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A quadratic interaction estimate for conservation laws: motivations, techniques and open problems*

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Abstract. In a series of joint works with S. Bianchini [3, 4, 5], we proved a quadratic interaction estimate for general systems of conservation laws. Aim of this paper is to present the results obtained in the three cited articles [3, 4, 5], discussing how they are related with the general theory of hyperbolic conservation laws. To this purpose, first we explain why this quadratic estimate is interesting, then we give a brief overview of the techniques we used to prove it and finally we present some related open problems.

Keywords: conservation laws, Glimm potentials, interaction estimates, Glimm scheme.

Mathematical subject classification: 35L65.

1 Introduction

A system of conservation laws in one space dimension (see [8]) is a system of PDEs of the form

$$u_t + f(u)_x = 0, \tag{1.1}$$

where $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}^n$ is the unknown and $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth (C^3) map, called *flux*, defined on a neighborhood Ω of the origin and satisfying the strict hyperbolicity condition, i.e. the Jacobian $Df(u)$ of f has n distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_n(u)$$

in each point $u \in \Omega$ of its domain. Throughout this paper, we will assume without loss of generality (w.l.o.g.) that

$$\lambda_k(u) \in [0, 1] \qquad \text{for any } k \text{ and for any } u. \tag{1.2}$$

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This can always be achieved by a change of variable in the (t, x)-plane. As it is customary, denote by $r_1(u), \dots, r_n(u)$ the right eigenvalues (normalized to 1) associated to $\lambda_1(u), \dots, \lambda_n(u)$ respectively:

$$Df(u)r_k(u) = \lambda_k(u)r_k(u), \qquad \text{for any } k = 1, \dots, n \text{ and for any } u \in \Omega.$$

Equation (1.1) is usually coupled with an initial datum

$$u(t = 0) = \bar{u}, \tag{1.3}$$

where $\bar{u} : \mathbb{R} \to \mathbb{R}^n$ is a given map, with sufficiently small total variation. W.l.o.g. we assume also that $\bar{u}$ has compact support.

It is well known that classical (smooth) solutions to the Cauchy problem (1.1), (1.3) are in general not defined on the whole time interval $[0, \infty)$, even if the initial datum is smooth, because they develop discontinuities in finite time. On the other side, the notion of distributional solution is too weak to guarantee the uniqueness. For this reasons the notion of solution which is typically used is the following one.

Definition 1. *A map $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}^n$ belonging to L^1_{loc} is said to be* a weak solution *of the Cauchy problem* (1.1)*,* (1.3) *if:*

(1) *u satisfies equation* (1.1) *in the sense of distributions;*

(2) *u is continuous as a map $[0, \infty) \to L^1_{loc}(\mathbb{R}; \mathbb{R}^n)$;*

(3) *at time $t = 0$, $u(0, x) = \bar{u}(x)$;*

(4) *u satisfies some additional admissibility criteria, which come from physical or stability considerations and guarantee the uniqueness of the solution.*

Many admissibility criteria have been proposed in the literature: just to name a few, the Lax-Liu condition on shocks (see [14, 16, 17]), the entropy condition (see [15]), the vanishing viscosity criterion (see [2]). We do not want to enter into details: the interested reader can refer to the cited literature.

The existence of solutions to the Cauchy problem (1.1), (1.3) can be proved through several approximation algorithms, for example the wavefront tracking algorithm and the Glimm scheme. One of the main problems in the construction of the approximate solutions and in the proof of the convergence of the approximate solutions to the exact one is to find suitable bounds on some quantities, usually called *amounts of interaction*, which, roughly speaking, measure how strong the interaction between two colliding wavefronts is. In particular, new results were recently obtained by S. Bianchini and the author [3, 4, 5], about a

quadratic interaction estimate (see Theorem 1 below) on the change in speed of the waves in a Glimm approximate solution. This quadratic estimate can be used to prove sharp results on the convergence rate of the Glimm scheme and could have applications in regularity and stability analysis.

Aim of this paper is to present the results obtained in the three cited articles [3, 4, 5], discussing how they are related with the general theory of hyperbolic conservation laws. To this purpose, first we explain why this quadratic estimate is interesting, then we give a brief overview of the techniques we used to prove it and finally we present some related open problems. In particular, after introducing some notations about the Riemann problem in Section 2, in Section 3 we describe the main results on the convergence of the Glimm scheme in the genuinely non linear/linearly degenerate case. In Section 4 we see how these results can be extended to the case in which no assumption on f is made except its strict hyperbolicity. In Section 5 we summarize what is done in the three papers [3, 4, 5], stating our main result, namely Theorem 1, and presenting the main techniques used in the proof. Finally Section 6 concludes the paper with some open problems for which the new tools introduced in [3, 4, 5] could be of help.

2 The Riemann problem

The basic ingredient to solve the Cauchy problem (1.1), (1.3) is the solution of the Riemann problem, i.e. the Cauchy problem when the initial datum has the simple form

$$u(0,x) = \bar{u}(x) = \begin{cases} u^L & \text{if } x < 0, \\ u^R & \text{if } x \geq 0. \end{cases} \tag{2.1}$$

The solution of the Riemann problem (1.1)-(2.1) was obtained first by P. Lax in 1957 [14], under the assumption that each characteristic field is either *genuinely non linear* (GNL), i.e. $\nabla\lambda_k(u)\cdot r_k(u) \neq 0$ for any u or *linearly degenerate* (LD), i.e. $\nabla\lambda_k(u)\cdot r_k(u) = 0$ for any u. In this case, if $|u^R - u^L| \ll 1$, using Implicit Function Theorem, one can find intermediate states $u^L = \omega_0, \omega_1, \ldots, \omega_n = u^R$ such that each pair of adjacent states (ω_{k-1}, ω_k) can be connected by either a shock or a rarefaction wave or a contact discontinuity of the k-th family. The complete solution is now obtained by piecing together the solutions of the n Riemann problems (ω_{k-1}, ω_k) on different sectors of the (t, x)-plane.

