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A priori error estimates for upwind finite volume schemes for two-dimensional linear convection diffusion problems

Dietmar Kröner* and Mirko Rokyta**

Abstract. It is still an open problem to prove a priori error estimates for finite volume schemes of higher order MUSCL type, including limiters, on unstructured meshes, which show some improvement compared to first order schemes. In this paper we use these higher order schemes for the discretization of convection dominated elliptic problems in a convex bounded domain Ω in $\mathbb{R}^2$ and we can prove such kind of an a priori error estimate. In the part of the estimate, which refers to the discretization of the convective term, we gain $h^{1/2}$. Although the original problem is linear, the numerical problem becomes nonlinear, due to MUSCL type reconstruction/limiter technique.

Keywords: linear convection dominated diffusion equation in 2D, upwind finite volume scheme, first and higher order finite volume schemes, a priori error estimates, MUSCL type reconstruction/limiter.

Mathematical subject classification: 65N15, 35J25, 76M25.

1 Introduction

There are many Finite Volume and Discontinuous Galerkin schemes for solving elliptic convection dominated problems and nonlinear conservation laws on unstructured grids in multi dimensions, like

$$\begin{aligned} \partial_t v + \operatorname{div} f(v) &= 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+ \\ v(x,0) &= u_0(x) \qquad \text{on } \mathbb{R}^n. \end{aligned} \tag{1.1}$$

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*Corresponding author.

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While for strongly elliptic problems like

$$\begin{aligned} -\varepsilon\Delta v + \operatorname{div}(bv) + cv &= f \qquad \text{in } \Omega \\ v(x) &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

with dominating diffusion ($\varepsilon = 1$) no stabilization is necessary for numerical schemes, we need some upwinding or, for higher order schemes, a suitable stabilization for convection dominated problems (with small ε). The same statement holds also for nonlinear conservation laws as in (1.1). In this case the stabilization is obtained e.g. by reconstruction technique with so called limiters. They make the scheme nonlinear, even in cases where the underlying partial differential equation (1.1) is linear. For finite volume schemes, the reconstruction with limiters can be realized in a very easy way even on unstructured grids, e.g. by MUSCL type discretizations. However, the theoretical background for these schemes, in particular when applied to conservation laws, is not yet satisfactorily developed. Concerning the convergence of both first and higher order schemes, there are results in the case of nonlinear scalar conservation laws [6], [19], [18], [8], and in the case of weakly coupled systems of conservation laws [23]. For conservation laws as in (1.1) a priori error estimates of the form

$$||v - u_h||_{L^\infty(L^1)} \leq ch^{\frac{1}{4}} + \text{approximation error of data} \tag{1.3}$$

are available [5], [25], [6], [1], [4]. Here, v denotes the exact solution of the underlying partial differential equation and u_h the approximative numerical solution obtained by a first order finite volume scheme in multi dimensions on unstructured grids.

From numerical experiments one would expect $h^{\frac{1}{2}}$ in (1.3), but the proof for this on unstructured grids is an open question. For smooth solutions of the linear transport equation one gets [1] $||v - u_h||_p \leq ch$.

There are also no error estimates $||v - u_h|| \leq ch^\beta$ for higher order finite volume schemes for conservation laws in multi dimensions on unstructured grids including limiters with $\beta > \frac{1}{4}$. To get results in this direction, concerning nonlinear hyperbolic conservation laws, seems to be very difficult. Theoretically justified error analysis for upwind finite volume schemes of higher order, which would also indicate the higher order convergence rate, remains an open problem. See for example [5], [6], [7], [8], [25].

Therefore in this paper we apply the higher order finite volume schemes with limiters to a linear convection dominated stationary diffusion equations like (1.2) in multi dimensions on partially unstructured grids. We can show that we gain $h^{\frac{1}{2}}$ in the error estimate for the term, which refers to the discretization of the convective term, compared to first order schemes.

Let us briefly mention some related results. In [15] a convection diffusion equation like (1.2) with $\varepsilon = 1$ and general elliptic part is considered. They prove error estimates of the form $||v - u_h||_{L^2} \leq ch$ for finite volume schemes of first order. Further results for elliptic and parabolic equations for finite volume schemes are obtained in [2], [11] and the results of an interesting benchmark problem are published in [12]. Lube considered in [21] discretizations of (1.2) and proved

$$||v - u_h|| \leq c_\varepsilon h^k (\varepsilon^{\frac{1}{2}} + h^{\frac{1}{2}})$$

for the streamline diffusion method. Here k is the degree of the local polynomials. A-priori error estimates of the type (1.3), e.g. with the $\varepsilon|| \cdot ||_{H^{1,2}} + || \cdot ||_{L^2}$-norm are also known for the streamline diffusion shock capturing method applied to the linear transport equation with $h^{\frac{3}{2}}$, cf. [16].

