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On the decay rate of the Gauss curvature for isometric immersions

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Abstract. We address the problem of global embedding of a two dimensional Riemannian manifold with negative Gauss curvature into three dimensional Euclidean space. A theorem of Efimov states that if the curvature decays too slowly to zero then global smooth immersion is impossible. On the other hand a theorem of J.-X. Hong shows that if decay is sufficiently rapid (roughly like $t^{-(2+\delta)}$ for $\delta > 0$) then global smooth immersion can be accomplished. Here we present recent results on applying the method of compensated compactness to achieve a non-smooth global immersion with rough data and we give an emphasis on the role of decay rate of the Gauss curvature.

Keywords: isometric immersion problem, Gauss curvature, Gauss-Codazzi system, systems of balance laws, weak solutions, compensated compactness.

Mathematical subject classification: Primary: 53C42, 53C21, 53C45, 58J32, 35L65, 35M10; Secondary: 35L45, 57R40, 57R42, 76H05, 76N10.

1 Introduction

This article serves as a survey of recent applications [2, 1, 3, 4] of the method of compensated compactness to prove the global isometric immersion of a two dimensional Riemannian manifold with slowly decaying negative Gauss curvature into three dimensional Euclidean space. The immersion will lie in the Sobolev space $W^{2,\infty}_{loc}$ and hence will be locally in $C^{1,1}$, cf. Evans [9, Chapter 5]. This means that the immersion is smooth enough so that the Gauss curvature is well defined. The main difference in these results [2, 1, 3, 4] is the rate of the Gauss curvature considered in each work and as it is mentioned later the case of the slower decay rate $t^{-(2+\delta)}$ of Hong [13] is the one promoted here.

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The analysis of isometric immersions of two dimensional Riemannian manifolds with negative Gauss curvature into $\mathbb{R}^3$ has a long history going back to the classical result of Hilbert of 1901 [12]. The story is well documented in the recent monograph of Han and Hong [11] and we borrow freely from their discussion. Specifically Hilbert proved that the hyperbolic plane cannot be isometrically immersed into $\mathbb{R}^3$. In that case the Gauss curvature is a negative constant. Hilbert's result was generalized in 1963 by Efimov [8] to complete manifolds where the negative Gauss curvature has a slow rate of decay. Efimov's result [8] reads

Theorem 1.1 (Han and Hong [11, Theorem 10.0.1]). *Let $(\mathcal{M}, g)$ be a complete negative curved smooth Riemannian manifold with Gauss curvature K satisfying*

$$\limsup_{p\to\infty}\left|D\left(\frac{1}{\sqrt{|K|}}\right)(p)\right| < \frac{1}{2}\sqrt{\sqrt{41}-5},$$

for $p \in \mathcal{M}$ with D denoting the gradient. Then $(\mathcal{M}, g)$ admits no C^3 isometric immersion into $\mathbb{R}^3$.

Consider for example the case given by Han and Hong [11, Ex. 10.1.10], in which a negatively curved manifold has the metric in geodesic polar coordinates given by

$$g = d\rho^2 + B^2(\rho)d\theta^2, \qquad \rho > 0,\ \theta \in [0, 2\pi]$$

where B is defined by $\partial_{\rho\rho}B = -KB$ with $B(0) = 0$, $\partial_\rho B(0) = 1$, and the Gauss curvature $K = K(\rho)$ is a negative smooth even function satisfying

$$K = -a^2/(\rho^2 - 1),$$

if $|\rho| \geq 2$ for a positive constant a. We see $\rho^2|K(\rho)|$ decreases as ρ increases when $|\rho| \geq 2$ but the decay rate as measured by

$$D(1/\sqrt{|K|}) = \frac{1}{a}\partial_\rho\sqrt{(\rho^2-1)} = \frac{\rho}{a\sqrt{\rho^2-1}}$$

has the limit $1/a$ as $\rho \to \infty$. Hence in this case Theorem 1.1 asserts there is no C^3 isometric immersion if a is sufficiently large.