In the general case (here and in the rest of the paper, by *general case* we mean that no assumption on f is made besides strict hyperbolicity) the solution to the Riemann problem (u^L, u^R) was obtained by S. Bianchini and A. Bressan in [2]. They first construct, for any left state u^L and for any family $k = 1, \ldots, n$, a curve $s \to T_s^k u^L$ of *admissible right states*, defined for $s \in \mathbb{R}$ small enough,

such that the Riemann problem $(u^L, T_s^k u^L)$ can be solved by (countable many) admissible shocks (in the sense of limit of viscosity approximations), contact discontinuities and rarefactions waves. Then, as in the GNL/LD case, the global solution of (u^L, u^R) is obtained by piecing together the solutions of n Riemann problems, one for each family: $u^R = T_{s_n}^n \circ \cdots \circ T_{s_1}^1 u^L$.

3 Glimm approximate solutions in the GNL/LD case

The first result about existence of solutions to the Cauchy problem (1.1), (1.3) can be found in the celebrated paper by J. Glimm [13] in 1965, in which the existence of solutions is proved under the assumption that each characteristic field is either GNL or LD. In [13], for any $\varepsilon > 0$ an approximate solution $u^\varepsilon(t, x)$ is constructed by recursion as follows. First of all, take any sampling sequence $\{\vartheta_i\}_{i\in\mathbb{N}} \subseteq [0, 1]$. The algorithm starts choosing, at time $t = 0$, an approximation $\bar{u}^\varepsilon$ of the initial datum $\bar{u}$, such that $\bar{u}^\varepsilon$ is compactly supported, right continuous, piecewise constant with jumps located at point $x = m\varepsilon, m \in \mathbb{Z}$. We can thus separately solve the Riemann problems located at $(t, x) = (0, m\varepsilon)$, $m \in \mathbb{Z}$. Thanks to (1.2), the solution $u^\varepsilon(t, x)$ can now be prolonged up to time $t = \varepsilon$. At $t = \varepsilon$ a restarting procedure is used. The value of u^ε at time ε is redefined as

$$u^\varepsilon(\varepsilon+, x) := u^\varepsilon(\varepsilon-, m\varepsilon + \vartheta_1\varepsilon), \qquad \text{if } x \in [m\varepsilon, (m + 1)\varepsilon). \tag{3.1}$$

The solution $u(\varepsilon, \cdot)$ is now again piecewise constant, with discontinuities on points of the form $x = m\varepsilon, m \in \mathbb{Z}$. If the sizes of the jumps are sufficiently small, we can again solve the Riemann problem at each point $(t, x) = (\varepsilon, m\varepsilon)$, $m \in \mathbb{Z}$ and thus prolong the solution up to time 2ε, where again the restarting procedure (3.1) is used, with ϑ_2 instead of ϑ_1. The above procedure can be repeated on any time interval $[i\varepsilon, (i + 1)\varepsilon]$, $i \in \mathbb{N}$, as far as the size of the jump at each point $(i\varepsilon, m\varepsilon)$, $i \in \mathbb{N}$, $m \in \mathbb{Z}$, remains small enough, or, in other words, as far as

$$\text{Tot.Var.}(u^\varepsilon(t); \mathbb{R}) \ll 1. \tag{3.2}$$

In order to prove (3.2), Glimm introduces a uniformly bounded decreasing functional $t \mapsto Q^{\text{Glimm}}(t) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u})^2$, such that at any time $i\varepsilon$, $i \in \mathbb{N}$,

$$\begin{aligned} &\text{Tot.Var.}(u^\varepsilon(i\varepsilon+); \mathbb{R}) - \text{Tot.Var.}(u(i\varepsilon-); \mathbb{R}) \\ &\quad \leq \mathcal{O}(1)\big(Q^{\text{Glimm}}(i\varepsilon-) - Q^{\text{Glimm}}(i\varepsilon+)\big). \end{aligned} \tag{3.3}$$

Here and in the following $\mathcal{O}(1)$ denotes a constant which depends only on the flux f. As an immediate consequence, we get

$$\text{Tot.Var.}(u^\varepsilon(t); \mathbb{R}) \leq \mathcal{O}(1)\text{Tot.Var.}(u^\varepsilon(0); \mathbb{R}) \ll 1$$

and thus the solution $u^\varepsilon(t,x)$ can be defined on the whole (t,x)-plane $[0,\infty)\times\mathbb{R}$. The uniform bound on the Tot.Var.$(u^\varepsilon(t);\mathbb{R})$ yields a compactness on the family $\{u^\varepsilon\}_\varepsilon$: we can thus extract a converging subsequence, which turns out to be, for almost every sampling sequence $\{\vartheta_i\}_i$, a weak admissible solution of the Cauchy problem (1.1), (1.3).