For dominating diffusion problems there are error estimates for first order schemes (cf. [15] for stationary case), which show $||v - u_h||_{L^2} \leq ch$. In [13], [22] convergence for a first order combined finite volume-finite element method in the non-stationary case was proved.

For second order TVD Rung-Kutta Discontinuous Galerkin methods with piecewise polynomials of order k in space a priori error estimates of the form $||u - u_h||_{L_2} \leq ch^{k+\frac{1}{2}}$ for smooth solutions u of (1.1) have been proved in [27]. More advanced results for (also hybridized) Discontinuous Galerkin methods can be found e.g. in [9], [26]. For further finite element approximations of convection diffusion problems we refer to [3], [24].

In this paper we omit all the proofs – they can be found in [20].

2 The problem

Consider the following boundary value problem

$$Lv := -\varepsilon \Delta v + \mathrm{div}(bv) + cv = f \qquad \text{in } \Omega\,, \tag{2.1}$$

$$v = 0 \qquad \text{on } \partial\Omega \tag{2.2}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ and $b(x)$, $c(x)$, $f(x)$ are functions which are sufficiently smooth on $\overline{\Omega}$ and such that $0 < c_0 \leq c(x) \leq c_1$, $\mathrm{div}\, b = 0$ in $\overline{\Omega}$. Moreover we suppose that the diffusion parameter ε is a positive constant, $0 < \varepsilon \leq 1$.

We consider that $\overline{\Omega} = \bigcup_j \overline{T}_j$, where $T_j \in \mathcal{T}_h$ are open triangles, $h := \sup_j \mathrm{diam}(T_j)$, $0 < h < h_0$. Furthemore, all boundary triangles are mirrored by the boundary of Ω to get a corresponding ghost triangles. The set of all ghost triangles will be denoted by $\mathcal{T}_G$, $\mathcal{T}_G \cap \mathcal{T}_h = \emptyset$.

Notation 2.1. *We denote by*

(i) $|T_j|$: *the volume of triangle* T_j; $T_j \in \mathcal{T}_h \cup \mathcal{T}_G$

(ii) x_j: *the center of gravity of* T_j *(i.e.,* x_j *is the center of the inscribed circle to the triangle* T_j*)*

(iii) $\overline{x_j}$: *intersection of the perpendicular bisectors of* T_j *(i.e.,* $\overline{x_j}$ *is the center of the circumscribed circle to the triangle* T_j*)*

(iv) $v_j := v(\overline{x_j})$

(v) N_j: *the set of the numbers of the neighboring triangles to* T_j, $T_j \in \mathcal{T}_h$

(xi) $T_{j\ell} := T_j \cup T_\ell$

(xii) $S_{j\ell}$, $\ell \in N_j$: *the joint edge of* T_j *and* T_ℓ *with length* $|S_{j\ell}|$, *where* $T_j \in \mathcal{T}_h$, $T_\ell \in \mathcal{T}_h \cup \mathcal{T}_G$

(xiii) $x_{j\ell}$: *the midpoint of* $S_{j\ell}$

(ix) $d_{j\ell} := |x_\ell - x_j|$

(x) $\overline{d_{j\ell}} := |\overline{x_\ell} - \overline{x_j}|$

(xi) $\gamma_{j\ell} := \frac{|S_{j\ell}|}{\overline{d_{j\ell}}}$; $\gamma := \min \gamma_{j\ell}$;

(xii) $n_{j\ell}$: *the outward unit normal to* $T_j \in \mathcal{T}_h$ *in the direction of* T_ℓ, $\ell \in N_j$.

We assume that there exists an $\eta > 0$ such that all angles of all triangles $T_j \in \mathcal{T}_h$ are less than $\frac{\pi}{2} - \eta$. Therefore, both x_j and $\overline{x_j}$ lie strictly inside of T_j for all j and there is a constant $c_\eta > 0$ independent of h such that $\gamma > c_\eta$.

Moreover we assume that $\mathcal{T}_h = \mathcal{T}_R \cup \mathcal{T}_S$, such that $\mathcal{T}_R \cap \mathcal{T}_S = \emptyset$, where

$$\mathcal{T}_R = \{T_j \in \mathcal{T}_h;\, T_j \text{ is equalsided } \textbf{and } T_\ell \text{ is equalsided } \forall \ell \in N_j\}\,, \tag{2.3}$$

and

$$\mathcal{T}_S = \{T_j \in \mathcal{T}_h;\, T_j \text{ is not equalsided } \textbf{or } \exists \ell \in N_j \text{ s.t. } T_\ell \text{ is not equalsided}\}\,. \tag{2.4}$$

The triangles in $\mathcal{T}_R$ and $\mathcal{T}_S$ are called regular and singular, respectively.

