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Lax entwining structures, groupoid algebras and cleft extensions

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Abstract. For an algebra A, a coalgebra C and a lax entwining structure (A, C, ψ), in this paper we introduce the notions of lax C-Galois extension with normal basis and lax C-cleft extension and we prove that these notions are equivalent if the functor $A \otimes -$ preserve coequalizers.

Keywords: lax, partial and weak entwining structure, weak Hopf algebra, groupoid algebra, Galois object, cleft extension, normal basis.

Mathematical subject classification: 16W30, 18D10, 20L05.

Introduction

The modern notion of Galois extension asssociated to a Hopf algebra H was introduced by Kreimer and Takeuchi [19] in the following way: let H be a Hopf algebra and A be a right H-comodule algebra with coaction $\rho_A(a) = a_{(0)} \otimes a_{(1)}$, then the extension $B \hookrightarrow A$, being $B = A^{coH} = \{a \in A \; ; \; \rho_A(a) = a \otimes 1_H\}$ the subalgebra of coinvariant elements, is H-Galois if the canonical morphism $\gamma_A : A \otimes_B A \to A \otimes H$, defined by $\gamma_A(a \otimes b) = ab_{(0)} \otimes b_{(1)}$, is an isomorphism. This definition has its origin in the approach to Galois theory of groups acting on commutative rings developed by Chase, Harrison and Rosenberg and in the extension of this theory to coactions of a Hopf algebra H acting on a commutative k-algebra A over a commutative ring k developed in 1969 by Chase and Sweedler [11]. An interesting class of H-Galois extensions has been provided by those for which there exists a convolution invertible right H-comodule morphism $h : H \to A$ called the cleaving morphism. These extensions were called

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cleft and it is well known that, using the notion of normal basis introduced by Kreimer and Takeuchi in [19], Doi and Takeuchi proved in [16] that $B \hookrightarrow A$ is a cleft extension if and only if it is H-Galois with normal basis, i.e., the H-Galois extension $B \hookrightarrow A$ is isomorphic to $B \otimes H$ as left B-module and right H-comodule.

In [18] the result obtained by Doi and Takeuchi was generalized to H-Galois extensions for Hopf algebras living in a symmetric monoidal closed category C and in [7] Brzeziński proved that if A is an algebra, C is a coalgebra and (A, C, ψ) is an entwining structure such that A is an entwined module, the existence of a convolution invertible C-comodule morphism $h : C \to A$ is equivalent to the properties that A is a Galois extension by the coalgebra C (see [6] for the definition) and A is isomorphic to the tensor product of the coinvariant subalgebra B with C as left B-modules and right C-comodules.

A more general result was proved in [2] for weak Galois extensions associated to the weak entwining structures introduced by Caenepeel and De Groot in [8]. In [1] the notion of weak cleft extension was introduced and Theorem 2.11 of [2] states that for a weak entwining structure (A, C, ψ) such that A is an entwined module, if the functor $A \otimes -$ preserves coequalizers, the algebra A is a weak C-cleft extension of the coinvariant subalgebra if and only if it is a weak C-Galois extension and the normal basis property, defined in [2], holds. Since Galois extensions associated to weak Hopf algebras are examples of weak Galois extensions, the characterization of weak cleft extensions in terms of weak Galois extensions satisfying the normal basis condition can be applied to them.

The main motivation of this paper is to extend the previous results to the theory of lax and partial weak entwining structures. These notions were defined by Caenepeel and Janssen in [9] and [10] with the aim of to introduce a theory of partial actions and coactions of Hopf algebras and then to obtain a Hopf-Galois theory in this setting. The notion of a partial group action on an algebra A over a commutative ring k has been introduced by Exel [17] in the context of operator algebras and the algebraic interest of these structures comes from the results proved in [13] by Dokuchaev, Exel and Piccione, in [14] by Dokuchaev and Exel and in [15] where a generalization of Galois theory over commutative rings to partial group actions was given by Dokuchaev, Ferrero and Paques.

The paper is organized as follows: In section one we review some of the standard facts about weak, partial and lax entwining structures proving that it is possible to obtain non trivial examples of lax entwining structures working with lax comodule algebras associated to a groupoid algebra. In particular, if the groupoid algebra is a group algebra we have examples of partial entwining structures. In the second section we have compiled the basic facts about Galois extensions in a lax setting and we introduce the notion of lax Galois extension with normal basis. Section 3 is devoted to study the notion of cleft extension in a lax context and contains the main theorem of this paper, i.e., under the mild assumption that the functor $A \otimes -$ preserves coequalizers, there exists an equivalence between the notions of lax Galois extension with normal basis and lax cleft extension. As a consequence, using that every partial or weak entwining structure is lax, when we particularize this result to the weak case we obtain Theorem 2.11 of [2] and in the partial case we obtain the characterization of partial cleft extensions as partial Galois extensions with normal basis. Finally, it is important to emphasize that the main motivation for the definition of lax cleft extension introduced in this section comes from the following fact: if A is an algebra, C is a coalgebra and (A, C, ψ) is a lax entwining structure such that A is a lax entwined module with coaction ρ_A, the existence of a comodule morphism $h : C \to A$ satisfying that there exists a morphism $h^{-1} : C \to A$ with convolution $h^{-1} \wedge h = e$, being $e = (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes \eta_A)$, implies that (A, C, ψ) is a weak entwining structure. As a consequence, the notion of partial cleft extension introduced in [5] is a classical cleft extension for an entwining structure.

1 Weak Hopf algebras and lax entwining structures

Throughout the paper C denotes a strict monoidal category with tensor product $\otimes$ and base object K. Given objects A, B, D and a morphism $f : B \to D$, we write $A \otimes f$ for $id_A \otimes f$ and $f \otimes A$ for $f \otimes id_A$ where id_A is the identity morphism for the object A. Also we assume that there exists coequalizers and equalizers. The existence of equalizers guarantees that every idempotent splits, i.e., for every morphism $\nabla : Y \to Y$, such that $\nabla = \nabla \circ \nabla$, there exist an object Z and morphisms $i : Z \to Y$ and $p : Y \to Z$ satisfying $\nabla = i \circ p$ and $p \circ i = id_Z$.

A braided monoidal category C means a monoidal category in which there is, for all M and N in C, some natural isomorphism $c_{M,N} : M \otimes N \to N \otimes M$, called the braiding, satisfying the Hexagon Axiom. If the braiding satisfies $c_{N,M} \circ c_{M,N} = id_{M\otimes N}$, the category C will be called symmetric.

As for prerequisites, the reader is expected to be familiar with the notions of algebra (monoid), coalgebra (comonoid), module and comodule in the monoidal setting. Given an algebra A and a coalgebra D, we let $\eta_A : K \to A$, $\mu_A : A \otimes A \to A$, $\varepsilon_D : D \to K$, and $\delta_D : D \to D \otimes D$ denote the unity, the product, the counity, and the coproduct respectively. Also, for two morphism $f, g : D \to A$, the symbol $\wedge$ denotes the usual convolution product in the category C, i.e., $f \wedge g = \mu_A \circ (f \otimes g) \circ \delta_D$.

If the category $\mathcal{C}$ is symmetric and A, B are algebras in $\mathcal{C}$, the object $A \otimes B$ is an algebra in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. In a dual way when D, E are coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$. Finally, A^{op} denotes the algebra with the opposite product $\mu_{A^{op}} = \mu_A \circ c_{A,A}$ and D^{cop} is the coalgebra with the coopposite coproduct $\delta_{D^{cop}} = c_{D,D} \circ \delta_D$.

Definition 1.1. *A lax entwining structure on $\mathcal{C}$ consists of a triple (A, C, ψ), where A is an algebra, C a coalgebra, and $\psi : C \otimes A \to A \otimes C$ a morphism (the entwining morphism) satisfying the relations*

$$\psi \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A), \tag{1}$$

$$(A \otimes \varepsilon_C) \circ \psi = \mu_A \circ (e \otimes A), \tag{2}$$

$$(\nabla_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A), \tag{3}$$

$$\psi \circ (C \otimes \eta_A) = \nabla_{A\otimes C} \circ (e \otimes C) \circ \delta_C, \tag{4}$$

where $e : C \to A$ is the morphism defined by $e = (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes \eta_A)$ and $\nabla_{A\otimes C} : A \otimes C \to A \otimes C$ is the idempotent morphism

$$\nabla_{A\otimes C} = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (A \otimes C \otimes \eta_A). \tag{5}$$

Then, by (1), we have

$$\nabla_{A\otimes C} \circ \psi = \psi \tag{6}$$

and the morphism e satisfies the equality

$$e \wedge e = e. \tag{7}$$

Indeed:

$$\begin{aligned} e \wedge e &= ((\mu_A \circ (e \otimes A)) \otimes \varepsilon_C) \circ (C \otimes \psi) \circ (\delta_C \otimes \eta_A) \\ &= (A \otimes \varepsilon_C) \circ \nabla_{A\otimes C} \circ (e \otimes C) \circ \delta_C = e \end{aligned}$$

where the first equality follows by (2), the second one by the definition of the idempotent morphism $\nabla_{A\otimes C}$, and the third one by (4).

On the other hand, if ψ satisfies the equality

$$(A \otimes \varepsilon_C) \circ \psi = \varepsilon_C \otimes A, \tag{8}$$

the morphism e defined previously is $e = \varepsilon_C \otimes \eta_A$ and, as a consequence, the identity (4) is irrelevant. In this case the entwining structure is called partial.

If the morphism ψ satisfies the conditions (1), (2) and

$$(A \otimes \delta_C) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A), \tag{9}$$

$$\psi \circ (C \otimes \eta_A) = (e \otimes C) \circ \delta_C, \tag{10}$$

the triple (A, C, ψ) is called a weak entwining structure. Finally, if the conditions (1), (8), (9), and

$$\psi \circ (C \otimes \eta_A) = \eta_A \otimes C, \tag{11}$$

hold, we recover the classical notion of entwining structure.

Obviously every partial or weak entwining structure is a lax entwining structure. Lax entwining structures have been introduced by Caenepeel and Janssen in [9] as a generalization of weak entwining structures defined in [8]. The notion of partial entwining structures arise in the context of partial group actions (see [9], [10] and [15]) and has its origins in the pioneering work of Exel [17] were partial actions were considered in the context of operator algebras.

Lemma 1.2. *Let (A, C, ψ) be a lax entwining structure and let*

$$\Delta_{A\otimes C} : A \otimes C \to A \otimes C$$

be the morphism

$$\Delta_{A\otimes C} = (\mu_A \otimes C) \circ (A \otimes ((e \otimes C) \circ \delta_C)).$$

Then

(i) $\Delta_{A\otimes C} = (((A \otimes \varepsilon_C) \circ \nabla_{A\otimes C}) \otimes C) \circ (A \otimes \delta_C)$.

(ii) *The morphism $\Delta_{A\otimes C}$ is idempotent.*

(iii) $\nabla_{A\otimes C} = \Delta_{A\otimes C} \circ \nabla_{A\otimes C} = \nabla_{A\otimes C} \circ \Delta_{A\otimes C}$.

(iv) $\Delta_{A\otimes C} \circ \psi = \psi$.

Proof. The equality (i) is a consequence of the definition of $\nabla_{A\otimes C}$. Secondly, the morphism $\Delta_{A\otimes C}$ is idempotent because using the equality (7), the associativity of the product defined in A and the co-associativity of δ_C, we have:

$$\Delta_{A\otimes C} \circ \Delta_{A\otimes C} = (\mu_A \otimes C) \circ (A \otimes (e \wedge e) \otimes C) \circ (A \otimes \delta_C) = \Delta_{A\otimes C}.$$

The proof for (iii) is the following:

$$\begin{aligned}
&\nabla_{A\otimes C}\circ\Delta_{A\otimes C}\\
=\;&(\mu_A\otimes C)\circ(A\otimes(\mu_A\circ(e\otimes A))\otimes C)\circ(A\otimes C\otimes\psi)\circ(A\otimes\delta_C\otimes\eta_A)\\
=\;&(\mu_A\otimes\varepsilon_C\otimes C)\circ(A\otimes\psi\otimes C)\circ(A\otimes C\otimes\psi)\circ(A\otimes\delta_C\otimes\eta_A)\\
=\;&(\mu_A\otimes\varepsilon_C\otimes C)\circ(A\otimes\nabla_{A\otimes C}\otimes C)\circ(A\otimes A\otimes\delta_C)\circ(A\otimes(\psi\circ(C\otimes\eta_A)))\\
=\;&\Delta_{A\otimes C}\circ\nabla_{A\otimes C},
\end{aligned}$$

where the first equality follows by the associativity of μ_A, the second one by (2), the third one by (3) and the fourth one by (i) and the associativity of μ_A.

Moreover, by (4)

$$\begin{aligned}
&\nabla_{A\otimes C}\circ\Delta_{A\otimes C}\\
=\;&(\mu_A\otimes C)\circ(A\otimes(\mu_A\circ(e\otimes A))\otimes C)\circ(A\otimes C\otimes\psi)\circ(A\otimes\delta_C\otimes\eta_A)\\
=\;&(\mu_A\otimes C)\circ(A\otimes(\nabla_{A\otimes C}\circ(e\otimes C)\circ\delta_C))\\
=\;&\nabla_{A\otimes C}.
\end{aligned}$$

Finally, (iv) holds because by (6) and the previous identities we have

$$\Delta_{A\otimes C}\circ\psi=\Delta_{A\otimes C}\circ\nabla_{A\otimes C}\circ\psi=\nabla_{A\otimes C}\circ\psi=\psi. \qquad \square$$

Remark 1.3. Notice that if (A, C, ψ) is a partial entwining structure the morphism $\Delta_{A\otimes C}$ is the identity of $A\otimes C$.

Theorem 1.4. *A lax entwining structure (A, C, ψ) is a weak entwining structure if and only if $\Delta_{A\otimes C}=\nabla_{A\otimes C}$. Moreover, if (A, C, ψ) is partial and weak then it is an entwining structure.*

Proof. Trivially, if the entwining structure is weak we obtain that $\Delta_{A\otimes C}=\nabla_{A\otimes C}$. Conversely, if (A, C, ψ) is lax and $\Delta_{A\otimes C}=\nabla_{A\otimes C}$, by (3) and (6) we have

$$\begin{aligned}
&(\psi\otimes C)\circ(C\otimes\psi)\circ(\delta_C\otimes A)\\
=\;&(\Delta_{A\otimes C}\otimes C)\circ(A\otimes\delta_C)\circ\psi\\
=\;&(A\otimes\delta_C)\circ\Delta_{A\otimes C}\circ\psi\\
=\;&(A\otimes\delta_C)\circ\psi
\end{aligned}$$

and (9) holds. Moreover, $\Delta_{A\otimes C}=\nabla_{A\otimes C}$ and (7) gives $\nabla_{A\otimes C}\circ(e\otimes C)\circ\delta_C=(e\otimes C)\circ\delta_C$ and (10) follows from (4). Thus, the entwining structure is weak.