Since K in the above is $O(\rho^{-2})$, a natural next step would be to try the case when K is $O(\rho^{-2-\delta})$ as $\rho \to \infty$. In fact this is the content of the theorem of Hong [13]:

Theorem 1.2 (Han and Hong [11, Theorem 10.2.2]). *For a complete simply connected two dimensional Riemannian manifold $(\mathcal{M}, g)$ with negative Gauss curvature K with metric $g = dt^2 + B^2(x, t)dx^2$. Assume for some constant $\delta > 0$*

(i) $t^{2+\delta}\,|K|$ *is decreasing in* $|t|$, $|t| > T$*;*

(ii) $\partial_x^i \ln|K|$*, for* $i = 1, 2$ *and* $t\partial_t\partial_x \ln|K|$ *are bounded;*

(iii) K *is periodic in* x *with period* 2π.

Then $(\mathcal{M}, g)$ *admits a smooth isometric immersion in* $\mathbb{R}^3$.

Now, the question raised is whether one can capture such a result in the setting of non-smooth immersions. A search for "corrugated immersions" that ask the data to be "rough" and not in C^1 is the issue pursued in the recent articles [2, 1, 3, 4]. In particular, we focus our discussion on data in L^∞ for which the method of compensated compactness has been successfully applied.

2 An Exposition on Non-Smooth Immersions

In two papers Chen, Slemrod and Wang [2] and Cao, Huang and Wang [1] have used the method of compensated compactness to establish global isometric immersions into $\mathbb{R}^3$ for two dimensional Riemannian manifolds for rough data. In the examples considered the Gauss curvature was negative and decayed at least as t^{-4} where initial data was given at $t = 0$. Needless to say that leaves open the question as whether the compensated compactness method will work for a slower rate of decay. Certainly based on the paper of Hong [13] we expect the result to be true for decay of order $t^{-2-\delta/2}$ where δ is between 0 and 4. This is the decay rate considered in [4] in the context of non-smooth immersions. To be precise, the Gauss curvature is chosen to be

$$K = -\frac{C}{(1+|t|)^{2+\frac{\delta}{2}}}, \quad C > 0 \tag{2.1}$$

as taken in Hong [13] and $\delta \in (0, 4)$.

The proof of Hong is a careful study of the hyperbolic system of two balance laws (the two Codazzi equations) and one closure relation (the Gauss equation) and requires two separate steps. The first step is to establish existence of smooth solutions to the balance laws for small, smooth data prescribed at a large enough time $t = T_1$. The reason for this part is that is only after large time that the decay of the Gauss curvature may be exploited to obtain the relevant C^1 a priori estimates. The second part of the proof is rather standard and simply asks for the initial data at $t = 0$ to be sufficiently small and smooth to enable us to get a solution up to $t = T_1$. Here no reference is given in Han and Hong but a standard existence, uniqueness theorem for quasi-linear hyperbolic systems will suffice.

Such a theorem may be found in Janenko and Rozdestvenskii [14, Chapter 1, Sect. 8] where the growth in C^1 of solutions is governed by a coupled pair of ordinary differential equations, one which is of Ricatti type. Hence just as in the classical theory of ordinary differential equations small data allows for a longer time of existence.

Now, in [4] as compliment to Hong's result we reconsider the first part of Hong's program and show that in fact that rough L^∞ data suffices at the initial time $t = T_1$ and that the method of compensated compactness will yield existence of weak solutions to the Gauss-Codazzi system for $t > T_1$. Of course one may wonder how to reach time $t = T_1$ from $t = 0$ by avoiding the classical smooth solution existence, uniqueness theorem and establishing a new global result in the non-smooth case. The answer for this step is to use the local existence result of weak solutions of Dafermos and Hsiao [6] to reach time $t = T_1$. Nevertheless we believe that the new application of the compensated compactness method is appealing and is of independent interest and in this survey we promote this analysis of [4] with the slow decay rate of Hong.

It should be mentioned that for discontinuous data of bounded variation, isometric immersions have been established using a different method in [3], the so-called random choice method of Glimm [10], but again with decay rate at least as t^{-4}. Both methods, the compensated compactness and the random choice, were introduced and developed in the context of the theory of hyperbolic balance laws. An exposition of the current state of the theory of systems of balance laws can be found in the book [5].

