

A note on integrability and finite orbits for subgroups of $\text{Diff}(\mathbb{C}^n, 0)$

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Abstract. In this note we extend to arbitrary dimensions a couple of results due respectively to Mattei-Moussu and to Camara-Scardua in dimension 2. We also provide examples of singular foliations having a Siegel-type singularity and answering a question independently raised by Abate and by Genzmer.

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1 Introduction

This note concerns certain recently investigated aspects of higher dimensional generalizations of Mattei-Moussu's celebrated topological characterization of integrable holomorphic foliations in dimension 2, cf. [13]. If \mathcal{F} denotes a singular holomorphic foliation defined about the origin of \mathbb{C}^2 , then the fundamental issue singled out in [13] is the fact that the existence of holomorphic first integrals for \mathcal{F} can be read off its topological dynamics. In particular, the existence of these first integrals can be detected at the level of the topological dynamics of the holonomy pseudogroup of \mathcal{F} . An immediate consequence of their criterion is that the existence of (non-constant) holomorphic first integrals for singular foliations as above is a property invariant by topological conjugation.

One basic question concerning higher dimensional generalizations of Mattei-Moussu's result was to know whether a (local) 1-dimensional singular holomorphic foliation on $(\mathbb{C}^n, 0)$ topologically conjugate to another holomorphic foliation possessing $n - 1$ independent holomorphic first integrals should possess $n - 1$ independent holomorphic first integrals as well. This type of questions recently began to be investigated in [5], [7] partly due to much recent progress

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in the study of the local dynamics of *parabolic* diffeomorphisms of $(\mathbb{C}^2, 0)$, see [1], [2], [3]. In fact, a high point in Mattei-Moussu's argument is their proof that a pseudogroup of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ having *finite orbits* must itself be finite and the main theorem in [5] is an extension of this to $(\mathbb{C}^2, 0)$ under certain additional conditions. Similarly, in [7], the authors considered the same question for foliations with Siegel singular points on $(\mathbb{C}^3, 0)$ which is justified by the fact that these Siegel foliations are the fundamental building blocks for the general case. Here it is convenient to remind the reader that a singular point of a 1-dimensional holomorphic foliation is said to be of Siegel type (or a Siegel singular point) if all the eigenvalues of the foliation at this singular point are different from zero and, moreover, $0 \in \mathbb{C}$ belongs to the convex hull defined by these eigenvalues. By resorting to a celebrated result due to M. Abate, Camara and Scardua affirmatively answered the question provided that the associated holonomy maps have isolated fixed points.

Very recently, a general counterexample was found in [16]. More precisely in [16] S. Pinheiro and the second author exhibited two topologically conjugate foliations on $(\mathbb{C}^3, 0)$ such that one admits two independent holomorphic first integrals but not the other. It so became clear that extending Mattei-Moussu theorem to higher dimensions is a subtle problem and it prompted to a deeper analysis of the basic ingredients in the two-dimensional argument namely, the nature of pseudogroups of $(\mathbb{C}^n, 0)$ having finite orbits and the corresponding consequences for foliations having Siegel singular points. The purpose of this Note is to contribute to the understanding of these questions by proposing an elementary generalization of Mattei-Moussu result for pseudogroups of $(\mathbb{C}^n, 0)$, which also yields a higher dimensional version of the main theorem in [7], and by answering in the *negative* the general question for Siegel singular points left open in [7].

To state our main results, let us work in the context of pseudogroups of $\text{Diff}(\mathbb{C}^n, 0)$ (the reader may check Section 2 for definitions and terminology). Our first result is a simple elaboration of the corresponding statement in [13] that turns out to generalize the corresponding result in [7] since it dispenses with the use of the deep theorem on the existence of parabolic domains due to M. Abate [1] and valid only in dimension 2.

Theorem A. *Let $G \subset \text{Diff}(\mathbb{C}^n, 0)$ be a finitely generated pseudogroup on a small neighborhood of the origin in \mathbb{C}^n . Given $g \in G$, let $\text{Dom}(g)$ denote the domain of definition of g as element of the pseudogroup in question. Suppose that for every $g \in G$ and $p \in \text{Dom}(g)$ satisfying $g(p) = p$, one of the following holds: either p is an isolated fixed point of g or g coincides with the identity on a neighborhood of p . Then the pseudogroup G has finite orbits on a neighborhood of the origin if and only if G itself is finite.*

Remark. When G is a subgroup of $\text{Diff}(\mathbb{C}, 0)$ the assumption of Theorem A is automatically verified so that the statement is reduced to Mattei-Moussu's corresponding result in [13]. On the other hand, it is proved in [13] that a subgroup of $\text{Diff}(\mathbb{C}, 0)$ is not only finite but also cyclic. In full generality, the second part of the statement cannot be generalized to higher dimensions since every finite group embeds into a matrix group of sufficiently high dimension. In Section 2 the reader will find simple examples showing that, in fact, the group need not be cyclic already in dimension 2 and even if the assumption of Theorem A about "isolated fixed points" is satisfied.

A two-dimensional variant of Theorem A is proved in [7] by resorting to Abate's theorem in [1] (cf. Section 3 for a detailed comparison between the statement in [7] and Theorem A). The authors of [7] then go ahead to apply their result to the problem of "complete integrability" of differential equations. Our Theorem A enables us to conduct a similar discussion in higher dimensions as it will be seen in Section 3. In particular, in Section 3 we shall also state and prove Theorem 5 which extends the main theorem in [7] to arbitrary dimensions.

For the time being, however, it suffices to consider the situation discussed in [7] namely, \mathcal{F} is a foliation on $(\mathbb{C}^3, 0)$ having a Siegel singular point at the origin and leaving invariant the three "coordinate axes". Roughly speaking, the question addressed by Camara-Scardua is to decide whether or not \mathcal{F} admits two independent holomorphic first integrals provided that the holonomy map associated to a certain invariant axis has finite orbits (cf. Section 3 for accurate statements). When a foliation as above admits two independent holomorphic first integrals it is said to be *completely integrable*. Concerning the formulation of the result in [7] about the existence of these first integrals, it is however convenient to mention an issue already pointed out by Y. Genzmer and by M. Abate in their reviews for MathSciNet and for Zentralblatt, respectively. Namely, whereas the methods of [7] clearly require the corresponding holonomy map to have isolated fixed points, the authors have failed to explicitly mention this condition in the statement of their main theorem. The question raised by Abate and by Genzmer concerns precisely the validity of Camara-Scardua result when no assumption involving isolated fixed points is put forward. In other words, the question is whether or not a Siegel singularity giving rise to a holonomy map with finite orbits must be *completely integrable*. Though an affirmative answer to the latter question was generally expected, Theorem B below shows that this is not the case.

