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Characterizations and integral formulae for generalized m-quasi-Einstein metrics

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Abstract. The aim of this paper is to present some structural equations for generalized m-quasi-Einstein metrics $(M^n, g, \nabla f, \lambda)$, which was defined recently by Catino in [11]. In addition, supposing that M^n is an Einstein manifold we shall show that it is a space form with a well defined potential f. Finally, we shall derive a formula for the Laplacian of its scalar curvature which will give some integral formulae for such a class of compact manifolds that permit to obtain some rigidity results.

Keywords: Ricci soliton, quasi-Einstein metrics, Bakry-Emery Ricci tensor, scalar curvature.

Mathematical subject classification: Primary: 53C25, 53C20, 53C21; Secondary: 53C65.

1 Introduction and statement of the main results

In recent years, much attention has been given to classification of Riemannian manifolds admitting an Einstein-like structure, which are natural generalization of the classical Ricci solitons. For instance, Catino in [11] introduced a class of special Riemannian metrics which naturally generalizes the Einstein condition. More precisely, he defined that a complete Riemannian manifold (M^n, g), $n \geq 2$, is a *generalized quasi-Einstein metric* if there exist three smooth functions f, λ and μ on M, such that

$$Ric + \nabla^2 f - \mu df \otimes df = \lambda g, \tag{1.1}$$

where Ric denotes the Ricci tensor of (M^n, g), while ∇^2 and $\otimes$ stand for the Hessian and the tensorial product, respectively.

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As a particular case of (1.1) we shall consider the following.

Definition 1. *We say that* $(M^n,\ g)$ *is a* generalized m-quasi-Einstein metric *if there exist two smooth functions* f *and* λ *on* M *satisfying*

$$Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{1.2}$$

where $0 < m \leq \infty$ *is an integer. The tensor* $Ric_f = Ric + \nabla^2 f - \frac{1}{m} df \otimes df$ *is called Bakry-Emery Ricci tensor.*

In particular, we have

$$Ric(\nabla f, \nabla f) + \langle \nabla_{\nabla f} \nabla f, \nabla f \rangle = \frac{1}{m} |\nabla f|^4 + \lambda |\nabla f|^2, \tag{1.3}$$

where $\langle\ ,\ \rangle$ and $|\ |$ stand for the metric g and its associated norm, respectively.

Moreover, if R stands for the scalar curvature of $(M^n,\ g)$, then taking trace of both members of equation (1.2) we deduce

$$R + \Delta f - \frac{1}{m} |\nabla f|^2 = \lambda n. \tag{1.4}$$

Thereby we derive

$$\langle \nabla f, \nabla R \rangle + \langle \nabla f, \nabla \Delta f \rangle = \frac{1}{m} \langle \nabla f, \nabla |\nabla f|^2 \rangle + n \langle \nabla \lambda, \nabla f \rangle. \tag{1.5}$$

One notices that combining equations (1.2) and (1.4) we infer

$$\nabla^2 f - \frac{\Delta f}{n} g = \frac{1}{m} \Big(df \otimes df - \frac{1}{n} |\nabla f|^2 g \Big) - \Big(Ric - \frac{R}{n} g \Big). \tag{1.6}$$

It is important to point out that if $m = \infty$ and λ is constant, equation (1.2) reduces to one associated to a gradient Ricci soliton, for a good survey in this subject we recommend the work due to Cao in [8], as well as if λ is only constant and m is a positive integer, it corresponds to m-quasi-Einstein metrics that are exactly those n-dimensional manifolds which are the base of an $(n + m)$-dimensional Einstein warped product, for more details see [5, 9, 10, 14]. The 1-quasi-Einstein metrics satisfying $\Delta e^{-f} + \lambda e^{-f} = 0$ are more commonly called *static metrics*, for more details see [12]. Static metrics have been studied extensively for their connection to scalar curvature, the positive mass theorem and general relativity, see e.g. [1, 2, 12]. In [14] it was given some classification for m-quasi-Einstein metrics where the base has non-empty boundary. Moreover, they have proved a characterization for m-quasi-Einstein metric when the base is locally conformally flat. In addition, considering $m = \infty$ in equation (1.2) we obtain the almost Ricci soliton equation, for more details see [4, 16]. We also point out that, Catino [11] has proved that around any regular point of f a generalized m-quasi Einstein metric $\big(M^n, g, \nabla f, \lambda\big)$ with harmonic Weyl tensor and $W(\nabla f, \ldots, \nabla f) = 0$ is locally a warped product with $(n-1)$-dimensional Einstein fibers.

A generalized m-quasi-Einstein manifold $\big(M^n, g, \nabla f, \lambda\big)$ will be called *trivial* if the potential function f is constant. Otherwise, it will be called *nontrivial*.

We observe that the triviality definition implies that M^n is an Einstein manifold, but the converse is not true. Meanwhile, we shall show in Theorem 1 that when $(M^n, g, \nabla f, \lambda)$, $n \geq 3$, is Einstein, but not trivial, it will be isometric to a space form with a well defined potential f. Introducing the function $u = e^{-\frac{f}{m}}$ on M we immediately have $\nabla u = -\frac{u}{m}\nabla f$, moreover the next relation, which can be found in [9], is true

$$\nabla^2 f - \frac{1}{m} df \otimes df = -\frac{m}{u}\nabla^2 u. \tag{1.7}$$

In particular, ∇u is a conformal vector field, i.e. $\frac{1}{2}\mathcal{L}_{\nabla u} g = \rho\, g$, for some smooth function ρ defined on M, if and only if M^n is an Einstein manifold. Hence, on a surface M^2, ∇u is always a conformal vector field.

