BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 47(1), 359-372

ISSN: 1678-7544 (Print) / 1678-7714 (Online)

Relative entropy based error estimates for discontinuous Galerkin schemes

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Abstract. These notes give an overview on how the relative entropy stability framework can be employed to derive a posteriori error estimates for semi-(spatially)-discrete discontinuous Galerkin schemes approximating systems of hyperbolic conservation laws endowed with one strictly convex entropy. We also show how these methods can be extended as to cover a related, higher order, model for compressible multiphase flows with non-convex energy.

Keywords: hyperbolic conservation law, discontinuous Galerkin method, a posteriori error analysis, compressible multiphase flows, relative entropy.

Mathematical subject classification: 35L60, 65M60, 76T10.

1 Introduction

Discontinuous Galerkin (dG) schemes are frequently used for the numerical approximation of (systems of) hyperbolic conservation laws since they allow the resolution of discontinuities as well as high order accuracy in places in which the solution is smooth [16, 18, 12]. We present an *a posteriori* error analysis for a class of dG schemes approximating systems of hyperbolic conservation laws in one space dimension

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0 \tag{1}$$

with periodic boundary conditions. In (1) we search for $\mathbf{u}$ taking values in some open set $U \subset \mathbb{R}^d$, called state space, while $\mathbf{f} : U \to \mathbb{R}^d$ is a given flux function. It is well known that solutions to (1) – in general – develop discontinuities in finite time even for smooth initial data. Easy examples show that weak solutions to (1) are not always unique. Motivated by the second law of thermodynamics attention is restricted to entropy solutions, in the following way: A pair

Received 27 March 2015.

of functions $(\eta, q) : U^2 \to \mathbb{R}^2$ is called an entropy/entropy flux pair of (1) provided

$$\mathrm{D}\,\eta\,\mathrm{D}\mathbf{f} = \mathrm{D}\,q,$$

where D denotes the Jacobian of a function or vector-field with respect to $\mathbf{u}$. In particular, η is called an entropy and q is called an entropy flux. A weak solution of (1) is called an entropy solution with respect to the pair (η, q) provided it satisfies

$$\partial_t \eta(\mathbf{u}) + \partial_x q(\mathbf{u}) \leq \mathbf{0} \tag{2}$$

in the sense of distributions. It is easy to check that strong solutions to (1) satisfy (2) as an equality.

Most systems of hyperbolic conservation laws are equipped with one convex entropy, while there are some systems, e.g. equations of isothermal elastodynamics, which do not allow for a convex entropy. This situation is very different from scalar hyperbolic conservation laws which posses infinitely many convex entropies, which gives rise to the rather strong L_1-contraction framework of Kruzkov, [17]. It should be noted that entropy solutions are unique for scalar problems while there are important systems, as Euler's equations, for which Cauchy problems in multiple space dimensions admit infinitely many entropy solutions, [5].

There are mainly two stability theories for systems of hyperbolic conservation laws. Firstly, there is the wave front tracking approach developed by Bressan and coworkers and, secondly, there is the relative entropy on which we will focus here. The latter goes back to [3, 4] where it was used to prove weak-strong-uniqueness for hyperbolic conservation laws endowed with one convex entropy. While it has been subsequently extended to cover systems only possessing quasi- or poly-convex entropies we will restrict ourselves to the case of systems endowed with at least one strictly convex entropy in the first part of this work. Our a posteriori analysis employs a reconstruction approach which allows us to view a computable reconstruction $\hat{\mathbf{u}}$ of the numerical solution as a Lipschitz continuous solution to a perturbed version of (1), i.e.,

$$\partial_t \hat{\mathbf{u}} + \partial_x \mathbf{f}(\hat{\mathbf{u}}) = \mathbf{R}, \tag{3}$$

with computable residual $\mathbf{R}$. Then, the relative entropy can be used in order to bound the difference between the possibly discontinuous entropy solution $\mathbf{u}$ to (1) and $\hat{\mathbf{u}}$. Proofs of all assertions made in the first part of this note can be found in [9].

Let us give a short account on other a posteriori results for dG approximations of hyperbolic conservation laws. Error analysis for finite volume and dG schemes for scalar problems (exploiting Kruzkov's theory) were given in [14, 8, 6]. Linear systems were studied by [13] and a posteriori estimates using dually weighted residuals were given by [11]. A posteriori error indicators based on nodal super convergence were investigated in a series of papers, see [1, e.g.], while indicators based on the entropy dissipation of the numerical solution were suggested recently [22].