In 1977 T.-P. Liu [18] improved Glimm's result, showing that if the sampling sequence is *equidistributed*, that means that for any $\lambda\in[0,1]$,

$$\lim_{j\to\infty}\frac{\sharp\{i\in\mathbb{N}\mid 1\le i\le j \text{ and } \vartheta_i\in[0,\lambda]\}}{j}=\lambda,$$

then the subsequence extracted from $\{u^\varepsilon\}_\varepsilon$ converges to a weak admissible solution of (1.1), (1.3).

A different approach which relies on results about the stabilty of the solution of (1.1), (1.3) with respect to (w.r.t.) the initial datum $\bar u$ led to the introduction of the notion of *standard Riemann semigroup*.

Definition 2. *A* standard Riemann semigroup *for the system of conservation laws* (1.1) *is a map $S:\mathcal{D}\times[0,\infty)\to\mathcal{D}$, defined on a domain $\mathcal{D}\subseteq L^1(\mathbb{R};\mathbb{R}^n)$ containing all functions with sufficiently small total variation, with the following properties:*

1. *for some Lipschitz constants L, L',*

$$\begin{gathered}\|S_t\bar u-S_s\bar v\|_1\le L\|\bar u-\bar v\|_1+L'|t-s|,\\ \text{for any}\quad \bar u,\bar v\in\mathcal{D},\qquad t,s\ge 0;\end{gathered}\tag{3.4}$$

2. *if $\bar u\in\mathcal{D}$ is piecewise constant, then for $t>0$ sufficiently small $S_t\bar u$ coincides with the solution of* (1.1), (1.3)*, which is obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.*

In the GNL/LD case it is proved (see, among others, [9], [20], [11]) that any system of conservation laws admits a standard Riemann semigroup and that at any time $t\ge 0$ the solution $u(t)$ obtained as limit of Glimm approximations $u^\varepsilon(t)$, for the initial datum $\bar u$, coincides with the semigroup $S_t\bar u$. We will discuss in the next section the general case.

Relying on the existence of the standard Riemann semigroup for GNL/LD systems, in 1998 A. Bressan and A. Marson [12] further improved the Glimm sampling method, constructing an equidistributed sequence $\{\vartheta_i\}$, satisfying the

additional assumption:

$$\sup_{\lambda\in[0,1]}\left|\lambda-\frac{\sharp\{i\in\mathbb{N}\mid j_1\le i<j_2 \text{ and } \vartheta_i\in[0,\lambda]\}}{j_2-j_1}\right|\le C\cdot\frac{1+\log(j_2-j_1)}{j_2-j_1}. \tag{3.5}$$

Using this sequence, they are able to prove that the rate of convergence of the Glimm approximate solutions $u^\varepsilon(t)$ to the exact weak admissible solution $u(t)=S_t\bar{u}$ at any time t is given by

$$\lim_{\varepsilon\to 0}\frac{\left\|u^\varepsilon(t,\cdot)-S_t\bar{u}\right\|_{L^1}}{\left|\log\varepsilon\right|\sqrt{\varepsilon}}=0. \tag{3.6}$$

The technique used in [12] to prove (3.6) is as follows. Thanks to the Lipschitz property of the semigroup (3.4), in order to estimate the distance

$$\left\|u^\varepsilon(t,\cdot)-u(t,\cdot)\right\|_{L^1}=\left\|u^\varepsilon(t,\cdot)-S_t\bar{u}\right\|_{L^1},$$

we can partition the time interval $[0,t]$ in subintervals $J_r:=[t_r,t_{r+1}]$ and estimate the error

$$\|u^\varepsilon(t_{r+1})-S_{t_{r+1}-t_r}u^\varepsilon(t_r)\|_{L^1} \tag{3.7}$$

on each interval J_r. The error (3.7) on J_r comes from two different sources:

1. first of all there is an error due to the fact that in a Glimm approximate solution, roughly speaking, we give each wave either speed 0 or speed 1 (according to the sampling sequence $\{\vartheta_i\}_i$), while in the exact solution it would have a speed in [0, 1], but not necessarily equal to 0 or 1;

2. secondly, there is an error due to the fact that some waves can be created at times $t>t_r$, some waves can be canceled at times $t<t_{r+1}$ and, above all, some waves, which are present both at time t_r and at time t_{r+1}, can change their speeds, when they interact with other waves.

The first error source is estimated by choosing the intervals J_r sufficiently large in order to use estimate (3.5) with $j_2-j_1\gg 1$.

The second error source can be estimated (choosing the intervals J_r not too large) if we are able to bound the change in speed of the waves present in the approximate solution. In the GNL/LD case, this was achieved by Liu in [18], where he provides a wave tracing algorithm which splits each wavefront in the approximate solution into a finite number of discrete waves, whose trajectories

can be traced and whose changes in speed at any interaction time are bounded by the corresponding decrease of the functional Q^{Glimm}.

As $\varepsilon \to 0$, it is convenient to choose the asymptotic size of the intervals J_r in such a way that the errors in (1) and (2) have approximately the same order of magnitude. In particular, the estimate (3.6) is obtained by choosing $|J_r| \approx \sqrt{\varepsilon} \log |\log \varepsilon|$.

4 Glimm approximate solutions in the general case

All the results in the previous section were obtained under the assumption that each characteristic field is either GNL or LD. In this section we consider now the general case, when this assumption is removed and the only property of f is its strict hyperbolicity.