Before to announce our main result we present a family of nontrivial examples on a space form. Let us start with a standard sphere $(\mathbb{S}^n, g_0)$, where g_0 is its canonical metric.

Example 1. On the standard unit sphere $(\mathbb{S}^n, g_0)$, $n \geq 2$, we consider the following function

$$f = -m \ln\left(\tau - \frac{h_v}{n}\right), \tag{1.8}$$

where τ is a real parameter lying in $(1/n, +\infty)$ and h_v is some height function with respect to a fixed unit vector $v \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, here we are considering $\mathbb{S}^n$ as a hypersurface in $\mathbb{R}^{n+1}$ and $h_v : \mathbb{S}^n \to \mathbb{R}$ is given by $h_v(x) = \langle x, v\rangle$. Taking into account that $\nabla^2 h_v = -h_v g_0$ and $u = e^{-\frac{f}{m}} = \tau - \frac{h_v}{n}$, we deduce from (1.7) that

$$\nabla^2 f - \frac{1}{m} df \otimes df = -m\frac{\tau - u}{u} g_0. \tag{1.9}$$

Since the Ricci tensor of $(\mathbb{S}^n, g_0)$ is given by $Ric = (n-1)g_0$, it is enough to consider $\lambda = (n-1) - m\frac{\tau - u}{u}$ in order to build a desired non trivial such structure on $(\mathbb{S}^n, g_0)$.

We now present a similar example as before on the Euclidean space $(\mathbb{R}^n, g_0)$, where g_0 is its canonical metric.

Example 2. On the Euclidean space $(\mathbb{R}^n, g_0)$, $n \geq 2$, we consider the following function

$$f = -m \ln\big(\tau + |x|^2\big), \tag{1.10}$$

where τ is a positive real parameter and $|x|$ is the Euclidean norm of x. Taking into account that $\nabla^2|x|^2 = 2g_0$ and $u = e^{-\frac{f}{m}} = \tau + |x|^2$, we deduce from (1.7) that

$$\nabla^2 f - \frac{1}{m} df \otimes df = -2\frac{m}{u} g_0. \tag{1.11}$$

Since the Ricci tensor of $(\mathbb{R}^n, g_0)$ is flat, it is enough to consider $\lambda = -2\frac{m}{u}$ in order to obtain a desired non trivial structure on $(\mathbb{R}^n, g_0)$.

On the other hand, concerning to hyperbolic space we have the following.

Example 3. Regarding the hyperbolic space $\mathbb{H}^n(-1) \subset \mathbb{R}^{n,1} : \langle x, x\rangle_0 = -1$, $x_1 > 0$, where $\mathbb{R}^{n,1}$ is the Euclidean space $\mathbb{R}^{n+1}$ endowed with the inner product $\langle x, x\rangle_0 = -x_1^2 + x_2^2 + \cdots + x_{n+1}^2$. We now follow the argument used on $\mathbb{S}^n$. First, we fixe a vector $v \in \mathbb{H}^n(-1) \subset \mathbb{R}^{n,1}$ and we consider a hight function $h_v : \mathbb{H}^n(-1) \to \mathbb{R}$ given by $h_v(x) = \langle x, v\rangle_0$. In this case, we have $\nabla^2 h_v = h_v g_0$. Then, taking

$$u = e^{-\frac{f}{m}} = \tau + h_v,\ \tau > -1 \tag{1.12}$$

we have from (1.7)

$$\nabla^2 f - \frac{1}{m} df \otimes df = -m\frac{u-\tau}{u} g_0. \tag{1.13}$$

Reasoning as in the spherical case it is enough to consider $\lambda = -(n-1) - m\frac{\tau - u}{u}$ in order to build a non trivial such structure on $(\mathbb{H}^n, g_0)$.

Now we announce the main theorem.

Theorem 1. *Let $\big(M^n,\ g,\ \nabla f,\ \lambda\big)$ be a non trivial generalized m-quasi-Einstein metric with $n \geq 3$. Suppose that either $(M^n,\ g)$ is an Einstein manifold or ∇u is a conformal vector field. Then one the following statements holds:*

(1) *M^n is isometric to a standard sphere $\mathbb{S}^n(r)$. Moreover, f is, up to constant, given by* (1.8).

(2) *M^n is isometric to a Euclidean space $\mathbb{R}^n$. Moreover, f is, up to change of coordinates, given by* (1.10).

(3) *M^n is isometric to a hyperbolic space $\mathbb{H}^n$, provided u has only one critical point. Moreover, f is, up to constant, given according to* (1.12).

As a consequence of this theorem we obtain the following corollary.

Corollary 1. *Let $\big(M^n,\ g,\ \nabla f,\ \lambda\big)$, $n \geq 3$, be a compact non trivial generalized m-quasi-Einstein metric such that $\int_M Ric(\nabla u, \nabla u)d\mu \geq \frac{n-1}{n}\int_M (\Delta u)^2 d\mu$, where $d\mu$ stands for the Riemannian measure associated to g. Then M^n is isometric to a standard sphere $\mathbb{S}^n(r)$. Moreover, the potential f is the same of identity* (1.8).