In the second part of this note, we describe a similar approach for proving *a posteriori* error estimates for dG schemes approximating the following one dimensional model problem for compressible visco-capillary multi-phase flows

$$\begin{aligned} \partial_t \tau - \partial_x v &= 0 \\ \partial_t v - \partial_x W'(\tau) &= \mu \partial_{xx} v - \gamma \partial_{xxx} \tau, \end{aligned} \tag{4}$$

where τ is the specific volume, v is the velocity, W is the non-convex energy density and μ, γ are positive coefficients modelling viscosity and capillarity, which was studied in [2, 23, e.g.]. Strong solutions to (4) satisfy the local balance of energy

$$\begin{aligned} &\partial_t (W(\tau) + \frac{1}{2} v^2 + \frac{\gamma}{2} (\partial_x \tau)^2) \\ &+ \partial_x (-v W(\tau) + \gamma v \partial_{xx} \tau - \gamma \partial_x v \partial_x \tau - \mu v \partial_x v) = -\mu (\partial_x v)^2. \end{aligned} \tag{5}$$

A particular, difficulty in the study of (4) stems from the fact that W cannot be expected to be convex. We are going to explain how the higher order terms in the model (4), i.e., those scaling with γ, μ, compensate the non-convexity of W such that a relative entropy like technique can be used to obtain stability results. Proofs of all assertions made in the second part of this note can be found in [10].

2 Hyperbolic conservation laws

2.1 Relative entropy

The relative entropy and relative entropy flux between two states $\mathbf{u}, \mathbf{v} \in U$ are given by

$$\begin{aligned} \eta(\mathbf{u}|\mathbf{v}) &:= \eta(\mathbf{u}) - \eta(\mathbf{v}) - \mathrm{D}\,\eta(\mathbf{v})(\mathbf{u} - \mathbf{v}) \\ q(\mathbf{u}|\mathbf{v}) &:= q(\mathbf{u}) - q(\mathbf{v}) - \mathrm{D}\,\eta(\mathbf{v})(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v})). \end{aligned}$$

The relative entropy stability framework is comprised in the following Theorem, which also holds in several space dimensions.

Theorem 2.1 (Dafermos, di Perna '79). *Let* $\mathbf{u}$ *be an entropy solution of* (1) *and* $\mathbf{v}$ *a Lipschitz solution of* (1) *on the flat one-dimensional torus* $\mathbb{T}$*, which both* *take values in some compact, convex $K \subset U$. Then, there exist $a, b > 0$ such that for $t > 0$*

$$\|\mathbf{u}(t,\cdot) - \mathbf{v}(t,\cdot)\|^2_{L_2(\mathbb{T})} \leq a\|\mathbf{u}_0 - \mathbf{v}_0\|^2_{L_2(\mathbb{T})} e^{bt},$$

with b depending on the Lipschitz constant of $\mathbf{v}$.

We will not give the proof of the Theorem but mention that it is based on verifying the weak form of the inequality

$$\partial_t \eta(\mathbf{u}|\mathbf{v}) + \partial_x q(\mathbf{u}|\mathbf{v}) \leq -(\partial_x \mathbf{v})^T \,\mathrm{D}^2\,\eta(\mathbf{v})\big(\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v}) - \mathrm{D}\,\mathbf{f}(\mathbf{v})(\mathbf{u}-\mathbf{v})\big), \quad (6)$$

where D^2 denotes the Hessian.

2.2 Discontinuous Galerkin schemes

In order to state the dG schemes under consideration, we fix some notation. We choose $0 = x_0 < x_1 < \cdots < x_N = 1$. By $h_n := x_{n+1} - x_n$ we denote the size of the n-th subinterval. In addition, $h := \max h_n$ and we denote by $\mathcal{T}$ the set of all cells (x_j, x_{j+1}) and by $\mathcal{E}$ the set of all common interfaces of $\mathcal{T}$. Let $\mathbb{P}_p(I)$ denote the space of polynomials of degree less than or equal to p on I, then we define

$$\mathbb{V}_p := \left\{\mathbf{g} : I \to \mathbb{R}^d \,:\, (g_i)|_K \in \mathbb{P}_p(K) \text{ for } i = 1,\ldots,d,\ K \in \mathcal{T}\right\}, \quad (7)$$

where $\mathbf{g} = (g_1,\ldots,g_d)^T$, is the usual space of piecewise p-th degree polynomials for vector valued functions over I. We define jumps by

$$[\mathbf{g}] := \mathbf{g}^- - \mathbf{g}^+ := \lim_{s\searrow 0} \mathbf{g}(\cdot - s) - \lim_{s\searrow 0} \mathbf{g}(\cdot + s), \quad (8)$$

so that $[\mathbf{g}]$ is defined on $\mathcal{E}$.