Finally, if (A, C, ψ) is partial and weak, trivially, we have (11) and (A, C, ψ) is an entwining structure. $\square$

A way to construct examples of lax entwining structures that are not partial or weak, is to work with lax comodule algebras associated to a weak bialgebra or a weak Hopf algebra in a symmetric monoidal category C. The notions of weak bialgebra and weak Hopf algebra in a symmetric monoidal setting are a generalization of the ones defined by Böhm, Nill and Szlachányi in [4]. The definition is the following:

Definition 1.5. *A weak bialgebra in a symmetric monoidal category C with symmetry isomorphism c, is an object in C with an algebra structure (H, η_H, μ_H) and a coalgebra structure $(H, \varepsilon_H, \delta_H)$ satisfying:*

(i) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H)$.

(ii) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes \delta_H \otimes H)$

$= ((\varepsilon_H \circ \mu_H) \otimes (\varepsilon_H \circ \mu_H)) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H)$.

(iii) $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$

$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ ((\delta_H \circ \eta_H) \otimes (\delta_H \circ \eta_H))$.

If moreover, the following conditions hold,

(iv) *There exists a morphism $\lambda_H : H \to H$ in C (called the antipode of H) satisfying:*

(iv-1) $id_H \wedge \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$,

(iv-2) $\lambda_H \wedge id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$,

(iv-3) $\lambda_H \wedge id_H \wedge \lambda_H = \lambda_H$,

the weak bialgebra H is a weak Hopf algebra in the symmetric monoidal category C.

As a consequence of this definition it is an easy exercise to prove that a weak Hopf algebra is a Hopf algebra if an only if the morphism δ_H (coproduct) is unit-preserving (i.e. $\eta_H \otimes \eta_H = \delta_H \circ \eta_H$) and if and only if the counit is a homomorphism of algebras (i.e. $\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$).

If H is a weak bialgebra it is possible to define the endomorphisms of H, Π_H^L (target morphism), Π_H^R (source morphism), by

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)),$$

and $\overline{\Pi}_H^L$, $\overline{\Pi}_H^R$ by

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H),$$

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)).$$

It is straightforward to show that they are idempotent (Proposition 2.9 of [3]), and if H is a weak Hopf algebra the antipode is antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant (see Proposition 2.20 of [3]).

Let $H = (H, \eta_H, \mu_H, \varepsilon_H, \delta_H, \lambda_H)$ be a weak Hopf algebra in C such that the antipode is an isomorphism. Then

$$H^{op} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, \delta_H, \lambda_H^{-1}) \quad \text{and}$$
$$H^{cop} = (H, \eta_H, \mu_H, \varepsilon_H, c_{H,H} \circ \delta_H, \lambda_H^{-1})$$

are weak Hopf algebras in C. Therefore $(H^{op})^{cop} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, c_{H,H} \circ \delta_H, \lambda_H)$ and $(H^{cop})^{op} = (H, \eta_H, \mu_H \circ c_{H,H}, \varepsilon_H, c_{H,H} \circ \delta_H, \lambda_H)$ are weak Hopf algebras in C. Moreover the weak Hopf algebras H, $(H^{op})^{cop}$ and $(H^{cop})^{op}$ are isomorphic. The isomorphisms are $\lambda_H : (H^{op})^{cop} \to H$ and $\lambda_H : H \to (H^{cop})^{op}$. Finally, note that $\overline{\Pi}_H^L = \Pi_{H^{op}}^R = \Pi_{H^{cop}}^L$ and $\overline{\Pi}_H^R = \Pi_{H^{cop}}^R = \Pi_{H^{op}}^L$.

Definition 1.6. *Let H be a weak bialgebra in a symmetric monoidal category C and let $\tau_{H,H} : H \otimes H \to H \otimes H$ be the morphism defined by $\tau_{H,H} = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H)$. An algebra A is said to be a lax right H-comodule algebra if the exists a morphism $\rho_A : A \to A \otimes H$ called the coaction such that*

$$\rho_A \circ \mu_A = \mu_{A \otimes H} \circ (\rho_A \otimes \rho_A), \tag{12}$$

$$(\rho_A \otimes H) \circ \rho_A = (\mu_A \otimes \tau_{H,H}) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes (\rho_A \circ \eta_A)), \tag{13}$$

$$(A \otimes \Pi_H^L) \circ \rho_A = ((\mu_A \circ c_{A,A}) \otimes \Pi_H^L) \circ (A \otimes (\rho_A \circ \eta_A)), \tag{14}$$

$$\rho_A \circ \eta_A = (A \otimes \mu_H) \circ (((A \otimes \Pi_H^L) \circ \rho_A) \otimes H) \circ \rho_A \circ \eta_A. \tag{15}$$

Note that if H is a bialgebra, we have $\Pi_H^L = \eta_H \otimes \varepsilon_H$ and condition (14) is equivalent to

$$(A \otimes \varepsilon_H) \circ \rho_A = ((\mu_A \circ c_{A,A}) \otimes \varepsilon_H) \circ (A \otimes (\rho_A \circ \eta_A)), \tag{16}$$

and the equality (15) is

$$\rho_A \circ \eta_A = ((A \otimes \varepsilon_H) \circ \rho_A) \otimes H) \circ \rho_A \circ \eta_A. \tag{17}$$

Then, if C is a category of modules over a commutative ring with unit and H is a bialgebra in C, this definition is the one introduced by Caenepeel and Janssen in Proposition 2.5 of [10]. Following [10] we will say that a bialgebra H, in a symmetric monoidal category C, coacts partially on an algebra A or that A is a right partial H-comodule algebra if there exists a morphism $\rho_A : A \to A \otimes H$ such that (12), (13) and

$$(A \otimes \varepsilon_H) \circ \rho_A = id_A \tag{18}$$

hold. Then, in this setting, (16) and (17) are trivial.

In the following Proposition we prove that every lax H-comodule algebra over a weak bialgebra H in a symmetric monoidal category C provides an example of a lax entwining structure.

Proposition 1.7. *Let H be a weak bialgebra in a symmetric monoidal category C. Let A be a lax right H-comodule algebra with coaction $\rho_A : A \to A \otimes H$. Under these conditions the triple $(A, H, \psi = (A\otimes\mu_H)\circ(c_{H,A}\otimes H)\circ(H\otimes\rho_A) : H \otimes A \to A \otimes H)$ is a lax entwining structure.*

Moreover, if H is a bialgebra and A is a right partial H-comodule algebra, the previous triple is a partial entwining structure.

Proof. First note that by (12), the naturality of the braiding and the associativity of the product in H we have

$$\begin{aligned} & \psi \circ (H \otimes \mu_A) \\ = \;& (A \otimes \mu_H) \circ (\mu_{A\otimes H} \otimes H) \circ (c_{H,A} \otimes c_{H,A} \otimes H) \circ (H \otimes \rho_A \otimes \rho_A) \\ = \;& (\mu_A \otimes H) \circ (A \otimes \psi) \circ (\psi \otimes A) \end{aligned}$$

and then (1) holds.

The proof for (2) is the following:

$$\begin{aligned} & \mu_A \circ (e \otimes A) \\ = \;& (\mu_A \otimes \varepsilon_H) \circ (A \otimes c_{H,A}) \circ ((\psi \circ (H \otimes \eta_A)) \otimes A) \\ = \;& (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\mu_A \circ c_{A,A}) \otimes H) \circ \\ & (H \otimes A \otimes (\rho_A \circ \eta_A)) \\ = \;& (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,A} \otimes H) \circ (H \otimes (\mu_A \circ c_{A,A}) \otimes \Pi_H^L) \circ \\ & (H \otimes A \otimes (\rho_A \circ \eta_A)) \\ = \;& (A \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{A,H} \otimes \Pi_H^L) \circ (H \otimes \rho_A) \\ = \;& (A \otimes \varepsilon_H) \circ \psi, \end{aligned}$$

where the first and the second equalities follow by the naturality of c and the definition of ψ, the third and the fifth ones by $\varepsilon_H \circ \mu_H = \varepsilon_H \circ \mu_H \circ (H \otimes \Pi_H^L)$ and the fourth one by (14).

On the other hand, by (13) and the naturality of c we obtain (3). Indeed:

$$\begin{aligned}
& (\psi \otimes H) \circ (H \otimes \psi) \circ (\delta_H \otimes A) \\
= \;& (A \otimes (\mu_{H\otimes H} \circ (\delta_H \otimes H \otimes H))) \circ (c_{H,A} \otimes H \otimes H) \\
& \circ (H \otimes ((\rho_A \otimes H) \circ \rho_A)) \\
= \;& (A \otimes (\mu_{H\otimes H} \circ (\delta_H \otimes H \otimes H))) \circ (c_{H,A} \otimes H \otimes H) \circ \\
& (H \otimes ((\mu_A \otimes \tau_{H,H}) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes (\rho_A \circ \eta_A)))) \\
= \;& (A \otimes ((\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \mu_H) \otimes H))) \circ \\
& (c_{H,A} \otimes H \otimes H) \circ (H \otimes ((\mu_A \otimes H \otimes H) \circ (A \otimes c_{H,A} \otimes H) \circ \\
& (\rho_A \otimes (\rho_A \circ \eta_A)))) \\
= \;& (\mu_{A\otimes H} \otimes H) \circ (A \otimes H \otimes A \otimes c_{H,H}) \circ (A \otimes H \otimes c_{H,A} \otimes H) \circ \\
& (A \otimes \delta_H \otimes (\rho_A \circ \eta_A)) \circ \psi \\
= \;& (\nabla_{A\otimes H} \otimes H) \circ (A \otimes \delta_H) \circ \psi.
\end{aligned}$$

The proof for (4) follows by

$$\begin{aligned}
& \nabla_{A\otimes H} \circ (e \otimes H) \circ \delta_H \\
= \;& (((A \otimes \varepsilon_H) \circ \psi) \otimes \mu_H) \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes (\rho_A \circ \eta_A)) \\
= \;& (A \otimes ((\varepsilon_H \otimes H) \circ \mu_{H\otimes H} \circ (\delta_H \otimes H \otimes H))) \circ \\
& (c_{H,A} \otimes H \otimes H) \circ (H \otimes ((\rho_A \otimes H) \circ \rho_A \circ \eta_A)) \\
= \;& (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes ((A \otimes (\mu_H \circ (\Pi_H^L \otimes H))) \circ \\
& (\rho_A \otimes H) \circ \rho_A \circ \eta_A)) \\
= \;& \psi \circ (H \otimes \eta_A),
\end{aligned}$$

where the first equality is a consequence of (2), the second one follows by the naturality of c, the third one by the identity $((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H) = \mu_H \circ (H \otimes \Pi_H^L)$ as well as the associativity of the product in H, and the fourth one by (15).

Finally, if H is a bialgebra and A a right partial H comodule algebra it is immediate to obtain (8), and then (A, H, ψ) is a partial entwining structure. $\square$

Example 1.8. As group algebras and their duals are the natural examples of Hopf algebras, groupoid algebras and their duals provide examples of weak Hopf algebras. Recall that a groupoid $\mathcal{G}$ is simply a small category in which every morphism is an isomorphism. In this example, we consider groupoids with a finite number of morphisms. The set of objects of $\mathcal{G}$ will be denoted by $\mathcal{G}_0$ and the set of morphisms by $\mathcal{G}_1$. The identity morphism on $x \in \mathcal{G}_0$ will also be denoted by id_x and for a morphism $\sigma : x \to y$ in $\mathcal{G}_1$, we write $s(\sigma)$ and $t(\sigma)$, respectively for the source and the target of σ. Finally, $E(x)$ denotes the set of endomorphisms of $x \in \mathcal{G}_0$.

Let $\mathcal{G}$ be a groupoid, and R a commutative ring with unit. The groupoid algebra is the direct sum

$$R\mathcal{G} = \bigoplus_{\sigma \in \mathcal{G}_1} R\sigma$$

with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise, i.e. $\sigma\tau = \sigma \circ \tau$ if $s(\sigma) = t(\tau)$ and $\sigma\tau = 0$ if $s(\sigma) \neq t(\tau)$. The unit element is $1_{R\mathcal{G}} = \sum_{x \in \mathcal{G}_0} id_x$. The algebra $R\mathcal{G}$ is a cocommutative weak Hopf algebra, with coproduct $\delta_{R\mathcal{G}}$, counit $\varepsilon_{R\mathcal{G}}$ and antipode $\lambda_{R\mathcal{G}}$ given by the formulas:

$$\delta_{R\mathcal{G}}(\sigma) = \sigma \otimes \sigma, \quad \varepsilon_{R\mathcal{G}}(\sigma) = 1_R, \quad \lambda_{R\mathcal{G}}(\sigma) = \sigma^{-1}.$$

For the weak Hopf algebra $R\mathcal{G}$ the target and source morphisms are respectively,

$$\Pi^L_{R\mathcal{G}}(\sigma) = id_{t(\sigma)}, \quad \Pi^R_{R\mathcal{G}}(\sigma) = id_{s(\sigma)}.$$

Let $\mathcal{G}$ a groupoid with $2 \leq |\mathcal{G}_0|$ and such that there exists an $x \in \mathcal{G}_0$ with $n_x = |E(x)|$ invertible in R. Define for an R-algebra A the R-linear morphism $\rho_A : A \to A \otimes R\mathcal{G}$ by

$$\rho_A(a) = \frac{1}{n_x}\, a \otimes h \tag{19}$$

where $h = \sum_{\sigma \in E(x)} \sigma$ and the unadorned tensor product denotes the tensor product over R. Then, A with the coaction ρ_A is an example of lax right $R\mathcal{G}$-comodule algebra. To show that ρ_A satisfies (12), (13), (14) and (15), first we prove that

$$\tau_{R\mathcal{G},R\mathcal{G}}(h \otimes h) = h \otimes h. \tag{20}$$

Indeed, using the fact that $\sigma h = h$ for all $\sigma \in E(x)$ we have

$$\tau_{R\mathcal{G},R\mathcal{G}}(h \otimes h) = ((\mu_{R\mathcal{G}} \otimes R\mathcal{G}) \circ (R\mathcal{G} \otimes c_{R\mathcal{G},R\mathcal{G}}))\Big(\sum_{\sigma \in E(x)} \sigma \otimes \sigma \otimes h\Big)$$

$$= (\mu_{R\mathcal{G}} \otimes R\mathcal{G})\Big(\sum_{\sigma \in E(x)} \sigma \otimes h \otimes \sigma\Big) = \sum_{\sigma \in E(x)} \sigma h \otimes \sigma = \sum_{\sigma \in E(x)} h \otimes \sigma$$

$$= h \otimes \sum_{\sigma \in E(x)} \sigma = h \otimes h.$$

For ρ_A condition (12) follows by:

$$(\mu_{A\otimes RG} \circ (\rho_A \otimes \rho_A))(a \otimes b) = \mu_{A\otimes RG}\left(\frac{1}{n_x^2} \sum_{\sigma,\tau\in E(x)} a \otimes \sigma \otimes b \otimes \tau\right)$$

$$= \frac{1}{n_x^2} \sum_{\sigma,\tau\in E(x)} ab \otimes \sigma\tau = \frac{1}{n_x^2}\, ab \otimes \sum_{\sigma\in E(x)} \sigma h$$

$$= \frac{1}{n_x^2}\, ab \otimes n_x h = \frac{1}{n_x}\, ab \otimes h = \rho_A(ab).$$