We also assume that the triangulation is locally irregular in the sense of Heinrich (cf. [14, par. 2.2.2, p. 27]), i.e. that the set $\mathcal{T}_S$ consists of the finite number of strips of triangles, each being of the width of $O(h)$.

Assumption 2.2. *For the solution v of (2.1) and (2.2) we assume $v \in W^{2,2}(\Omega)$ and that v can be extended onto a small strip ω_d of the width of $O(h)$ outside of Ω such that we have $v(\overline{x_\ell}) = -v(\overline{x_j})$ if $T_\ell \subset \omega_d$ is the mirrored ghost triangle to T_j. For the continuation v_d of v we assume*

$$||v_d||_{W^{2,2}(\Omega_d)} \leq c||v||_{W^{2,2}(\Omega)},$$

where the constant c is independent of v and $\Omega_d := \Omega \cup \omega_d$.

In the context of the locally irregular grid we will also use the following result (cf. [14, p. 189] and the references there):

Theorem 2.3. *Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and $\omega_h \subset \Omega$ be the strip of the width of $O(h)$, $0 < h < h_0$. Then there is a constant $c > 0$ independent of h such that for all $v \in W^{j+1,2}(\Omega)$, $j = 0, 1, 2$, we have*

$$\|v\|_{W^{j,2}(\omega_h)} \leq ch^{\frac{1}{2}} \|v\|_{W^{j+1,2}(\Omega)}, \qquad j = 0, 1, 2. \tag{2.5}$$

We split the set $\mathcal{E}$ of the edges $S_{j\ell} = \overline{T_j} \cap \overline{T_\ell}$, with $T_j, T_\ell \in \mathcal{T}_h$, into three parts, $\mathcal{E} = \mathcal{E}_R \cup \mathcal{E}_S \cup \mathcal{E}_M$, where

$$\begin{aligned}
\mathcal{E}_R &:= \{S_{j\ell};\ S_{j\ell} \not\subset \partial\Omega;\ \text{both } T_j \text{ and } T_\ell \text{ are regular}\}, \\
\mathcal{E}_S &:= \{S_{j\ell};\ S_{j\ell} \not\subset \partial\Omega;\ \text{both } T_j \text{ and } T_\ell \text{ are singular}\}, \\
\mathcal{E}_M &:= \{S_{j\ell};\ S_{j\ell} \not\subset \partial\Omega;\ T_j \text{ and } T_\ell \text{ are of different type (regular, singular)}\}.
\end{aligned} \tag{2.6}$$

and call them regular, singular and mixed edges, respectively. We also denote by

$$\mathcal{E}_B := \{S_{j\ell};\ S_{j\ell} \subset \partial\Omega; \} \tag{2.7}$$

and call them the boundary edges.

Furthermore we denote

$$N_{jI} := \{\ell \mid T_\ell \text{ is neighboring triangle to } T_j,\ \text{and } T_\ell \subset \Omega\},$$

and

$$\begin{aligned}
N_{jR} &:= \{\ell \mid T_\ell \text{ is neighboring triangle to } T_j,\ \text{and } T_\ell \in \mathcal{T}_R\}, \\
N_{jS} &:= \{\ell \mid T_\ell \text{ is neighboring triangle to } T_j,\ \text{and } T_\ell \in \mathcal{T}_S\}. \\
N_{jG} &:= \{\ell \mid T_\ell \text{ is neighboring triangle to } T_j,\ \text{and } T_\ell \in \mathcal{T}_G\},
\end{aligned}$$

3 The scheme

Let $c_h(x) := c_j$, $f_h(x) := f_j$ for $x \in T_j \in \mathcal{T}_h$, be piecewise constant approximants of c, f, respectively, defined by

$$c_j := \frac{1}{|T_j|} \int_{T_j} c, \qquad f_j := \frac{1}{|T_j|} \int_{T_j} f. \tag{3.1}$$

Let $u_h(x) = u_j$ for $x \in T_j \in \mathcal{T}_h \cup T_G$, be a piecewise constant solution of the discrete problem

$$\begin{aligned}(L_h u_h)_j &= f_j\,, && \text{if } T_j \in \mathcal{T}_h, && (3.2)\\ u_\ell &= -u_j\,, && \text{if } S_{j\ell} \subset \partial\Omega, \text{ and } T_\ell \in \mathcal{T}_G \text{ is the ghost triangle} \\ & && \text{to } T_j \subset \Omega, && (3.3)\end{aligned}$$

where the discrete operator is given by

$$(L_h u_h)_j := -\frac{\varepsilon}{|T_j|}\sum_{\ell\in N_j}(u_\ell - u_j)\gamma_{j\ell} + \frac{1}{|T_j|}\sum_{\ell\in N_j} g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) + c_j u_j\,. \quad (3.4)$$