Theorem B. *Let \mathcal{F} denote the foliation associated with the vector field*

$$X = x(1 + x^2yz^3)\frac{\partial}{\partial x} + y(1 - x^2yz^3)\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$$

which has a Siegel singular point at the origin. This foliation \mathcal{F} does not have two independent holomorphic first integrals (though it possesses one non-constant holomorphic first integral). Besides the holonomy map associated with the axis $\{x = y = 0\}$ has finite orbits whereas it does not generate a finite subgroup of $\text{Diff}(\mathbb{C}^2, 0)$.

In the above example, the reader will note that the restriction of X to the invariant plane $\{z = 0\}$ yields the radial vector field $x\partial/\partial x + y\partial/\partial y$ which admits the meromorphic first integral y/x . This also answers in the negative a refinement of the Abate-Genzmer question for Siegel singular point that had been speculated by Mattei, namely whether the existence of a meromorphic first integral on the transverse plane plus the assumption of finite orbits for the holonomy map might force the holonomy map in question to have finite order.

The simplest example of an element of $\text{Diff}(\mathbb{C}^2, 0)$ having finite orbits is obtained by setting $F(x, y) = (x + f(y), y)$ with $f(0) = 0$. For example, let $f(y) = 2\pi iy$. For this choice of f , the resulting diffeomorphism F can be realized as the holonomy map of the foliation \mathcal{F} associated with the vector field

$$X = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial z}$$

with respect to the invariant curve $\{x = y = 0\}$. The foliation \mathcal{F} does not admit two independent holomorphic first integrals whereas the following holds:

- The foliation \mathcal{F} admits one non-constant holomorphic first integral, namely the projection $\Pi_2(x, y, z) = y$.
- The holonomy map associated with the invariant axis $\{x = y = 0\}$ has finite orbits.

However the reader will also note that all the leaves of \mathcal{F} are locally closed as there follows easily from the explicit parameterizations $(x_0, y_0, z_0) \mapsto (y_0 t, y_0, z_0 e^t)$ for the integral curves of X . Nonetheless the foliation \mathcal{F} does not satisfy the assumption of [7] (or more generally the assumption in our Theorem 5, cf. Section 3) since the singular point is not of Siegel type (one of its eigenvalues is equal to zero).

Before proceeding further, let us justify our claim that \mathcal{F} does not admit two independent holomorphic first integrals. For this, note first that $\Pi_2(x, y, z) = y$ is already non-constant holomorphic first integral for \mathcal{F} . Next, consider the restriction $\mathcal{F}_{\{y=0\}}$ of \mathcal{F} to the invariant plane given by $\{y = 0\}$. The restricted foliation $\mathcal{F}_{\{y=0\}}$ is associated with the integral curves of the vector field $z\partial/\partial z$.

so that it admits $h(x, z) = x$ as holomorphic first integral. Now assume aiming at a contradiction that \mathcal{F} admits two independent holomorphic first integrals. Then the invariance of $\{z = 0\}$ by \mathcal{F} has to be identifiable from the two first integrals in question. Therefore at least one of these (non-constant) first integrals should be divisible by z . Since Π_2 is not divisible by z , there must exist another first integral $g(x, y, z)$ having the form

$$g(x, y, z) = z\bar{g}(x, y, z)$$

for some holomorphic function $\bar{g}(x, y, z)$. The function $\bar{g}(x, y, z)$ can be chosen so as not to be divisible by y . In fact, if $\bar{g}(x, y, z) = y^l \tilde{g}(x, y, z)$ for $l \geq 1$ and some holomorphic function $\tilde{g}(x, y, z)$, then the alternative first $z\tilde{g}(x, y, z)$ can be considered in the place of the initial function g . From this, we conclude that $\bar{g}(x, 0, z)$ is not identically zero. Since h , defined by $h(x, z) = x$, is a holomorphic first integral for $\mathcal{F}_{\{y=0\}}$ and $\bar{g}(x, 0, z)$ is not identically zero, there follows that $g(x, 0, z) = z\bar{g}(x, 0, z)$ coincides with a holomorphic function of h , i.e. there is some holomorphic function H defined around $0 \in \mathbb{C}$ such that $g(x, 0, z) = H \circ h(x, z) = H(x)$. In other words, the function $g(x, 0, z) = z\bar{g}(x, 0, z)$ cannot depend on z which clearly contradicts the holomorphic nature of $\bar{g}(x, 0, z)$.

This short paper is organized as follows. Section 2 contains the proof of Theorem A along with the relevant definitions. As mentioned, Theorem A is a simple elaboration of the arguments in [13]. Section 3 contains a brief digression on Siegel singular points which enables us to state and prove Theorem 5 extending to higher dimensions the result in [7]. The proof of Theorem 5, in turn, amounts to a simple combination of Theorem A and some useful results due to P. Elizarov-II'yashenko and to Reis, [8], [18] connecting the linearization problem of these singular foliations to the same question for certain holonomy maps. Finally, in Section 4, a few interesting examples of local dynamics of diffeomorphisms tangent to the identity will be provided. These local diffeomorphisms will also be realized as local holonomy maps of suitable foliations and, by building on these examples, the proof of Theorem B will quickly be derived.

2 Proof of Theorem A

In the sequel, G denotes a finitely generated subgroup of $\text{Diff}(\mathbb{C}^n, 0)$, where $\text{Diff}(\mathbb{C}^n, 0)$ stands for the group of germs of local holomorphic diffeomorphisms of \mathbb{C}^n fixing the origin. Assume that G is generated by the elements $h_1, \dots, h_k \in \text{Diff}(\mathbb{C}^n, 0)$. A natural way to make sense of the local dynamics

of G consists of choosing representatives for h_1, \dots, h_k as local diffeomorphisms fixing $0 \in \mathbb{C}$. These representatives are still denoted by h_1, \dots, h_k and, once this choice is made, G itself can be identified to the *pseudogroup* generated by these local diffeomorphisms on a (sufficiently small) neighborhood of the origin. It is then convenient to begin by briefly recalling the notion of *pseudogroup*. For this, consider a small neighborhood V of the origin where the local diffeomorphisms h_1, \dots, h_k , along with their inverses $h_1^{-1}, \dots, h_k^{-1}$, are defined and one-to-one. The pseudogroup generated by h_1, \dots, h_k (or rather by $h_1, \dots, h_k, h_1^{-1}, \dots, h_k^{-1}$ if there is any risk of confusion) on V is defined as follows. Every element of G has the form $F = F_s \circ \dots \circ F_1$ where each $F_i, i \in \{1, \dots, s\}$, belongs to the set $\{h_i^{\pm 1}, i = 1, \dots, k\}$. The element $F \in G$ should be regarded as an one-to-one holomorphic map defined on a subset of V . Indeed, denoting by $\text{Dom}_V(h)$ the domain of definition of $F = F_s \circ \dots \circ F_1$ as element of the pseudogroup in question, there follows that $\text{Dom}_V(h)$ consists of those points $x \in V$ such that for every $1 \leq l < s$ the point $F_l \circ \dots \circ F_1(x)$ belongs to V . Since the origin is fixed by all the diffeomorphisms h_1, \dots, h_k , we conclude that every element F in this pseudogroup possesses a non-empty open domain of definition. This domain of definition may however be disconnected. Whenever no misunderstanding is possible, the pseudogroup defined above will also be denoted by G and we are allowed to shift back and forward from G viewed as pseudogroup or as group of germs.