Before to announce the next results we point out that they are generalizations of ones found in [15] and [3] for Ricci solitons, [4] for almost Ricci solitons and [9] for quasi-Einstein metrics. First, we have the following theorem.

Theorem 2. *Let $\big(M^n,\ g,\ \nabla f,\ \lambda\big)$ be a compact generalized m-quasi-Einstein metric. Then M^n is trivial provided:*

(1) $\int_M Ric(\nabla f, \nabla f)d\mu \leq \frac{2}{m}\int_M |\nabla f|^2 \Delta f d\mu - (n-2)\int_M \langle \nabla\lambda, \nabla f\rangle d\mu.$
(2) $R \geq \lambda n$ *or* $R \leq \lambda n.$

Now, if $\big(M^n,\ g,\ \nabla f,\ \lambda\big)$ is a generalized m-quasi-Einstein metric and m is finite, we shall present conditions in order to obtain $\nabla f \equiv 0$.

Theorem 3. *Let $\big(M^n,\ g,\ \nabla f,\ \lambda\big)$ be a complete generalized m-quasi-Einstein metric with m finite. Then $\nabla f \equiv 0$, if one of the following conditions holds:*

(1) *M^n is non compact, $n\lambda \geq R$ and $|\nabla f| \in \mathrm{L}^1(M^n)$. In particular, M^n is an Einstein manifold.*
(2) *$(M^n,\ g)$ is Einstein and ∇f is a conformal vector filed.*

2 Preliminaries

In this section we shall present some preliminaries which will be useful for the establishment of the desired results. The first one is a general lemma for a vector field $X \in \mathfrak{X}(M^n)$ on a Riemannian manifold M^n.

Lemma 1. *Let $(M^n,\ g)$ be a Riemannian manifold and $X \in \mathfrak{X}(M^n)$. Then the following statements hold:*

(1) *If $\big(X^\flat \otimes X^\flat\big) = \rho g$ for some smooth function $\rho : M \to \mathbb{R}$, then $\rho = |X|^2 =$ 0. In particular, the unique solution of the equation $df \otimes df = \rho g$ is f constant.*

(2) *If M^n is compact and X is a conformal vector field, then*

$$\int_M |X|^2 div\, X d\mu = 0.$$

In particular, if $X = \nabla f$ is a gradient conformal vector field, then $\int_M |\nabla f|^2 \Delta f d\mu = 0$.

Proof. Since $\left(X^\flat \otimes X^\flat\right)$ is a degenerate (0, 2) tensor the first statement is trivial. Taking into account that X is a conformal vector field we have $\frac{1}{2}\mathcal{L}_X g = \rho\, g$, where $\rho = \frac{1}{n} div\, X$. From which we obtain

$$|X|^2 div\, X = n\langle \nabla_X X, X\rangle. \tag{2.1}$$

On the other hand, since $div\,(|X|^2 X) = |X|^2 div\, X + 2\langle \nabla_X X, X\rangle$, one has

$$div\left(|X|^2 X\right) = \frac{n+2}{n}|X|^2 div\, X, \tag{2.2}$$

which allows us to complete the proof of the lemma. □

The following formulae from [15] will be useful: on a Riemannian manifold $(M^n,\ g)$ we have

$$div\,(\mathcal{L}_X g)(X) = \frac{1}{2}\Delta|X|^2 - |\nabla X|^2 + Ric\,(X, X) + D_X div\, X, \tag{2.3}$$

$$div\,(\mathcal{L}_{\nabla f} g)(Z) = 2Ric\,(Z, \nabla f) + 2D_Z div\, \nabla f, \tag{2.4}$$

or on (1, 1)-tensorial notation

$$div\, \nabla^2 f = Ric\,(\nabla f) + \nabla\Delta f \tag{2.5}$$

and

$$\frac{1}{2}\Delta\, |\nabla f|^2 = |\nabla^2 f|^2 + D_{\nabla f} div \nabla f + Ric(\nabla f, \nabla f). \tag{2.6}$$

Taking into account that $div(\lambda I)(X) = \langle \nabla\lambda, X\rangle$, where λ is a smooth function on M^n and $X \in \mathfrak{X}(M)$, equation (2.3) allows us to deduce the following lemma.

Lemma 2. *Let $(M^n,\ g,\ \nabla f,\ \lambda)$ be a generalized m-quasi-Einstein metric. Then we have*

(1) $\dfrac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 - Ric(\nabla f, \nabla f) + \dfrac{2}{m}|\nabla f|^2\Delta f - (n-2)\langle \nabla\lambda, \nabla f\rangle.$

(2) $\dfrac{1}{2}\nabla R = \frac{m-1}{m} Ric(\nabla f) + \dfrac{1}{m}(R - (n-1)\lambda)\nabla f + (n-1)\nabla\lambda.$

(3) $\nabla(R + |\nabla f|^2 - 2(n-1)\lambda) = 2\lambda\nabla f + \dfrac{2}{m}\{\nabla_{\nabla f}\nabla f + (|\nabla f|^2 - \Delta f)\nabla f\}.$