In [4] there are two main ideas that will be used in the analysis. The first following Hong is to write the Gauss-Codazzi system in Rozhdestvenskii-Poznyak form and prove the global existence of entropy admissible weak solutions for this system via a viscous approximate scheme for which we adopt the compensated compactness framework. Secondly, having obtained a weak solution for this form of the Gauss-Codazzi equations we have to be sure that a version of the fundamental theorem of surface theory (see for example do Carmo [7] as well as Han and Hong [11]) applies to yield the non-smooth immersion. Fortunately such a result exists and has been given by S. Mardare [15]. Thus we will obtain a global $W^{2,\infty}_{loc}$ immersion which hence implies a $C^{1,1}$ regularity locally.

It should be noted that the relevant Gauss-Codazzi system can be written as a linearly degenerate system or what is termed "weakly non-linear quasi-linear" system in the monograph of Janenko and Rozdestvenskii [14] and is discussed in Chap. 4, Sec. 4 of that book. They note that such systems possess the property that uniform boundedness of solutions on $0 \leq t \leq T$ and strict hyperbolicity imply uniform boundedness of first derivatives on $0 \leq t \leq T$ if these derivatives are initially bounded. Thus it appears that the crucial estimates will be a uniform bound on the dependent variables and in addition proof that the strict hyperbolicity is not lost. It was this path that was followed by Hong [13].

The paper is organized as follows: In Section 3 we set up the isometric immersion problem, present Mardare's theorem on non-smooth immersions and write the Gauss-Codazzi system for geodesically complete Riemannian manifolds. In Section 4 we give a viscous approximation scheme for resolving the relevant Gauss-Codazzi system and in Section 5 we state the main result of [4] for the slow decay rate (2.1) under consideration. Finally Section 6 pursues the issue as to whether the decay rate of the Gauss curvature can be further reduced.

3 Set Up of the Problem

Let $\Omega \subset \mathbb{R}^2$ be an open set. Consider a map $\mathbf{y} : \Omega \to \mathbb{R}^3$ having the tangent plane of the surface $\mathbf{y}(\Omega) \subset \mathbb{R}^3$ at $\mathbf{y}(x_1, x_2)$ spanned by the vectors $\{\partial_1\mathbf{y}, \partial_2\mathbf{y}\}$. Then, the corresponding metric is

$$ds^2 = (\partial_1\mathbf{y} \cdot \partial_1\mathbf{y})(dx_1)^2 + 2(\partial_1\mathbf{y} \cdot \partial_2\mathbf{y})dx_1\, dx_2 + (\partial_2\mathbf{y} \cdot \partial_1\mathbf{y})(dx_2)^2\,. \quad (3.1)$$

The isometric immersion problem is an *inverse* problem: Given (g_{ij}) $i, j = 1, 2$ functions in Ω, with $g_{12} = g_{21}$, find a map $\mathbf{y} : \Omega \to \mathbb{R}^3$ so that

$$\partial_1\mathbf{y} \cdot \partial_1\mathbf{y} = g_{11}, \qquad \partial_1\mathbf{y} \cdot \partial_2\mathbf{y} = g_{12}, \qquad \partial_2\mathbf{y} \cdot \partial_2\mathbf{y} = g_{22} \quad (3.2)$$

with a linearly independent set $\{\partial_1\mathbf{y}, \partial_2\mathbf{y}\}$ in $\mathbb{R}^3$. Hence, the isometric immersion problem is fully nonlinear in the three unknowns being the three components of the map $\mathbf{y}$.

We recall that a two dimensional manifold $(\mathcal{M}, g)$ parametrized by Ω with associated metric $g = (g_{ij})$ admits two fundamental forms: the first fundamental form I for $\mathcal{M}$ on Ω is

$$I \doteq g_{11}(dx_1)^2 + 2g_{12}dx_1dx_2 + g_{22}(dx_2)^2 \quad (3.3)$$

and the second fundamental form II is

$$II \doteq -d\mathbf{n} \cdot d\mathbf{y} = h_{11}(dx_1)^2 + 2h_{12}dx_1\, dx_2 + h_{22}(dx_2)^2 \quad (3.4)$$

with $\mathbf{n}$ being the unit normal vector to $\mathcal{M}$. By equating the cross-partial derivatives of $\mathbf{y}$, the isometric immersion problem as stated above reduces to the Gauss-Codazzi system