Let us continue with some definitions that will be useful throughout the text. Suppose we are given local holomorphic diffeomorphisms $h_1, \dots, h_k, h_1^{-1}, \dots, h_k^{-1}$ fixing the origin of \mathbb{C}^n . Let V be a neighborhood of the origin where all these local diffeomorphisms are defined and one-to-one. From now on, let G be viewed as the pseudogroup acting on V generated by these local diffeomorphisms.

Definition 1. *The V_G -orbit $\mathcal{O}_V^G(p)$ of a point $p \in V$ is the set of points in V obtained from p by taking its image through every element of G whose domain of definition (as element of G) contains p . In other words,*

$$\mathcal{O}_V^G(p) = \{q \in V ; q = h(p), h \in G \text{ and } p \in \text{Dom}_V(h)\}.$$

Fixed $h \in G$, the V_h -orbit of p can be defined as the $V_{\langle h \rangle}$ -orbit of p , where $\langle h \rangle$ denotes the subgroup of $\text{Diff}(\mathbb{C}^n, 0)$ generated by h .

We can now define “pseudogroups with finite orbits”. Note that neighborhoods of the origin in \mathbb{C}^n are always sufficiently small to ensure that $h_1, \dots, h_k, h_1^{-1}, \dots, h_k^{-1}$ are well-defined injective maps on V .

Definition 2. A pseudogroup $G \subseteq \text{Diff}(\mathbb{C}^n, 0)$ is said to have finite orbits if there exists a sufficiently small open neighborhood V of $0 \in \mathbb{C}^n$ such that the set $\mathcal{O}_V^G(p)$ is finite for every $p \in V$. Analogously, $h \in G$ is said to have finite orbits if the pseudogroup $\langle h \rangle$ generated by h has finite orbits.

Fixed $h \in G$, the number of iterations of p by h is the cardinality of the set $\{n \in \mathbb{Z}; p \in \text{Dom}_V(h^n)\}$, where $\text{Dom}_V(h^n)$ stands for the domain of definition of h^n as element of the pseudogroup in question. The number of iterations of p by h is denoted by $\mu_V^h(p)$ and belongs to $\mathbb{N} \cup \{\infty\}$. The lemma below is attributed to Lewowicz and can be found in [13].

Lemma 3 (Lewowicz). Let K be a compact connected neighborhood of $0 \in \mathbb{R}^n$ and h a homeomorphism from K onto $h(K) \subseteq \mathbb{R}^n$ verifying $h(0) = 0$. Then there exists a point p on the boundary ∂K of K whose number of iterations in K by h is infinite, i.e. p satisfies $\mu_K^h(p) = \infty$. □

Fixed an open set V , note that the existence of points in V such that $\mu_K^h(p) = \infty$ does not imply that p is a point with infinite orbit, i.e. there may exist points p in V such that $\mu_V^h(p) = \infty$ but $\#\mathcal{O}_V^{(h)}(p) < \infty$, where $\#$ stands for the cardinality of the set in question. These points are called *periodic for h on V* . A local diffeomorphism is said to be *periodic* if there is $k \in \mathbb{N}^*$ such that f^k coincides with the identity on a neighborhood of the origin. Clearly periodic diffeomorphisms possess finite orbits. To prove Theorem A, we first need to show the following.

Proposition 4. Suppose that $G \subseteq \text{Diff}(\mathbb{C}^n, 0)$ is a group satisfying the condition of isolated fixed points of Theorem A. Let h be an element of G and assume that h has only finite orbits. Then h is periodic.

Assuming that Proposition 4 holds, Theorem A can be derived as follows:

Proof of Theorem A. We want to prove that G is finite (for example at the level of germs). So, let us consider the homomorphism $\sigma : G \rightarrow GL(n, \mathbb{C})$ assigning to an element $h \in G$ its derivative D_0h at the origin. The image $\sigma(G)$ of G is a finitely generated subgroup of $GL(n, \mathbb{C})$ all of whose elements have finite order. According to Schur’s theorem concerning the affirmative solution of Burnside problem for linear groups, the group $\sigma(G)$ must be finite, cf. [19]. Therefore, to conclude that G is itself finite, it suffices to check that σ is one-to-one or, equivalently, that its kernel is reduced to the identity. Hence suppose that $h \in G$ lies in the kernel of σ , i.e. D_0h coincides with the identity. To show that h itself coincides with the identity, note that h must be periodic since it has

finite orbits, cf. Proposition 4. Therefore h is conjugate to its linear part at the origin (cf. [4]), i.e. it is conjugate to the identity map. Thus h coincides with the identity on a neighborhood of the origin of \mathbb{C}^n . Theorem A is proved. \square

Before proving Proposition 4, let us make some comments concerning the proof of Theorem A. When $n = 1$, Leau theorem immediately implies that the above considered homomorphism σ is one-to-one so that G will be abelian and, indeed, cyclic. This fact does not carry over higher dimensions since, as already mentioned, every finite group can be realized as a matrix group and therefore as a pseudogroup of $\text{Diff}(\mathbb{C}^n, 0)$ having finite orbits. Yet, in general, groups obtained in this manner contain non-trivial elements with non-isolated fixed points. In any event, since every finite subgroup of $\text{Diff}(\mathbb{C}^n, 0)$ is linearizable, the classification of these finite subgroups is equivalent to the classification of all finite subgroups of $\text{GL}(n, \mathbb{C})$. In the 2-dimensional case, the situation is fully understood since the classification of finite subgroups of $\text{GL}(2, \mathbb{C})$, and hence of $\text{Diff}(\mathbb{C}^2, 0)$, is a well-known result going back to F. Klein, see [11].

Let us now prove Proposition 4. As already pointed out, the proof amounts to a careful reading of the argument supplied in [13] for the case $n = 1$.

Proof of Proposition 4. Let h be a local diffeomorphism in $\text{Diff}(\mathbb{C}^n, 0)$ whose periodic points are isolated unless the corresponding power of h coincides with the identity on a neighborhood of the mentioned periodic point. Let us assume that h is not periodic. To prove the statement, we are going to show the existence of an open neighborhood U of $0 \in \mathbb{C}^n$ such that the set of points $x \in U$ with infinite $U_{\langle h \rangle}$ -orbit is uncountable and has the origin as an accumulation point. It will then result that h cannot have finite orbits, thus proving the proposition.