We study the following class of spatially-discrete discontinuous Galerkin schemes: $\mathbf{u}_h \in C^1([0,T), \mathbb{V}_p)$ is determined by the system of ODEs

$$0 = \sum_{K\in\mathcal{T}} \int_K \partial_t(\mathbf{u}_h)\cdot\boldsymbol{\phi} - \mathbf{f}(\mathbf{u}_h)\cdot\partial_x\boldsymbol{\phi}\,\mathrm{d}x + \int_{\mathcal{E}} \mathbf{F}(\mathbf{u}_h^-, \mathbf{u}_h^+)\cdot[\boldsymbol{\phi}]\ \forall\,\boldsymbol{\phi}\in\mathbb{V}_p \quad (9)$$

and an appropriate discretisation of the initial data. In the sequel we will assume that (9) is uniquely solvable and, in particular, that $\mathbf{u}_h$ takes values in U. We account for the periodic boundary conditions by setting

$$\mathbf{u}_h^-(x_0) := \mathbf{u}_h^-(x_N), \qquad \text{and} \qquad \mathbf{u}_h^+(x_N) := \mathbf{u}_h^+(x_0). \quad (10)$$

In (9) we denote the numerical flux by $\mathbf{F} : U^2 \subset \mathbb{R}^{2d} \to \mathbb{R}^d$. We restrict our attention to a certain class of numerical flux functions, satisfying more than the classical consistency and Lipschitz conditions. We impose existence of a function

$$\mathbf{w} : U \times U \to U \text{ such that } \mathbf{F}(\mathbf{u}, \mathbf{v}) = \mathbf{f}(\mathbf{w}(\mathbf{u}, \mathbf{v})) \tag{11}$$

and a constant $L > 0$ such that $\mathbf{w}$ satisfies

$$|\mathbf{w}(\mathbf{u}, \mathbf{v}) - \mathbf{u}| + |\mathbf{w}(\mathbf{u}, \mathbf{v}) - \mathbf{v}| \leq L|\mathbf{u} - \mathbf{v}| \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathbb{R}^d. \tag{12}$$

Remark 2.2. The condition on the flux functions, in general, restricts our analysis to fluxes of Godunov type. We require this condition in the definition of the reconstructions in Section 2.3. If we do not have this restriction we may still define reconstructions (differently) but we would no longer expect the error estimate to be of optimal order for smooth solutions of (1).

Note that both the numerical solution $\mathbf{u}_h$ and the entropy solution $\mathbf{u}$ to (1) may be discontinuous, in general. This causes the need to introduce an intermediate quantity, the reconstruction $\hat{\mathbf{u}}$, which will be Lipschitz continuous such that it can be used in the relative entropy stability framework. To be more specific, we will use the relative entropy framework for estimating the difference between $\hat{\mathbf{u}}$ and the entropy solution $\mathbf{u}$, even if $\mathbf{u}$ is discontinuous. Once we obtained such an estimate we can estimate the error of the numerical scheme by

$$\|\mathbf{u} - \mathbf{u}_h\|_{L_\infty(0,T;L_2(\mathbb{T}))} \leq \|\mathbf{u} - \hat{\mathbf{u}}\|_{L_\infty(0,T;L_2(\mathbb{T}))} + \|\hat{\mathbf{u}} - \mathbf{u}_h\|_{L_\infty(0,T;L_2(\mathbb{T}))}. \tag{13}$$

2.3 Reconstructions

Our error estimate for the schemes (9) is based on reconstructions $\hat{\mathbf{u}}$ of the solution and $\hat{\mathbf{f}}$ of the flux. The latter is similar to reconstructions employed for dG schemes in time, [21]. For brevity we omit the time dependency of all quantities in this section.

Definition 2.3 (Reconstruction of the flux). *The reconstruction* $\hat{\mathbf{f}}$ *is the unique element of* $\mathbb{V}_{p+1}$ *satisfying*

$$\begin{aligned} \sum_{K \in \mathcal{T}} \int_K \partial_x \hat{\mathbf{f}} \cdot \boldsymbol{\phi} \,\mathrm{d}x \;&=\; -\sum_{K \in \mathcal{T}} \int_K \mathbf{f}(\mathbf{u}_h) \cdot \partial_x \boldsymbol{\phi} \,\mathrm{d}x \\ &\quad + \int_{\mathcal{E}} \mathbf{f}(\mathbf{w}(\mathbf{u}_h^-, \mathbf{u}_h^+)) \cdot [\boldsymbol{\phi}] \; \forall\, \boldsymbol{\phi} \in \mathbb{V}_p \end{aligned} \tag{14}$$

and

$$\hat{\mathbf{f}}^+ = \mathbf{f}(\mathbf{w}(\mathbf{u}_h^-, \mathbf{u}_h^+)) \text{ on } \mathcal{E}. \tag{15}$$