The problem of finding a suitable decreasing potential to bound the increase of $t \mapsto \text{Tot.Var.}(u^\varepsilon(t); \mathbb{R})$ for a Glimm approximate solution u^ε (see (3.3)) was solved first by T.-P. Liu in [19] for fluxes with a finite number of inflection points. Later, in [1], Bianchini solved the problem for general hyperbolic fluxes, introducing the cubic functional

$$t \mapsto Q^{\text{cubic}}(t) := \iint |\sigma(t,s) - \sigma(t,s')| ds ds' \leq \mathcal{O}(1)\text{Tot.Var.}(u^\varepsilon(t))^3, \quad (4.1)$$

where s, s' are two waves in the approximate solution at time t and $\sigma(t,s)$, $\sigma(t,s')$ denote their speed. In [2] Bianchini and Bressan also proved that any strictly hyperbolic f admits a standard Riemann semigroup (see Definition 2 above) of *vanishing viscosity solutions* with small total variation obtained as the (unique) limits of solutions to the viscous parabolic approximations

$$u_t + f(u)_x = \mu u_{xx}, \quad (4.2)$$

when the viscosity $\mu \to 0$.

About the rate of convergence of the Glimm scheme when an equidistibuted sequence satisfying (3.5) is used, the main problem in extending the proof by Bressan and Marson to the general case is to partition each wavefront in the approximate solution in waves whose trajectories can be traced and whose changes in speed are bounded by some suitable decreasing functional. Indeed, if such a functional is provided, considerations similar to the ones in previous section can be made (see Points (1)-(2) above and the consequent analysis). While Glimm potential Q^{Glimm} fits appropriately in the GNL/LD case to control the change in speed of the waves, it is not apt anymore in the general case. We try to explain

the problem as follows. At any interaction time $t = i\varepsilon$, the change in speed of the waves, integrated over all the possible waves,

$$\int_{\text{waves}} \Big|\sigma\big(i\varepsilon, s\big) - \sigma\big((i-1)\varepsilon, s\big)\Big| ds \tag{4.3}$$

satisfies

$$\int_{\text{waves}} \Big|\sigma\big(i\varepsilon, s\big) - \sigma\big((i-1)\varepsilon, s\big)\Big| ds \leq \text{Lip}(f)\text{Tot.Var.}(u^\varepsilon(t))^2$$

i.e. it is quadratic w.r.t. the total variation of the solution at time t, thanks to the Lipschitzianity of f. Glimm potential Q^{Glimm} is also quadratic w.r.t. Tot.Var.$(u^\varepsilon(t))$ and thus it is reasonable to think that it can be used to bound not only the increase of the Total Variation (3.3), but also the change in speed of the waves (4.3). On the other hand, the potential Q^{cubic} in (4.1) is cubic w.r.t. Tot.Var.$(u^\varepsilon(t))$ and thus there is no hope to use it in order to bound a quadratic quantity such (4.3).

5 A new quadratic interaction functional

In a series of joint works with S. Bianchini ([3] for the scalar case, [4] for a simple system case, [5] for the general system case, see also [21] for a summary of the work done in the three cited papers), we construct a new quadratic functional $t \mapsto Q^{\text{quadr}}(t)$, defined for any Glimm approximate solution $u^\varepsilon(t, x)$, whose decrease at each interaction controls the change in speed (4.3) of the waves.

More precisely, we first construct a wave tracing algorithm (which we call *Lagrangian representation*) which splits each wavefront in the approximate solution u^ε into infinite many infinitesimal waves as follows:

- for any family $k = 1, \ldots, n$, we define a set of real numbers $\mathcal{W}_k \subseteq \mathbb{R}$, called *the set of (infinitesimal) waves*;
- for any wave $w \in \mathcal{W}_k$, we define its sign $S(w) \in \{-1, 1\}$, its *time of creation* $\mathtt{t}^{\text{cr}}(w) \in [0, \infty)$ and its *time of cancellation* $\mathtt{t}^{\text{canc}}(w) \in (0, \infty]$;
- for any wave $w \in \mathcal{W}_k$ and any time $t \in [\mathtt{t}^{\text{cr}}(w), \mathtt{t}^{\text{canc}}(w))$ we define the *position* $\mathtt{x}(t, w)$ of the wave w at time t and the *speed* $\sigma(t, w)$ of the wave w at time t.

The speed $\sigma(t, w)$ and the position $\mathtt{x}(t, w)$ are defined by recursion on time intervals $[i\varepsilon, (i+1)\varepsilon]$, $i \in \mathbb{N}$ and are related to each other as described below:

- the speed $\sigma(t, w)$ is the speed given to the wave w by the Riemann problem located at point $(t, \mathtt{x}(t, w))$;
- the position $\mathtt{x}(t, w)$ of the wave w at time t is related to the speed $\sigma(t, w)$ by the formula

$$\text{for any } t \in [i\varepsilon, (i+1)\varepsilon),\ \mathtt{x}(t, w) =$$

$$\begin{cases} \mathtt{x}(i\varepsilon, w) & \text{if } \sigma(i\varepsilon, w) \le \vartheta_{i+1} \\ \mathtt{x}(i\varepsilon, w) + t - i\varepsilon & \text{if } \sigma(i\varepsilon, w) > \vartheta_{i+1}, \end{cases}$$

which takes into account the sampling method used to construct the Glimm approximate solution u^ε.