Let U be an arbitrarily small open neighborhood of $0 \in \mathbb{C}^n$ contained in the domains of definition of h, h^{-1} . Suppose also that h, h^{-1} are one-to-one on U . Consider $\rho_0 > 0$ such that $D_{\rho_0} \subseteq U$, where D_{ρ_0} stands for the closed ball of radius ρ_0 centered at the origin. Following [13], we define the following sets

$$\begin{aligned} \mathbf{P} &= \{x \in D_{\rho_0} : \mu_{D_{\rho_0}}(x) = \infty, \#\mathcal{O}_{D_{\rho_0}}^{(h)}(x) < \infty\}, \\ \mathbf{F} &= \{x \in D_{\rho_0} : \mu_{D_{\rho_0}}(x) < \infty, \#\mathcal{O}_{D_{\rho_0}}^{(h)}(x) < \infty\}, \\ \mathbf{I} &= \{x \in D_{\rho_0} : \mu_{D_{\rho_0}}(x) = \infty, \#\mathcal{O}_{D_{\rho_0}}^{(h)}(x) = \infty\}. \end{aligned}$$

In other words, \mathbf{P} is the set of periodic points in D_{ρ_0} for h , \mathbf{F} denotes the set of points leaving D_{ρ_0} after finitely many iterations and \mathbf{I} stands for the set of non-periodic points with infinite orbit. Naturally, $D_{\rho_0} = \mathbf{P} \cup \mathbf{F} \cup \mathbf{I}$ and

Lewowicz’s lemma implies that

$$(\mathbf{P} \cup \mathbf{I}) \cap \partial D_\rho \neq \emptyset.$$

for every $\rho \leq \rho_0$. Thus, at least one between \mathbf{P} and \mathbf{I} is uncountable. In what follows, the diffeomorphism h is supposed to be non-periodic. With this assumption, our purpose is to show that \mathbf{I} must be uncountable and the origin is accumulation point of \mathbf{I} .

For $n \geq 0$, let A_n denote the domain of definition of h^n viewed as an element of the pseudogroup generated on D_{ρ_0} . Clearly $A_{n+1} \subseteq A_n$. Next, let C_n be the connected (compact) component of A_n containing the origin and pose

$$C = \bigcap_{n \in \mathbb{N}} C_n.$$

Note that C is the intersection of a decreasing sequence of compact connected sets. Therefore C is non-empty and connected.

Claim. *Without loss of generality, we can assume that C is a countable set.*

Proof of the Claim. Suppose that C is uncountable. The reader is reminded that our aim is to conclude that \mathbf{I} is uncountable provided that h is not periodic. Therefore we suppose for a contradiction that \mathbf{I} is countable. Since \mathbf{I} is countable so is $\mathbf{I} \cap C$. Consider now $C \cap \mathbf{P}$ and note that this intersection must be uncountable, since $C \subset \mathbf{P} \cup \mathbf{I}$. Let

$$C \cap \mathbf{P} = \bigcup_{n \in \mathbb{N}} P_n,$$

where P_n is the set of points $x \in C \cap \mathbf{P}$ of period n . Note that there exists a certain $n_0 \in \mathbb{N}$ such that P_{n_0} is infinite, otherwise all of the P_n would be finite and $C \cap \mathbf{P}$ would be countable. Being infinite, P_{n_0} has a non-trivial accumulation point p in C_{n_0} . The map h^{n_0} is holomorphic on an open neighborhood of C_{n_0} and it is the identity on $P_{n_0} \cap C_{n_0}$. Since p is not an isolated fixed point of h^{n_0} , it follows that h^{n_0} coincides with the identity map on C_{n_0} , i.e. on the connected component of the domain of definition of h^{n_0} that contains the origin. This contradicts the assumption of non-periodicity of h (modulo reducing the neighborhood of the origin). Hence \mathbf{I} is uncountable. Moreover the closure of \mathbf{I} contains the origin since, otherwise, there is a small disc D about the origin such that $D \cap \mathbf{I} = \emptyset$. If this is the case, it suffices to repeat the above procedure with $C \cap D$ to obtain a contradiction. □

In view of the preceding, in the sequel C will be supposed to consist of countably many points. The purpose is still to conclude that the set \mathbf{I} is uncountable (and the origin belongs to its closure). Since, in addition, C is connected, it must be reduced to the origin itself. Then, for every $\rho < \rho_0$, we have $C \cap \partial D_\rho = \emptyset$. Now note that, for a fixed $\rho > 0$, the sets

$$C_1 \cap \partial D_\rho, (C_1 \cap C_2) \cap \partial D_\rho, (C_1 \cap C_2 \cap C_3) \cap \partial D_\rho, \dots$$

form a decreasing sequence of compact sets. Hence the intersection

$$\bigcap_{n \in \mathbb{N}} C_n \cap \partial D_\rho$$

is non-empty, unless there exists $n_0 \in \mathbb{N}$, such that $C_{n_0} \cap \partial D_\rho = \emptyset$. The latter case must occur since $C \cap \partial D_\rho = \emptyset$. However, the value of n_0 for which the mentioned intersection becomes empty may depend on ρ .

Fix $\rho > 0$ and let n_0 be as above. Let K be a compact connected neighborhood of C_{n_0} that does not intersect the other connected components of A_{n_0} , if they exist. The set K can be chosen so that $\partial K \cap A_{n_0} = \emptyset$. The inclusion $A_{n+1} \subseteq A_n$ guarantees that ∂K does not intersect A_n , for every $n \geq n_0$. Therefore

$$\partial K \cap \mathbf{P} = \emptyset.$$

In fact, if there were a periodic point x of D_{ρ_0} on ∂K , then x would belong to every set A_n . In particular, it would belong to A_{n_0} , hence leading to a contradiction. Nonetheless, Lewowicz's lemma guarantees the existence of a point x on the boundary ∂K of K such that the number of iterations in K is infinite, i.e. such that $\mu_K(x) = \infty$. Since $K \subseteq D_\rho \subseteq D_{\rho_0}$, it follows that

$$\partial K \cap \mathbf{I} \neq \emptyset.$$

By construction, it is clear that a compact set K satisfying the above conditions is not unique. Indeed, for K as before, denote by K_ε the compact connected neighborhood of K whose boundary has distance to ∂K equal to ε . Then, there exists $\varepsilon_0 > 0$ such that K_ε satisfies the same properties as K for every $0 \leq \varepsilon \leq \varepsilon_0$ with respect to A_{n_0} . In particular,

$$\partial K_\varepsilon \cap \mathbf{I} \neq \emptyset$$

for all $0 \leq \varepsilon \leq \varepsilon_0$. Therefore \mathbf{I} must be uncountable. Finally, it remains to prove that $0 \in \mathbb{C}^n$ is an accumulation point of \mathbf{I} . This is, however, a simple consequence of the fact that a compact set $K \subseteq D_\rho$ as above can be considered for all $\rho > 0$. This completes the proof of Proposition 4. \square

3 Siegel singular points and an extension of a result by Camara-Scardua

We can now move on to state and prove Theorem 5 (see below). The proof of this theorem follows from the combination of our Theorem A with the results in [8] or in [18].

