bar{\rho}\bar{c}_f^2 \operatorname{div} u = -\bar{p}_{e_1} \chi/\bar{\tau}, \\ u_t + \nabla p/\bar{\rho} = 0, \\ \chi_t + \bar{a} \operatorname{div} u = -(1 + \bar{Q}_{e_1})\chi/\bar{\tau}, \end{cases} \tag{2.8}$$

$$s_t = 0. \tag{2.9}$$

As in the case of one dimensional flow, the linear entropy equation (2.9) is completely decoupled from the others. The 5×5 system (2.8), however, still violates assumption (iii). To see this we write (2.8) as

$$\tilde{w}_t + A_1 \tilde{w}_{x_1} + A_2 \tilde{w}_{x_2} + A_3 \tilde{w}_{x_3} = B\tilde{w}, \tag{2.10}$$

where $\tilde{w} = (p, u, \chi)^t$, and A_i, $1 \le i \le 3$, and B are the constant coefficient matrices. Assumption (iii) now reads: The null space of B contains no eigenvectors of $\sum_{i=1}^3 \nu_i A_i$ for all unit vectors $\nu = (\nu_1, \nu_2, \nu_3)^t \in \mathbb{R}^3$. From (2.8) it is clear that

$$\sum_{i=1}^{3} \nu_i A_i = \begin{pmatrix} 0 & \nu_1 \bar{\rho}\bar{c}_f^2 & \nu_2 \bar{\rho}\bar{c}_f^2 & \nu_3 \bar{\rho}\bar{c}_f^2 & 0 \\ \nu_1/\bar{\rho} & 0 & 0 & 0 & 0 \\ \nu_2/\bar{\rho} & 0 & 0 & 0 & 0 \\ \nu_3/\bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \nu_1 \bar{a} & \nu_2 \bar{a} & \nu_3 \bar{a} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & -\bar{p}_{e_1}/\bar{\tau} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(1 + \bar{Q}_{e_1})/\bar{\tau} \end{pmatrix}. \tag{2.11}$$

Equation (2.11) implies that the null space of B is $\left\{ \begin{pmatrix} \eta \\ 0 \end{pmatrix} \middle| \eta \in \mathbb{R}^4 \right\}$, which contains the following eigenvectors of $\sum_{i=1}^{3} \nu_i A_i$:

$$(0, \nu_2, -\nu_1, 0, 0)^t, \quad (0, \nu_3, 0, -\nu_1, 0)^t, \quad \nu_1 \neq 0 \text{ or } \nu_2, \nu_3 \neq 0, \tag{2.12}$$

where $\nu = (\nu_1, \nu_2, \nu_3)^t$ is an unit vector in $\mathbb{R}^3$. That is, the null space of B contains at least one, and in many cases, two eigenvectors of $\sum_{i=1}^{3} \nu_i A_i$.

The above discussion indicates that although the non-decaying entropy wave is weakly coupled in the solution of (1.1) (coupled through nonlinear terms), there is another non-decaying part strongly coupled in the solution (coupled by linear terms). As to be revealed in next section, this part is the rotation in the velocity. At this moment, the following special solution of (1.1) is relevant:

$$\begin{aligned} &\rho = \text{positive constant}, \quad u_1 = \cos\big(\theta(x_3)\big), \quad u_2 = \sin\big(\theta(x_3)\big), \quad u_3 = 0, \\ &\quad e_1 = \text{positive constant}, \quad q = Q(1/\rho, e_1) = \text{positive constant}. \end{aligned} \tag{2.13}$$

3 Wave Pattern of Three Dimensional Flows

To understand the wave pattern of the three dimensional flow we consider the Green's function for the Cauchy problem of the linearization (2.8), (2.9) of (1.1). This is the solution matrix $\tilde{G}(x,t)$ that satisfies the initial condition $\tilde{G}(x,0) = \delta(x) I_{6\times 6}$, where $\delta(x)$ is the Dirac δ-function. Clearly, $\tilde{G}(x,t) = \text{diag}(G(x,t), \delta(x))$, where $G(x,t)$ is the Green's function of the Cauchy problem for (2.8). Therefore, we focus on G, i.e., on the pressure and velocity, expecting that χ is a higher order term. Here we announce a recent result about G:

Theorem 3.1. *Let assumption* (1.8) *be true and* $G(x,t)$ *be the Green's function of the Cauchy problem of* (2.8)*. Let* $M > \bar{c}_f$ *and* $0 < c_1, c_2 < \bar{c}$ *be constants, where* $\bar{c}_f$ *and* $\bar{c}$ *are the frozen sound speed and equilibrium sound speed, respectively, taken at the constant equilibrium state, see* (1.9)*. Then there is a constant* $C > 0$ *such that for* $x \in \mathbb{R}^3$*,* $t \in \mathbb{R}^+$ *we have the following estimates: For* $|x| \leq M(t+1)$*,*

$$\begin{aligned} G(x,t) \; = \; & \text{char}\{t \geq 1\} G^*(x,t) + O(1)(t+1)^{-\frac{5}{2}} e^{-\frac{(|x|-\bar{c}t)^2}{C(t+1)}} \\ & + \text{char}\{|x| \leq \bar{c}t\} \,\text{diag}(0, O(1)(t+1)^{-3}, 0) \\ & - \text{char}\{|x| \leq c_1(t+1)\} \,\text{diag}(0, \triangle^{-1}\nabla\nabla^t \delta(x), 0) \\ & + \delta(x) \,\text{diag}(0, I_{3\times 3}, 0) \\ & + \{\textit{exponentially decaying distributions along} \\ & \qquad \textit{the particle path and the frozen sound cone}\}, \end{aligned} \tag{3.1}$$

and for $|x| > M(t+1)$,

$$\begin{aligned} G(x,t) \;&= O(1)e^{-(|x|+t)/C} + \delta(x)\,\mathrm{diag}(0, I_{3\times 3}, 0) \\ &\quad + \{\textit{exponentially decaying distributions along} \\ &\qquad \textit{the particle path and the frozen sound cone}\}. \end{aligned} \tag{3.2}$$

Here $\triangle^{-1}$ *is the inverse Laplace operator,* $\mathrm{char}\{\mathcal{D}\}$ *is the characteristic function of a set* $\mathcal{D}$*, and* G^* *is defined as*

$$\begin{aligned} G^*(x,t) \;&= \mathrm{char}\{|x| \geq c_2(t+1)\}\frac{1}{2(2\pi\bar{\alpha}t)^{3/2}}\frac{|x|-\bar{c}t}{t} \\ &\quad \times e^{-\frac{(|x|-\bar{c}t)^2}{2\bar{\alpha}t}} \begin{pmatrix} \frac{1}{\bar{c}} & \frac{\bar{c}\bar{\rho}t}{|x|^2}x^t & -\frac{\bar{p}_{e_1}}{\bar{c}\bar{\zeta}} \\ \frac{t}{\bar{c}\bar{\rho}|x|^2}x & \frac{t}{|x|^3}xx^t & -\frac{\bar{p}_{e_1}t}{\bar{c}\bar{\rho}\bar{\zeta}|x|^2}x \\ 0 & 0_{1\times 3} & 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

On the right-hand side of (3.3)*, we recall that the bar accent is used for thermodynamic variables taken at the constant equilibrium state, and we define*

$$\zeta \equiv 1 + Q_{e_1} > 0, \quad \alpha \equiv \frac{\tau\omega'(T_1)T_1 p_{e_1}^2}{\rho^2\zeta^2} > 0.$$

The proof of Theorem 3.1 and the explicit formulation of the exponentially decaying distributions in (3.1) and (3.2) are given in an upcoming paper [14]. Here we give a brief discussion on (3.1). First, G^* is the leading term of the decaying part. From (3.3) it is a higher order term of the heat kernel along the equilibrium sound cone by a factor $(|x| - \bar{c}t)/t$, which is equivalent to $t^{-1/2}$. The second term is a higher order term of G^* (with faster decay in t). The third term is an algebraic decay restricted to the inside of the equilibrium sound cone. It affects u only since the middle block of the Green's matrix is for the u components. The fourth and fifth terms are the non-decaying part, affecting u only as well. It is interesting to observe that in the one dimensional case, the double Riesz transform becomes one, and these two terms cancel each other. This explains that in one space dimension there is no non-decaying part in the velocity. The last term is for exponentially decaying distributions, along the particle path and the frozen sound cone.

What we have discussed can be illustrated by Figure 1. The cone in dash lines is the frozen sound cone, along which exponentially decaying distributions propagate. The shaded region is where G^* and its higher order terms are. This region expends in the rate of $t^{1/2}$ because it is diffusion. Inside the cone it is filled for u with a decay rate t^{-3} while it is empty (exponential decay) for p.