Then, for any grid point $(i\varepsilon, m\varepsilon) \in \mathbb{N}\varepsilon \times \mathbb{Z}\varepsilon$, we define the *change in speed* of the waves which at time $i\varepsilon$ have position $m\varepsilon$ and which exist also at time $(i-1)\varepsilon$:

$$\Delta\sigma(i\varepsilon, m\varepsilon) := \sum_{k=1}^{n} \int_{\mathcal{W}_k(i\varepsilon, m\varepsilon) \cap \mathcal{W}_k((i-1)\varepsilon)} \big|\sigma(i\varepsilon, w) - \sigma((i-1)\varepsilon, w)\big| dw$$

where for any time t and point x

$$\mathcal{W}_k(t) := \big\{ w \in \mathcal{W}_k \;\big|\; t \in [\mathtt{t}^{\mathrm{cr}}(w), \mathtt{t}^{\mathrm{canc}}(w)) \big\},$$
$$\mathcal{W}_k(t, x) := \big\{ w \in \mathcal{W}_k(t) \;\big|\; \mathtt{x}(t, w) = x \big\}.$$

The theorem we prove in [5] is the following.

Theorem 1. *It holds*

$$\sum_{i=1}^{+\infty} \sum_{m \in \mathbb{Z}} \Delta\sigma(i\varepsilon, m\varepsilon) \le \mathcal{O}(1) \mathit{Tot.Var.}(\bar{u}; \mathbb{R})^2. \tag{5.1}$$

Recall that $\mathcal{O}(1)$ is a constant which depends only on the flux f.

The proof of Theorem 1 follows a classical approach used in hyperbolic systems of conservation laws in one space dimension.

We first prove a *local* estimate. For the couple of Riemann problems (u^L, u^M), (u^M, u^R), we define the quantity

$$\mathbb{A}(u^L, u^M, u^R) := \mathbb{A}^{\mathrm{trans}}(u^L, u^M, u^R)$$
$$+ \sum_{h=1}^{n} \Big(\mathbb{A}_h^{\mathrm{quadr}}(u^L, u^M, u^R) + \mathbb{A}_h^{\mathrm{canc}}(u^L, u^M, u^R) + \mathbb{A}_h^{\mathrm{cubic}}(u^L, u^M, u^R) \Big), \tag{5.2}$$

which we will call the *global amount of interaction* of the two merging RPs (u^L, u^M), (u^M, u^R). Three of the terms in the r.h.s. of (5.2) have already been introduced in the literature, namely

- $\mathtt{A}^{\text{trans}}(u^L, u^M, u^R)$ is the *transversal amount of interaction* (see [13] and Definition 2.5 in [5]) and measures the strength of the interactions between waves of different families;
- $\mathtt{A}_h^{\text{canc}}(u^L, u^M, u^R)$ is the *amount of cancellation* of the h-th family (see Definition 2.8 in [5]) and measures how many waves of the h-th family are canceled, if two waves of the h-th family with different sign interact;
- $\mathtt{A}_h^{\text{cubic}}(u^L, u^M, u^R)$ is the *cubic amount of interaction* of the h-th family (see [1] and Definition 2.6 in [5]) and measures how many waves of the h-th family are canceled and created in the interaction because of the full nonlinearity of the flux f.

The term

- $\mathtt{A}_h^{\text{quadr}}(u^L, u^M, u^R)$ is the *quadratic amount of interaction* of the h-th family (see Definition 3.1 in [5]) and it is introduced for the first time in [5]; it measures the change in speed of the waves when two wavefronts of the same family and having the same sign interact.

The local estimate we prove is the following: at any grid point $(i\varepsilon, m\varepsilon)$, the change in speed is bounded by

$$\Delta\sigma_k(u^L, u^M, u^R) \leq \mathcal{O}(1)\mathtt{A}(u^{i,m-1}, u^{i-1,m-1}, u^{i,m}) =: \mathtt{A}(i\varepsilon, m\varepsilon), \tag{5.3}$$

where, for any $j \in \mathbb{N}$ and $r \in \mathbb{Z}$, $u^{j,r} := u^\varepsilon(j\varepsilon, r\varepsilon)$. See Section 3 in [5].

Next we show a *global* estimate, based on a new interaction potential. For any grid point $(i\varepsilon, m\varepsilon)$ define $\mathtt{A}^{\text{trans}}(i\varepsilon, m\varepsilon)$, $\mathtt{A}_h^{\text{canc}}(i\varepsilon, m\varepsilon)$, $\mathtt{A}_h^{\text{cubic}}(i\varepsilon, m\varepsilon)$, $\mathtt{A}_h^{\text{quadr}}(i\varepsilon, m\varepsilon)$ as the transversal amount of interaction, the amount of cancellation, the cubic amount of interaction and the quadratic amount of interaction respectively of the two Riemann problems $(u^{i,m-1}, u^{i-1,m-1})$, $(u^{i-1,m-1}, u^{i,m})$ interacting at the grid point $(i\varepsilon, m\varepsilon)$.

We introduce a new interaction potential Υ with the following properties:

(a) it is uniformly bounded at time $t = 0$: in fact,

$$\Upsilon(0) \leq \mathcal{O}(1)\text{Tot.Var.}(\bar{u}; \mathbb{R})^2;$$

(b) it is constant on time intervals $[(i-1)\varepsilon, i\varepsilon)$;

(c) at any time $i\varepsilon$, it decreases at least of $\frac{1}{2}\sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon)$.

It is fairly easy to see that Points a, b, c above, together with inequality (5.3), imply Theorem 1.