To begin with, let \mathcal{F} be a singular 1-dimensional foliation on $(\mathbb{C}^n, 0)$ and let X be a representative of \mathcal{F} , i.e. X is a holomorphic vector field tangent to \mathcal{F} and whose singular set has codimension at least 2. Suppose that the origin is a singular point of \mathcal{F} and denote by $\lambda_1, \dots, \lambda_n$ the corresponding eigenvalues of DX at the origin. Assume the following holds:

- (1) \mathcal{F} has an isolated singularity the origin.
- (2) The singularity of \mathcal{F} is of Siegel type.
- (3) The eigenvalues $\lambda_1, \dots, \lambda_n$ are all different from zero and there exists a straight line through the origin, in the complex plane, separating λ_1 from the remainder eigenvalues.
- (4) Up to a change of coordinates, $X = \sum_{i=1}^n \lambda_i x_i (1 + f_i(x)) \partial / \partial x_i$, where $x = (x_1, \dots, x_n)$ and $f_i(0) = 0$ for all i .

Note that condition (4) is equivalent to the existence of n invariant hyperplanes through the origin. This condition, as well as condition (3), is always verified when $n = 3$ provided that the singular point is of *strict Siegel type* cf. [6]. Here the reader is reminded that, in dimension 3, the singular point is said to be of strict Siegel type if $0 \in \mathbb{C}$ is contained in the interior of the convex hull of $\{\lambda_1, \lambda_2, \lambda_3\}$.

In general, the *eigenvalues of a foliation* \mathcal{F} are nothing but the eigenvalues of the differential of a representative vector field X for \mathcal{F} . These eigenvalues are therefore defined only up to a multiplicative constant and this definition allows us to avoid passing through the choice of a representative vector field when dealing with foliations belonging to the Siegel domain.

In any event, when it comes to problems of “complete integrability”, the relevant Siegel singular points are not of strict Siegel type. Indeed, in the case of strict Siegel singular point, the corresponding holonomy maps have a “hyperbolic part” i.e. they are partially hyperbolic and therefore cannot have finite orbits. Similarly the corresponding foliation cannot be “completely integrable”. In these cases, whereas conditions (3) and (4) may fail to hold, they are essentially always satisfied in the cases of interest (at least in dimension 3). This is due to the basic properties of the standard reduction procedure by means of blow-up maps for singularities in dimension 3 (which is known to the expert to hold in dimension 3 provided that natural topological conditions are satisfied). However, to avoid making the discussion needlessly long, we shall proceed as

in [7] and assume once and for all that, up to a rotation, there is an eigenvalue λ_1 lying in \mathbb{R}_+ whereas the remaining eigenvalues $\lambda_2, \dots, \lambda_n$ lie in \mathbb{R}_- . We can now state Theorem 5.

Theorem 5. *Let \mathcal{F} be a singular foliation on $(\mathbb{C}^n, 0)$ possessing n invariant pairwise transverse hyperplanes and denote by $\lambda_1, \dots, \lambda_n$ its eigenvalues. Suppose also that $\lambda_1 \in \mathbb{R}_+$ while $\lambda_2, \dots, \lambda_n$ are all negative reals. Denote by h_1 the local holonomy map associated to the axis x_1 (corresponding to the eigenvalue λ_1) and suppose that h_1 has isolated fixed points (in the sense of Theorem A) and that it has finite orbits. Then \mathcal{F} admits $n - 1$ independent holomorphic first integrals.*

To prove Theorem 5, Theorem 6 below will be needed. The latter theorem generalizes to higher dimensions a unpublished result of Mattei which, in turn, improved on an earlier version appearing in [13].

Theorem 6 ([EI], [Re]). *Let X and Y be two vector fields satisfying conditions (1), (2), (3) and (4) above. Denote by h^X (resp. h^Y) the holonomy of X (resp. Y) relatively to the separatrix of X (resp. Y) tangent to the eigenspace associated to the first eigenvalue. Then h^X and h^Y are analytically conjugate if and only if the foliations associated to X and Y are analytically equivalent.*

Remark 7. In the statement of both Theorem 5 and Theorem 6, the condition (4) concerning a special normal form for the vector field X is truly indispensable. In particular, this condition ensures that the linear part of X is diagonalizable. In general, when the linear part of X is allowed to have “Jordan blocks” then further obstructions to topological equivalence are known to arise as first pointed out by N. Ladis and by Camacho-Kuiper-Palis; see [6], [12] as well as Naishul’s paper [15] for the non-linear case.

In our context, the following example was suggested by the referee. Consider the vector fields Y_1, Y_2, Y_3 defined on \mathbb{C}^5 with coordinates (x_1, \dots, x_5) as follows:

$$Y_1 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \quad Y_2 = (x_3 + x_4) \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}, \quad \text{and} \quad Y_3 = x_5 \frac{\partial}{\partial x_5}.$$

Now for $a > 0$ fixed, let X_1 and X_2 be the vector fields defined by

$$X_1 = (a + 1)Y_1 + aY_2 - Y_3 \quad \text{and} \quad X_2 = aY_1 + (a + 1)Y_2 - Y_3.$$

The axis x_5 is invariant by both vector fields X_1 and X_2 . Moreover, the holonomy maps arising from this invariant axis are conjugate. Namely they are conjugate to the linear transformation having eigenvalue $e^{2i\pi a}$ with multiplicity 4

and Jordan block decomposition $4 = 1 + 2 + 1$. Yet, the 2×2 Jordan block is associated to different eigenvalues, namely to a in the case of X_1 and to $a + 1$ in the case of X_2 . Therefore a homeomorphism conjugating these holonomy maps cannot be extended to the plane $\{x_5 = 0\}$.

The proof of Theorem 6 can be found in either [8] or [18], a particularly detailed exposition appears in [17]. With this theorem in hand, the proof of Theorem 5 goes as follows.