$$\begin{aligned} \partial_1 M - \partial_2 L &= \Gamma^{(2)}_{22} L - 2\Gamma^{(2)}_{12} M + \Gamma^{(2)}_{11} N \\ \partial_1 N - \partial_2 M &= -\Gamma^{(1)}_{22} L + 2\Gamma^{(1)}_{12} M - \Gamma^{(1)}_{11} N \end{aligned} \quad (3.5)$$

with the condition

$$LN - M^2 = K\,, \tag{3.6}$$

where

$$L = \frac{h_{11}}{\sqrt{|g|}}, \qquad M = \frac{h_{12}}{\sqrt{|g|}}, \qquad N = \frac{h_{22}}{\sqrt{|g|}} \tag{3.7}$$

and $|g| \doteq \det(g_{ij}) = g_{11}g_{22} - g_{12}^2$. The Gauss curvature $K = K(x_1, x_2)$ is given by

$$K(x_1, x_2) = \frac{R_{1212}}{|g|}\,, \tag{3.8}$$

where R_{ijkl} is the curvature tensor

$$R_{ijkl} = g_{lm}\left(\partial_k \Gamma_{ij}^{(m)} - \partial_j \Gamma_{ik}^{(m)} + \Gamma_{ij}^{(n)}\Gamma_{nk}^{(m)} - \Gamma_{ik}^{(n)}\Gamma_{nj}^{(m)}\right)\,, \tag{3.9}$$

and $\Gamma_{ij}^{(k)}$ is the Christoffel symbol

$$\Gamma_{ij}^{(k)} \doteq \frac{1}{2} g^{kl}\left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right)\,. \tag{3.10}$$

Here, the indices $i, j, k, l = 1, 2$, $(\partial_1, \partial_2) = (\partial_{x_1}, \partial_{x_2})$ and the summation convention is used. Also, (g^{kl}) is the inverse of (g_{ij}).

The fundamental theorem of surface theory states that given forms I and II with (g_{ij}) being positive definite and smooth coefficients, (g_{ij}) and (h_{ij}) that satisfy the Gauss-Codazzi system (3.5)-(3.7), then there exists a surface embedded into $\mathbb{R}^3$ with first and second fundamental forms I and II. This result has been extended by S. Mardare [15] when $(h_{ij}) \in L^\infty_{loc}(\Omega)$ for given $(g_{ij}) \in W^{1,\infty}_{loc}(\Omega)$ and then, the surface immersed is $C^{1,1}(\Omega)$ locally. Thus, the isometric immersion problem reduces to solving the Gauss-Codazzi system (3.5)-(3.7) for $(h_{ij}) \in L^\infty_{loc}(\Omega)$ with a given positive definite metric $(g_{ij}) \in W^{1,\infty}_{loc}(\Omega)$ and then, immediately, we recover the immersion surface $\mathbf{y}(\Omega)$, which is $C^{1,1}$ locally.

Now let us recall the following definition.

Definition 3.1. $(\mathcal{M}, g)$ *is a geodesically complete Riemannian manifold if and only if every geodesic can be extended indefinitely.*

Under the assumption that our two dimensional manifold is geodesically complete and simply connected we can simplify the structure of our metric. This is actually the structure used in [13, 4]. The exact result is as follows and is essentially due to Hadamard but we use the presentation given in Han and Hong [11].

Lemma 3.2 (Han and Hong [11, Lemma 10.2.1]). *Let $(\mathcal{M}, g)$ be a geodesically complete simply connected smooth two dimensional Riemannian manifold with non-positive Gauss curvature. Then there exists a global geodesic coordinate system (x, t) in $\mathcal{M}$ with metric*

$$g = dt^2 + B^2(x,t)dx^2 \tag{3.11}$$

where B is a smooth function satisfying $B(x,0) = 1$ and $\partial_t B(x,0) = 0$ for $x \in \mathbb{R}$.