The first term in (3.4) approximates the value of the diffusion term $-\varepsilon\Delta v$ in $\overline{x_j}$, while $\sum_{\ell\in N_j} g_{j\ell}\big(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}\big)$ approximates the values of the convective term $\operatorname{div}(bv)$ along $S_{j\ell}$. Here, $g_{j\ell}$ stands for an upwind finite volume flux, and,

$$\mathcal{U}_{j\ell} = \mathcal{U}_{j\ell}(u_j, u_\ell), \qquad \mathcal{U}_{\ell j} = \mathcal{U}_{\ell j}(u_\ell, u_j) \quad (3.5)$$

will be defined more precisely later. A particular scheme of the type (3.2)-(3.5) is then chosen by the particular choice of functions $\mathcal{U}_{j\ell}$ and $g_{j\ell}$.

Example 3.1 (General numerical flux). In general, we suppose that the upwind finite volume flux $g_{j\ell}(u, v)$ is a *Lipschitz continuous* function, i.e., we suppose that there is a constant $c > 0$ such that

$$|g_{j\ell}(u, v) - g_{j\ell}(u', v')| \le c\,h\,\big(|u - u'| + |v - v'|\big)\,. \quad (3.6)$$

Furthermore we suppose that $g_{j\ell}$ satisfies the following three basic properties:

$$\begin{aligned} g_{j\ell}(u, u) &= u\int_{S_{j\ell}} b n_{j\ell}\, ds\,, && (3.7)\\ g_{j\ell}(u, v) &= -g_{\ell j}(v, u)\,, && (3.8)\\ \frac{\partial}{\partial u} g_{j\ell}(u, v) \ge 0 &\ge \frac{\partial}{\partial v} g_{j\ell}(u, v)\,, && (3.9)\end{aligned}$$

which are referred to as *consistency*, *conservativity*, and *monotonicity* of the numerical flux $g_{j\ell}$, respectively. (See [18] or [19] for more discussion on general upwind finite volume numerical fluxes.) Moreover, due to (3.7) and $\operatorname{div} b(x) = 0$, we have that (cf. (3.13)):

$$\sum_{\ell\in N_j} g_{j\ell}(u, u) = 0 \quad \text{for all } j\,. \quad (3.10)$$

Example 3.2 (First order Engquist-Osher scheme). As a particular example of the numerical flux we choose the Engquist-Osher type upwind finite volume flux $g_{j\ell}$ defined by

$$g_{j\ell}(u,v) := b_{j\ell}^{+} u + b_{j\ell}^{-} v\,, \qquad b_{j\ell}^{\pm} := \int_{S_{j\ell}} (bn_{j\ell})^{\pm}\, ds\,. \tag{3.11}$$

It can be easily shown that this particular numerical flux satisfies (3.6)-(3.10). The easiest choice of $\mathcal{U}_{j\ell}$ in (3.5), namely

$$\mathcal{U}_{j\ell} := u_j\,, \qquad \mathcal{U}_{\ell j} := u_\ell\,, \tag{3.12}$$

used together with (3.11) in (3.4) defines a *first order* numerical scheme.

Remark 3.3. Due to the properties of b we have for all $T_j \in \mathcal{T}_h$

$$\begin{aligned}\sum_{\ell\in N_j}(b_{j\ell}^{+}+b_{j\ell}^{-}) &= \sum_{\ell\in N_j} b_{j\ell} = \sum_{\ell\in N_j}\int_{S_{j\ell}} bn_{j\ell}\,ds\\ &= \int_{\partial T_j} bn_{j\ell}\,ds = \int_{T_j} \operatorname{div} b\, dx = 0\,,\end{aligned} \tag{3.13}$$

and, for all $S_{j\ell} \in \mathcal{E}$,

$$b_{\ell j} = -b_{j\ell}\,, \quad b_{\ell j}^{+} = -b_{j\ell}^{-}\,, \quad b_{\ell j}^{-} = -b_{j\ell}^{+}\,. \tag{3.14}$$

Example 3.4 (Higher order scheme using MUSCL type reconstruction). Let T_k, T_ℓ, T_m be all neighboring triangles to T_j with centers of gravity x_k, x_ℓ, x_m, x_j, respectively. Let $w \in L^\infty(\Omega)$ with $w|_{T_j} \in C^0(T_j)$ and $w_i := w(x_i)$ for $i = k, \ell, m, j$, respectively. Let

$$\begin{aligned}
&R_k^w \text{ be a plane passing through } (x_\ell, w_\ell), (x_m, w_m), (x_j, w_j)\,,\\
&R_\ell^w \text{ be a plane passing through } (x_k, w_k), (x_m, w_m), (x_j, w_j)\,,\\
&R_m^w \text{ be a plane passing through } (x_k, w_k), (x_\ell, w_\ell), (x_j, w_j)\,.
\end{aligned}$$