Condition (13) is checked by applying (20):

$$((\mu_A \otimes \tau_{RG,RG}) \circ (A \otimes c_{RG,A} \otimes H) \circ (\rho_A \otimes (\rho_A \circ \eta_A)))(a)$$

$$= ((\mu_A \otimes \tau_{RG,RG}) \circ (A \otimes c_{RG,A} \otimes H))\left(\frac{1}{n_x^2}\, a \otimes h \otimes 1_A \otimes h\right)$$

$$= \frac{1}{n_x^2}\, a \otimes \tau_{RG,RG}(h \otimes h) = \frac{1}{n_x^2}\, a \otimes h \otimes h = (\rho_A \otimes RG) \circ \rho_A(a).$$

Condition (14) is proven by the properties of the target morphism:

$$(((\mu_A \circ c_{A,A}) \otimes \Pi^L_{RG}) \circ (A \otimes (\rho_A \circ \eta_A)))(a)$$

$$= ((\mu_A \circ c_{A,A}) \otimes \Pi^L_{RG})\left(\frac{1}{n_x}\, a \otimes 1_A \otimes h\right)$$

$$= \frac{1}{n_x}\, a \otimes \Pi^L_{RG}(h) = (A \otimes \Pi^L_{RG}) \circ \rho_A(a).$$

Finally, condition (15) is verified using the identity $\Pi^L_{RG}(h) = n_x id_x$:

$$((A \otimes \mu_{RG}) \circ (((A \otimes \Pi^L_{RG}) \circ \rho_A) \otimes RG) \circ \rho_A)(1_A)$$

$$= (A \otimes \mu_H)\left(\frac{1}{n_x^2}\, 1_A \otimes \Pi^L_{RG}(h) \otimes h\right) = \frac{1}{n_x^2}\, 1_A \otimes n_x id_x h$$

$$= \frac{1}{n_x}\, 1_A \otimes h = \rho_A(1_A).$$

Then, as a consequence of Proposition 1.7 we obtain an example of a lax entwining structure where $\psi : R\mathcal{G} \otimes A \to A \otimes R\mathcal{G}$ is defined by

$$\psi\left(\sum_{i=1}^{n} r_i\sigma_i \otimes a_i\right) = \frac{1}{n_x} \sum_{i=1}^{n} r_i a_i \otimes \sigma_i h \tag{21}$$

and

$$e(\sigma) = ((A \otimes \varepsilon_{R\mathcal{G}}) \circ \psi)(\sigma \otimes 1_A) = \begin{cases} 0 & \text{if} \quad s(\sigma) \neq x \\ 1_A & \text{if} \quad s(\sigma) = x. \end{cases}$$

This lax entwining structure is not partial because if $\omega \in \mathcal{G}_1$ and $s(\omega) \neq x$ we have

$$e(\omega) = 0 \neq 1_A = (\varepsilon_{R\mathcal{G}} \otimes A)(\omega \otimes 1_A).$$

Finally, note that ρ_A satisfies the identity $((A \otimes \varepsilon_{R\mathcal{G}}) \circ \rho_A)(a) = a$ for all $a \in A$ because:

$$(A \otimes \varepsilon_{R\mathcal{G}}) \circ \rho_A(a) = \frac{1}{n_x}\, a \otimes \sum_{\sigma \in E(x)} \varepsilon_{R\mathcal{G}}(\sigma) = \frac{1}{n_x}\, a \otimes n_x 1_R = a.$$

If G is a finite group and $\mathcal{G}$ is the groupoid associated to G, $R\mathcal{G}$ is the group algebra of G denoted by RG which is a cocommutative Hopf algebra. In this case, for an R-algebra A, the coaction defined in (19) is

$$\rho_A(a) = \frac{1}{|G|}\, a \otimes h, \tag{22}$$

where $h = \sum_{g \in G} g$ and the entwining morphism is

$$\psi\left(\sum_{i=1}^{n} r_i g_i \otimes a_i\right) = \frac{1}{|G|} \sum_{i=1}^{n} r_i a_i \otimes g_i h = \frac{1}{|G|} \sum_{i=1}^{n} r_i a_i \otimes h. \tag{23}$$

Then, in this particular case, we have

$$((A \otimes \varepsilon_{RG}) \circ \psi)(g \otimes a) = (A \otimes \varepsilon_{RG})\Big(\frac{1}{|G|}\, a \otimes gh\Big)$$
$$= (A \otimes \varepsilon_{RG})\Big(\frac{1}{|G|}\, a \otimes h\Big) = \frac{1}{|G|}\, a \otimes |G| 1_R = a = (\varepsilon_{RG} \otimes A)(g \otimes a)$$

and therefore (A, RG, ψ) is a partial entwining structure.

By the following Theorem these partial entwining structures can be used to provide examples of lax entwining structures that are not partial.

Theorem 1.9. *Let (A, C, ψ) be a lax entwining structure and let $d : K \to A$ be a morphism such that $\mu_A \circ (d \otimes d) = d$ and $\mu_A \circ (d \otimes A) = \mu_A \circ (A \otimes d)$. If f_d denotes the morphism $f_d = \mu_A \circ (d \otimes A) : A \to A$ and $(f_d \otimes C) \circ \psi = \psi \circ (C \otimes f_d)$, the triple $(A, C, \psi_{f_d} = (f_d \otimes C) \circ \psi)$ is a lax entwining structure. Moreover, if (A, C, ψ) is partial and $f_d \neq id_A$, (A, C, ψ_{f_d}) is lax and not partial.*

Proof. First note that f_d is an idempotent morphism because by the associativity of the product μ_A we have:

$$f_d \circ f_d = \mu_A \circ ((\mu_A \circ (d \otimes d)) \otimes A) = f_d.$$

Also, it is easy to show that $f_d \circ \eta_A = d$ and

$$f_d \circ \mu_A = \mu_A \circ (f_d \otimes f_d) = \mu_A \circ (f_d \otimes A) = \mu_A \circ (A \otimes f_d). \tag{24}$$

Then, by the conditions of this Theorem and the equalities (24), and (1) we obtain

$$\begin{aligned}\psi_{f_d} \circ (C \otimes \mu_A) &= \psi \circ (C \otimes (\mu_A \circ (f_d \otimes f_d)))\\ &= (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A) \circ (C \otimes f_d \otimes f_d)\\ &= (\mu_A \otimes C) \circ (A \otimes \psi_{f_d}) \circ (\psi_{f_d} \otimes A).\end{aligned}$$

Therefore, (1) holds for (A, C, ψ_{f_d}).

On the other hand, $e_{f_d} = (A \otimes \varepsilon_C) \circ \psi_{f_d} \circ (C \otimes \eta_A) = f_d \circ e$ and, as a consequence, using the associativity of μ_A and (2) we have:

$$\mu_A \circ (e_{f_d} \otimes A) = f_d \circ \mu_A \circ (e \otimes A) = (A \otimes \varepsilon_C) \circ \psi_{f_d}.$$

Then, (2) holds for (A, C, ψ_{f_d}).

Moreover, by the idempotent character of f_d and (3) we have

$$\begin{aligned}&(\psi_{f_d} \otimes C) \circ (C \otimes \psi_{f_d}) \circ (\delta_C \otimes A) = (f_d \otimes C \otimes C) \circ (\psi \otimes C) \circ\\ &(C \otimes \psi) \circ (\delta_C \otimes A) = (((f_d \otimes C) \circ \nabla_{A\otimes C}) \otimes C) \circ (A \otimes \delta_C) \circ \psi\\ &= (\nabla_{A\otimes C}^{f_d} \otimes C) \circ (A \otimes \delta_C) \circ \psi_{f_d}\end{aligned}$$

and then (3) holds for (A, C, ψ_{f_d}) because by (24),

$$\nabla_{A\otimes C}^{f_d} = \nabla_{A\otimes C} \circ (f_d \otimes C) = (f_d \otimes C) \circ \nabla_{A\otimes C} = (f_d \otimes C) \circ \nabla_{A\otimes C} \circ (f_d \otimes C).$$

Finally, by (4) and using that $(f_d \otimes C) \circ \nabla_{A\otimes C} = (f_d \otimes C) \circ \nabla_{A\otimes C} \circ (f_d \otimes C)$ we obtain

$$\psi_{f_d} \circ (C \otimes \eta_A) = (f_d \otimes C) \circ \nabla_{A\otimes C} \circ (e \otimes C) \circ \delta_C$$

$$= (f_d \otimes C) \circ \nabla_{A\otimes C} \circ ((f_d \circ e) \otimes C) \circ \delta_C = \nabla^{f_d}_{A\otimes C} \circ (e_{f_d} \otimes C) \circ \delta_C$$

and (4) holds for (A, C, ψ_{f_d}).

If (A, C, ψ) is partial and $f_d \neq id_A$, (A, C, ψ_{f_d}) is lax and not partial because:

$$(A \otimes \varepsilon_C) \circ \psi_{f_d} = \varepsilon_C \otimes f_d. \qquad \square$$

Example 1.10. Let $(A, R\mathcal{G}, \psi)$ be the lax entwining structure defined in example 1.8 in the groupoid setting. If d is a central idempotent element in A, the morphism $f_d : A \to A$ defined by $f_d(a) = da$ satisfies the conditions of the previous Theorem and, as a consequence, $(A, R\mathcal{G}, \psi_{f_d})$ is lax entwining structure where

$$\psi_{f_d}\left(\sum_{i=1}^{n} r_i\sigma_i \otimes a_i\right) = \frac{1}{n_x}\sum_{i=1}^{n} r_i da_i \otimes \sigma_i h. \tag{25}$$

The result proved by Connell in [12] assures that if R is a completely reducible associative ring with unit and H is a finite group such that $|H|$ is invertible in R, there exist a set of elements in RH, $\{e_1, \dots, e_n\}$, such that

(i) $e_i \neq 0$ is a central idempotent, $1 \leq i \leq n$.

(ii) If $i \neq j$ then $e_i e_j = 0$.

(iii) $1_{RH} = \sum_{i=1}^{n} e_i$.

(iv) The element e_i cannot be written as $e_i = e_i' + e_i''$ where e_i' and e_i'' are central idempotents such that $e_i', e_i'' \neq 0$ and $e_i' e_i'' = 0$, $1 \leq i \leq n$.

Then for all e_k the morphism

$$\psi_{f_{e_k}}\left(\sum_{i=1}^{n} r_i g_i \otimes h_i\right) = \frac{1}{|G|}\sum_{i=1}^{n} r_i e_k h_i \otimes h. \tag{26}$$

induces a lax entwining structure $(RH, RG, \psi_{f_{e_k}})$, for every finite group G with $|G|$ invertible in R, that is not partial.

In the final part of this section we will study the dual notion of lax right H-comodule algebra.

Definition 1.11. Let H be a weak bialgebra in a symmetric monoidal category C. An algebra A is said to be a lax left H-module algebra if there exists a morphism $\varphi_A : H \otimes A \to A$ called the action such that

$$\mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A) = \varphi_A \circ (H \otimes \mu_A), \quad (27)$$

$$\varphi_A \circ (H \otimes \varphi_A) = \mu_A \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes (\varphi_A \circ (\mu_H \otimes A))) \circ (\delta_H \otimes H \otimes A), \quad (28)$$

$$\mu_A \circ (A \otimes (\varphi_A \circ (\overline{\Pi}_H^L \otimes \eta_A))) = \varphi_A \circ c_{A,H} \circ (A \otimes \overline{\Pi}_H^L), \quad (29)$$

$$\varphi_A \circ (H \otimes \eta_A) = \varphi_A \circ (H \otimes \varphi_A) \circ ((c_{H,H} \circ (H \otimes \overline{\Pi}_H^L) \circ \delta_H) \otimes \eta_A). \quad (30)$$

Note that if H is a bialgebra we have $\overline{\Pi}_H^L = \eta_H \otimes \varepsilon_H$ and condition (29) is equivalent to

$$\varphi_A \circ (\eta_H \otimes A) = \mu_A \circ (A \otimes (\varphi_A \circ (\eta_H \otimes \eta_A))), \quad (31)$$

and the equality (30) is

$$\varphi_A \circ (H \otimes \eta_A) = \varphi_A \circ (\eta_H \otimes (\varphi_A \circ (H \otimes \eta_A))). \quad (32)$$

Then, if C is a category of modules over a commutative ring with unit and H is a bialgebra in C, this definition includes the one introduced by Caenepeel and Janssen in Proposition 4.4 of [10]. We will say that a bialgebra H, in a symmetric monoidal category C, acts partially on an algebra A or that A is a left partial H-module algebra if there exists a morphism $\varphi_A : H \otimes A \to A$ such that (27), (28) and

$$\varphi_A \circ (\eta_H \otimes A) = id_A \quad (33)$$

hold. Then, in this setting, (31) and (32) are trivial.

If we work with finite weak Hopf algebras in C it is possible to obtain a relation between right lax comodule algebras and left module algebras. First, we recall some definitions and results about finite objects and finite weak Hopf algebras in C.

Definition 1.12. *An object P in C is said to be finite if there exists $P^* \in C$ such that*

$$(P \otimes -, P^* \otimes -, \alpha_P, \beta_P)$$

is an adjoint pair.

If $(P \otimes -, P^* \otimes -, \alpha_P, \beta_P)$ is an adjoint pair then $(P^* \otimes -, P \otimes -, \alpha_{P^*}, \beta_{P^*})$ with $\alpha_{P^*} = (c_{P,P^*} \otimes -) \circ \alpha_P$ and $\beta_{P^*} = \beta_P \circ (c_{P^*,P} \otimes -)$ is an adjoint pair. Thus, if P is a finite object, P^* is finite with adjunction $(P^* \otimes -, P \otimes -, \alpha_{P^*},$

β_{P^*}), $\alpha_{P^*} = (c_{P,P^*} \otimes -) \circ \alpha_P$ and $\beta_{P^*} = \beta_P \circ (c_{P^*,P} \otimes -)$. As a consequence, when P is finite, $P^{**} = P$. If $f : M \to N$ is a morphism between finite objects we denote by $f^* : N^* \to M^*$ the dual morphism defined by

$$f^* = (M^* \otimes \beta_N(K)) \circ (((M^* \otimes f) \circ \alpha_M(K)) \otimes N^*).$$

Let H be a finite weak Hopf algebra in C. We define

$$H^* = (H^*, \eta_{H^*}, \mu_{H^*}, \varepsilon_{H^*}, \delta_{H^*}, \lambda_{H^*})$$

where

$$\eta_{H^*} = (H^* \otimes \varepsilon_H) \circ \alpha_H(K),$$

$$\mu_{H^*} = (H^* \otimes \beta_H(K)) \circ (H^* \otimes H \otimes \beta_H(K) \otimes H^*) \circ (H^* \otimes \delta_H \otimes H^* \otimes H^*) \circ (\alpha_H(K) \otimes H^* \otimes H^*),$$

$$\varepsilon_{H^*} = \beta_H(K) \circ (\eta_H \otimes H^*),$$

$$\delta_{H^*} = (H^* \otimes H^* \otimes \beta_H(K)) \circ (H^* \otimes H^* \otimes \mu_H \otimes H^*) \circ (H^* \otimes \alpha_H(K) \otimes H \otimes H^*) \circ (\alpha_H(K) \otimes H^*),$$

$$\lambda_{H^*} = \lambda_H^*.$$

Then, H^* is a weak Hopf algebra.