Proof of Theorem 5. Consider a foliation \mathcal{F} as in the statement of Theorem 5. The local holonomy map h_1 is defined on a suitable local section and it can also be identified to a local diffeomorphism fixing the origin of \mathbb{C}^{n-1} . By assumption, all iterates of h have isolated fixed points. Therefore Theorem A implies that the local orbits of h are finite if and only if h is periodic. Naturally, we may assume this to be the case. Let then N be the *period* of h , namely the smallest strictly positive integer for which h^N coincides with the identity on a neighborhood of the origin of \mathbb{C}^{n-1} (with the above mentioned identifications). Denote also by T the derivative of h at the origin, which is itself identified to a linear transformation of \mathbb{C}^{n-1} . The fact that h is periodic of period N ensures that T is also periodic with the same period N . In fact, h and T are analytically conjugate as already mentioned (i.e. h is linearizable). Next denote by \mathcal{F}_Z the foliation associated to the linear vector field

$$Z = \sum_{i=1}^n \lambda_i x_i \partial / \partial x_i .$$

It is immediate to check that the map T coincides with the holonomy map induced by \mathcal{F}_Z with respect to the axis x_1 . Therefore Theorem 6 implies that the foliations \mathcal{F} and \mathcal{F}_Z are analytically equivalent. However, since \mathcal{F}_Z is induced by a linear (diagonal) vector field, it becomes clear that the complete integrability of \mathcal{F}_Z is equivalent to the periodic character of the holonomy map T . Since \mathcal{F} and \mathcal{F}_Z are analytically equivalent, we conclude from what precedes that the condition of having a local holonomy h with finite orbits forces \mathcal{F} to be completely integrable. The converse is clear, since having \mathcal{F} completely integrable ensures at once that the holonomy map h must be periodic. Theorem 5 is proved. □

Remark 8. Concerning the argument given in [7] for the analogous statement in dimension 3, the authors have relied in Abate’s theorem to conclude that the corresponding holonomy map must be of finite order. Abate’s theorem however is no longer available once the dimension is 4 or larger.

4 Examples of local dynamics and proof of Theorem B

This section is split in two paragraphs. First we shall describe the local dynamics of certain local diffeomorphisms fixing the origin of \mathbb{C}^2 that happen to be tangent to the identity. These examples include families of diffeomorphisms having finite orbits whereas the diffeomorphism itself has infinite order. In the second paragraph we are going to prove Theorem B by showing that, in fact, the holonomy map associated to the axis $\{x = y = 0\}$ in the example described in Theorem B falls in one of the previously discussed classes of diffeomorphisms.

4.1 Local diffeomorphisms

Recall that $\text{Diff}_1(\mathbb{C}^2, 0)$ denotes the normal subgroup of $\text{Diff}(\mathbb{C}^2, 0)$ consisting of diffeomorphisms tangent to the identity. The simplest example of an element of $\text{Diff}_1(\mathbb{C}^2, 0)$ possessing finite orbits is obtained by setting $F(x, y) = (x + f(y), y)$, with $f(0) = f'(0) = 0$. This case, however, can be set aside in what follows since the foliation associated to the infinitesimal generator of F is regular. In particular, it cannot be realized as the holonomy map of a Siegel singular point as those considered in the context of Theorem 5 since it does not leave the axis $\{x = 0\}$ invariant. Some genuinely more interesting examples are listed below.

Example 1. Linear vector fields.

Consider the vector field Y given by $Y = x\partial/\partial x - \lambda y\partial/\partial y$ where $\lambda = n/m$ with $m, n \in \mathbb{N}^*$. The foliation associated to Y will be denoted by \mathcal{F} and it should be noted that the holomorphic function $(x, y) \mapsto x^n y^m$ is a first integral for \mathcal{F} . Let ϕ_Y denote the time-one map induced by Y . The local dynamics of ϕ_Y can easily be described as follows. The vector field Y can be projected on the axis $\{y = 0\}$ as the vector field $x\partial/\partial x$. Therefore the (real) integral curves of Y coincide with the lifts in the corresponding leaves of \mathcal{F} of the (real) trajectories of $x\partial/\partial x$ on $\{y = 0\}$. The latter trajectories are radial lines being emanated from $0 \in \{y = 0\} \simeq \mathbb{C}$ so that the local dynamics of ϕ_Y restricted to $\{y = 0\}$ is such that, whenever $x_0 \neq 0$, the sequence $\{\phi_Y^n(x_0)\}$ marches off a uniform neighborhood of $0 \in \{y = 0\} \simeq \mathbb{C}$ as $n \rightarrow \infty$ and it converges to $0 \in \{y = 0\} \simeq \mathbb{C}$ as $n \rightarrow -\infty$. Consider now the orbit of a point (x_0, y_0) , $x_0 y_0 \neq 0$, by ϕ_Y . Since this is simply the lift in the leaf of \mathcal{F} through (x_0, y_0) of the dynamics of $x_0 \in \{y = 0\} \simeq \mathbb{C}$, it follows that $\phi_Y^n(x_0, y_0)$ leaves a fixed neighborhood of $(0, 0) \in \mathbb{C}^2$ since the first coordinate increases to uniformly large values provided that $n \rightarrow \infty$. Similarly, when $n \rightarrow -\infty$, the first coordinate of $\phi_Y^n(x_0, y_0)$ must converge to zero so that the second coordinate becomes

“large” due to the first integral $x^n y^m$. Thus, fixed a (small) neighborhood U of $(0, 0) \in \mathbb{C}^2$, every orbit of ϕ_Y that is not contained in $\{x = 0\} \cup \{y = 0\}$ is bound to intersect U at finitely many points.

Clearly the time-one map induced by Y is not tangent to the identity. However, examples of time-one maps tangent to the identity and satisfying the desired conditions can be obtained, for example, by considering the vector field $X = x^n y^m Y$ and taking the time-one map ϕ_X induced by X . Clearly the linear part of X at $(0, 0)$ equals zero so that ϕ_X must be tangent to the identity. Furthermore, the multiplicative factor $x^n y^m$ annihilates the dynamics of ϕ_X over the coordinate axes so that only the orbits of points (x_0, y_0) with $x_0 y_0 \neq 0$ have to be considered. The leaf of \mathcal{F} through (x_0, y_0) will be denoted by L and $c \in \mathbb{C}$ will stand for the value of $x^n y^m$ on L . The restriction of X to L is nothing but the restriction of Y to L multiplied by the scalar $c \in \mathbb{C}$. Therefore the real orbits of X in L coincide with the lift to L of the real orbits of the vector field $cx\partial/\partial x$ defined on $\{y = 0\}$. The geometric nature of the orbits of $cx\partial/\partial x$ depends on the argument of $c \in \mathbb{C}$, i.e. setting $c = |c|e^{2\pi i\alpha}$, this geometry depends on $\alpha \in [0, 2\pi)$. If $\alpha = \pi/2$, then the orbits of $cx\partial/\partial x$ are contained in circles about the origin. After finitely many tours, these circles lift into the corresponding leaf (i.e. the leaf on which xy equals c) as closed paths invariant by ϕ_X . In addition for a “generic” choice of c satisfying $\alpha = \pi/2$, the resulting time-one map restricted to the corresponding invariant path will be conjugate to an irrational rotation. Thus Φ_X does not have finite orbits.

Let us now briefly discuss the slightly more general case where $X = x^a y^b Y$ with $a, b \in \mathbb{N}^*$. Setting $d = am - bn$, we can suppose without loss of generality that $d \geq 1$. Next, by considering the system

$$\begin{cases} \frac{dx}{dt} = mx^{a+1}y^b \\ \frac{dy}{dt} = -nx^a y^{b+1}, \end{cases}$$

we conclude that $dy/dx = -ny/mx$ so that $y = cx^{-n/m}$ in ramified coordinates. In turn, this yields $dx/dt = c^b mx^{1+d/m}$. Since $d \geq 1$ by construction, the orbits of the latter vector field defines the well-known “petals” associated to Leau flower in the case of periodic linear part, cf. [14]. For example, setting $m = 1$ to simplify, the orbits of the vector field $x^{1+d}\partial/\partial x$ consists of $d + 1$ “petals” in non-ramified coordinates.