Define an index i by

$$|\nabla R_i^w| = \min\left\{|\nabla R_k^w|, |\nabla R_\ell^w|, |\nabla R_m^w|\right\} \tag{3.15}$$

and put

$$G_j^w := \nabla R_i^w. \tag{3.16}$$

If $w_j \geq \max\{w_k, w_\ell, w_m\}$ or $w_j \leq \min\{w_k, w_\ell, w_m\}$, we say that w_j is a local extremum. Let the coefficients $\alpha_j = \alpha_j^w \in \{0, 1\}$ be such that

$$\alpha_j^w = \begin{cases} 0 & \text{if } w_j \text{ is the local extremum,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.17}$$

Then define

$$L_j^w(x) := w_j + \alpha_j^w G_j^w(x - x_j)\,. \tag{3.18}$$

Finally, the higher order MUSCL type Engquist-Osher scheme is defined by (3.4) with the numerical flux (3.11) and

$$\mathcal{U}_{j\ell} := L_j^u(x_{j\ell})\,, \qquad \mathcal{U}_{\ell j} := L_\ell^u(x_{j\ell})\,. \tag{3.19}$$

It can be shown that, on the regular grid, the reconstruction operator L_j^u defined by (3.18) has the following properties.

Lemma 3.1. *For all $T_j \in \mathcal{T}_R$ we have*

$$(a) \quad |L_j^u(x_\ell) - u_j| \leq |u_j - u_\ell| \qquad \textit{for all } \ell \in N_j\,, \tag{3.20}$$

$$(b) \quad |L_j^u(x_{j\ell}) - u_j| \leq \frac{1}{2}|u_j - u_\ell| \qquad \textit{for all } \ell \in N_j\,, \tag{3.21}$$

$$(c) \quad (u_\ell - L_j^u(x_\ell))(u_j - u_\ell) \leq 0 \qquad \textit{for all } \ell \in N_j\,. \tag{3.22}$$

4 Main result

We will use the scheme (3.2)-(3.4) with the following definition of the numerical flux:

- If $S_{j\ell} \in \mathcal{E}_R$ we use the higher order flux using MUSCL type reconstruction, i.e. we set (cf. (3.11), (3.18)-(3.19))

$$g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) := b_{j\ell}^+ L_j^u(x_{j\ell}) + b_{j\ell}^- L_\ell^u(x_{j\ell})\,. \tag{4.1}$$

- If $S_{j\ell} \in \mathcal{E}_M$ or $S_{j\ell} \in \mathcal{E}_S$ we use the first order flux, i.e. we set (cf. (3.11), (3.12))

$$g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) := b_{j\ell}^+ u_j + b_{j\ell}^- u_\ell\,. \tag{4.2}$$

- If $S_{j\ell} \in \mathcal{E}_B$ we use

$$g_{j\ell}(\mathcal{U}_{j\ell}, \mathcal{U}_{\ell j}) := b_{j\ell}^+ u_j + b_{j\ell}^- u_\ell \tag{4.3}$$

  where in this case u_ℓ is the value in the ghost cell of the cell T_j satisfying $u_\ell = -u_j$ (cf. (3.3)).

The main result of this paper is formulated in the following theorem.

Theorem 4.1. *Let $u_h(x) = u_j$ for $x \in T_j \in \mathcal{T}_h$, be a piecewise constant numerical solution of the discrete problem* (3.2)-(3.4) *with a numerical flux satisfying* (3.11)*,* (3.18)-(3.19)*,* (4.1)-(4.3) *and let the Assumption 2.2 hold. We define*

$$z_h := I_h v - u_h \tag{4.4}$$

where

$$I_h v(x) := v(\overline{x_j}) = v_j \quad \text{if } x \in T_j \in \mathcal{T}_h\,. \tag{4.5}$$

Then, defining

$$\|z_h\|_\varepsilon^2 := \varepsilon\gamma \sum_{\mathcal{E}\cup\mathcal{E}_B} (z_j - z_\ell)^2 + c_0 \sum_{T_j\in\mathcal{T}_h} z_j^2 |T_j|\,, \tag{4.6}$$

we have the following error estimate for any $\delta > 0$:

$$\|z_h\|_\varepsilon^2 \le c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{2,2}^2 + c\frac{h^4}{\varepsilon}\sum_{\mathcal{T}_j\in\mathcal{T}_R} R_j^2|T_j|\,. \tag{4.7}$$

If, moreover, $v \in W^{3,2}(\Omega)$, we have

$$\|z_h\|_\varepsilon^2 \;\le\; c\Big(\varepsilon h^3 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{3,2}^2 + c\frac{h^4}{\varepsilon}\sum_{\mathcal{T}_j\in\mathcal{T}_R} R_j^2|T_j|\,. \tag{4.8}$$

Here, $R_j := \frac{1}{|T_j|}\sum_{\ell\in N_j}(u_\ell - u_j)\gamma_{j\ell}$.