Theorem 1.13. *Let A be an algebra in C and let H be a finite weak bialgebra in C. Then, A is a left lax H-module algebra if and only if A^{op} is a right lax H^{*cop}-comodule algebra.*

Proof. First, assume that A is a left lax H-module algebra with action φ_A. Define the coaction $\rho_{A^{op}} : A \to A \otimes H^*$ by

$$\rho_{A^{op}} = c_{H^*,A} \circ (H^* \otimes \varphi_A) \circ (\alpha_H(K) \otimes A). \tag{34}$$

Then, $(A^{op}, \rho_{A^{op}})$ is a right lax H^{*cop}-comodule algebra. To prove the previous assertion, we have to obtain that (A^{op}, ρ_A) satisfies (12), (13), (14) and (15) for the weak Hopf algebra H^{*cop}. Indeed:

$$\begin{aligned} & (\mu_{A^{op}} \otimes \mu_{H^{*cop}}) \circ (A \otimes c_{H^*,A} \otimes A) \circ (\rho_{A^{op}} \otimes \rho_{A^{op}}) \\ = \; & ((\mu_A \circ c_{A,A} \circ (\varphi_A \otimes \varphi_A)) \otimes H^*) \circ (H \otimes A \otimes H \otimes c_{H^*,A}) \\ & \circ (H \otimes A \otimes H \otimes \mu_{H^*} \otimes A) \circ (c_{A,H} \otimes c_{H^*,H} \otimes H^* \otimes A) \\ & \circ (A \otimes \alpha_{H^*}(K) \otimes \alpha_{H^*}(K) \otimes A) \end{aligned}$$

$$
\begin{aligned}
&= ((\mu_A \circ (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)) \otimes H^*) \\
&\quad \circ (H \otimes A \otimes c_{H^*,A}) \circ (H \otimes c_{H^*,A} \otimes A) \circ (\alpha_{H^*}(K) \otimes c_{A,A}) \\
&= ((\varphi_A \circ (H \otimes \mu_{A^{op}})) \otimes H^*) \circ (c_{A,H} \otimes c_{H^*,A}) \circ (A \otimes \alpha_{H^*}(K) \otimes A) \\
&= \rho_{A^{op}} \circ \mu_{A^{op}},
\end{aligned}
$$

where the first and the fourth equalities follow by the naturality of c, the second one by

$$
\begin{aligned}
&(H \otimes H \otimes \mu_{H^{*cop}}) \circ (H \otimes c_{H^*,H} \otimes H^*) \circ (\alpha_{H^*}(K) \otimes \alpha_{H^*}(K)) \\
&= ((c_{H,H} \circ \delta_H) \otimes H^*) \circ \alpha_{H^*}(K)
\end{aligned}
$$

as well as the naturality of c and the third one by (27).

$$
\begin{aligned}
&(\mu_{A^{op}} \otimes \tau_{H^{*cop},H^{*cop}}) \circ (A \otimes c_{H^*,A} \otimes H^*) \circ (\rho_{A^{op}} \otimes (\eta_{A^{op}} \circ \rho_{A^{op}})) \\
&= ((\mu_A \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes \varphi_A) \circ (H \otimes c_{A,H}) \circ \\
&\quad (c_{A,H} \otimes H)) \otimes \tau_{H^{*cop},H^{*cop}}) \circ (A \otimes ((H \otimes \alpha_{H^*}(K) \otimes H^*) \circ \alpha_{H^*}(K))) \\
&= ((\mu_A \circ ((\varphi_A \circ (H \otimes \eta_A)) \otimes \varphi_A) \circ (H \otimes \mu_H \otimes A) \circ (\delta_H \otimes H \otimes A) \\
&\quad \circ (H \otimes c_{A,H})) \otimes H^* \otimes H^*) \circ (c_{A,H} \otimes c_{H^*,H} \otimes H^*) \\
&\quad \circ (A \otimes \alpha_{H^*}(K) \otimes \alpha_{H^*}(K)) \\
&= ((\varphi_A \circ (H \otimes (\varphi_A \circ c_{A,H}))) \otimes H^* \otimes H^*) \circ \\
&\quad (c_{A,H} \otimes c_{H^*,H} \otimes H) \circ (A \otimes \alpha_{H^*}(K) \otimes \alpha_{H^*}(K)) \\
&= ((\varphi_A \circ (H \otimes \varphi_A)) \otimes H^* \otimes H^*) \circ (H \otimes H \otimes c_{H^*,A} \otimes H^*) \circ \\
&\quad (H \otimes c_{H^*,H} \otimes c_{H^*,H}) \circ (\alpha_{H^*}(K) \otimes \alpha_{H^*}(K) \otimes A) \\
&= (\rho_{A^{op}} \otimes H^{*cop}) \circ \rho_{A^{op}},
\end{aligned}
$$

where the first, the fourth and the fifth equalities follow by the naturality of c, the second one by

$$
\begin{aligned}
&(H \otimes H \otimes \tau_{H^{*cop},H^{*cop}}) \circ (H \otimes \alpha_{H^*}(K) \otimes H^*) \circ \alpha_{H^*}(K) \\
&= (((H \otimes \mu_H) \circ (\delta_H \otimes H)) \otimes H^* \otimes H^*) \circ (H \otimes c_{H^*,H} \otimes H^*) \\
&\quad \circ (\alpha_{H^*}(K) \otimes \alpha_{H^*}(K))
\end{aligned}
$$

and the third one by (28).

$$
\begin{aligned}
&((\mu_{A^{op}} \circ c_{A,A}) \otimes \Pi^L_{H^{*cop}}) \circ (A \otimes (\rho_{A^{op}} \circ \eta_{A^{cop}})) \\
&= (\mu_A \circ (A \otimes (\varphi_A \circ (\overline{\Pi}^L_H \otimes \eta_A))) \otimes H^*) \circ (A \otimes \alpha_{H^*}(K)) \\
&= ((\varphi_A \circ c_{A,H} \circ (A \otimes \overline{\Pi}^L_H)) \otimes H^*) \circ (A \otimes \alpha_{H^*}(K))
\end{aligned}
$$

$$
\begin{aligned}
&= (\varphi_A \otimes H) \circ (\overline{\Pi}_H^L \otimes H) \circ (\alpha_{H^*}(K) \otimes A) \\
&= (A \otimes \Pi^L_{H^{*cop}}) \circ \rho_{A^{op}},
\end{aligned}
$$

where the first and the fourth equalities follow by

$$
\Pi^L_{H^{*cop}} = \overline{\Pi}_H^{L*} \tag{35}
$$

the second one by (29) and the third one by the naturality of c.

$$
\begin{aligned}
& (A \otimes (\mu_{H^{*cop}} \circ (\Pi^L_{H^{*cop}} \otimes H^*))) \circ (\rho_{A^{op}} \otimes H^*) \circ \rho_{A^{op}} \circ \eta_{A^{op}} \\
=\ & ((\varphi_A \circ (H \otimes \varphi_A) \circ ((c_{H,H} \circ (H \otimes \overline{\Pi}_H^L) \circ \delta_H) \otimes \eta_A)) \otimes H^*) \circ \alpha_{H^*}(K) \\
=\ & ((\varphi_A \circ (H \otimes \eta_A)) \otimes H^*) \circ \alpha_{H^*}(K) \\
=\ & \rho_{A^{op}} \circ \eta_{A^{op}},
\end{aligned}
$$

where the first equality follows by the naturality of c, the properties of α_H and β_H and by (35). The second one is a consequence of (30) and the last one follows by the naturality of c.

Conversely, if A^{op} is a right lax H^{*cop}-comodule algebra then A with the action defined by

$$
\varphi_A = (A \otimes \beta_H(K)) \circ (c_{H,A} \otimes H^*) \circ (H \otimes \rho_{A^{op}}) : H \otimes A \to A
$$

is a left lax H-module algebra. The proof is dual to the previous one and we leave the details to the reader. □

Remark 1.14. Obviously, the previous Theorem remains valid if we change the weak bialgebra by a usual bialgebra or the lax structure by a partial one. In this case we obtain as a corollary Theorem 4.7 of [10].

On the other hand, Theorem 1.13 admits an equivalent formulation in the following way: Let A be an algebra in C and let H be a finite weak bialgebra in C. Then, A is a right lax H-comodule algebra if and only if A^{op} is a left lax H^{*op}-module algebra.

Example 1.15. Let $\mathcal{G}$ be a groupoid with $\mathcal{G}_1$ finite. Then $R\mathcal{G}$ is free of a finite rank as a R-module, hence $R(\mathcal{G}) = (R\mathcal{G})^* = Hom_R(R\mathcal{G}, R)$ is a commutative weak bialgebra. As R-module

$$
R(\mathcal{G}) = \bigoplus_{\sigma \in \mathcal{G}_1} R f_\sigma
$$

with $f_\sigma(\tau) = \delta_{\sigma,\tau}$. Then, for all $f \in R(\mathcal{G})$ we have

$$g = \sum_{\sigma \in \mathcal{G}_1} g(\sigma) f_\sigma.$$

The algebra structure is given by the formulas $f_\sigma f_\tau = \delta_{\sigma,\tau} f_\sigma$ and $1_{R(\mathcal{G})} = \sum_{\sigma \in \mathcal{G}_1} f_\sigma$. The coalgebra structure is

$$\delta_{R(\mathcal{G})}(f_\sigma) = \sum_{\tau\rho=\sigma} f_\tau \otimes f_\rho = \sum_{\rho \in \mathcal{G}_1} f_{\sigma\rho^{-1}} \otimes f_\rho, \qquad \varepsilon_{R(\mathcal{G})}(f_\sigma) = \delta_{\sigma, id_{t(\sigma)}}.$$

Also $R(\mathcal{G})$ is a weak Hopf algebra where the antipode is given by

$$\lambda_{R(\mathcal{G})}(f_\sigma) = f_{\sigma^{-1}}.$$

As a consequence we have an adjoint pair $R\mathcal{G} \otimes - \dashv R(\mathcal{G}) \otimes -$ where the morphisms $\alpha_{R(\mathcal{G})}(R)$ and $\beta_{R(\mathcal{G})}(R)$ are defined by

$$\alpha_{R(\mathcal{G})}(R)(1_R) = \sum_{\sigma \in \mathcal{G}_1} f_\sigma \otimes \sigma, \qquad \beta_{R(\mathcal{G})}(R)(\tau \otimes f) = f(\tau)$$

respectively. Then, if A is an R-algebra and ρ_A the coaction defined in (19), A^{op} is a left lax $R(\mathcal{G})^{op}$-module algebra with action $\varphi_{A^{op}} : R(\mathcal{G}) \otimes A \to A$ defined by

$$\varphi_{A^{op}}(f \otimes a) = n_x^{-1}\left(\sum_{\sigma \in E(x)} f(\sigma)\right) a. \tag{36}$$

2 Entwined modules for lax entwining structures and Galois objects

Definition 2.1. *Let (A, C, ψ) be a lax entwining structure. We denote by $\mathcal{M}_A^C(\psi)$ the category whose objects are triples (M, ϕ_M, ρ_M), where (M, ϕ_M) is a right A-module, $\rho_M : M \to M \otimes C$ is a morphism satisfying*

$$(M \otimes \varepsilon_C) \circ \rho_M = id_M, \tag{37}$$

$$(\rho_M \otimes H) \circ \rho_M = (\nabla_{M\otimes C} \otimes C) \circ (M \otimes \delta_C) \circ \rho_M, \tag{38}$$

where $\nabla_{M\otimes C} : M \otimes C \to M \otimes C$ is the idempotent defined by

$$\nabla_{M\otimes C} = (\phi_M \otimes C) \circ (M \otimes (\psi \circ (C \otimes \eta_A))),$$

and the usual entwined module condition

$$\rho_M \circ \phi_M = (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\rho_M \otimes A). \tag{39}$$

The objects in $\mathcal{M}_A^C(\psi)$ *will be called right-right lax entwined modules (or just entwined modules if no confusion is likely) and the morphisms in* $\mathcal{M}_A^C(\psi)$ *are morphisms* $f : M \to N$ *of* A*-modules (i.e.* $f \circ \phi_M = \phi_N \circ (f \otimes A)$*) such that* $(f \otimes C) \circ \rho_M = \rho_N \circ f$.

Note that the identities (37) *and* (38) *imply that*

$$\nabla_{M\otimes C} \circ \rho_M = \rho_M \tag{40}$$

and, if M *is a lax entwined module, by* (37)*, the following identity*

$$\phi_M \circ (M \otimes e) \circ \rho_M = id_M \tag{41}$$

holds.

Also, it follows easily that if A *is a lax right* H*-comodule algebra for a weak bialgebra in a symmetric monoidal category* C*, then* A *is a lax entwining module for the entwining structure defined in Theorem* 1.7*. Moreover, in the particular case presented in Example* 1.8 *the additional condition* $(A \otimes \varepsilon_H) \circ \rho_A = id_A$ *holds.*

Finally, for a partial entwining structure, the category of partial entwined modules is defined as in the lax setting.

In the following definition we introduce the category of weak comodules for a coalgebra C to explain the meaning of condition (38).

Definition 2.2. *Let* C *be a coalgebra. With* $\mathcal{W}(\mathcal{M}^C)$ *we denote the category whose objects are triples* $(M, \nabla_{M\otimes C}, \rho_M)$ *such that:*

(i) M *is an object in* C*,*

(ii) $\nabla_{M\otimes C} : M \otimes C \to M \otimes C$ *is an idempotent morphism in* C*,*

(iii) $\rho_M : M \to M \otimes C$ *is a coaction satisfying* (37)*,* (38)*.*

The morphisms in $\mathcal{W}(\mathcal{M}^C)$ *are defined in the following way: we say that*

$$f : (M, \nabla_{M\otimes C}, \rho_M) \to (N, \nabla_{N\otimes C}, \rho_N)$$

is a morphism in $\mathcal{W}(\mathcal{M}^C)$ *if* $f : M \to N$ *is a morphism in* C *and the identities*

$$\rho_N \circ f = \nabla_{N\otimes C} \circ (f \otimes C) \circ \rho_M, \tag{42}$$

$$\nabla_{N\otimes C} \circ (f \otimes C) \circ \nabla_{M\otimes C} = \nabla_{N\otimes C} \circ (f \otimes C) \tag{43}$$

hold.