A direct substitution of (3.11) in (3.5) then yields that L, M, N satisfy the Gauss-Codazzi system in the form

$$\begin{aligned} \partial_t L - \partial_x M &= L\partial_t \ln B - M\partial_x \ln B + NB\partial_t B, \\ \partial_t M - \partial_x N &= -M\partial_t \ln B, \end{aligned} \tag{3.12}$$

with

$$LN - M^2 = KB^2, \tag{3.13}$$

where $\partial_{tt} B = -KB$ defines the Gauss curvature K in terms of the metric. In [4], we work with the scaled variables

$$l = \frac{L}{B^2\sqrt{|K|}}, \quad m = \frac{M}{B\sqrt{|K|}}, \quad n = \frac{N}{\sqrt{|K|}} \tag{3.14}$$

that satisfy the system

$$\begin{aligned} \partial_t l - \frac{1}{B}\partial_x m + (l-n)\partial_t \ln B + \frac{l}{2}\partial_t \ln|K| - \frac{m}{2B}\partial_x \ln|K| = 0, \\ \partial_t m - \frac{1}{B}\partial_x n + 2m\partial_t \ln B + \frac{m}{2}\partial_t \ln|K| - \frac{n}{2}\partial_x \ln|K| = 0, \end{aligned} \tag{3.15}$$

with

$$ln - m^2 = -1. \tag{3.16}$$

The eigenvalues associated with system (3.15) are

$$\lambda_1 = \frac{m-1}{lB}, \qquad \lambda_2 = \frac{m+1}{lB} \tag{3.17}$$

and we see that each characteristic field is linear degenerate. System (3.15) is *strictly hyperbolic* if $\lambda_1 < \lambda_2$, or equivalently if l is finite.

4 The Approximate Scheme

As we will be dealing with non-smooth data a natural approach is embed our initial value problem in viscous approximating system with viscosity $\mu > 0$ and attempt to recover our solution as limit for $\mu \to 0+$. Therefore, consider the viscous approximations (l^μ, m^μ, n^μ) that satisfy system

$$\begin{aligned}
&\partial_t l^\mu - \frac{1}{B}\partial_x m^\mu + (l^\mu - n^\mu)\partial_t \ln B + \frac{l^\mu}{2}\partial_t \ln|K| - \frac{m^\mu}{2B}\partial_x \ln|K| = \mu\partial_{xx} l^\mu, \\
&\partial_t m^\mu - \frac{1}{B}\partial_x n^\mu + 2m^\mu\partial_t \ln B + \frac{m^\mu}{2}\partial_t \ln|K| - \frac{n^\mu}{2}\partial_x \ln|K| = \mu\partial_{xx} m^\mu,
\end{aligned} \tag{4.1}$$

with

$$l^\mu n^\mu - (m^\mu)^2 = -1. \tag{4.2}$$

Here $\mu > 0$ is a constant "viscosity". Let (u, v) be the Riemann invariants given by

$$u = -\frac{m^\mu}{l^\mu} + \frac{1}{l^\mu}, \qquad v = -\frac{m^\mu}{l^\mu} - \frac{1}{l^\mu} \tag{4.3}$$

It is straightforward to compute the viscous equations of (u, v):

$$\begin{aligned}
&\partial_t u + \frac{v}{B}\partial_x u + v(1+u^2)\partial_t \ln B - \frac{(u-v)}{4}\left(\partial_t \ln|K| + \frac{u}{B}\partial_x \ln|K|\right) \\
&\quad = \frac{\mu}{(v-u)}\left\{((\partial_x u)^2 - (\partial_x v)^2) - \frac{2u}{v-u}(\partial_x u - \partial_x v)^2 - u\partial_{xx} u\right\},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&\partial_t v + \frac{u}{B}\partial_x v + u(1+v^2)\partial_t \ln B - \frac{(v-u)}{4}\left(\partial_t \ln|K| + \frac{v}{B}\partial_x \ln|K|\right) \\
&\quad = \frac{\mu}{(v-u)}\left\{((\partial_x u)^2 - (\partial_x v)^2) - \frac{2v}{v-u}(\partial_x u - \partial_x v)^2 + v\partial_{xx} v\right\}.
\end{aligned} \tag{4.5}$$

Now strict hyperbolicity in the (u, v) variables is equivalent to $u \neq v$ and the system is *uniformly strictly hyperbolic* if $v - u$ is uniformly bounded away from zero.