Proof. Since $Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g$ we use the second contracted Bianchi identity

$$\nabla R = 2 div\, Ric \tag{2.7}$$

as well as the next identity

$$div\,(df \otimes df) = \Delta f\, \nabla f + \nabla_{\nabla f} \nabla f \tag{2.8}$$

and (2.5) to deduce

$$\nabla R + 2Ric\,(\nabla f) + 2\nabla \Delta f - \frac{2}{m}\Delta f\, \nabla f - \frac{2}{m}\nabla_{\nabla f}\nabla f = 2\nabla\lambda. \tag{2.9}$$

In particular one deduces

$$\begin{aligned} &\langle \nabla R, \nabla f\rangle + 2Ric(\nabla f, \nabla f) + 2\langle \nabla\Delta f, \nabla f\rangle \\ &-\frac{2}{m}\Delta f |\nabla f|^2 - \frac{2}{m}\langle \nabla_{\nabla f}\nabla f, \nabla f\rangle = 2\langle \nabla\lambda, \nabla f\rangle. \end{aligned} \tag{2.10}$$

Next using (1.5) and (2.6) jointly with the last identity we conclude

$$\begin{aligned} \frac{1}{2}\Delta|\nabla f|^2 &= |\nabla^2 f|^2 - Ric(\nabla f, \nabla f) + \frac{2}{m}|\nabla f|^2 div\, \nabla f \\ &\quad - (n-2)\langle \nabla\lambda, \nabla f\rangle, \end{aligned} \tag{2.11}$$

which finishes the first statement of the lemma. On the other hand, substituting $\Delta f = -R + \lambda n + \frac{1}{m}|\nabla f|^2$ and remembering that $\nabla|\nabla f|^2 = 2\nabla_{\nabla f}\nabla f$ we use once more (2.9) to write

$$\begin{aligned} \frac{1}{2}\nabla R &= -Ric(\nabla f) - \nabla\left(-R + \lambda n + \frac{1}{m}|\nabla f|^2\right) \\ &\quad + \frac{1}{m}\Delta f \nabla f + \frac{1}{m}\nabla_{\nabla f}\nabla f + \nabla\lambda \\ &= -Ric(\nabla f) + \nabla R - \frac{1}{m}\nabla_{\nabla f}\nabla f + \frac{1}{m}\Delta f\nabla f - (n-1)\nabla\lambda. \end{aligned}$$

Of which we deduce

$$\frac{1}{2}\nabla R = Ric(\nabla f) - \frac{1}{m}\Delta f \nabla f + \frac{1}{m}\nabla_{\nabla f}\nabla f + (n-1)\nabla\lambda. \tag{2.12}$$

We now use the fundamental equation to write

$$\nabla_{\nabla f}\nabla f = \lambda\nabla f + \frac{1}{m}|\nabla f|^2\nabla f - Ric(\nabla f). \tag{2.13}$$

In particular, combining (2.12) and (2.13) we obtain

$$\begin{aligned}\frac{1}{2}\nabla R &= \frac{m-1}{m}Ric(\nabla f) + \frac{1}{m}\Big(\lambda + \frac{1}{m}|\nabla f|^2 - \Delta f\Big)\nabla f + (n-1)\nabla\lambda\\ &= \frac{m-1}{m}Ric(\nabla f) + \frac{1}{m}(R-(n-1)\lambda)\nabla f + (n-1)\nabla\lambda,\end{aligned}$$

which gives the second assertion.

Finally, noticing that $\frac{1}{2}\nabla R + \frac{1}{2}\nabla|\nabla f|^2 = \frac{1}{2}\nabla R + \nabla_{\nabla f}\nabla f$ we use the last equation and (2.13) to write

$$\begin{aligned}\frac{1}{2}\nabla R + \frac{1}{2}\nabla|\nabla f|^2 &= \frac{m-1}{m}Ric(\nabla f) + \frac{1}{m}(R-(n-1)\lambda)\nabla f + (n-1)\nabla\lambda\\ &\quad + \lambda\nabla f + \frac{1}{m}|\nabla f|^2\nabla f - Ric(\nabla f).\end{aligned}$$

Thus, using equation (1.4) once more, we achieve

$$\begin{aligned}&\nabla(R+|\nabla f|^2 - 2(n-1)\lambda) - 2\lambda\nabla f\\ &= \frac{2}{m}\big\{(|\nabla f|^2 + R - (n-1)\lambda)\nabla f - Ric(\nabla f)\big\}\\ &= \frac{2}{m}\big\{(|\nabla f|^2 + R - n\lambda + \lambda)\nabla f - Ric(\nabla f)\big\}\\ &= \frac{2}{m}\left\{(|\nabla f|^2 + \frac{1}{m}|\nabla f|^2 - \Delta f + \lambda)\nabla f - Ric(\nabla f)\right\}\\ &= \frac{2}{m}\big\{\nabla_{\nabla f}\nabla f + (|\nabla f|^2 - \Delta f)\nabla f\big\},\end{aligned}$$

which concludes the proof of the lemma. □

It is convenient to point out that for $m = \infty$ and λ constant, assertion (2) of the last lemma is a generalization of the classical Hamilton equation [13] for a gradient Ricci soliton: $R + |\nabla f|^2 - 2\lambda f = C$, where C is constant, as well as for the following relation: $\nabla(R+|\nabla f|^2 - 2(n-1)\lambda) = 2\lambda\nabla f$, that was proved in [4] for an almost Ricci soliton. Choosing $Z \in \mathfrak{X}(M)$, we deduce from the first assertion of Lemma 2 the following identity

$$\begin{aligned}\frac{1}{2}\langle\nabla R, Z\rangle &= \frac{m-1}{m}Ric(\nabla f, Z) + \frac{1}{m}(R-(n-1)\lambda)\langle\nabla f, Z\rangle\\ &\quad + (n-1)\langle\nabla\lambda, Z\rangle.\end{aligned} \tag{2.14}$$