It is easy to show that $\mathcal{W}(\mathcal{M}^C)$ is a category. We call it the category of right weak C-comodules. In a similar form it is possible to define the category of left weak C-comodules and also by duality the corresponding weak categories of modules. Note that if (M, ρ_M) is a right C-comodule we have that $(M, \nabla_{M\otimes C} = id_{M\otimes C}, \rho_M)$ is a right weak C-comodule. Also, every right C-comodule morphism $f : (M, \rho_M) \to (N, \rho_N)$ is a morphism in $\mathcal{W}(\mathcal{M}^C)$ between the objects $(M, \nabla_{M\otimes C} = id_{M\otimes C}, \rho_M)$ and $(N, \nabla_{N\otimes C} = id_{N\otimes C}, \rho_N)$. Finally, notice that if (A, C, ψ) is a lax entwining structure and (M, ϕ_M, ρ_M) is a lax entwined module, then the triple

$$(M, \nabla_{M\otimes C} = (\phi_M \otimes C) \circ (M \otimes (\psi \circ (C \otimes \eta_A))), \rho_M)$$

is an object in $\mathcal{W}(\mathcal{M}^C)$ and if $f : (M, \phi_M, \rho_M) \to (N, \phi_N, \rho_N)$ is a morphism in $\mathcal{M}_A^C(\psi)$ we obtain that

$$f : (M, \nabla_{M\otimes C}, \rho_M) \to (N, \nabla_{N\otimes C}, \rho_N)$$

is a morphism of weak right C-comodules because

$$\nabla_{N\otimes C} \circ (f \otimes C) = (f \otimes C) \circ \nabla_{M\otimes C}.$$

Proposition 2.3. *Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. If for all $(M, \phi_M, \rho_M) \in \mathcal{M}_A^C(\psi)$, we denote by M_C the equalizer of ρ_M and $\zeta_M = (\phi_M \otimes C) \circ (M \otimes (\rho_A \circ \eta_A))$ and by i_M the injection of M_C in M, we have the following:*

(i) *The triple $(A_C, \eta_{A_C}, \mu_{A_C})$ is an algebra in $\mathcal{C}$, where $\eta_{A_C} : K \to A_C$ and $\mu_{A_C} : A_C \otimes A_C \to A_C$ are the factorizations of η_A and $\mu_A \circ (i_A \otimes i_A)$ respectively, through the equalizer i_A.*

(ii) *The pair (M_C, ϕ_{M_C}) is a right A_C-module, where $\phi_{M_C} : M_C \otimes A_C \to M_C$ is the factorization of $\phi_M \circ (i_M \otimes i_A)$ through the equalizer i_M.*

Proof. The proof is an easy consequence of the identity (39) and we leave the details to the reader (see [1] for weak entwining structures). □

Example 2.4. Let (A, RG, ψ) the lax entwining structure introduced in example 1.8. In this setting for all $a \in A$ we have $\zeta_A(a) = \rho_A(a)$ and then $A_{RG} = A$.

2.5. If the conditions of Proposition 2.3 hold, it is obvious that $(A, \varphi_A = \mu_A \circ (i_A \otimes A))$ is a left A_C-module and $(A, \phi_A = \mu_A \circ (A \otimes i_A))$ is a right A_C-module.

If with $q_{A,A}$ we denote the coequalizer morphism of $A \otimes \varphi_A$ and $\phi_A \otimes A$ we have the coequalizer diagram

$$A \otimes A_C \otimes A \underset{\phi_A \otimes A}{\overset{A \otimes \varphi_A}{\rightrightarrows}} A \otimes A \xrightarrow{q_{A,A}} A \otimes_{A_C} A$$

2.6. Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. As a consequence of the idempotent character of $\nabla_{A\otimes C}$, there exist an object $A\Box C$ and morphisms $i_{A\otimes C} : A\Box C \to A \otimes C$ and $p_{A\otimes C} : A \otimes C \to A\Box C$ satisfying $\nabla_{A\otimes C} = i_{A\otimes C} \circ p_{A\otimes C}$ and $p_{A\otimes C} \circ i_{A\otimes C} = id_{A\Box C}$. The object $A\Box C$ is a lax entwined module with action $\phi_{A\Box C} = p_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \psi) \circ (i_{A\otimes C} \otimes A)$ and morphism $\rho_{A\Box C} = (p_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ i_{A\otimes C}$. Indeed, trivially

$$\phi_{A\Box C} \circ (A\Box C \otimes \eta_A) = p_{A\otimes C} \circ \nabla_{A\otimes C} \circ i_{A\otimes C} = id_{A\Box C}.$$

Moreover, by the identities (1) and

$$\nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \psi) = (\mu_A \otimes C) \circ (A \otimes \psi) \tag{44}$$

we obtain easily $\phi_{A\Box C} \circ (\phi_{A\Box C} \otimes A) = \phi_{A\Box C} \circ (A\Box C \otimes \mu_A)$ and then $A\Box C$ is a right A-module with action $\phi_{A\Box C}$. On the other hand, by

$$(\mu_A \otimes C) \circ (A \otimes \psi) \circ (\nabla_{A\otimes C} \otimes A) = (\mu_A \otimes C) \circ (A \otimes \psi) \tag{45}$$

and (3) we have

$$\begin{aligned}
& (\nabla_{A\Box C\otimes C} \otimes C) \circ (A\Box C \otimes \delta_C) \circ \rho_{A\Box C} \\
= \; & (p_{A\otimes C} \otimes C \otimes C) \circ (((\mu_A \otimes C) \circ (A \otimes \psi)) \otimes C \otimes C) \circ \\
& (\nabla_{A\otimes C} \otimes (((\psi \circ (C \otimes \eta_A)) \otimes C) \circ \delta_C)) \otimes (A \otimes \delta_C) \circ i_{A\otimes C} \\
= \; & p_{A\otimes C} \circ (\mu_A \otimes C \otimes C \otimes C) \circ (A \otimes ((\psi \otimes C \otimes C) \circ (C \otimes \psi \otimes C) \circ \\
& (\delta_C \otimes \eta_A \otimes C) \circ \delta_C)) \circ i_{A\otimes C} \\
= \; & p_{A\otimes C} \circ (\mu_A \otimes C \otimes C \otimes C) \circ (A \otimes ((\nabla_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ \\
& (\psi \circ (C \otimes \eta_A)) \otimes C)) \circ (A \otimes \delta_C) \circ i_{A\otimes C} \\
= \; & (\rho_{A\Box C} \otimes C) \circ \rho_{A\Box C}
\end{aligned}$$

and then (38) holds. Finally, using (3) and (45), we obtain (39) for $\rho_{A\square C}$ and $\phi_{A\square C}$.

Let $t_A : A \otimes A \to A \otimes C$ be the morphism defined by $t_A = (\mu_A \otimes C) \circ (A \otimes \rho_A)$. Then, by the same proof that we can find in 1.5 of [1] for weak entwining structures, we obtain that $\nabla_{A\otimes C} \circ t_A = t_A$, and therefore, there exists a unique morphism $r_A : A \otimes A \to A\square C$ such that $i_{A\otimes C} \circ r_A = t_A$. On the other hand, the morphism r_A satisfies $r_A \circ (A \otimes \varphi_A) = r_A \circ (\phi_A \otimes A)$ and, as a consequence, there exists a unique morphism (called the canonical morphism)

$$\gamma_A : A \otimes_{A_C} A \to A\square C \tag{46}$$

such that $\gamma_A \circ q_{A,A} = r_A$.

Suppose that $- \otimes A$ preserves coequalizers. Then $A \otimes_{A_C} A$ is a right A-module being the action $\phi_{A\otimes_{A_C} A} : (A \otimes_{A_C} A) \otimes A \to A \otimes_{A_C} A$ the factorization of $q_{A,A} \circ (A \otimes \mu_A)$ through the coequalizer $q_{A,A} \otimes A$, i.e., $\phi_{A\otimes_{A_C} A}$ is the unique morphism such that $\phi_{A\otimes_{A_C} A} \circ (q_{A,A} \otimes A) = q_{A,A} \circ (A \otimes \mu_A)$. Also, there exists a morphism $\rho_{A\otimes_{A_C} A} : A \otimes_{A_C} A \to (A \otimes_{A_C} A) \otimes C$ defined by the factorization of $(q_{A,A} \otimes C) \circ (A \otimes \rho_A)$ through the coequalizer $q_{A,A}$, or equivalently, $\rho_{A\otimes_{A_C} A}$ is the unique morphism such that $\rho_{A\otimes_{A_C} A} \circ q_{A,A} = (q_{A,A} \otimes C) \circ (A \otimes \rho_A)$. The triple $(A \otimes_{A_C} A, \phi_{A\otimes_{A_C} A}, \rho_{A\otimes_{A_C} A})$ is a lax entwining module because composing with the coequalizer $q_{A,A}$, using the entwined module condition of A and the properties of $\phi_{A\otimes_{A_C} A}$, we have

$$\begin{aligned}
& (\rho_{A\otimes_{A_C} A} \otimes C) \circ \rho_{A\otimes_{A_C} A} \circ q_{A,A} \\
= \;& (q_{A,A} \otimes C \otimes C) \circ (A \otimes ((\rho_A \otimes C) \circ \rho_A)) \\
= \;& (q_{A,A} \otimes C \otimes C) \circ (A \otimes ((\nabla_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ \rho_A)) \\
= \;& ((q_{A,A} \circ (A \otimes \mu_A)) \otimes C \otimes C) \circ \\
& (A \otimes (A \otimes (((\psi \circ (C \otimes \eta_A)) \otimes C) \circ \delta_C))) \circ (A \otimes \rho_A) \\
= \;& (\phi_{A\otimes_{A_C} A} \otimes C \otimes C) \circ (q_{A,A} \otimes (((\psi \circ (C \otimes \eta_A)) \otimes C) \circ \delta_C)) \circ (A \otimes \rho_A) \\
= \;& (\nabla_{(A\otimes_{A_C} A)\otimes C} \otimes C) \circ (q_{A,A} \otimes \delta_C) \circ (A \otimes \rho_A) \\
= \;& (\nabla_{(A\otimes_{A_C} A)\otimes C} \otimes C) \circ (A \otimes_{A_C} A \otimes \delta_C) \circ \rho_{A\otimes_{A_C} A} \circ q_{A,A},
\end{aligned}$$

i.e., (38) holds. Moreover, composing with the coequalizer $q_{A,A} \otimes A$ and by similar arguments to the previous ones we obtain

$$\begin{aligned}
& \rho_{A\otimes_{A_C} A} \circ \phi_{A\otimes_{A_C} A} \circ (q_{A,A} \otimes A) \\
= \;& (q_{A,A} \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \\
= \;& (q_{A,A} \otimes C) \circ (A \otimes ((\mu_A \otimes C) \circ (A \otimes \psi) \circ (\rho_A \otimes A))) \\
= \;& (\phi_{A\otimes_{A_C} A} \otimes C) \circ ((A \otimes_{A_C} A) \otimes \psi) \circ (\rho_{A\otimes_{A_C} A} \otimes A) \circ (q_{A,A} \otimes A),
\end{aligned}$$

i.e., (39) holds.

As a consequence, using the entwined module structures defined in the previous paragraphs, it is easy to show (see for example the similar proof in the weak setting contained in 1.5 of [2]) that γ_A is morphism of right A-modules. Moreover, γ_A satisfies the identity $(\gamma_A \otimes C) \circ \rho_{A\otimes_{A_C} A} = \rho_{A\square C} \circ \gamma_A$. Indeed, composing with the coequalizer $q_{A,A}$, applying (38) and

$$\nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A) = (\mu_A \otimes C) \circ (A \otimes \rho_A) \tag{47}$$

we have

$$\begin{aligned} & (\gamma_A \otimes C) \circ \rho_{A\otimes_{A_C} A} \circ q_{A,A} \\ = \; & ((p_{A\otimes C} \circ (\mu_A \otimes C)) \otimes A) \circ (A \otimes ((\rho_A \otimes C) \circ \rho_A)) \\ = \; & ((p_{A\otimes C} \circ (\mu_A \otimes C)) \otimes A) \circ (A \otimes ((\nabla_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ \rho_A)) \\ = \; & (p_{A\otimes C} \otimes C) \circ (\mu_A \otimes \delta_C) \circ (A \otimes \rho_A) \\ = \; & (p_{A\otimes C} \otimes C) \circ (A \otimes \delta_C) \circ \nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A) \\ = \; & \rho_{A\square C} \circ \gamma_A \circ q_{A,A}. \end{aligned}$$

Therefore, γ_A is a morphism of lax entwined modules.

Finally, if $A \otimes -$ preserves coequalizers, then γ_A is a morphism of left A-modules being $\varphi_{A\otimes_{A_C} A} : A \otimes (A \otimes_{A_C} A) \to A \otimes_{A_C} A$ the factorization of $q_{A,A} \circ (\mu_A \otimes A)$ through the coequalizer $A \otimes q_{A,A}$, i.e. $\varphi_{A\otimes_{A_C} A}$ is the unique morphism such that $\varphi_{A\otimes_{A_C} A} \circ (A \otimes q_{A,A}) = q_{A,A} \circ (\mu_A \otimes A)$, and $\varphi_{A\square C} : A \otimes A\square C \to A\square C$ is defined by $\varphi_{A\square C} = p_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes i_{A\otimes C})$ (see 1.5 of [2] for the proof).

Definition 2.5. *Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ such that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. We say that $A_C \hookrightarrow A$ is a lax C-Galois extension if the canonical morphism γ_A defined in* (46) *is an isomorphism.*

Notice that in (2.6) *we obtain that γ_A satisfies the identity $(\gamma_A \otimes C) \circ \rho_{A\otimes_{A_C} A} = \rho_{A\square C} \circ \gamma_A$ and, if the functor $- \otimes A$ preserves coequalizers, γ_A is a morphism of right A-modules and, as a consequence, is a morphism of lax entwined modules. Moreover, if $A \otimes -$ preserves coequalizers γ is a morphism of left A-modules. For example, if C is symmetric closed we have these properties.*

Using the fact that every partial entwining structure is a lax entwining structure we define the notion of partial C-Galois extension in a similar way.