In any event, the sequence of points consisting of the first coordinates of the full orbits of ϕ_X either marches straight off a neighborhood of $0 \in \{y = 0\} \simeq \mathbb{C}$ or it converges to $(0, 0)$ (by construction this sequence is contained in $\{y =$

0}). Converging to $(0, 0)$ will force the second coordinates of the points in the ϕ_X -orbit to increase uniformly so that the orbit in question must leave a fixed neighborhood of $(0, 0) \in \mathbb{C}^2$. Summarizing, we conclude:

Claim 1. *Fixed a neighborhood U of $(0, 0) \in \mathbb{C}^2$ and given $p = (x_0, y_0)$, $x_0 y_0 \neq 0$, the set*

$$U \cap \left\{ \bigcup_{n=-\infty}^{\infty} \phi_X^n(p) \right\}$$

is finite.

Example 2. Diffeomorphisms leaving the function $(x, y) \mapsto xy$ invariant.

Here we are going to see two examples of diffeomorphisms having a nature somehow similar to those discussed in Example 1 but having also the advantage that they can easily be realized as the holonomy maps of foliations with Siegel singular points. To begin with, let $F \in \text{Diff}_1(\mathbb{C}^2, 0)$ be given by

$$F(x, y) = [x(1 + xyf(xy)), y(1 + xyf(xy))^{-1}], \quad (1)$$

where $f(z)$ is a holomorphic function defined about $0 \in \mathbb{C}$ and satisfying $f(0) \neq 0$. Note that F leaves the function $(x, y) \mapsto xy$ invariant since the product of its first and second components equals xy .

Next, consider an initial point (x_0, y_0) with $x_0 y_0 = C \neq 0$. The orbit of (x_0, y_0) under F is hence contained in the curve defined by $\{xy = C\}$. Moreover, for a point (\tilde{x}, \tilde{y}) lying in $\{xy = C\}$, the value of $F(\tilde{x}, \tilde{y})$ takes on the form

$$F(\tilde{x}, \tilde{y}) = [\tilde{x}(1 + Cf(C)), \tilde{y}(1 + Cf(C))^{-1}].$$

In particular, those values of C for which $|1 + Cf(C)| = 1$ give rise to a rotation in the first coordinate. Therefore the lifts of these circles in the corresponding leaves are loops. Besides for a generic choice of C satisfying $|1 + Cf(C)| = 1$ the dynamics induced on one of these invariant loops is conjugate to an irrational rotation so that F does not have finite orbits.

Consider now the local diffeomorphism H which is given by

$$H(x, y) = [x(1 + x^2 y f(x^2 y)), y(1 + x^2 y f(x^2 y))^{-1}], \quad (2)$$

where f is as above. For this local diffeomorphism, we have:

Lemma 9. *The local diffeomorphism H given by Formula (2) possesses only finite orbits.*

Proof. Again the product of its first and second components of H equals xy so that H preserves the function $(x, y) \mapsto xy$. To check that H has finite orbits, we proceed as follows. Fix an initial point (x_0, y_0) with $x_0y_0 = C \neq 0$ so that the orbit of (x_0, y_0) under H is contained in the curve $\{xy = C\}$. Next note that, if (\tilde{x}, \tilde{y}) lies in $\{xy = C\}$, we have

$$H(\tilde{x}, \tilde{y}) = [\tilde{x}(1 + \tilde{x}Cf(\tilde{x}C)), \tilde{y}(1 + \tilde{x}Cf(\tilde{x}C))^{-1}].$$

The dynamics of the first component of H behaves now as the Leau flower since it is given by $x \mapsto x + x^2Cf(xC)$, where $C \neq 0$. Therefore, by resorting to an argument totally analogous to the one employed in Example 1 for $X = x^a y^b Y$ with $d = am - bn \neq 0$, we conclude that all the orbits of H are finite as desired. □

4.2 Singular foliations and holonomy

In this paragraph Theorem B will be proved and a couple of related examples will also be provided.

Let us begin by pointing out a simple observation showing that every element of $\text{Diff}_1(\mathbb{C}^2, 0)$ that is the time-one map of a vector field can be realized as a local holonomy map for some foliation on $(\mathbb{C}^3, 0)$. Indeed, consider a singular foliation \mathcal{F} on $(\mathbb{C}^3, 0)$ admitting a separatrix S through the origin and denote by h the holonomy map associated to \mathcal{F} , with respect to S . Assume that the foliation is locally given by the vector field $A(x, y, z)\partial/\partial x + B(x, y, z)\partial/\partial y + C(x, y, z)\partial/\partial z$. Assume furthermore that the separatrix S is given, in the same coordinates, by $\{x = 0, y = 0\}$. Setting $z = e^{2\pi it}$, the corresponding holonomy map can be viewed as the time-one map associated to the differential equation

$$\begin{cases} \frac{dx}{dt} = \frac{dx}{dz} \frac{dz}{dt} = 2\pi i e^{2\pi it} \frac{A(x, y, e^{2\pi it})}{C(x, y, e^{2\pi it})} \\ \frac{dy}{dt} = \frac{dy}{dz} \frac{dz}{dt} = 2\pi i e^{2\pi it} \frac{B(x, y, e^{2\pi it})}{C(x, y, e^{2\pi it})} \end{cases}.$$

In the particular case where A, B do not depend on z and C is reduced to $C(x, y, z) = z$, the holonomy map of \mathcal{F} with respect to S reduces to the time-one map induced by a vector field on $(\mathbb{C}^2, 0)$, namely by the vector field

$$2\pi i \left[A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} \right].$$

Considering a local diffeomorphism h possessing finite orbits and realizable as time-one map of a vector field Y , then to find a vector field on $(\mathbb{C}^3, 0)$ whose foliation has h as holonomy map, it suffices to take Y and “join” the term $2\pi iz\partial/\partial z$. Then the holonomy of the foliation associated to the vector field

$$X = Y + 2\pi iz \frac{\partial}{\partial z},$$

with respect to the z -axis is nothing but h itself.

Note that the vector field X above corresponds to a saddle-node vector field of codimension 2. This is equivalent to saying that its linear part admits exactly two eigenvalues equal to zero and a non-zero eigenvalue associated to the direction of the separatrix $\{x = y = 0\}$. The fact that the holonomy of $\{x = y = 0\}$ has finite orbits is a phenomenon without analogue for saddle-nodes in dimension 2.

Let us now consider an example of a foliation on $(\mathbb{C}^3, 0)$ possessing only eigenvalues different from zero, as in the case of Theorem B.