We now present the main result of this section. Taking in account that $u = e^{-\frac{f}{m}}$ we have the following lemma.

Lemma 3. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 3$, be a generalized m-quasi-Einstein metric. If, in addition M^n is Einstein, then we have*

$$\nabla^2 u = \left(-\frac{R}{n(n-1)}u + \frac{c}{m}\right) g, \tag{2.15}$$

where c is constant.

Proof. Since M^n is Einstein and $n \geq 3$ we have $Ric = \frac{R}{n}g$ with R constant. In particular, it follows from (1.7) that

$$\nabla^2 u = \frac{1}{m}\left(\frac{R}{n}u - \lambda u\right) g. \tag{2.16}$$

Whence, using (2.5) we deduce

$$Ric\,(\nabla u) + \nabla \Delta u = \frac{1}{m}\nabla\left(\frac{R}{n}u - \lambda u\right). \tag{2.17}$$

Therefore we infer

$$\frac{R}{n}\nabla u + \nabla \Delta u = \frac{R}{nm}\nabla u - \frac{1}{m}\nabla(\lambda u). \tag{2.18}$$

On the other hand, in accordance with (1.2) and (1.7) we deduce

$$\Delta u = \frac{R}{m}u - \frac{n}{m}\lambda u. \tag{2.19}$$

We now compare (2.18) and (2.19) to obtain

$$\nabla(\lambda u) = R\frac{(m+n-1)}{n(n-1)}\nabla u. \tag{2.20}$$

Therefore we deduce $\lambda u = R\frac{(m+n-1)}{n(n-1)}u - c$, where c is constant. Next we use this value of λu in (2.16) to complete the proof of the lemma. □

3 Proofs of the main results

3.1 Proof of Theorem 1

Proof. First of all, we notice that (1.7) gives that M^n is Einstein if and only if ∇u is a conformal vector field. Since f is not constant and we are supposing that ∇u is a non trivial conformal vector field, which enables us to write $\frac{1}{2}\mathcal{L}_{\nabla u}g =$

$\nabla^2 u = \frac{\Delta u}{n} g$, we deduce that M^n is Einstein. Moreover, using (1.2) and (1.7) we deduce

$$Ric = \left(\lambda + m\frac{\Delta u}{nu}\right) g.$$

Since $n \geq 3$, we have from Schur's Lemma that $R = n\lambda + m\frac{\Delta u}{u}$ is constant.

On the other hand, from Lemma 3 we have

$$\nabla^2 u = \left(-\frac{R}{n(n-1)}u + \frac{c}{m}\right) g$$

where c is constant. Therefore, we are in position to apply Theorem 2 due to Tashiro [17] to deduce that M^n is a space form.

If R is positive, we may assume that M^n is isometric to a unit standard sphere $\mathbb{S}^n$. Since $R = n(n-1)$ we deduce from Lemma 3 that $\Delta u + nu = kn$, where k is constant. Then, up to constant, u is a first eigenfunction of the Laplacian of $\mathbb{S}^n$. Therefore, we have $u = h_v(x) = \langle x, v\rangle + k$, where v is a linear combination of unit vectors in $\mathbb{R}^{n+1}$. Hence, f is, up to constant, given by (1.8).

Next, if $R = 0$ we have from (2.20) that c is not zero. In this case M^n is isometric to a Euclidean space $\mathbb{R}^n$. Using once more Lemma 3 we obtain $\Delta u = k$, where k is constant. Since u must be positive, up change of coordinates, we deduce that $u(x) = |x|^2 + \tau$, with $\tau > 0$.

Finally, if $R < 0$, it follows from Theorem 2 of [17] that M^n is isometric to a hyperbolic space, since we have only one critical point for u. Now let us suppose that M^n is isometric to $\mathbb{H}^n(-1)$. We can use the same argument due to Tashiro [17] to conclude that, up to constant, $u = h_v + \tau$, $\tau > -1$, with $v \in \mathbb{H}^n(-1)$, since in this case $\langle x, v\rangle_0 = -\cosh \eta(x, v)$, where $\eta(x, v)$ is the time-like angle between x and v, which is exactly the geodesic distance between them. Therefore, we complete the proof of the theorem. □

3.2 Proof of Corollary 1

Proof. On integrating Bochner's formula we obtain

$$\int_M \left|\nabla^2 u - \frac{\Delta u}{n} g\right|^2 d\mu = \frac{n-1}{n}\int_M (\Delta u)^2 d\mu - \int_M Ric(\nabla u, \nabla u) d\mu. \quad (3.1)$$