Remark 4.2.

- It follows from Lemma 8.2 that the sum $\sum_{\mathcal{T}_j\in\mathcal{T}_R} R_j^2|T_j|$ is of the same order in ε as $\|v\|_{2,2}^2(\Omega)$, cf. (8.2) and (8.4).
- Note that if T_ℓ is the mirrored ghost triangle to T_j, we have $v(\overline{x_\ell}) = -v(\overline{x_j})$ (see Assumption 2.2), and also $u_\ell = -u_j$ (see (3.3)). Therefore,

$$z_\ell = v_\ell - u_\ell = -v_j + u_j = -z_j\,, \tag{4.9}$$

  if T_ℓ is the mirrored ghost triangle to T_j.
- In the case when the first order scheme is used in the whole domain we get (for comparison) the following result:

$$\|z_h\|_\varepsilon^2 \le c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^2}{\varepsilon}\Big)\|v\|_{2,2}^2\,. \tag{4.10}$$

For the higher order MUSCL type scheme we thus gain $h^{1/2}$ inside the estimate of the norm of $\|z_h\|_\varepsilon$ for the term corresponding (as it shows) to the convective part of the equation, compared to the first order scheme: compare $\frac{h^3}{\varepsilon}$ to $\frac{h^2}{\varepsilon}$ in the estimate of the norm of $\|z_h\|^2_\varepsilon$.

- For the particular numerical calculation for which $\varepsilon \approx h$, we get, using (4.10) and (4.7), the error estimates of the order $O(\sqrt{h})$ and $O(h)$ in the cases of first order and higher order scheme, respectively. If $\varepsilon \approx \sqrt{h}$, we get in the corresponding cases the error estimates of the order $O(h^{3/4})$ and $O(h^{5/4})$, respectively.

5 The energy estimate

We prove the discrete energy estimate for the higher order scheme.

Lemma 5.1. *Let u_h be the numerical solution defined by the scheme* (3.2)-(3.4) *with a numerical flux satisfying* (3.11)*,* (3.18)-(3.19)*,* (4.1)-(4.3) *and let Assumption 2.2 hold. Then there is a constant $c > 0$ such that for all $\varepsilon > 0$ and all $h > 0$*

$$\varepsilon\gamma \sum_{\mathcal{E}\cup\mathcal{E}_B} (u_j - u_\ell)^2 + c_0 \sum_{T_j\in\mathcal{T}_h} u_j^2 |T_j| \le c \sum_{T_j\in\mathcal{T}_h} f_j^2 |T_j|. \tag{5.1}$$

6 The basic strategy in proving the main result

The main technical step in the whole proof is to consider the term $(L_h(I_h v) - L_h u_h, z_h) := \sum_j (L_h(I_h v) - L_h u_h)_j |T_j| z_j$ in the following form.

Lemma 6.1.

$$(L_h(I_h v) - L_h u_h, z_h) = (\Psi_H, z_h) + (\Psi_K, z_h) + (\Psi_N, z_h)\,, \tag{6.1}$$

where

$$\begin{aligned}
\Psi_{Hj} &= -\frac{\varepsilon}{|T_j|} \sum_{\ell\in N_j} |S_{j\ell}| \Big(\frac{v_\ell - v_j}{\overline{\overline{d_{j\ell}}}} - \frac{1}{|S_{j\ell}|} \int_{S_{j\ell}} \partial_n v \Big)\,,\\
\Psi_{Kj} &= \frac{1}{|T_j|} \sum_{\ell\in N_j} \Big(g_{j\ell}(\mathcal{V}_{j\ell}, \mathcal{V}_{\ell j}) - \int_{S_{j\ell}} n_{j\ell} b v \Big)\,,\\
\Psi_{Nj} &= \frac{1}{|T_j|} \int_{T_j} c(v_j - v)
\end{aligned}$$

where $\mathcal{V}_{j\ell} = \mathcal{U}_{j\ell}(v_j, v_\ell)$, $\mathcal{V}_{\ell j} = \mathcal{U}_{\ell j}(v_\ell, v_j)$, *and*

$$(\Psi_A, z_h) := \sum_j \Psi_{Aj} z_j |T_j|, \quad \textit{for} \quad A = H, K, N.$$