Example 3. Let \mathcal{F} denote the foliation associated to the vector field

$$X = x(1 + xyz^2) \frac{\partial}{\partial x} + y(1 - xyz^2) \frac{\partial}{\partial y} - z \frac{\partial}{\partial z}.$$

The z -axis corresponds to one of the separatrices of \mathcal{F} . Taking $z = e^{2\pi it}$, it follows that the holonomy map h associated to \mathcal{F} , with respect to the z -axis, is given by the time-one map associated to the vector field

$$\begin{cases} \frac{dx}{dt} = \frac{dx}{dz} \frac{dz}{dt} = -2\pi ix(1 + e^{4\pi it}xy) \\ \frac{dy}{dt} = \frac{dy}{dz} \frac{dz}{dt} = -2\pi iy(1 - e^{4\pi it}xy) \end{cases}. \tag{3}$$

To solve this system of differential equations, we should consider the series expansion of $(x(t), y(t))$ in terms of the initial condition. More precisely, if $(x(0), y(0)) = (x_0, y_0)$, then we should let $x(t) = \sum a_{ij}(t)x_0^i y_0^j$ and $y(t) = \sum b_{ij}(t)x_0^i y_0^j$. Clearly $a_{10}(0) = b_{01}(0) = 1$ and $a_{ij}(0) = b_{ij}(0) = 0$ in the other cases. Substituting the series expansion of $x(t)$ and $y(t)$ on (3) and comparing the same powers on the initial conditions, it can be said that the system (3) induces an infinite number of differential equations involving the functions a_{ij} , b_{ij} and their derivatives. Each one of the differential equation takes on the form

$$\begin{aligned} a'_{ij}(t) &= -2\pi i \left[a_{ij}(t) + \sum e^{4\pi it} a_{p_1q_1}(t) a_{p_2q_2}(t) b_{p_3q_3}(t) \right] \\ b'_{ij}(t) &= -2\pi i \left[b_{ij}(t) + \sum e^{4\pi it} a_{p_1q_1}(t) b_{p_2q_2}(t) b_{p_3q_3}(t) \right] \end{aligned}$$

where $p_1 + p_2 + p_3 = i$ and $q_1 + q_2 + q_3 = j$. In particular, the terms on the sum in the right hand side of the equation above involves only coefficients of the monomials $x_0^p y_0^q$ of degree less than $i + j$ and such that $p \leq i$ and $q \leq j$. Computing this holonomy map becomes much easier with the following lemma:

Lemma 10. *The holonomy map h preserves the function $(x, y) \mapsto xy$.*

Proof. To check that the level sets of $(x, y) \mapsto xy$ are preserved by h , consider the derivative of the product $x(t)y(t)$ with respect to t . This gives us

$$\begin{aligned} \frac{d}{dt}(xy) &= \frac{dx}{dt}y + x\frac{dy}{dt} \\ &= -[2\pi ix(1 + e^{4\pi it}xy)]y - x[2\pi iy(1 - e^{4\pi it}xy)] \\ &= -4\pi ixy. \end{aligned}$$

Thus, by integrating the previous differential equation with respect to the product xy , we obtain $(xy)(t) = x_0y_0e^{-4\pi it}$. Since the holonomy map corresponds to the time-one map of the system of differential equations (3) and since $e^{-4\pi it} = 1$ for all $t \in \mathbb{Z}$, it follows that the orbits of h are contained in the level sets of $(x, y) \mapsto xy$ as desired. □

Lemma 10 implies that it suffices to determine the first coordinate of h . By recovering the preceding non-autonomous system of differential equations, a simple induction argument on the value of $i + j$ shows that h has the form

$$h(x, y) = (x(1 + xyf(xy)), y(1 + xyf(xy))^{-1}), \tag{4}$$

where f represents a holomorphic function of one complex variable such that $f(0) = 2\pi i$ (the expression for the second coordinate of h is obtained from the first coordinate by means of Lemma 10). The resulting diffeomorphism h is clearly non-periodic but it does have invariant sets given by “circles”. Besides on some of these invariant “circles” the dynamics is conjugate to an irrational rotation, cf. Example 2.

We are now ready to prove Theorem B.

Proof of Theorem B. Let then \mathcal{F} denote the foliation associated to the vector field

$$X = x(1 + x^2yz^3)\frac{\partial}{\partial x} + y(1 - x^2yz^3)\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}.$$

Again the z -axis corresponds to one of the separatrices of \mathcal{F} . Taking $z = e^{2\pi it}$, it follows that the holonomy map h associated to \mathcal{F} , with respect to the z -axis, is given by the time-one map associated to the vector field

$$\begin{cases} \frac{dx}{dt} = \frac{dx}{dz} \frac{dz}{dt} = -2\pi ix(1 + e^{6\pi it} x^2 y) \\ \frac{dy}{dt} = \frac{dy}{dz} \frac{dz}{dt} = -2\pi iy(1 - e^{6\pi it} x^2 y) \end{cases} \quad (5)$$

The same argument employed in Lemma 10 shows that the holonomy map h in question preserves the level sets of the function $(x, y) \mapsto xy$. To solve the corresponding system of equations, we consider again the series expansion of $(x(t), y(t))$ in terms of the initial condition. Let then $x(t) = \sum a_{ij}(t)x_0^i y_0^j$ and $y(t) = \sum b_{ij}(t)x_0^i y_0^j$, where $a_{10}(0) = b_{01}(0) = 1$ and $a_{ij}(0) = b_{ij}(0) = 0$ in the remaining cases. It can immediately be checked that the functions a_{ij}, b_{ij} vanish identically for $2 \leq i + j \leq 3$. As to the monomials of degree 4, it can similar be checked that they all vanish identically except $a_{31}(t)$ and $b_{22}(t)$. In fact, the latter functions satisfy

$$\begin{cases} a'_{31}(t) = -2\pi i [a_{31}(t) + e^{6\pi i} a_{10}^3(t) b_{01}(t)] \\ b'_{22}(t) = -2\pi i [b_{22}(t) - e^{6\pi i} a_{10}^2(t) b_{01}^2(t)] \end{cases} \quad (6)$$

so that $a_{31}(t) = -2\pi i t e^{-2\pi it}$ whereas $b_{22}(t) = 2\pi i t e^{-2\pi it}$. In particular, $a_{31}(1) = -2\pi i$ and $b_{22}(1) = 2\pi i$. By using induction on $i + j$ (and keeping in mind that h preserves the function $(x, y) \mapsto xy$), it can be shown that h takes on the form

$$h(x, y) = (x(1 + x^2 y f(x^2 y)), y(1 + x^2 y f(x^2 y))^{-1}) \quad (7)$$

with $f(0) = 2\pi i$. It follows from Lemma 9 that this diffeomorphism possesses finite orbits whereas it is clearly non-periodic. This finishes the proof of Theorem B. \square

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