The potential Υ is constructed as follows. We define a quadratic functional $t \mapsto Q^{\mathrm{quadr}}(t)$, constant in the time intervals $[(i-1)\varepsilon, i\varepsilon)$ and bounded by $\mathcal{O}(1)\mathrm{Tot.Var.}(\bar{u};\mathbb{R})^2$ at $t=0$, which satisfies the following inequality:

$$\begin{aligned}
&Q^{\mathrm{quadr}}(i\varepsilon) - Q^{\mathrm{quadr}}((i-1)\varepsilon)\\
&\le -\sum_{m\in\mathbb{Z}}\sum_{h=1}^{n} \mathtt{A}_h^{\mathrm{quadr}}(i\varepsilon, m\varepsilon)\\
&\quad + \mathcal{O}(1)\mathrm{Tot.Var.}(\bar{u};\mathbb{R})\sum_{m\in\mathbb{Z}} \mathtt{A}(i\varepsilon, m\varepsilon)\\
&= -\big(1-\mathcal{O}(1)\mathrm{Tot.Var.}(\bar{u};\mathbb{R})\big)\sum_{m\in\mathbb{Z}}\sum_{h=1}^{n} \mathtt{A}_h^{\mathrm{quadr}}(i\varepsilon, m\varepsilon)\\
&\quad + \mathcal{O}(1)\mathrm{Tot.Var.}(\bar{u};\mathbb{R})\sum_{m\in\mathbb{Z}} \mathtt{A}^{\mathrm{trans}}(i\varepsilon, m\varepsilon)\\
&\quad + \mathcal{O}(1)\mathrm{Tot.Var.}(\bar{u};\mathbb{R})\sum_{m\in\mathbb{Z}}\sum_{h=1}^{n}\Big(\mathtt{A}_h^{\mathrm{canc}}(i\varepsilon, m\varepsilon) + \mathtt{A}_h^{\mathrm{cubic}}(i\varepsilon, m\varepsilon)\Big).
\end{aligned} \tag{5.4}$$

It is well known (see [13], [1]) that the decreasing potential

$$Q^{\mathrm{known}}(t) := c_1\mathrm{Tot.Var.}(u^\varepsilon(t)) + c_2 Q^{\mathrm{trans}}(t) + c_3 Q^{\mathrm{cubic}}(t)$$

at each time $i\varepsilon$ satisfies

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}}\left[\mathtt{A}^{\mathrm{trans}}(i\varepsilon, m\varepsilon) + \sum_{h=1}^{n}\Big(\mathtt{A}_h^{\mathrm{canc}}(i\varepsilon, m\varepsilon) + \mathtt{A}_h^{\mathrm{cubic}}(i\varepsilon, m\varepsilon)\Big)\right]\\
&\qquad \le Q^{\mathrm{known}}((i-1)\varepsilon) - Q^{\mathrm{known}}\big(i\varepsilon\big).
\end{aligned} \tag{5.5}$$

Here $Q^{\mathrm{trans}}(t)$ is the transversal part of the Glimm potential $Q^{\mathrm{Glimm}}(t)$ (see Section 2.4 in [5] for a precise definition). It is straightforward to see from (5.4), (5.5) that we can find a constant C big enough, such that the potential

$$\Upsilon(t) := Q^{\mathrm{quadr}}(t) + C Q^{\mathrm{known}}(t)$$

satisfies Properties (a)-(c) above, provided that $\mathrm{Tot.Var.}(\bar{u};\mathbb{R}) \ll 1$.

We would like to stress the main differences between our constructions (the Lagrangian representation and the quadratic functional Q^{quadr}) and all the wave tracing algorithms and functionals already present in the literature.

First of all, our Lagrangian representation is *continuous*: each wavefront in the approximate solution is partitioned into an infinite number of waves whose strength is infinitesimal, while all previous wave tracing algorithm are based on the partition provided by Liu in [18], where each wavefront in the approximate solution is partitioned into a finite number of discrete waves.

Secondly, our functional is *non-local* in time. All previous functionals

$$t \mapsto \text{Tot.Var.}(u(t)),\ Q^{\text{Glimm}}(t),\ Q^{\text{cubic}}(t)$$

are *local* in time: in order to compute their value at a fixed time $\bar{t}$ it is enough to know the approximate solution $u^\varepsilon(\bar{t})$ at time $\bar{t}$. On the contrary, to compute the value of our functional $Q^{\text{quadr}}(\bar{t})$ at time $\bar{t}$, one must know the behavior of the solution u^ε at any time $t \in [0, \infty)$. A non-local in time functional is needed for the following reason. In the GNL/LD case, the functional Q^{Glimm} fits appropriately to estimate the change in speed (4.3), because two waves w, w' which *interact* at time $\bar{t}$ (i.e. $\mathrm{x}(\bar{t}, w) = \mathrm{x}(\bar{t}, w')$), will have the same position at any time $t \geq \bar{t}$: $\mathrm{x}(t, w) = \mathrm{x}(t, w')$ for any $t \geq \bar{t}$. No splitting can occur. The main problem in the general case is that this property is not true anymore. Indeed two waves which interact at time $\bar{t}$ can split after time $\bar{t}$, due to an interaction with a wave of the same family but different sign, or due to an interaction with a wave of a different family. Later they can interact again, then they can split again and so on. This phenomenon can happen an unbounded number of times. The non-locality in time of Q^{quadr} takes into account this kind of behavior.