In what follows, we split the sums in (6.1) into two parts,

$$\sum_j = \sum_{T_j \in \mathcal{T}_R} + \sum_{T_j \in \mathcal{T}_S}.$$

In the "regular" part of the sum we have regular triangles and the higher order approximation, in the "singular" part of the sum ("on the strips") we have general triangles and the first order approximation. We thus get

$$\begin{aligned}(L_h(I_h v) - L_h u_h, z_h) &= (\Psi_H, z_h)_R + (\Psi_H, z_h)_S + (\Psi_K, z_h)_R \\ &\quad + (\Psi_K, z_h)_S + (\Psi_N, z_h)_R + (\Psi_N, z_h)_S\end{aligned}$$

and will proceed by estimating the terms on the right-hand side both from above and from below.

7 The estimates from above

7.1 The estimate of (Ψ_N, z_h) from above

For the estimate of (Ψ_N, z_h) (approximating the zero-order term) from above we obtain the following results.

Lemma 7.1. *We have on the regular triangles*

$$\sum_{T_j \in \mathcal{T}_R} |\Psi_{Nj}|^2 |T_j| \leq ch^4 ||v||^2_{2,2,R}, \tag{7.1}$$

$$(\Psi_N, z_h)_R \leq ch^4 \|v\|^2_{2,2,R} + \frac{c_0}{8} \sum_{T_j \in \mathcal{T}_R} z_j^2 |T_j| \tag{7.2}$$

where $\|v\|^2_{2,2,R} := \sum_{T_j \in \mathcal{T}_R} \|v\|^2_{W^{2,2}(T_j)}$.

Lemma 7.2. *We have on singular triangles*

$$\sum_{T_j \in \mathcal{T}_S} |\Psi_j|^2 |T_j| \leq ch^{4-2\delta} ||v||^2_{2,2,S} \tag{7.3}$$

$$(\Psi_N, z_h)_S \leq ch^{4-2\delta} \|v\|^2_{2,2,S} + \frac{c_0}{8} \sum_{T_j \in \mathcal{T}_S} z_j^2 |T_j| \tag{7.4}$$

for any $\delta > 0$, *where* $\|v\|^2_{2,2,S} := \sum_{T_j \in \mathcal{T}_S} \|v\|^2_{W^{2,2}(T_j)}$.

Putting the results of Lemmata 7.1, 7.2 together, we obtain

Lemma 7.3. *We have (on the whole domain)*

$$(\Psi_N, z_h) \quad \leq \quad c h^{4-2\delta} \|v\|^2_{2,2} + \frac{c_0}{8} \sum_{T_j \in \mathcal{T}_h} z_j^2 |T_j| \,. \tag{7.5}$$

7.2 The estimates of (Ψ_H, z_h) from above

For the estimate of (Ψ_H, z_h) (approximating the diffusion part) from above we obtain the following result.

Lemma 7.4. *For* $v \in W^{2,2}(\Omega)$ *we have*

$$(\Psi_H, z_h) \quad \leq \quad c\,\varepsilon\, h\, \|v\|_{2,2} \Big(\sum_{\mathcal{E} \cup \mathcal{E}_B} (z_j - z_\ell) \Big)^{1/2} \tag{7.6}$$

$$\leq \quad c\,\varepsilon\, h^2\, \|v\|^2_{2,2} + \frac{\varepsilon\gamma}{8} \sum_{\mathcal{E} \cup \mathcal{E}_B} (z_j - z_\ell)^2 \,. \tag{7.7}$$

If moreover $v \in W^{3,2}(\Omega)$, *we have*

$$(\Psi_H, z_h) \quad \leq \quad c\,\varepsilon\, h^3\, \|v\|^2_{3,2} + \frac{\varepsilon\gamma}{8} \sum_{\mathcal{E} \cup \mathcal{E}_B} (z_j - z_\ell)^2 \,. \tag{7.8}$$

7.3 The estimates of (Ψ_K, z_h) from above

For (Ψ_K, z_h) (approximating the convective term) we obtain the following lemma.

Lemma 7.5. *We have*

$$(\Psi_K, z_h)_R \quad \leq \quad c\, \frac{h^3}{\varepsilon} \|v\|^2_{2,2} + \frac{\varepsilon\gamma}{8} \sum_{\mathcal{E} \cup \mathcal{E}_B} (z_j - z_\ell)^2. \tag{7.9}$$

7.4 The final estimate from above

Putting together the estimates (7.5), (7.7), (7.8), (7.9), we get the following result.