Definition 2.4 (Reconstruction of the solution). *The reconstruction* $\hat{\mathbf{u}}$ *is the unique element of* $\mathbb{V}_{p+1}$ *satisfying*

$$\sum_{K\in\mathcal{T}} \int_K \hat{\mathbf{u}} \cdot \boldsymbol{\phi} \,\mathrm{d}\,x = \sum_{K\in\mathcal{T}} \int_K \mathbf{u}_h \cdot \boldsymbol{\phi} \,\mathrm{d}\,x \;\forall\, \boldsymbol{\phi} \in \mathbb{V}_{p-1} \tag{16}$$

and

$$\hat{\mathbf{u}}^+ = \mathbf{w}(\mathbf{u}_h^-, \mathbf{u}_h^+) \text{ and } \hat{\mathbf{u}}^- = \mathbf{w}(\mathbf{u}_h^-, \mathbf{u}_h^+) \quad \text{on } \mathcal{E}. \tag{17}$$

Lemma 2.5 (Properties of the reconstructions). *The reconstructions* $\hat{\mathbf{u}}$ *and* $\hat{\mathbf{f}}$ *are uniquely defined and continuous. Moreover, the reconstructions are explicitly and locally computable.*

Due to the specific reconstruction (14) and (9) we obtain

$$0 = \sum_{K\in\mathcal{T}} \int_K \partial_t \mathbf{u}_h \cdot \boldsymbol{\phi} - \partial_x \hat{\mathbf{f}} \cdot \boldsymbol{\phi} \,\mathrm{d}\,x \;\forall\, \boldsymbol{\phi} \in \mathbb{V}_p. \tag{18}$$

As $\partial_t \mathbf{u}_h$ and $\partial_x \hat{\mathbf{f}}$ are piecewise polynomials of degree p equation (18) gives rise to the *point-wise* equation

$$\partial_t \mathbf{u}_h + \partial_x \hat{\mathbf{f}} = \mathbf{0} \quad \text{a.e. in } \mathbb{T} \tag{19}$$

such that

$$\partial_t \hat{\mathbf{u}} + \partial_x \mathbf{f}(\hat{\mathbf{u}}) = \mathbf{R}_h := \partial_t \hat{\mathbf{u}} - \partial_t \mathbf{u}_h + \partial_x \mathbf{f}(\hat{\mathbf{u}}) - \partial_x \hat{\mathbf{f}}. \tag{20}$$

Remark 2.6. The reconstruction $\hat{\mathbf{u}}$ satisfies a perturbed version of (1), i.e. (20). As $\mathbf{f}(\hat{\mathbf{u}})$ and $\hat{\mathbf{f}}$ are continuous and piecewise differentiable, the residual satisfies $\mathbf{R}_h(t,\cdot) \in L_2(\mathbb{T})$ for all $t > 0$ and it is explicitly computable.

Remark 2.7. It might be expected that the x-derivatives appearing in the definition of $\mathbf{R}_h$ in (20) might lead to a suboptimal order of the residual. At this point assumption (11) is used in order to obtain an error estimate of optimal order. For the technical details we refer to [9].

Using an analogous balance equation to (6) we infer

Theorem 2.8 (A posteriori error estimate). *Let* $\mathbf{f} \in W_2^\infty(U, \mathbb{R}^d)$. *Let* $\mathbf{u}$ *be an entropy solution of* (1) *with periodic boundary conditions. Provided* $\mathbf{u}$ *and* *the reconstruction* $\hat{\mathbf{u}}$ *of the numerical solution* $\mathbf{u}_h$*, given by* (9)*, take values in some compact and convex* $K \subset U$*, then for any* $T > 0$ *there exists a computable constant* C *depending on* K *and* T *so that for* $0 \leq t \leq T$ *the error between* $\mathbf{u}_h$ *and* $\mathbf{u}$ *satisfies*

$$\begin{aligned} \|\mathbf{u}(t,\cdot) - \mathbf{u}_h(t,\cdot)\|^2_{L_2(\mathbb{T})} &\leq \|\hat{\mathbf{u}}(t,\cdot) - \mathbf{u}_h(t,\cdot)\|^2_{L_2(\mathbb{T})} \\ &+ C\Big(\|\mathbf{R}_h\|^2_{L_2((0,t)\times\mathbb{T})} + C\|\mathbf{u}_0 - \hat{\mathbf{u}}_0\|^2_{L_2(\mathbb{T})}\Big) \\ &\times \exp\Big(\int_0^t C^2 \|\hat{\mathbf{u}}_x(s,\cdot)\|_{L_\infty(\mathbb{T})} + C^3 \,\mathrm{d}\,s\Big) \end{aligned} \tag{21}$$

where $\mathbf{R}_h$ *is defined in* (20).