Let us sketch now briefly the definition of the functional $t \mapsto Q^{\text{quadr}}(t)$. We set

$$Q^{\text{quadr}}(t) := \sum_{k=1}^{n} \iint_{\mathcal{W}_k(t)\times\mathcal{W}_k(t)} \mathfrak{q}(t, w, w')dwdw',$$

where $\mathfrak{q}(t, w, w')$ is called *the weight of the pair of waves* (w, w') *at time* t and its value is determined by the past and future common history of (w, w'), from the last time $\mathfrak{t}^{\text{split}}(t, w, w') \leq t$ before time t when w, w' interact, to the first time $\mathfrak{t}^{\text{int}}(t, w, w') > t$ after time t when they interact again. More precisely, we define the interval of waves $\mathcal{I}(t, w, w')$ as the smallest interval which contains all the waves which have the same position as w, w' at time $\mathfrak{t}^{\text{split}}(t, w, w')$ and which have not been canceled up to time t. Associated to this interval, we define a partition $\mathcal{P}(t, w, w')$ of $\mathcal{I}(t, w, w')$: the elements of this partition are waves with the same past history, from the time $\mathfrak{t}^{\text{split}}(t, w, w')$. The weight $\mathfrak{q}(t, w, w')$ can thus be defined (roughly speaking) as

$$\mathfrak{q}_k(t, w, w') \approx \frac{\begin{array}{c}\text{difference in speed of } w, w', \text{ assigned through the partition } \mathcal{P}(t, w, w') \\ \text{of the interval } \mathcal{I}(t, w, w') \text{ with an appropriate flux function } \mathfrak{f}_k^{\text{eff}}\end{array}}{\text{length of the interval } \mathcal{I}(t, w, w')}.$$

The precise form is slightly more complicated, in order to minimize the oscillations of $\mathfrak{q}_k$ in time. The flux function $\mathfrak{f}_k^{\text{eff}}$ at time $\mathfrak{t}^{\text{split}}(t, w, w')$ is defined as the reduced flux function which splits w, w'; at any later time $t > \mathfrak{t}^{\text{split}}(t, w, w')$ it changes taking into account all the transversal interactions which occur in the time interval $(\mathfrak{t}^{\text{split}}(t, s, s'), t]$. See Section 6 in [5] for a precise definition.

6 Open problems

Even if Theorem 1 has been proved to extend formula (3.6) to a general system with a strictly hyperbolic flux f, it is also interesting by itself, since we think that it could be used to prove further results on conservation laws.

In particular, in a work under preparation [6], S. Bianchini and the author are constructing a *Lagrangian representation* for the exact solution $S_t\bar{u}$, passing to the limit the Lagrangian representation for the Glimm approximate solutions u^ε and using Theorem 1. This *limit* Lagrangian representation can be viewed as a sort of generalized characteristics method, or, in other words, as a *continuous* wavefront tracking, which leads to a deeper understanding of the fine structure of the solutions.

Both Theorem 1 and the *limit* Lagrangian representation could be used to investigate the following problems.

6.1 Regularity of solutions to systems of conservation laws

It is well known that the solution $u(t, x)$ is a BV function of the two variables (t, x). Hence is shares the regularity properties of general BV functions. In particular either u is approximately continuous or it has an approximate jump at each point (t, x), with the exception of a set $\mathcal{N}$ whose one-dimensional Hausdorff measure is zero.

However it seems that much stronger regularity properties hold. In the GNL/LD case it is proved (see, for instance, [10] and [7]) that the set $\mathcal{N}$ is countable and u is continuous (not just approximately continuous) outside $\mathcal{N}$ and outside a countable family Γ of Lipschitz shock curves; moreover at any point $\Gamma \setminus \mathcal{N}$, the solution has left and right limits (not just approximate limits).

We think that the Lagrangian representation we introduce in [6] can be of great help to investigate similar regularity properties for a solution to a general Cauchy problem (1.1), (1.3), without any assumption on f except the strict hyperbolicity.

6.2 Structural stability of solutions to systems of conservation laws

Another interesting problem, reletated to the previous one concerns the *structural stability* of the solutions to system of conservation laws: one expects that the strength and the location of a large shock is not affected too much by a small perturbation of the initial datum. Various results in this direction, but still in the GNL/LD case, can be found in [10, 7]. It would be interesting to extend such results to the general case.

6.3 L^1 stability of solutions to system of conservation laws via hyperbolic methods

As observed in Section 4, the paper [2] by Bianchini and Bressan provides also the L^1 stability of the solutions to systems of conservation laws. However, in their work the authors use stability estimates on the parabolic equation (4.2) to get the stability result (3.4) for the hyperbolic system.

In several previous works by different people (see, among many, [9], [20]), the stability of genuinely non linear/linearly degenerate systems is proved using purely hyperbolic techniques. One of these techniques relies on a functional $\Phi(t) = \Phi(u(t), v(t))$ defined on pairs of solutions, equivalent to the L^1 norm, with the fundamental property that it is "almost decreasing" in time. This functional works only in the GNL/LD setting, while it is not clear how to define a similar functional in the general case. The tools we developed in [3, 4, 5] could be of help in answering this question.

6.4 Quadratic interaction estimate for viscous conservation laws

Finally, a very interesting question concerns the proof of a quadratic estimate similar to (5.1) for the conservation law (4.2) with a viscosity term μu_{xx}, in which we can assume w.l.o.g. $\mu = 1$ (see [2]).