Theorem 2.6. *Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$.*

Suppose that $A_C \hookrightarrow A$ is a lax C-Galois extension such that the functor $-\otimes A$ preserves coequalizers. Then,

$$\psi = i_{A\otimes C} \circ \gamma_A \circ \phi_{A\otimes_{A_C} A} \circ ((\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C)) \otimes A). \qquad (48)$$

Proof. Composing with the coequalizer $q_{A,A} \otimes A$ we obtain

$$\begin{aligned} & i_{A\otimes C} \circ \gamma_A \circ \phi_{A\otimes_{A_C} A} \circ (q_{A,A} \otimes A) \\ = \; & \nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \\ = \; & (\mu_A \otimes C) \circ (\mu_A \otimes \psi) \circ (A \otimes \rho_A \otimes A) \\ = \; & (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((\nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A)) \otimes A) \\ = \; & (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((i_{A\otimes C} \circ \gamma_A) \otimes A) \circ (q_{A,A} \otimes A), \end{aligned}$$

where the first equality follows the properties of the action and the canonical morphism, the second one by (44) and the entwined module condition (39) for A, the third one by (44) and the fourth one by the properties of the canonical morphism.

Therefore,

$$i_{A\otimes C} \circ \gamma_A \circ \phi_{A\otimes_{A_C} A} = (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((i_{A\otimes C} \circ \gamma_A) \otimes A) \qquad (49)$$

and, as a consequence of this identity, we have

$$\begin{aligned} & i_{A\otimes C} \circ \gamma_A \circ \phi_{A\otimes_{A_C} A} \circ ((\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C)) \otimes A) \\ = \; & (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((\nabla_{A\otimes C} \circ (\eta_A \otimes C)) \otimes A) \\ = \; & (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((\psi \circ (C \otimes \eta_A)) \otimes A) = \psi. \end{aligned}$$

□

2.9. Let (M, ϕ_M, ρ_M) be a lax entwined module for a lax entwining structure (A, C, ψ). The morphism $\Delta_{M\otimes C} : M \otimes C \to M \otimes C$ defined by

$$\Delta_{M\otimes C} = (\phi_M \otimes C) \circ (M \otimes ((e \otimes C) \circ \delta_C))$$

is idempotent because by (7) we have

$$\Delta_{M\otimes C} \circ \Delta_{M\otimes C} = (\phi_M \otimes C) \circ (M \otimes (((e \wedge e) \otimes C) \circ \delta_C)) = \Delta_{M\otimes C}.$$

Then, there exist an object $M \times C$ and morphisms $j_{M\otimes C} : M \times C \to M \otimes C$ and $q_{M\otimes C} : M \otimes C \to M \times C$ satisfying $\Delta_{M\otimes C} = j_{M\otimes C} \circ q_{M\otimes C}$ and $q_{M\otimes C} \circ j_{M\otimes C} = id_{M\times C}$.

If there exists a morphism $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$, for the morphism t_A defined in 2.6 we obtain

$$\Delta_{A\otimes C} \circ t_A = t_A. \tag{50}$$

Indeed, by (40) and (iii) of Lemma 1.2 we have

$$\begin{aligned} & \Delta_{A\otimes C} \circ t_A \\ = \;& (\mu_A \otimes C) \circ (A \otimes (\Delta_{A\otimes C} \circ \rho_A)) \\ = \;& (\mu_A \otimes C) \circ (A \otimes (\Delta_{A\otimes C} \circ \nabla_{A\otimes C} \circ \rho_A)) \\ = \;& (\mu_A \otimes C) \circ (A \otimes (\nabla_{A\otimes C} \circ \rho_A)) \\ = \;& t_A. \end{aligned}$$

Therefore, there exists a unique morphism $r'_A = q_{A\otimes C} \circ t_A : A \otimes A \to A \times C$ such that $j_{A\otimes C} \circ r'_A = t_A$. On the other hand, the morphism r'_A satisfies $r'_A \circ (A \otimes \varphi_A) = r'_A \circ (\phi_A \otimes A)$ and, as a consequence, there exists a unique morphism (called the second canonical morphism)

$$\beta_A : A \otimes_{A_C} A \to A \times C \tag{51}$$

such that $\beta_A \circ q_{A,A} = r'_A$.

Let

$$\Omega_1 = q_{A\otimes C} \circ i_{A\otimes C} : A\Box C \to A \times C \quad \text{and}$$

$$\Omega_2 = p_{A\otimes C} \circ j_{A\otimes C} : A \times C \to A\Box C$$

be the morphisms defined using the injections and the projections associated to the idempotents $\nabla_{A\otimes C}$ and $\Delta_{A\otimes C}$. Then,

$$\Omega_1 \circ \gamma_A \circ q_{A,A} = q_{A\otimes C} \circ \nabla_{A\otimes C} \circ t_A = q_{A\otimes C} \circ t_A = r'_A = \beta_A \circ q_{A,A},$$

$$\Omega_2 \circ \beta_A \circ q_{A,A} = p_{A\otimes C} \circ \Delta_{A\otimes C} \circ t_A = p_{A\otimes C} \circ t_A = r_A = \gamma_A \circ q_{A,A}$$

and, as a consequence we obtain the following relations between the canonical morphism and the second canonical morphism

$$\Omega_1 \circ \gamma_A = \beta_A, \qquad \Omega_2 \circ \beta_A = \gamma_A. \tag{52}$$

As in Theorem 2.6 if the functor $- \otimes A$ preserves coequalizers and β_A is an isomorphism it it possible to obtain an expression of ψ involving β_A and its inverse. To prove this assertion, first we obtain the identity

$$j_{A\otimes C} \circ \beta_A \circ \phi_{A\otimes_{A_C} A} = (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((j_{A\otimes C} \circ \beta_A) \otimes A). \tag{53}$$

Indeed, composing with the coequalizer $q_{A,A} \otimes A$ we have

$$\begin{aligned} & j_{A\otimes C} \circ \beta_A \circ \phi_{A\otimes_{A_C} A} \circ (q_{A,A} \otimes A) \\ = \;& \Delta_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \\ = \;& (\mu_A \otimes C) \circ (\mu_A \otimes (\Delta_{A\otimes C} \circ \psi)) \circ (A \otimes \rho_A \otimes A) \\ = \;& (\mu_A \otimes C) \circ (\mu_A \otimes \psi) \circ (A \otimes \rho_A \otimes A) \\ = \;& (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((\Delta_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A)) \otimes A) \\ = \;& (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((j_{A\otimes C} \circ \beta_A) \otimes A) \circ (q_{A,A} \otimes A), \end{aligned}$$

where the first equality follows by the properties $\phi_{A\otimes_{A_C} A}$ and β_A, the second one by the entwined module condition for A, the associativity of the product μ_A as well as the left A-linearity of $\Delta_{A\otimes C}$ being $\varphi_{A\otimes C} = \mu_A \otimes C$, i.e.

$$(\mu_A \otimes C) \circ (A \otimes \nabla_{A\otimes C}) = \nabla_{A\otimes C} \circ (\mu_A \otimes C). \tag{54}$$

The third one follows by (iv) of Lemma 1.2, the fourth one by (50) and finally, the fifth one by the properties of β_A. Therefore (53) holds and then applying this identity, the equalities (2), (3) and (i) and (iv) of Lemma 1.2, we obtain

$$\begin{aligned} & j_{A\otimes C} \circ \beta_A \circ \phi_{A\otimes_{A_C} A} \circ ((\beta_A^{-1} \circ q_{A\otimes C} \circ (\eta_A \otimes C)) \otimes A) \\ = \;& (\mu_A \otimes C) \circ (A \otimes \psi) \circ ((\Delta_{A\otimes C} \circ (\eta_A \otimes C)) \otimes A) \\ = \;& (\mu_A \otimes C) \circ (e \otimes \psi) \circ (\delta_C \otimes A) \\ = \;& (((A \otimes \varepsilon_C) \circ \psi) \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A) \\ = \;& (((A \otimes \varepsilon_C) \circ \nabla_{A\otimes C}) \otimes C) \circ (A \otimes \delta_C) \circ \psi \\ = \;& \Delta_{A\otimes C} \circ \psi \\ = \;& \psi. \end{aligned}$$

The following Theorem clarify the implications of assuming the isomorphism condition for β_A.

Theorem 2.7. *Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Suppose that the functor $- \otimes A$ preserves coequalizers. Then if β_A is an isomorphism, the canonical morphism γ_A is an isomorphism and (A, C, ψ) is a weak entwining structure.*

Proof. If β_A is an isomorphism, γ_A is an isomorphism with inverse $\gamma_A^{-1} = \beta_A^{-1} \circ \Omega_1$ where Ω_1 is the morphism defined in 2.9. Indeed,

$$\gamma_A \circ \beta_A^{-1} \circ \Omega_1 = \Omega_2 \circ \beta_A \circ \beta_A^{-1} \circ \Omega_1 = \Omega_2 \circ \Omega_1 = p_{A\otimes C} \circ \Delta_{A\otimes C} \circ i_{A\otimes C} = id_{A\square C}$$

where the last equality follows by (iii) of Lemmma 1.2. On the other hand, composing with the coequalizer $q_{A,A}$, we have

$$\begin{aligned}\beta_A^{-1} \circ \Omega_1 \circ \gamma_A \circ q_{A,A} &= \beta_A^{-1} \circ q_{A\otimes C} \circ \nabla_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A)\\ &= \beta_A^{-1} \circ q_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \rho_A)\\ &= \beta_A^{-1} \circ r'_A = \beta_A^{-1} \circ \beta_A \circ q_{A,A} = q_{A,A}\end{aligned}$$

and therefore $\beta_A^{-1} \circ \Omega_1 \circ \gamma_A = id_{A\otimes_{A_C} A}$.

Finally, if γ_A and β_A are isomorphisms by (48) and the similar equality obtained in 2.9 we have

$$\begin{aligned}\psi &= i_{A\otimes C} \circ \gamma_A \circ \phi_{A\otimes_{A_C} A} \circ ((\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C)) \otimes A)\\ &= j_{A\otimes C} \circ \beta_A \circ \phi_{A\otimes_{A_C} A} \circ ((\beta_A^{-1} \circ q_{A\otimes C} \circ (\eta_A \otimes C)) \otimes A).\end{aligned}$$

Then, composing with $C \otimes \eta_A$ we prove that

$$\psi \circ (C \otimes \eta_A) = \nabla_{A\otimes C} \circ (\eta_A \otimes C) = \Delta_{A\otimes C} \circ (\eta_A \otimes C),$$

and therefore,

$$\psi \circ (C \otimes \eta_A) = (e \otimes C) \circ \delta_C,$$

i.e., (A, C, ψ) is a weak entwining structure (see Theorem 1.4). □

Example 2.8. Let $(A, R\mathcal{G}, \psi)$ the lax entwining structure introduced in example 1.8. Then, it is easy to show that

$$\beta_A(a \otimes_{A_{R\mathcal{G}}} b) = n_x \gamma_A(a \otimes_{A_{R\mathcal{G}}} b)$$

and then, β_A is an isomorphism if and only if γ_A is an isomorphism. As a consequence, by Theorem 2.7, γ_A and β_A are not isomorphisms.

Lemma 2.9. *Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Let $\sigma : A \to A_C \otimes C$ be a morphism of left A_C-modules for the actions $\varphi_{A_C\otimes C} = \mu_{A_C} \otimes C$ and $\varphi_A = \mu_A \circ (i_A \otimes A)$. Then, there exists an unique morphism $m_\sigma : A \otimes_{A_C} A \to A$ such that*

$$m_\sigma \circ q_{A,A} = \mu_A \circ (A \otimes ((i_C^A \otimes \varepsilon_C) \circ \sigma)).$$

Moreover, if $A \otimes -$ preserves coequalizers, m_σ is a left A-module morphism.

Proof. The proof is similar to the one developed in Lemma 1.9 of [2]. □

Definition 2.10. *Let $A_C \hookrightarrow A$ be a lax C-Galois extension. We will say that $A_C \hookrightarrow A$ satisfies the normal basis property (or $A_C \hookrightarrow A$ is a lax C-Galois extension with normal basis) if there exists an idempotent morphism of left A_C-modules $\Omega_A : A_C \otimes C \to A_C \otimes C$, for the action $\varphi_{A_C\otimes C} = \mu_{A_C} \otimes C$, and an isomorphism $b_A : A_C \boxtimes C \to A$, where $A_C \boxtimes C$ is the image of Ω_A, satisfying the following conditions:*

(i) *b_A is an isomorphism of of left A_C-modules, where $\varphi_{A_C\boxtimes C} = r_{A_C\otimes C} \circ (\mu_{A_C} \otimes C) \circ (A_C \otimes s_{A_C\otimes C})$, and $s_{A_C\otimes C} : A_C \boxtimes C \to A_C \otimes C$, $r_{A_C\otimes C} : A_C \otimes C \to A_C \boxtimes C$ are the morphisms such that $s_{A_C\otimes C} \circ r_{A_C\otimes C} = \Omega_A$, $r_{A_C\otimes C} \circ s_{A_C\otimes C} = id_{A_C\boxtimes C}$.*

(ii) *If $\omega_A = b_A \circ r_{A_C\otimes C} : A_C \otimes C \to A$ and $\omega'_A = s_{A_C\otimes C} \circ b_A^{-1} : A \to A_C \otimes C$, the following equalities hold*

$$\Omega_A = (((A_C \otimes \varepsilon_C) \circ \omega'_A) \otimes C) \circ \rho_A \circ \omega_A, \tag{55}$$

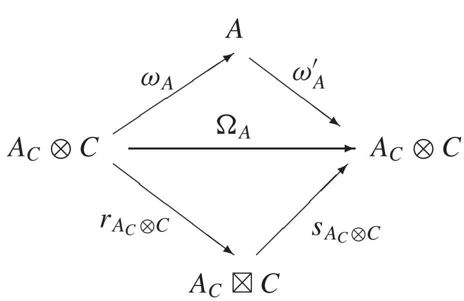
$$\rho_A \circ \omega_A = \nabla_{A\otimes C} \circ (\omega_A \otimes C) \circ (A_C \otimes \delta_C). \tag{56}$$

(iii) *If $f = \omega_A \circ (\eta_{A_C} \otimes C)$ and $f' = m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C)$ the equality*

$$f' \wedge f = \mu_A \circ (A \otimes (f' \wedge f)) \circ \psi \circ (C \otimes \eta_A) \tag{57}$$

holds where $m_{\omega'_A}$ is the morphism introduced in Lemma 2.9.