Remark 2.9. All the terms on the right hand side of (21) are explicitly computable. Provided $\|\hat{\mathbf{u}}_x(s,\cdot)\|_{L_\infty(\mathbb{T})}$ is uniformly bounded in h we expect the right hand side of (21) to be of optimal order. This is confirmed by numerical experiments, given in [9], in case of an at least Lipschitz continuous entropy solution.

If, however, the entropy solution is discontinuous, then we observe that $\|\hat{\mathbf{u}}_x(s,\cdot)\|_{L_\infty(\mathbb{T})}$ scales like $\frac{1}{h}$ such that the right hand side of (21) diverges for $h \to 0$. This is not only a consequence of the employed stability framework but also reflects the fact that entropy solutions to systems of hyperbolic conservation laws are not necessarily unique.

We like to mention recent results obtained by Vasseur and coworkers [19, e.g.] who managed to establish relative entropy results "with a shift" for solutions with one shock. Their analysis also reveals that after shock formation an L_2 framework without shift is ill-suited for stability estimates even for scalar conservation laws.

3 The multi-phase flow model

In this section we show how a similar approach, i.e., a combination of a reconstruction technique with relative entropy stability, can be employed in the study of numerical approximations of a model problem for compressible liquid vapour flows.

3.1 Reduced relative entropy

Due to the non-convexity of W the relative entropy is ill-suited to measure the difference between two solutions of (4), i.e., two solutions might have relative entropy zero while they are far away from each other in L_2. Thus, we define the *reduced relative entropy* by

$$\eta_R(t) := \frac{1}{2}\int_{\mathbb{T}} (v(t,\cdot)-\hat{v}(t,\cdot))^2 + \gamma(\partial_x \tau(t,\cdot) - \partial_x \hat{\tau}(t,\cdot))^2 \\ + \frac{\mu}{4}\int_0^t |v(s,\cdot)-\hat{v}(s,\cdot)|^2_{H^1(\mathbb{T})}\,\mathrm{d}s. \tag{22}$$

Theorem 3.1 (Reduced relative entropy bound). *Let (τ, v) be a strong solution to (4) and suppose $(\hat{\tau}, \hat{v})$ is a strong solution to the perturbed problem*

$$\begin{aligned} \partial_t \hat{\tau} - \partial_x \hat{v} &= 0 \\ \partial_t \hat{v} - \partial_x W'(\hat{\tau}) &= \mu \partial_{xx}\hat{v} - \gamma \partial_{xxx}\hat{\tau} + \mathfrak{R} \end{aligned} \tag{23}$$

where $\mathfrak{R}$ is some residual and $\gamma > 0, \mu \geq 0$. Assume that $\hat{\tau}(0,\cdot) = \tau(0,\cdot)$, $\hat{v}(0,\cdot) = v(0,\cdot)$ and that

$$\overline{M} := \max\left(|\tau\|_{L_\infty((0,\infty)\times\mathbb{T})}, \|\hat{\tau}\|_{L_\infty((0,\infty)\times\mathbb{T})}\right) < \infty. \tag{24}$$

Then, the reduced relative entropy between (τ, v) and $(\hat{\tau}, \hat{v})$ satisfies

$$\eta_R(t) \leq (\eta_R(0) + \|\mathfrak{R}\|^2_{L_2((0,t)\times\mathbb{T})})\exp\left(\int_0^t K[\hat{\tau}](s)\,\mathrm{d}s\right)\ \forall t, \tag{25}$$

where

$$K[\hat{\tau}](t) := \max\left\{\frac{2C_P^2\overline{W}^2}{\gamma}\|\partial_x\hat{\tau}(t,\cdot)\|^2_{L_\infty(\mathbb{T})} + \frac{2\overline{W}^2}{\gamma}, \frac{3}{2}\right\}, \tag{26}$$

$\overline{W} := \|W\|_{C^3[-\overline{M},\overline{M}]}$, and C_P is the Poincaré constant on $\mathbb{T}$.

The proof of Theorem 3.1 is given in [7].

3.2 Discontinuous Galerkin scheme

Let us describe the discretisation of (4) under consideration.