In particular, from our assumption we conclude that

$$\int_M \left|\nabla^2 u - \frac{\Delta u}{n} g\right|^2 d\mu = 0. \quad (3.2)$$

Whence, we deduce that ∇u is a non trivial conformal vector field. Then, for $n \geq 3$, we can apply Theorem 1 to conclude the proof of the corollary. □

3.3 Proof of Theorem 2

Proof. First we integrate the identity derived in Lemma 2 and we use Stokes' formula to infer

$$\begin{aligned}\int_M |\nabla^2 f|^2 d\mu \;=\; &\int_M Ric\,(\nabla f, \nabla f)d\mu - \frac{2}{m}\int_M |\nabla f|^2 \Delta f d\mu \\ &+ (n-2)\int_M \langle \nabla\lambda, \nabla f\rangle d\mu.\end{aligned} \tag{3.3}$$

On the other hand, since we are assuming that the right hand of above identity is less than or equal to zero, we obtain $\nabla^2 f = 0$. Therefore, $\Delta f = 0$, which implies by Hopf's theorem that f is constant and we finish the establishment of the first assertion.

Proceeding one notices that for $m = \infty$, using equation (1.4) the result follows. On the other hand, for m finite, considering once more the auxiliary function $u = e^{-\frac{f}{m}}$, as we already saw $\Delta u = \frac{u}{m}(R - \lambda n)$. Since M^n is compact, $u > 0$ and $(\mathrm{R} - \mathrm{n}\lambda) \geq 0\,(\leq 0)$, we can use once more Hopf's theorem to deduce that u is constant and so is f. From which we complete the proof of the theorem. □

3.4 Proof of Theorem 3

Proof. Taking into account identity (1.4) we obtain

$$m div\,\nabla f = |\nabla f|^2 + m(n\lambda - R). \tag{3.4}$$

By one hand $m div\,\nabla f \geq 0$, since $(n\lambda - R) \geq 0$. On the other hand, if $|\nabla f| \in L^1(M^n)$, we may invoke Proposition 1 in [7], which is a generalization of a result due to Yau [18] for subharmonic functions, to derive that $div\,\nabla f = 0$. Next, we may use equation (3.4) to conclude that $\nabla f \equiv 0$, as well as $n\lambda = R$. Therefore, f is constant and M^n is an Einstein manifold, which gives the first assertion. Now let us suppose that $(M^n,\ g)$ is an Einstein manifold, in particular a surface has this propriety. If ∇f is a conformal vector field with conformal factor ρ, here we can have a Killing vector field, then $\nabla^2 f = \rho g$, where $\rho = \frac{1}{n} div\,\nabla f$. Since $Ric = \frac{R}{n} g$ we deduce from equation (1.6) that

$$\frac{1}{m}(df \otimes df) = |\nabla f|^2 g. \tag{3.5}$$

But, using that m is finite, we can apply Lemma 1 to conclude that $\nabla f \equiv 0$, which completes the proof of the theorem. □

4 Integral formulae for generalized m-quasi-Einstein metrics

In this section we shall introduce some integral formulae for a compact generalized m-quasi-Einstein metric. Before, we present the next result which is a natural extension of one obtained for an almost Ricci soliton in [4], as well as a similar one in [16].

Lemma 4. *Let $\big(M^n,\ g,\ \nabla f,\ \lambda\big)$ be a generalized m-quasi-Einstein metric. Then we have*

$$\frac{1}{2}\Delta R = -\left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \left\{\frac{m+n}{nm}\right\}(\Delta f)^2 - \frac{n}{2}\langle\nabla f, \nabla\lambda\rangle + \langle\nabla f, \nabla R\rangle$$
$$+ \left\{\frac{m-2}{2m}\right\}\langle\nabla f, \nabla\Delta f\rangle + \frac{1}{m}div\left(\nabla_{\nabla f}\nabla f\right) + (n-1)\Delta\lambda + \lambda\Delta f.$$

Proof. Initially by using assertion (2) of Lemma 2 to compute the divergence of ∇R we obtain

$$\Delta R + \Delta|\nabla f|^2 - 2(n-1)\Delta\lambda = 2div\,(\lambda\nabla f)$$
$$+\frac{2}{m}\Big\{\langle\nabla(|\nabla f|^2 - \Delta f), \nabla f\rangle + (|\nabla f|^2 - \Delta f)\Delta f + div\,(\nabla_{\nabla f}\nabla f)\Big\}.$$

We now use $|\nabla^2 f - \frac{\Delta f}{n}g|^2 = |\nabla^2 f|^2 - \frac{1}{n}(\Delta f)^2$ with Bochner's formula to write

$$\frac{1}{2}\Delta R = -Ric\,(\nabla f, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \frac{1}{n}(\Delta f)^2 - \langle\nabla\Delta f, \nabla f\rangle$$
$$+ (n-1)\Delta\lambda + div\,(\lambda\nabla f) + \frac{2}{m}\langle\nabla_{\nabla f}\nabla f, \nabla f\rangle$$
$$+ \frac{1}{m}\Big\{(|\nabla f|^2 - \Delta f)\Delta f - \langle\nabla\Delta f, \nabla f\rangle + div\,(\nabla_{\nabla f}\nabla f)\Big\}.$$