Observe that ω_A and ω'_A are morphisms of left A_C-modules and then we can apply Lemma 2.9 *for $\sigma = \omega'_A$. Moreover, $\varphi_{A_C\boxtimes C}$ is a well defined structure of left A_C-module because Ω_A is a morphism of left A_C-modules. Also, we have a commutative diagram*



Also, notice that equality (56) *says that ω_A is a morphism in $\mathcal{W}(\mathcal{M}^C)$ between the objects $(A_C \otimes C, \nabla_{A_C\otimes C\otimes C} = id_{A_C\otimes C\otimes C}, \rho_{A_C\otimes C} = A_C \otimes \delta_C)$ and $(A, \nabla_{A\otimes C}, \rho_A)$*

This definition is a generalization of the one introduced in [2] *for weak entwining structures. In this setting* (A, ρ_A) *is a right* C*-comodule and* Ω_A *and* b_A *are morphisms of right* C*-comodules. As a consequence,* ω_A *and* ω'_A *are also morphisms of right* C*-comodules and the equalities* (55) *and* (56) *hold trivially. Moreover, in the weak setting* (57) *holds (see the proof of Theorem* 2.11 *of* [2] *).*

Note that if C *is a category of modules over a commutative ring, this notion of normal basis in the lax setting says that* A *is a direct summand of* $A_C \otimes C$ *in the category of left* A_C*-modules and as well as the image of an idempotent morphism that satisfies some identities closely related with the right* C*-comodule condition.*

Finally, if $A_C \hookrightarrow A$ *is a partial* C*-Galois extension, the definition of partial* C*-Galois extension with normal basis is the one obtaining when we replace in Definition* 2.10 *lax by partial.*

3 The characterization of lax C-Galois extensions with normal basis

In this section, for a lax entwinig structure, we introduce the notion of lax C-cleft extension A and we obtain a characterization of C-Galois extensions with normal basis in as lax C-cleft extensions.

Definition 3.1. *Let* (A, C, ψ) *be a lax entwining structure. By* $Reg^{WR}(C, A)$ *we denote the set of morphisms* $h \in Hom_C(C, A)$ *such that there exists a morphism* $h^{-1} \in Hom_C(C, A)$*, called the left weak inverse of* h*, satisfying* $h^{-1} \wedge h = e$*.*

Remark 3.2. Suppose that (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Then if $h \in Hom_C(C, A)$ is a morphism such that

$$\rho_A \circ h = \nabla_{A\otimes C} \circ (h \otimes C) \circ \delta_C, \tag{58}$$

i.e. h is a morphism in $\mathcal{W}(\mathcal{M}^C)$ between the objects $(C, \nabla_{C\otimes C} = id_{C\otimes C}, \rho_C = \delta_C)$ and $(A, \nabla_{A\otimes C}, \rho_A)$, by the entwined module condition for A, it is easy to show that

$$h \wedge e = h. \tag{59}$$

Moreover, if $h \in Reg^{WR}(C, A)$, it is possible to obtain an explicit expression for ψ as

$$\psi = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A). \tag{60}$$

Indeed,

$$\begin{aligned}
& \psi \\
= \; & \Delta_{A\otimes C} \circ \psi \\
= \; & (((A \otimes \varepsilon_C) \circ \nabla_{A\otimes C}) \otimes C) \circ (A \otimes \delta_C) \circ \psi \\
= \; & ((A \otimes \varepsilon_C) \otimes C) \circ (\psi \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A) \\
= \; & ((\mu_A \circ (e \otimes A)) \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A) \\
= \; & ((\mu_A \circ ((h^{-1} \wedge h) \otimes A)) \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A) \\
= \; & (\mu_A \otimes C) \circ (A \otimes (((\mu_A \circ (h \otimes A)) \otimes C) \circ (C \otimes \psi))) \circ \\
& (h^{-1} \otimes \delta_C \otimes A) \circ (\delta_C \otimes A) \\
= \; & (\mu_A \otimes C) \circ (A \otimes ((\mu_A \otimes C) \circ (A \otimes \psi))) \circ \\
& (h^{-1} \otimes (\nabla_{A\otimes C} \circ (h \otimes C) \circ \delta_C) \otimes A) \circ (\delta_C \otimes A) \\
= \; & (\mu_A \otimes C) \circ (A \otimes ((\mu_A \otimes C) \circ (A \otimes \psi))) \circ \\
& (h^{-1} \otimes (\rho_A \circ h) \otimes A) \circ (\delta_C \otimes A) \\
= \; & (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_C) \otimes A),
\end{aligned}$$

where the first equality follows by (iv) of Lemma 1.2, the second one by (i) of Lemma 1.2, the third one by (3) and the fourth one by (2). To obtain the fifth one we used that $h \in Reg^{WR}(C, A)$, while the sixth one is a consequence of the associativity of μ_A and the coassociativity of δ_C. The seventh one follows by (45), the eighth one by (58) and finally the ninth one by the entwined module condition for A.

Notice that if we replace the equality (58) by

$$\rho_A \circ h = (h \otimes C) \circ \delta_C,$$

that is, if we assume that h is a right C-comodule morphism, we can obtain the same explicit expression for ψ and then, composing with $C \otimes \eta_A$, we have

$$\psi \circ (C \otimes \eta_A) = (\mu_A \otimes C) \circ (A \otimes \rho_A) \circ (h^{-1} \otimes h) \circ \delta_C = (e \otimes C) \circ \delta_C,$$

or, equivalently, (A, C, ψ) is a weak entwining structure.

Example 3.3. Let R be a commutative ring, $G = \{g_i \mid i = 0, \dots, n\}$ a finite group with g_0 the identity element and A a R-algebra. Following [10], a left partial action of G on A consists of a set of idempotents $\{e_{g_i} \mid i = 0, \dots, n\} \subset A$, and a set of isomorphisms $\alpha_{g_i} : e_{g_i^{-1}} A \to e_{g_i} A$ such that $e_{g_0} = 1_A$, $\alpha_{g_0} =$

id_A and

$$e_{g_i}\alpha_{g_i g_j}(e_{g_j^{-1}g_i^{-1}}a) = \alpha_{g_i}(e_{g_i^{-1}}\alpha_{g_j}(e_{g_j^{-1}}a)), \tag{61}$$

$$\alpha_{g_i}(e_{g_i^{-1}}ab) = \alpha_{g_i}(e_{g_i^{-1}}a)\alpha_{g_i}(e_{g_i^{-1}}b), \tag{62}$$

$$\alpha_{g_i}(e_{g_i^{-1}}) = e_{g_i} \tag{63}$$

for all g_i, g_j in G and a, b in A.

By Proposition 4.9 of [10] we know that there exists a bijective correspondence between left partial G-actions and structures of left partial RG-module algebra over A. Then, A with the partial action

$$\varphi_A(g_i \otimes a) = \alpha_{g_i}(e_{g_i^{-1}}a)$$

is a left partial RG-module algebra. Therefore, A^{op} with the coaction defined in (34) is a partial right $R(G)^{cop}$-comodule algebra. In this case

$$\rho_{A^{op}}(a) = \sum_{g_i \in G} \varphi_A(g_i \otimes a) \otimes f_{g_i}$$

where $\{f_{g_i} \mid i = 0, \ldots, n\}$ is the dual basis of $\{g_i \mid i = 0, \ldots, n\}$ and, as a consequence, $\rho_{A^{op}}$ is the coaction determined by the grouplike element

$$\sum_{g_i \in G} f_{g_i}$$

(see Proposition 6.3 of [5]).

Then if we assume as in [5] that a plausible definition of cleft extension in the partial setting involves the existence of a convolution invertible right colinear morphism from $R(G)^{cop}$ to A, by the arguments presented in the previous remark, we obtain that the partial entwining structure associated to $\rho_{A^{op}}$, i.e.

$$(A^{op}, R(G)^{cop}, \psi = (A \otimes \mu_{R(G)^{cop}}) \circ (c_{R(G),A} \otimes R(G)) \circ (R(G) \otimes \rho_{A^{op}})),$$

is weak. Therefore, by Theorem 1.4 we prove that $(A^{op}, R(G)^{cop}, \psi)$ is an entwining structure and, as a consequence, $\varphi_A \circ (RG \otimes \eta_A) = \varepsilon_{RG} \otimes \eta_A$, or equivalently, (A, φ_A) is a usual left RG-module algebra.

Motivated for this problem, in the following definition we introduce a new notion of cleft extension in a lax setting.

Definition 3.4. *Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ satisfying that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$.*

We will say that $A_C \hookrightarrow A$ *is a lax* C*-cleft extension if there exists a morphism* $h : C \to A$ *in* $Reg^{WR}(C, A)$*, called the cleaving morphism, satisfying the equality* (58) *and*

$$\mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C} = \mu_A \circ (A \otimes h^{-1}), \tag{64}$$

$$\psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ h^{-1}, \tag{65}$$

where ζ_A *is the morphism defined in Proposition* 2.3.

This definition is a generalization of the one introduced for the weak entwining setting in [1] *and the one used by Brzeziński in* [7] *for entwining structures. Notice that, while in the case of a cleft extension for a weak entwining structure* h *is required to be a right comodule morphism* (A *is a right* C*-comodule), here this condition is replaced by* (58)*. Also, in this definition appears a new condition* (64) *that, in the weak setting, is a consequence of* $\nabla_{A\otimes C} = \Delta_{A\otimes C}$ *and*

$$e \wedge h^{-1} = h^{-1}. \tag{66}$$

Finally, as in the previous definitions, the notion of partial C*-cleft extension is introduced in a similar way.*

Lemma 3.5. *Let* $A_C \hookrightarrow A$ *be a lax* C*-cleft extension with cleaving morphism* h*. Then, for the left inverse of* h *the equality* (66) *holds.*

Proof. Check

$$e \wedge h^{-1} = (A \otimes \varepsilon_C) \circ \psi \circ (C \otimes h^{-1}) \circ \delta_C = (A \otimes \varepsilon_C) \circ \zeta_A \circ h^{-1} = h^{-1}$$

where the first equality follows by (2), the second on by (65) and the last one by the identity $id_A = (A \otimes \varepsilon_C) \circ \rho_A$. □

Remark 3.6. In the conditions of the previous Lemma, the inverse of the cleaving morphism can be obtained as

$$\begin{aligned} h^{-1} &= e \wedge h^{-1} = \mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C} \circ (e \otimes A) \circ \delta_C \\ &= \mu_A \circ (A \otimes h^{-1}) \circ \psi \circ (C \otimes \eta_A) \end{aligned}$$

where the second equality follows by (64) and the last one by (4). Then, as a consequence, the following identity

$$\phi_M \circ (M \otimes h^{-1}) \circ \nabla_{M\otimes C} = \phi_M \circ (M \otimes h^{-1}) \tag{67}$$

holds for any lax entwined module M.

In the followig proposition we give a characterization of lax C-cleft extensions.

Proposition 3.7. *Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ such that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Let $h \in Reg^{WR}(C, A)$ satisfying the equalities* (58) *and* (64). *Then the following assertions are equivalent:*

(i) *For every lax entwined module the morphism*

$$g_M = \phi_M \circ (M \otimes h^{-1}) \circ \rho_M : M \to M$$

factors through the equalizer of ρ_M and ζ_M, that is, if $i_M : M_C \to M$ is the equalizer morphism defined in Proposition 2.3, *there exists an unique morphism $p_M : M \to M_C$ such that $i_M \circ p_M = g_M$.*

(ii) *The morphism*

$$g_A = \mu_A \circ (A \otimes h^{-1}) \circ \rho_A : A \to A$$

factors through the equalizer of ρ_A and ζ_A, that is, if $i_A : A_C \to A$ is the equalizer morphism defined in Proposition 2.3, *there exists an unique morphism $p_A : A \to A_C$ such that $i_A \circ p_A = g_A$.*

(iii) *$A_C \hookrightarrow A$ is a lax C-cleft extension with cleaving morphism h.*

Proof. (i) $\Rightarrow$ (ii) Just consider $M = A$.

(ii) $\Rightarrow$ (iii) To prove this assertion, first we show that

$$g_A \circ h = h \wedge h^{-1}. \tag{68}$$

Indeed, by (58) and (64) we have

$$h \wedge h^{-1} = \mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C} \circ (h \otimes C) \circ \delta_C = g_A \circ h.$$

Then,

$$\begin{aligned}
& \psi \circ (C \otimes h^{-1}) \circ \delta_C \\
= \;& (\mu_A \otimes C) \circ (h^{-1} \otimes (\rho_A \circ (h \wedge h^{-1}))) \circ \delta_C \\
= \;& (\mu_A \otimes C) \circ (h^{-1} \otimes (\rho_A \circ g_A \circ h))) \circ \delta_C \\
= \;& (\mu_A \otimes C) \circ (h^{-1} \otimes (\zeta_A \circ g_A \circ h))) \circ \delta_C \\
= \;& \zeta_A \circ (h^{-1} \wedge (g_A \circ h)) \\
= \;& \zeta_A \circ (h^{-1} \wedge (h \wedge h^{-1})) \\
= \;& \zeta_A \circ (e \wedge h^{-1}) \\
= \;& \zeta_A \circ h^{-1},
\end{aligned}$$

where the first equality follows by (60) and the coassocitivity of δ_C and the second one by (68). To obtain the third one we used that g_A factors through i_A and the fourth one follows by the associativity of μ_A. The fifth equality is a consequence of (68) and the sixth one follows by the associativity of the convolution product as well as the condition $h \in Reg^{WR}(C, A)$. Finally, the last one follows by (66).

Therefore, $A_C \hookrightarrow A$ is a lax C-cleft extension with cleaving morphism h.

(iii) $\Rightarrow$ (i) Let M be a lax entwined module and assume that $A_C \hookrightarrow A$ is a lax C-cleft extension with cleaving morphism h. Then

$$
\begin{aligned}
& \rho_M \circ g_M \\
= \;& (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\rho_M \otimes h^{-1}) \circ \rho_M \\
= \;& (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\nabla_{M\otimes C} \otimes h^{-1}) \circ (M \otimes \delta_C) \circ \rho_M \\
= \;& (\phi_M \otimes C) \circ (M \otimes (\psi \circ (C \otimes h^{-1}) \circ \delta_C)) \circ \rho_M \\
= \;& (\phi_M \otimes C) \circ (M \otimes (\zeta_A \circ h^{-1})) \circ \rho_M \\
= \;& \zeta_M \circ g_M,
\end{aligned}
$$

where the first equality follows by the entwined module condition for M, the second one by (38), the third one by

$$(\phi_M \otimes C) \circ (M \otimes \psi) \circ (\nabla_{M\otimes C} \otimes A) = (\phi_M \otimes C) \circ (M \otimes \psi), \qquad (69)$$

the fourth one by (65) and the last one by the right A-module condition for M.

Therefore, g_M factors through the equalizer of ρ_M and ζ_M. □

Lemma 3.8. *Let $A_C \hookrightarrow A$ be a lax C-cleft extension with cleaving morphism h. Then for all lax entwined module M the following equality holds.*

$$p_M \circ \phi_M \circ (i_M \otimes A) = \phi_{M_C} \circ (M_C \otimes p_A). \qquad (70)$$

Proof. To prove this equality check

$$
\begin{aligned}
& i_M \circ p_M \circ \phi_M \circ (i_M \otimes A) \\
= \;& \phi_M \circ (\phi_M \otimes h^{-1}) \circ (M \otimes \psi) \circ ((\rho_M \circ i_M) \otimes A) \\
= \;& \phi_M \circ (M \otimes (\mu_A \circ (A \otimes h^{-1}) \circ \psi)) \circ ((\rho_M \circ i_M) \otimes A) \\
= \;& \phi_M \circ (i_M \otimes (\mu_A \circ (\mu_A \otimes h^{-1}) \circ (A \otimes \psi) \circ ((\rho_A \circ \eta_A) \otimes A))) \\
= \;& \phi_M \circ (i_M \otimes g_A)
\end{aligned}
$$

$$
\begin{aligned}
&= \phi_M \circ (i_M \otimes (i_A \circ p_A)) \\
&= i_M \circ \phi_{M_C} \circ (M_C \otimes p_A),
\end{aligned}
$$

where the first equality follows by the entwined module condition for M, the second one by the A-module condition for M, the third one is a consequence of $\rho_M \circ i_M = \zeta_M \circ i_M$, the A-module condition for M as well as the associativity of μ_A. In the fourth one we use the entwined module condition for A and the unit properties. The fifth one follows by the factorization properties of g_A and finally, the last one, by (ii) of Proposition 2.3.