To this end we use the same decomposition of the spatial domain and notation as in Section 2.2. In addition to the jump operator defined in (8) we define the average operator

$$\{v\} := \frac{1}{2}(v^+ + v^-) := \frac{1}{2}\Big(\lim_{s\searrow 0} v(\cdot + s) + \lim_{s\searrow 0} v(\cdot - s)\Big). \tag{27}$$

Note that for $v \in \mathbb{V}_p$ it holds $\{v\} \in L_2(\mathcal{E})$. We also define discrete gradient operators $G^\pm : H^1(\mathcal{T}) \to \mathbb{V}_p$ by

$$\int_{\mathbb{T}} G^\pm[\psi]\Phi = \sum_{K\in\mathcal{T}} \int_K \partial_x \psi \Phi - \int_{\mathcal{E}} [\psi]\Phi^\pm \;\forall \Phi \in \mathbb{V}_p, \tag{28}$$

where $H^1(\mathcal{T})$ is a so-called broken Sobolev space. Note that if $\psi \in H^1(\mathbb{T})$ then $G^\pm[\psi]$ is the L_2 orthogonal projection of $\partial_x\psi$ onto $\mathbb{V}_p$.

We study the following class of semi-discrete numerical schemes where $(\tau_h,\ v_h,\ \kappa_h) \in C^1(0,T;\mathbb{V}_p) \times C^1(0,T;\mathbb{V}_p) \times C^0(0,T;\mathbb{V}_p)$ are determined by

$$\begin{aligned}
0 &= \int_{\mathbb{T}} \partial_t \tau_h \Phi - G^-[v_h]\Phi \quad \forall\, \Phi \in \mathbb{V}_p \\
0 &= \int_{\mathbb{T}} \partial_t v_h \Psi - G^+[\kappa_h]\Psi + \mu G^-[v_h] G^-[\Psi] \quad \forall\, \Psi \in \mathbb{V}_p \\
0 &= \gamma \mathcal{A}_h(\tau_h, Z) + \int_{\mathbb{T}} \kappa_h Z - W'(\tau_h) Z \quad \forall\, Z \in \mathbb{V}_p,
\end{aligned} \tag{29}$$

where $\mathcal{A}_h : \mathbb{V}_p \times \mathbb{V}_p \to \mathbb{R}$ is a symmetric bilinear form representing a consistent discretisation of the Laplacian. We impose that it is coercive and continuous with respect to the dG seminorm on $\mathbb{V}_p$. Solutions to the semi-discrete scheme exist and admit a monotonously decreasing energy functional.

Our subsequent approach is similar to what we did for hyperbolic conservation laws. We determine a reconstruction of the numerical solution, which is a *strong* solution to a perturbed equation. A certain difficulty arises from the fact that strong solutions to (4) are far more regular than strong solutions to hyperbolic conservation laws. Therefore, we need two reconstruction approaches. The first approach is analogous to what we did in Section 2. The second approach is elliptic reconstruction, see [20]. While the reconstruction is explicitly computable in the first approach, called *discrete reconstruction*, we need to use elliptic a posteriori estimates for controlling the error in the second reconstruction step.

Definition 3.2 (Discrete reconstruction). *We define the* discrete reconstruction operator $\mathrm{D}^\pm : \mathbb{V}_p \to \mathbb{V}_{p+1}$ *by requiring*

$$0 = \int_{\mathbb{T}} \partial_x(\mathrm{D}^\pm[\Psi])\Phi - G^\pm[\Psi]\Phi \;\forall\, \Phi \in \mathbb{V}_p \;\textit{and}\; (\mathrm{D}^\pm[\Psi])^\pm = \Psi^\mp \;\textit{on}\; \mathcal{E} \tag{30}$$

for every $\Psi \in \mathbb{V}_p$.

Remark 3.3 (Continuity of discrete reconstruction). Note that for any $\Psi \in \mathbb{V}_p$ it holds $\mathrm{D}^\pm[\Psi] \in \mathbb{V}_{p+1} \cap C^0(\mathbb{T})$. In addition, the following approximation properties hold,

$$\|\Psi - \mathrm{D}^\pm[\Psi]\|^2_{L_2(\mathbb{T})} \lesssim \|\sqrt{h}[\Psi]\|^2_{L_2(\mathcal{E})} \tag{31}$$

$$|\Psi - \mathrm{D}^\pm[\Psi]|^2_{\mathrm{dG}} \lesssim \|\sqrt{h^{-1}}[\Psi]\|^2_{L_2(\mathcal{E})}. \tag{32}$$

These estimates are proven in [21].