Next, we invoke equation (1.4) to write $\langle\nabla\Delta f, \nabla f\rangle = \langle\nabla\big(n\lambda + \frac{1}{m}|\nabla f|^2 - R\big), \nabla f\rangle$. Then the last relation becomes

$$\frac{1}{2}\Delta R = -Ric\,(\nabla f, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \frac{m+n}{nm}(\Delta f)^2 + (n-1)\Delta\lambda$$
$$- \left\langle\nabla\left(\frac{1}{m}|\nabla f|^2 - R + \lambda n\right), \nabla f\right\rangle + \frac{2}{m}\langle\nabla_{\nabla f}\nabla f, \nabla f\rangle$$

$$
\begin{aligned}
&+ div\,(\lambda\nabla f) + \frac{1}{m}\Big\{|\nabla f|^2\Delta f - \langle\nabla\Delta f, \nabla f\rangle + div\,(\nabla_{\nabla f}\nabla f)\Big\}\\
= \quad & -\big(Ric\,(\nabla f, \nabla f) + (n-1)\langle\nabla\lambda, \nabla f\rangle\big) - \Big|\nabla^2 f - \frac{\Delta f}{n}g\Big|^2\\
&- \frac{m+n}{nm}(\Delta f)^2 + (n-1)\Delta\lambda + \lambda\Delta f + \langle\nabla R, \nabla f\rangle\\
&+ \frac{1}{m}\Big\{|\nabla f|^2\Delta f - \langle\nabla\Delta f, \nabla f\rangle + div\,(\nabla_{\nabla f}\nabla f)\Big\}.
\end{aligned}
$$

On the other hand, using (2.14) we can write

$$
\begin{aligned}
&Ric(\nabla f, \nabla f) + (n-1)\langle\nabla\lambda, \nabla f\rangle\\
&= \frac{1}{2}\langle\nabla R, \nabla f\rangle + \frac{1}{m}Ric(\nabla f, \nabla f) - \frac{1}{m}(R - (n-1)\lambda)|\nabla f|^2.
\end{aligned} \tag{4.1}
$$

Therefore, we compare the last two equations to obtain

$$
\begin{aligned}
\frac{1}{2}\Delta R \; = \; & \frac{1}{2}\langle\nabla R, \nabla f\rangle - \Big|\nabla^2 f - \frac{\Delta f}{n}g\Big|^2 - \frac{m+n}{nm}(\Delta f)^2 + (n-1)\Delta\lambda + \lambda\Delta f\\
&+ \frac{1}{m}\Big\{- Ric(\nabla f, \nabla f) + \big(\Delta f + R - n\lambda\big)|\nabla f|^2 + \lambda|\nabla f|^2\Big\}\\
&+ \frac{1}{m}\Big\{- \langle\nabla\Delta f, \nabla f\rangle + div\,(\nabla_{\nabla f}\nabla f)\Big\}\\
= \; & \frac{1}{2}\langle\nabla R, \nabla f\rangle - \Big|\nabla^2 f - \frac{\Delta f}{n}g\Big|^2 - \frac{m+n}{nm}(\Delta f)^2 + (n-1)\Delta\lambda + \lambda\Delta f\\
&+ \frac{1}{m}\Big\{\langle\nabla_{\nabla f}\nabla f, \nabla f\rangle - \langle\nabla\Delta f, \nabla f\rangle + div\,(\nabla_{\nabla f}\nabla f)\Big\}\\
= \; & \frac{1}{2}\langle\nabla R, \nabla f\rangle - \Big|\nabla^2 f - \frac{\Delta f}{n}g\Big|^2 - \frac{m+n}{nm}(\Delta f)^2 + (n-1)\Delta\lambda + \lambda\Delta f\\
&+ \frac{1}{2}\langle\nabla R, \nabla f\rangle + \frac{1}{2}\langle\nabla f, \nabla\Delta f\rangle - \frac{n}{2}\langle\nabla\lambda, \nabla f\rangle\\
&- \frac{1}{m}\langle\nabla\Delta f, \nabla f\rangle + \frac{1}{m}div\,(\nabla_{\nabla f}\nabla f).
\end{aligned}
$$

We now group terms to arrive at the desired result, hence we complete the proof of the lemma. □

As a consequence of this lemma we obtain the following integral formulae.

Theorem 4. *Let* $\big(M^n,\ g,\ \nabla f,\ \lambda\big)$ *be a compact orientable generalized m-quasi-Einstein metric. Then we have*

(1) $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 d\mu + \frac{n+2}{2n}\int_M (\Delta f)^2 d\mu$
$= \int_M \langle \nabla f, \nabla R\rangle d\mu - \frac{n+2}{2}\int_M \langle \nabla f, \nabla \lambda\rangle d\mu.$

(2) $\int_M \big(Ric\,(\nabla f, \nabla f) + \langle \nabla f, \nabla R\rangle\big) d\mu$
$= \frac{3}{2}\int_M (\Delta f)^2 d\mu + \frac{n+2}{2}\int_M \langle \nabla f, \nabla \lambda\rangle d\mu.$

(3) M^n *is trivial, provided* $\int_M \langle \nabla R, \nabla f\rangle d\mu \leq \frac{n+2}{2}\int_M \langle \nabla f, \nabla\lambda\rangle d\mu.$