While the building block of the solution of the hyperbolic equation (1.1) is the Riemann problem, in the viscous case the building block is the *viscous traveling profile*, i.e. a solution of the form

$$u(t, x) = U(x - \lambda t) \tag{6.1}$$

which satisfies the second order ODE

$$U'' = (Df(U) - \lambda)U'.$$

In this case the velocity of the viscous profile is λ and it holds

$$\lambda = -\frac{u_t}{u_x}$$

In the inequality (5.1) the l.h.s. is the sum over all grid points $(i\varepsilon, m\varepsilon)$, $i \in \mathbb{N}$, $m \in \mathbb{Z}$ on the (t, x) plane of the change in speed of the wavefronts present at point $(i\varepsilon, m\varepsilon)$ multiplied by their strength. Hence its equivalent in the viscous setting is the integral over the (t, x) plane of the change in speed multiplied by the strength of the viscous profile at point (t, x), i.e.

$$\iint_{[0,\infty)\times\mathbb{R}} \left|\partial_t \frac{u_t}{u_x}\right| |u_x| dx dt = \iint_{[0,\infty)\times\mathbb{R}} \left|u_{tt} - \frac{u_t}{u_x} u_{tx}\right| dx dt \qquad (6.2)$$

What would be nice to obtain is an uniform estimate of (6.2) in terms of the total variation of the initial datum. However, though the term to estimate can be written down in a very clean form (6.2), it is not clear at all which technique can be used in order to estimate it. Indeed, the method we use to prove (5.1) relies heavily on the wave tracing algorithm we developed, in particular on the notion of *position of a given wave w at a given time t*: in the viscous case it is not clear any more what this concept means, since the traveling profiles (6.1) are not localized in space, and thus it is not at all obvious how to define a suitable functional which can play the role that Q^{quadr} has in the hyperbolic case. It would be thus nice to develop an analog theory on viscous conservation laws which leads to estimate the term (6.2).

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References

[1] S. Bianchini. *Interaction Estimates and Glimm Functional for General Hyperbolic Systems*. Discrete and Continuous Dynamical Systems, **9** (2003), 133–166.

[2] S. Bianchini and A. Bressan. *Vanishing viscosity solutions of nonlinear hyperbolic systems*. Annals of Mathematics, **161** (2005), 223–342.

[3] S. Bianchini and S. Modena. *On a quadratic functional for scalar conservation laws*. Journal of Hyperbolic Differential Equations, **11**(2) (2014), 355–435.

[4] S. Bianchini and S. Modena. *Quadratic interaction functional for systems of conservation laws: a case study*. Bulletin of the Institute of Mathematics, Academia Sinica (New Series), **9**(3) (2014), 487–546.

[5] S. Bianchini and S. Modena. *Quadratic interaction functional for general systems of conservation laws*, accepted for publication on Comm. Math. Phys., (2015).

[6] S. Bianchini and S. Modena. *Lagrangian representation for solution to general systems of conservation laws*, in preparation.

[7] S. Bianchini and L. Yu. *Global Structure of Admissible* BV *Solutions to Piecewise Genuinely Nonlinear, Strictly Hyperbolic Conservation Laws in One Space Dimension*. Comm. PDE, **39**(2) (2014), 244–273.

[8] A. Bressan. *Hyperbolic Systems of Conservation Laws. The One Dimensional Cauchy problem*. Oxford University Press (2000).

[9] A. Bressan, G. Crasta and B. Piccoli. *Well posedness of the Cauchy problem for $n \times n$ conservation laws*. Amer. Math. Soc. Memoir, **694** (2000).

[10] A. Bressan and P. LeFloch. *Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws*. Indiana Univ. Math. J., **48** (1999), 43–84.

[11] A. Bressan, T.-P. Liu and T. Yang. *L^1 stability estimates for $n \times n$ conservation laws*. Arch. Rat. Mech. Anal., **149** (1999), 1–22.

[12] A. Bressan and A. Marson. *Error Bounds for a Deterministic Version of the Glimm Scheme*. Arch. Rational Mech. Anal., **142** (1998), 155–176.

[13] J. Glimm. *Solutions in the Large for Nonlinear Hyperbolic Systems of Equations*. Comm. Pure Appl. Math., **18** (1965), 697–715.

[14] P.D. Lax. *Hyperbolic systems of conservation laws II*. Comm. Pure Appl. Math., **10** (1957), 537–566.

[15] P.D. Lax. *Shock waves and entropy*, in Contribution to Nonlinear Functional Analysis, E. Zarantonello Ed., Academic Press, New York, (1971), 603–634.

[16] T.P. Liu. *The Riemann problem for general 2×2 conservation laws.* Trans. Amer. Math. Soc., **199** (1974), 89–112.

[17] T.P. Liu. *The Riemann problem for general systems of conservation laws*. J. Diff. Eqs., **18** (1975), 218–234.

[18] T.-P. Liu. *The deterministic version of the Glimm scheme*. Comm. Math. Phys., **57** (1977), 135–148.

[19] T.-P. Liu. *Admissible solutions of hyperbolic conservation laws*. Amer. Math. Soc. Memoir, **240** (1981).

[20] T.-P. Liu and T. Yang. *A new entropy functional for scalar conservation laws*. Comm. Pure Appl. Math., **52** (1999), 1427–1442.

[21] S. Modena. *Quadratic Interaction Estimate for hyperbolic conservation laws, an overview*, accepted for publication on the "Proceedings of the DFDE2014 Conference" in Contemporary Mathematics. Fundamental Directions (2015).

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