Definition 3.4 (Continuous projection operator). *We define $P^C_p : L_2(\mathbb{T}) \to \mathbb{V}_p \cap C^0(\mathbb{T})$ to be the $L_2(\mathbb{T})$ orthogonal projection operator satisfying*

$$\int_{\mathbb{T}} P^C_p(w)\Phi = \int_{\mathbb{T}} w\Phi \ \forall\ \Phi \in \mathbb{V}_p \cap C^0(\mathbb{T}). \tag{33}$$

It is straightforward to verify the $L_2(\mathbb{T})$-stability of P^C_p, i.e., $\|P^C_p(w)\|_{L_2(\mathbb{T})} \leq \|w\|_{L_2(\mathbb{T})}$, and the optimal approximation properties of P^C_p, i.e.,

$$\|P^C_p(w) - w\|_{L_2(\mathbb{T})} \lesssim h^{p+1}\|w\|_{H^{p+1}(\mathbb{T})}. \tag{34}$$

Definition 3.5 (Continuous reconstruction operators). *We define three continuous reconstruction operators, $\mathcal{R}_1[\tau_h] \in H^3(\mathbb{T})$, $\mathcal{R}_2[\tau_h] \in H^2(\mathbb{T})$ and $\mathcal{R}[v_h] \in H^2(\mathbb{T})$ to be solutions of*

$$\begin{aligned} 0 &= \gamma\partial_{xx}\mathcal{R}_1[\tau_h] - P^C_{p+1}(W'(\tau_h)) + \mathrm{D}^+[\kappa_h] \\ 0 &= \gamma\partial_{xx}\mathcal{R}_2[\tau_h] - W'(\tau_h) + \kappa_h \\ 0 &= \partial_{xx}\mathcal{R}[v_h] - \partial_{xt}\mathcal{R}_1[\tau_h], \end{aligned} \tag{35}$$

respectively, such that each of the problems has matching mean value with the discrete solution, that is

$$0 = \int_{\mathbb{T}} \mathcal{R}_1[\tau_h] - \tau_h = \int_{\mathbb{T}} \mathcal{R}_2[\tau_h] - \tau_h = \int_{\mathbb{T}} \mathcal{R}[v_h] - v_h. \tag{36}$$

The reconstruction $\mathcal{R}_2[\tau_h]$ is the *elliptic reconstruction* of τ_h, see [20]. We will assume existence of an optimal order elliptic a posteriori estimate controlling $\|\tau_h - \mathcal{R}_2[\tau_h]\|_{\mathrm{dG}}$, that is, there exists a functional H_1 depending only upon τ_h and the problem data such that

$$\|\tau_h - \mathcal{R}_2[\tau_h]\|_{\mathrm{dG}} \lesssim H_1\left[\tau_h, \frac{1}{\gamma}(\kappa_h - W'(\tau_h))\right] \sim \mathcal{O}(h^p). \tag{37}$$

Example 3.1 (A posteriori control for the interior penalty discretisation). Taking $f := \kappa_h - W'(\tau_h)$, if $\mathcal{A}_h$ is an interior penalty discretisation, i.e.,

$$\mathcal{A}_h(\tau_h, Z) = \int_{\mathbb{T}} \partial_x \tau_h \partial_x Z - \int_{\mathcal{E}} [\tau_h]\{\partial_x Z\} + [Z]\{\partial_x \tau_h\} - \frac{\sigma}{h}[\tau_h][Z], \tag{38}$$

where σ is the penalty parameter and is chosen large enough to guarantee coercivity, we may use estimates of the form

$$\begin{aligned} H_1[\tau_h, f]^2 &= \sum_{K \in \mathcal{T}} h_K^2 \|f - \partial_{xx}\tau_h\|_{L_2(K)}^2 \\ &+ \sum_{e \in \mathcal{E}} \Big(h \|[\partial_x \tau_h]\|_{L_2(e)}^2 + \sigma^2 h^{-1} \|[\tau_h]\|_{L_2(e)}^2 \Big). \end{aligned} \tag{39}$$

See for example [15, Thm 3.1].