(4) $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 d\mu = \frac{n-2}{2n}\int_M \langle \nabla f, \nabla R\rangle d\mu - \frac{n+2}{2nm}\int_M |\nabla f|^2 \Delta f\, d\mu.$

Proof. Since M^n is compact we use Lemma 4 and Stokes' formula to infer

$$\begin{aligned}\int_M \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 d\mu \ &= -\Big(\frac{m+n}{nm}\Big)\int_M (\Delta f)^2 d\mu \\ &\quad - \Big(\frac{m-2}{2m}\Big)\int_M (\Delta f)^2 d\mu - \frac{n}{2}\int_M \langle \nabla\lambda, \nabla f\rangle d\mu \\ &\quad - \int_M \langle \nabla\lambda, \nabla f\rangle d\mu + \int_M \langle \nabla f, \nabla R\rangle d\mu.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\int_M \left(\left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 + \frac{n+2}{2n}(\Delta f)^2\right) d\mu \\ &= \int_M \langle \nabla f, \nabla R\rangle d\mu - \frac{n+2}{2}\int_M \langle \nabla f, \nabla\lambda\rangle d\mu,\end{aligned} \tag{4.2}$$

which gives the first statement.

Next, we integrate Bochner's formula to get

$$\int_M Ric\,(\nabla f, \nabla f) d\mu + \int_M |\nabla^2 f|^2 d\mu + \int_M \langle \nabla f, \nabla \Delta f\rangle d\mu = 0. \tag{4.3}$$

Since

$$\int_M \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 d\mu = \int_M |\nabla^2 f|^2 d\mu - \frac{1}{n}\int_M (\Delta f)^2 d\mu$$

we use Stokes' formula once more to deduce

$$\int_M Ric\,(\nabla f, \nabla f) d\mu + \int_M \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 d\mu = \frac{n-1}{n}\int_M (\Delta f)^2 d\mu. \quad (4.4)$$

Now, comparing (4.2) with (4.4) we obtain

$$\int_M \big(Ric\,(\nabla f, \nabla f) + \langle \nabla f, \nabla R\rangle\big) d\mu = \frac{3}{2}\int_M (\Delta f)^2 d\mu + \frac{n+2}{2}\int_M \langle \nabla f, \nabla \lambda\rangle d\mu,$$

that was to be proved.

On the other hand, if $\int_M \langle \nabla R, \nabla f\rangle d\mu \leq \frac{n+2}{2}\int_M \langle \nabla f, \nabla\lambda\rangle d\mu$, in particular this occurs if R and λ are both constant, we deduce from the first assertion

$$\int_M \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 d\mu + \frac{n+2}{2n}\int_M (\Delta f)^2 d\mu = 0, \quad (4.5)$$

which implies that f must be constant, so M^n is trivial.

Finally, from (1.4) we can write

$$\int_M \langle \nabla f, \nabla \lambda\rangle d\mu = \frac{1}{n}\int_M \langle \nabla f, \nabla \left(R + \Delta f - \frac{1}{m}|\nabla f|^2\right)\rangle d\mu.$$

Hence, by using equation (4.2) we infer

$$\int_M \left(\left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 + \frac{n+2}{2n}(\Delta f)^2\right) d\mu$$

$$= \frac{n-2}{2n}\int_M \langle \nabla f, \nabla R\rangle d\mu + \frac{n+2}{2n}\int_M (\Delta f)^2 d\mu + \frac{n+2}{2nm}\int_M \langle \nabla f, \nabla |\nabla f|^2\rangle d\mu.$$

Therefore, after cancelations and using Stokes' formula, we deduce

$$\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 d\mu = \frac{n-2}{2n}\int_M \langle \nabla f, \nabla R\rangle d\mu - \frac{n+2}{2nm}\int_M |\nabla f|^2 \Delta f d\mu,$$

which completes the proof of the theorem. □

Now we remember that for a conformal vector field X on a compact Riemannian manifold M^n we have $\int_M \mathcal{L}_X R d\mu = \int_M \langle X, \nabla R\rangle d\mu = 0$, see e.g. [6]. On the other hand, from Lemma 1 we also have $\int_M |X|^2 div X d\mu = 0$. Hence, using the last item of the above theorem we deduce that the converse of those two results are true for a gradient vector field. More exactly, we have the following corollary.

Corollary 2. *Let $\big(M^n,\ g,\ \nabla f,\ \lambda\big)$ be a compact orientable generalized m-quasi-Einstein metric with m finite. Then we have:*

(1) *If* $n \geq 3$, $\int_M \langle \nabla f, \nabla R \rangle d\mu = 0$ *and* $\int_M |\nabla f|^2 \Delta f d\mu = 0$, *then* ∇f *is a conformal vector field.*

(2) *If* $n = 2$ *and* $\int_M |\nabla f|^2 \Delta f d\mu = 0$, *then* f *is constant.*

Proof. For the first statement we use the last item of Theorem 4 to deduce $\nabla^2 f = \frac{\Delta f}{n} g$, which gives that ∇f is conformal. Next, we notice that for $n = 2$, it is enough to suppose $\int_M |\nabla f|^2 \Delta f d\mu = 0$ to conclude that ∇f is conformal. But, using Theorem 3 we conclude that f is constant, which completes the proof of the corollary. □

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