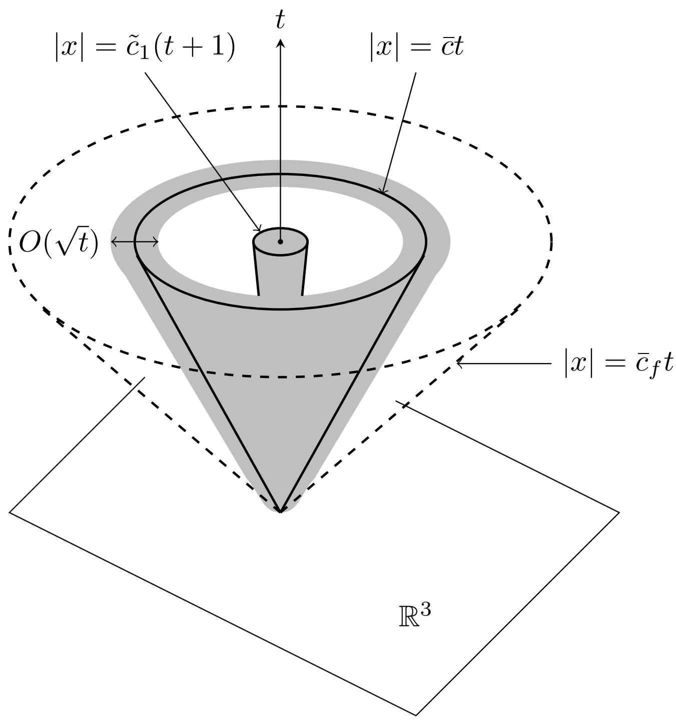

Figure 1: Wave pattern in Green's function $G(x, t)$ in three space dimensions.

Near the center, which is the particle path, we have non-decaying distributions for u. In the center there are exponentially decaying distributions for the whole solution.

References

[1] S. Bianchini, B. Hanouzet and R. Natalini. *Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*. Comm. Pure Appl. Math., **60**(11) (2007), 1559–1622.

[2] G.Q. Chen, C.D. Levermore and T.-P. Liu. *Hyperbolic conservation laws with stiff relaxation terms and entropy*. Comm. Pure. Appl. Math., **47**(6) (1994), 787–830.

[3] K.O. Friedrichs. *Symmetric hyperbolic linear differential equations*. Comm. Pure. Appl. Math., **7** (1954), 345–392.

[4] B. Hanouzet and R. Natalini. *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*. Arch. Ration. Mech. Anal., **169**(2) (2003), 89–117.

[5] L. Hsiao and T.-P. Liu. *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping*. Comm. Math. Phys., **143**(3) (1992), 599–605.

[6] T. Kato. *The Cauchy problem for quasi-linear symmetric hyperbolic systems*. Arch. Ration. Mech. Anal., **58** (1975), 181–205.

[7] S. Kawashima and W.-A. Yong. *Decay estimates for hyperbolic balance laws*. Z. Anal. Anwend., **28**(1) (2009), 1–33.

[8] T.-P. Liu. *Hyperbolic conservation laws with relaxation*. Comm. Math. Phys., **108**(1) (1987), 153–175.

[9] T. Nishida. *Nonlinear hyperbolic equations and related topics in fluid dynamics*. Publications Mathématiques D'Orsay, Département de Mathématique, Université de Paris-Sud, Orsay, 78–02 (1978).

[10] Y. Shizuta and S. Kawashima. *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*. Hokkaido Math. J., **14** (1985), 249–275.

[11] W.-A. Yong. *Entropy and global existence for hyperbolic balance laws*. Arch. Ration. Mech. Anal., **172**(2) (2004), 247–266.

[12] Y. Zeng. *Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation*. Arch. Rational Mech. Anal., **150**(3) (1999), 225–279.

[13] Y. Zeng. *Gas flows with several thermal nonequilibrium modes*. Arch. Rational Mech. Anal., **196** (2010), 191–225.

[14] Y. Zeng. *Thermal non-equilibrium flows in three space dimensions*. Arch. Rational Mech. Anal. (DOI) 10.1007/s00205-015-0892-8.

Yanni Zeng
Department of Mathematics
University of Alabama at Birmingham
USA

E-mail: zeng@math.uab.edu