Lemma 3.6 (Reconstructed PDE system). *The reconstructions defined in Definition 3.5 satisfy the following perturbed version of* (4)

$$\begin{aligned} \partial_t \mathcal{R}_1[\tau_h] - \partial_x \mathcal{R}[v_h] &= 0 \\ \partial_t \mathcal{R}[v_h] - \partial_x W'(\mathcal{R}_1[\tau_h]) + \gamma \partial_{xxx} \mathcal{R}_1[\tau_h] - \mu \partial_{xx} \mathcal{R}[v_h] &= E, \end{aligned} \tag{40}$$

where

$$\begin{aligned} E := \partial_t(\mathcal{R}[v_h] - v_h) - \partial_x(W'(\mathcal{R}_1[\tau_h]) - P_{p+1}^C(W'(\tau_h))) \\ - \mu \partial_x(\partial_t \mathcal{R}_1[\tau_h] - \mathrm{D}^+[\partial_t \tau_h]). \end{aligned} \tag{41}$$

We obtain an a posteriori error estimate by combining Theorem 3.1 with Lemma 3.6 and an a posteriori estimate of E in Lemma 3.6. The upshot of our analysis is

Theorem 3.7 (A posteriori control of the reduced relative entropy). *Let*

$$\begin{aligned} (\tau, v) \in C^1(0, T; H^1(\mathbb{T})) \cap C^0(0, T; H^3(\mathbb{T})) \\ \times C^1(0, T; L_2(\mathbb{T})) \cap C^0(0, T; H^2(\mathbb{T})) \end{aligned}$$

solve the model problem (4) and $(\tau_h, v_h, \kappa_h) \in C^1([0, T), \mathbb{V}_p) \times C^1([0, T), \mathbb{V}_p) \times C^0([0, T), \mathbb{V}_p)$ *be the semi-discrete approximations generated by the scheme (29) then given the reduced relative entropy*

$$\begin{aligned} \eta_R(t) := \int_{\mathbb{T}} \frac{\gamma}{2}(\partial_x \tau - \partial_x \mathcal{R}_1[\tau_h])^2 \\ + \frac{1}{2}(v - \mathcal{R}[v_h])^2 + \frac{\mu}{4} \int_0^t |v - \mathcal{R}[v_h]|_{H^1(\mathbb{T})}, \end{aligned} \tag{42}$$

we have that

$$\eta_R(t) \lesssim \Big(\eta_R(0) + \int_0^t \mathfrak{E}_s[\tau_h(s), v_h(s), \kappa_h(s)]^2 \,\mathrm{d}\, s\Big) \exp\Big(\int_0^t K[\mathcal{R}_1[\tau_h]](s)\,\mathrm{d}\, s\Big)$$

with

$$\begin{aligned}
&\mathfrak{E}_t[\tau_h, v_h, \kappa_h]^2 \\
&:= H_1[\tau_h, \frac{1}{\gamma}(\kappa_h - W'(\tau_h))]^2 + \mu H_1[\partial_t \tau_h, \frac{1}{\gamma}(\partial_t \kappa_h - \partial_t W'(\tau_h))]^2 \\
&\quad + H_1[\partial_{tt}\tau_h, \frac{1}{\gamma}(\partial_{tt}\kappa_h - \partial_{tt} W'(\tau_h))]^2 + \|\sqrt{h^{-1}}[\tau_h]\|^2_{L_2(\mathcal{E})} \\
&\quad + \mu\|\sqrt{h^{-1}}[\partial_t \tau_h]\|^2_{L_2(\mathcal{E})} + \sum_{K\in\mathcal{T}} h^{2q}\|W'(\tau_h)\|^2_{H^{q+1}(K)} \\
&\quad + \frac{h}{\gamma^2}\Big(\|[\kappa_h]\|^2_{L_2(\mathcal{E})} + \|[\partial_{tt}\kappa_h]\|^2_{L_2(\mathcal{E})} + \|[\tau_h]\|^2_{L_2(\mathcal{E})} + \|[[\partial_{tt} W'(\tau_h)]\|^2_{L_2(\mathcal{E})}\Big) \\
&\quad + h\Big(\frac{\mu}{\gamma^2}\|[\partial_t\kappa_h]\|^2_{L_2(\mathcal{E})} + \frac{\mu}{\gamma^2}\|[\partial_t W'(\tau_h)]\|^2_{L_2(\mathcal{E})} + \|[\partial_t v_h]\|^2_{L_2(\mathcal{E})}\Big) \\
&\quad + \frac{1}{\gamma^2}\sum_{K\in\mathcal{T}} h^{2q+2}\Big(\|\partial_t W'(\tau_h)\|^2_{H^{q+1}(K)} + \|W'(\tau_h)\|^2_{H^{q+1}(K)} \\
&\qquad\qquad + \|\partial_{tt} W'(\tau_h)\|^2_{H^{q+1}(K)}\Big),
\end{aligned}$$

where the right hand side is evaluated at time t.

Numerical experiments, presented in [10], show that the error estimator given in Theorem 3.7 is of the same order as the exact error.

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