Theorem 7.1 (Estimate from above). *Under the assumptions of Theorem 4.1 there exists a constant $c > 0$ independent of ε such that for $v \in W^{2,2}(\Omega)$ we have*

$$
\begin{aligned}
(L_h I_h v - L_h u_h, z_h) \;\le\; & c\Big(\varepsilon h^2 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{2,2}^2 \\
& + \frac{\varepsilon\gamma}{4}\sum_{\mathcal{E}\cup\mathcal{E}_B}(z_j - z_\ell)^2 + \frac{c_0}{8}\sum_{T_j\in\mathcal{T}_h} z_j^2|T_j|\,.
\end{aligned}
\tag{7.10}
$$

If moreover $v \in W^{3,2}(\Omega)$, we have

$$
\begin{aligned}
(L_h I_h v - L_h u_h, z_h) \;\le\; & c\Big(\varepsilon h^3 + h^{4-2\delta} + \frac{h^3}{\varepsilon}\Big)\|v\|_{3,2}^2 \\
& + \frac{\varepsilon\gamma}{4}\sum_{\mathcal{E}\cup\mathcal{E}_B}(z_j - z_\ell)^2 + \frac{c_0}{8}\sum_{T_j\in\mathcal{T}_h} z_j^2|T_j|\,.
\end{aligned}
\tag{7.11}
$$

8 The estimates from below

In this part of the paper we will prove an estimate from below.

Theorem 8.1 (Estimate from below). *Under the assumptions of Theorem 4.1 there exists a constant $c > 0$ independent of ε such that for $v \in W^{2,2}(\Omega)$ we have*

$$
\begin{aligned}
(L_h I_h v - L_h u_h, z_h) \;\ge\; & \frac{\varepsilon\gamma}{2}\sum_{\mathcal{E}}(z_\ell - z_j)^2 + 2\varepsilon\gamma\sum_{\mathcal{E}_B} z_j^2 \\
& + c_0\sum_j z_j^2|T_j| + \frac{1}{2}\sum_{\mathcal{E}_M\cup\mathcal{E}_S}(b_{j\ell}^+ - b_{j\ell}^-)(z_j - z_\ell)^2 \\
& - c\,\frac{h^4}{\varepsilon}\|v\|_{W^{2,2}(\Omega)}^2 - c\,\frac{h^4}{\varepsilon}\sum_{\mathcal{T}_R} R_j^2|T_j|
\end{aligned}
\tag{8.1}
$$

where $\gamma = \min \gamma_{j\ell}$ *and* $R_j := \frac{1}{|T_j|}\sum_{\ell\in N_j}(u_\ell - u_j)\gamma_{j\ell}$.

Lemma 8.1. *Let $R_j = \frac{1}{|T_j|}\sum_{\ell\in N_j}(u_\ell - u_j)\gamma_{j\ell}$ for $T_j \in \mathcal{T}_R$. Then there is a constant $c > 0$ independent of ε and h, such that*

$$
\sum_{\mathcal{T}_R} R_j^2|T_j| \;\le\; \frac{c}{\varepsilon^3}\,\|f\|_{L^2(\Omega)}^2.
\tag{8.2}
$$

Remark 8.2. The terms

$$\|v\|^2_{W^{2,2}(\Omega)} \quad \text{and} \quad \sum_{\mathcal{T}_R} R_j^2 |T_j| \tag{8.3}$$

contained on the right-hand side of the estimate (8.1) depend of course on ε. As we will see in the following Lemma, the sum $\sum_{\mathcal{T}_R} R_j^2|T_j|$ is of the same order in ε as is the norm $\|v\|^2_{W^{2,2}(\Omega)}$.

Lemma 8.2. *Let $v \in W^{2,2}(\Omega)$ be the solution of* (2.1)-(2.2). *Then*

$$\|v\|^2_{W^{2,2}(\Omega)} \leq \frac{c}{\varepsilon^3} \|f\|^2_{L^2(\Omega)}, \tag{8.4}$$

9 The final estimate

Putting together the estimates (7.10), (8.1) and using the definition of $\|z_h\|_\varepsilon$ (see (4.6)), we obtain the main estimates (4.7) and (4.8) of Theorem 4.1. The result for the first order scheme (4.10) can be obtained using only the parts of the estimates (7.10), (8.1) which corresponds to the first order parts of the scheme.

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Dietmar Kröner
Universität Freiburg
Institut für Angewandte Mathematik
Hermann–Herder–Str. 10
79104 Freiburg
GERMANY

E-mail: dietmar@mathematik.uni-freiburg.de

Mirko Rokyta
Charles University
Faculty of Mathematics and Physics
Department of Mathematical Analysis
Sokolovská 83, 186 75 Praha
CZECH REPUBLIC

E-mail: rokyta@karlin.mff.cuni.cz