5 A $C^{1,1}$ Isometric Immersion

We consider a special class of metrics of the form (3.11) just to keep our ideas simple and clear. Of course the method could be generalized beyond this case at the cost of greater complications.

Choose the Gauss curvature K to be given by (2.1) and set $h \doteq B > 0$ and $k^* \doteq |K|$. Furthermore assume that k^*, h are taken to be independent of x. Then h and k^* satisfy

$$\partial_{tt}h = k^*h, \quad h(0) = 1, \quad \partial_t h(0) = 0. \tag{5.1}$$

Consider the domain $\Omega = [T_1, T_2] \times \mathbb{R}^3$ with T_1 sufficiently large and any value $T_2 > T_1$. The corrugated immersion contructed below is over this domain. If one would like to see a global immersion, then using the local existence result of [6] one can reach time $t = T_1$ from $t = 0$ starting at $t = 0$ with small BV data depending on the size of T_1.

The main result of [4] is:

Theorem 5.1. *Let* $(\mathcal{M}, g)$ *be a geodesically complete simply connected and smooth two dimensional Riemannian manifold with non-positive Gauss curvature* K *and a metric of the form* (3.11)*. Assume that* $h \doteq B$ *and* $k^* = |K|$ *are independent of* x *satisfying* (5.1) *and* k^* *is given by* (2.1)*. There exists* $\mathbf{y} \in W^{2,\infty}_{loc}(\Omega)$ *satisfying the embedding equations.*

Proof. The heart of the matter is to establish uniform L^∞ bounds on the solutions to the viscous system (4.1)-(4.2). Under the choice of curvature (2.1) and a careful analysis, we can derive the so called *sign-switch property* when $0 < \delta < 4$ as well as crucial estimates on h and $\partial_t h$. Combining these with a comparison argument, we establish an invariant region for (u, v) and hence apriori L^∞ estimates for the viscous approximations (l^μ, m^μ, n^μ). It should be mentioned that the proof of the invariant region of (u, v) is motivated by the one given in Han-Hong [11]. The advantage of the one given in [4] is its relative simplicity and the precise estimates for (u, v) from above, below (respectively). Next, we show a crucial H^{-1}_{loc} estimate which is needed to apply the method of compensated compactness following a standard argument say as given in the paper of Cao, Huang and Wang [1] and use the entropy, entropy flux pair

$$\eta = -\frac{m^{\mu 2} + 1}{l^\mu}, \qquad q = \frac{m^{\mu 3} - m^\mu}{h l^{\mu 2}}. \tag{5.2}$$

Having these, we observe that the compensated compactness framework as described in Theorem 4.1 of [2] is applicable. This implies that there exists a subsequence, still labeled (l^μ, m^μ, n^μ) that converges weak* in $L^\infty(\Omega)$ to $(\tilde{l}, \tilde{m}, \tilde{n})$ as $\mu \to 0$ and the limit $(\tilde{l}, \tilde{m}, \tilde{n})$ is a bounded weak solution of the Gauss-Codazzi system in the domain Ω. By applying Mardare's theorem [15], we conclude the result.

All the details of the proof can be found in [4]. □

6 A Weaker Decay Rate

Immediate inspection of our proofs in [4] raise a natural question of whether we can produce k^*, h with weaker decay than given by (2.1) and still satisfy the strict hyperbolicity condition as well as $k^*, sk^* \in L^1[0, \infty)$. In fact the answer is yes for the example:

$$k^* = \frac{1}{(3+t)^2(\ln(3+t))^p}, \qquad t > 0, \; p > 1. \tag{6.1}$$

In other words, for this special choice, it is shown that the preservation of strict hyperbolicity is retained. Hence any lack of non-smooth embedding must be due to lack of L^∞ bounds on two of the three components on the second fundamental form (the third component is a priori bounded). For the details, we refer the reader to [4, Section 7].

A summary of our observations is given in the following theorem.

Theorem 6.1. *Assume that h and k^* are independent of x satisfying* (5.1) *and k^* is as given by* (6.1)*. Then if T_1 is sufficiently large, then there is an invariant region for (u, v) to remain away from zero for all times $t > T_1$. Hence strict hyperbolicity of* (3.15)-(3.16) *would not be lost.*

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