Therefore (70) holds because i_M is an equalizer. □

If $A_C \hookrightarrow A$ is a lax C-cleft extension, in the following Proposition, for all lax entwined module M, we obtain an idempotent morphism satisfying suitable conditions.

Proposition 3.9. *Let $A_C \hookrightarrow A$ be a lax C-cleft extension with cleaving morphism h. Let M be a lax entwined module. Define:*

$$\omega_M = \phi_M \circ (i_M \otimes h) : M_C \otimes C \to M$$

and

$$\omega'_M = (p_M \otimes C) \circ \rho_M : M \to M_C \otimes C.$$

Then, the following assertions hold:

(i) *The morphisms ω_M and ω'_M satisfy the equality*

$$\omega_M \circ \omega'_M = id_M \tag{71}$$

and ω_M is a morphism in $\mathcal{W}(\mathcal{M}^C)$ between the objects

$$(M_C \otimes C, \nabla_{M_C \otimes C \otimes C} = id_{M_C \otimes C \otimes C}, \rho_{M_C \otimes C} = M_C \otimes \delta_C)$$

and $(M, \nabla_{M \otimes C}, \rho_M)$, i.e.

$$\rho_M \circ \omega_M = \nabla_{M \otimes C} \circ (\omega_M \otimes C) \circ (M_C \otimes \delta_C). \tag{72}$$

As a consequence $\Omega_M = \omega'_M \circ \omega_M : M_C \otimes C \to M_C \otimes C$ is an idempotent morphism and

$$\Omega_M = (((M_C \otimes \varepsilon_C) \circ \omega'_M) \otimes C) \circ \rho_M \circ \omega_M. \tag{73}$$

(ii) *In the particular case of* $M = A$, *we have that* $\omega_A = \mu_A \circ (i_A \otimes h)$ *and* $\omega'_A = (p_A \otimes C) \circ \rho_A$ *are morphism of left* A_C*-modules for the actions* $\varphi_{A_C \otimes C} = \mu_{A_C} \otimes C$ *and* $\varphi_A = \mu_A \circ (i_A \otimes A)$.

Proof. The equality (71) holds because:

$$
\begin{aligned}
& \omega_M \circ \omega'_M \\
= \;& \phi_M \circ ((\phi_M \circ (M \otimes h^{-1}) \circ \rho_M) \otimes h) \circ \rho_M \\
= \;& \phi_M \circ ((\phi_M \circ (M \otimes h^{-1}) \circ \nabla_{M \otimes C}) \otimes h) \circ (M \otimes \delta_C) \circ \rho_M \\
= \;& \phi_M \circ (\phi_M \otimes A) \circ (M \otimes ((h^{-1} \otimes h) \circ \delta_C)) \circ \rho_M \\
= \;& \phi_M \circ (M \otimes e) \circ \rho_M \\
= \;& id_M,
\end{aligned}
$$

where the first equality follows by definition, the second one by (38), the third one by (67), the fourth one by the A-module condition for M and finally, the fifth one by (41).

On the other hand,

$$
\begin{aligned}
& \rho_M \circ \omega_M \\
= \;& (\phi_M \otimes C) \circ (M \otimes \psi) \circ ((\rho_M \circ i_M) \otimes H) \\
= \;& (\phi_M \otimes C) \circ (i_M \otimes ((\mu_A \otimes C) \circ (A \otimes \psi) \circ ((\rho_A \otimes \eta_A) \otimes h))) \\
= \;& (\phi_M \otimes C) \circ (i_M \otimes (\rho_A \circ h)) \\
= \;& (\phi_M \otimes C) \circ (i_M \otimes (\nabla_{A \otimes C} \circ (h \otimes C) \circ \delta_C)) \\
= \;& \nabla_{M \otimes C} \circ (\omega_M \otimes C) \circ (M_C \otimes \delta_C),
\end{aligned}
$$

where the first equality follows by the entwined module condition for M, the second one by the properties of the equalizer morphism i_M as well as the A-module condition for M, the third one the entwined module condition for A, the fourth one by (58) and the last one by

$$\nabla_{M \otimes C} \circ (\phi_M \otimes C) = (\phi_M \otimes C) \circ (M \otimes \nabla_{A \otimes C}). \tag{74}$$

Therefore, (72) holds. Finally, the identity (73) is a trivial consequence of $id_M = (M \otimes \varepsilon_C) \circ \rho_M$.

In the particular case of $M = A$ we obtain that ω_A and ω'_A are morphisms of left A_C modules because by (i) of Proposition 2.3 and the associativity of

μ_A we have

$$\begin{aligned} & \omega_A \circ \varphi_{A_C\otimes C} \\ = \; & \mu_A \circ ((i_A \circ \mu_{A_C}) \otimes h) \\ = \; & \mu_A \circ ((\mu_A \circ (i_A \otimes i_A) \otimes h) \\ = \; & \varphi_A \circ (A_C \otimes \omega_A) \end{aligned}$$

and by the entwined module condition for A, the properties of i_A and (70) we obtain

$$\begin{aligned} & \omega'_A \circ \varphi_A \\ = \; & ((p_A \circ \mu_A) \otimes C) \circ (A \otimes \psi) \circ ((\rho_A \circ i_A) \otimes A) \\ = \; & ((p_A \circ \mu_A) \otimes C) \circ (i_A \otimes ((\mu_A \otimes C) \circ (A \otimes \psi) \circ ((\rho_A \circ \eta_A) \otimes A))) \\ = \; & ((p_A \circ \mu_A) \otimes C) \circ (i_A \otimes \rho_A) \\ = \; & \varphi_{A_C\otimes C} \circ (A_C \otimes \omega'_A). \end{aligned}$$

Remark 3.10. Note that in the conditions of the previous Proposition we have an isomorphism $b_M : M_C \boxtimes C \to M$ where $M_C \boxtimes C$ denotes the image of the idempotent Ω_M. If we denote by $s_{M_C\otimes C}$ and $r_{M_C\otimes C}$ the injection and the projection associated to Ω_M, we have that $b_M = \omega_M \circ s_{M_C\otimes C}$ and therefore $b_M^{-1} = r_{M_C\otimes C} \circ \omega'_M$.

In the case $M = A$, the idempotent Ω_A is a left A_C-module morphism and b_A is also a left A_C-module morphism for $\varphi_{A_C\boxtimes C} = r_{A_C\otimes C} \circ \varphi_{A_C\otimes C} \circ (A_C \otimes s_{A_C\otimes C})$.

Finally, note that if ω'_M is a morphism in $\mathcal{W}(\mathcal{M}^C)$ between the objects $(M, \nabla_{M\otimes C}, \rho_M)$ and $(M_C \otimes C, id_{M_C\otimes C\otimes C}, \rho_{M_C\otimes C} = M_C \otimes \delta_C)$ we obtain that $\nabla_{M\otimes C} = id_{M\otimes C}$ and then (M, ρ_M) is right C-comodule. On the other hand, if $\nabla_{M\otimes C} = id_{M\otimes C}$, we have that (M, ρ_M) is right C-comodule and ω'_M is a morphism of right C-comodules.

Now we can prove the main theorem of this section.

Theorem 3.11. *Let (A, C, ψ) be a lax entwining structure such that there exists a morphism $\rho_A : A \to A \otimes C$ such that (A, μ_A, ρ_A) belongs to $\mathcal{M}_A^C(\psi)$. Consider the following assertions:*

(i) *$A_C \hookrightarrow A$ is a lax C-cleft extension.*

(ii) *$A_C \hookrightarrow A$ is a lax C-Galois extension with normal basis.*

Then (i) $\Rightarrow$ (ii). *If $A \otimes -$ preserves coequalizers* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) Let $A_C \hookrightarrow A$ be a lax C-cleft extension and take

$$\gamma'_A = q_{A,A} \circ (\mu_A \otimes A) \circ (A \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C) \circ i_{A\otimes C} : A\square C \to A \otimes_{A_C} A$$

Then $\gamma_A \circ \gamma'_A = id_{A\square C}$ because

$$\begin{aligned} & i_{A\otimes C} \circ \gamma_A \circ \gamma'_A \\ = \; & t_A \circ (\mu_A \otimes A) \circ (A \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C) \circ i_{A\otimes C} \\ = \; & (\mu_A \otimes C) \circ (A \otimes \mu_A \otimes C) \circ (A \otimes h^{-1} \otimes (\nabla_{A\otimes C} \circ \\ & (h \otimes C) \circ \delta_C)) \circ (A \otimes \delta_C) \circ i_{A\otimes C} \\ = \; & \nabla_{A\otimes C} \circ \Delta_{A\otimes C} \circ i_{A\otimes C} \\ = \; & i_{A\otimes C}, \end{aligned}$$

where the first equality follows by the properties of γ_A, the second one by (58) and the associativity of μ_A, the third one by (54) and the last one by (iii) of Lemma 1.2.

On the other hand, $\gamma'_A \circ \gamma_A = id_{A\otimes_{A_C} A}$ because

$$\begin{aligned} & \gamma'_A \circ \gamma_A \circ q_{A,A} \\ = \; & q_{A,A} \circ (\mu_A \otimes A) \circ (A \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C) \circ t_A \\ = \; & q_{A,A} \circ (\mu_A \otimes A) \circ (A \otimes (\mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C}) \otimes h) \circ \\ & (A \otimes A \otimes \delta_C) \circ (A \otimes \rho_A) \\ = \; & q_{A,A} \circ (\mu_A \otimes A) \circ (A \otimes g_A \otimes h) \circ (A \otimes \rho_A) \\ = \; & q_{A,A} \circ (A \otimes (\mu_A \circ (g_A \otimes h) \circ \rho_A)) \\ = \; & q_{A,A} \circ (A \otimes (\mu_A \circ ((\mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C}) \otimes h) \circ (A \otimes \delta_C) \circ \rho_A)) \\ = \; & q_{A,A} \circ (A \otimes (\mu_A \circ (A \otimes e) \circ \rho_A)) \\ = \; & q_{A,A}, \end{aligned}$$

where the first equality follows by the properties of γ_A, the second and the sixth ones by the associativity of μ_A and (64), the third and the fifth ones by (38), the fourth one by the properties of $q_{A,A}$ and finally, the last one by (41).

Therefore γ_A is an isomorphism with inverse $\gamma_A^{-1} = \gamma'_A$ and, as a consequence, $A_C \hookrightarrow A$ is a lax C-Galois extension. Finally, by (i) of Proposition 3.9 and Remark 3.10 we obtain that $A_C \hookrightarrow A$ satisfies the normal basis condition because in this setting (see (ii) of Proposition 3.9) we have that $\omega_A = \mu_A \circ (i_A \otimes h)$, $\omega'_A = (p_A \otimes C) \circ \rho_A$ and then

$$f = \omega_A \circ (\eta_{A_C} \otimes C) = \mu_A \circ ((i_A \circ \eta_{A_C}) \otimes h) = h$$

and

$$\begin{aligned}
& f' \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C) \\
= \;& m_{\omega'_A} \circ q_{A,A} \circ (\mu_A \otimes A) \circ (A \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C) \circ \nabla_{A\otimes C} \circ (\eta_A \otimes C) \\
= \;& \mu_A \circ (A \otimes (h^{-1} \wedge (g_A \circ h))) \circ \nabla_{A\otimes C} \circ (\eta_A \otimes C) \\
= \;& \mu_A \circ (A \otimes (h^{-1} \wedge (h \wedge h^{-1}))) \circ \nabla_{A\otimes C} \circ (\eta_A \otimes C) \\
= \;& \mu_A \circ (A \otimes ((h^{-1} \wedge h) \wedge h^{-1})) \circ \nabla_{A\otimes C} \circ (\eta_A \otimes C) \\
= \;& \mu_A \circ (A \otimes (e \wedge h^{-1})) \circ \nabla_{A\otimes C} \circ (\eta_A \otimes C) \\
= \;& \mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C} \circ (\eta_A \otimes C) \\
= \;& \mu_A \circ (A \otimes h^{-1}) \circ (\eta_A \otimes C) \\
= \;& h^{-1},
\end{aligned}$$

where the first and the second equalities follow by definition, the third one by Lemma 2.9 ($\sigma = \omega'_A$) and the associativity of μ_A, the fourth one by (68), the fifth one by the associativity of the convolution product, the sixth one by the properties of h^{-1}, the seventh one by (66), the eighth one by (64) and finally the last one by the unit properties.

Therefore,

$$f' \wedge f = e = \mu_A \circ (A \otimes (f' \wedge f)) \circ \psi \circ (C \otimes \eta_A).$$

(ii) ⇒ (i) Let $A_C \hookrightarrow A$ be a weak C-Galois extension satisfying the normal basis condition and suppose that $A \otimes -$ preserves coequalizers. Take $h = f$ and and $h^{-1} = f'$.

Then by (56) we obtain (58). The proof for (64) is the following:

$$\begin{aligned}
& \mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C} \\
= \;& m_{\omega'_A} \circ \varphi_{A\otimes_{A_C} A} \circ (A \otimes (\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C))) \circ \nabla_{A\otimes C} \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ \varphi_{A\Box C} \circ (A \otimes (p_{A\otimes C} \circ (\eta_A \otimes C))) \circ \nabla_{A\otimes C} \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes (\nabla_{A\otimes C} \circ (\eta_A \otimes C))) \circ \nabla_{A\otimes C} \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes (\nabla_{A\otimes C} \circ (\eta_A \otimes C))) \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ \varphi_{A\Box C} \circ (A \otimes (p_{A\otimes C} \circ (\eta_A \otimes C))) \\
= \;& m_{\omega'_A} \circ \varphi_{A\otimes_{A_C} A} \circ (A \otimes (\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C))) \\
= \;& \mu_A \circ (A \otimes h^{-1}),
\end{aligned}$$

where in the first equality we used that $m_{\omega'_A}$ is a morphism of left A-modules, the second one by the same property for γ_A^{-1}, the third one is a consequence of the definition of $\varphi_{A\square C}$ and the fourth one follows by (54). The fifth, sixth, seventh and eighth ones follow by the same arguments.

To show that $h^{-1} \wedge h = e$ we need some preliminary steps. First we prove that

$$\mu_A \circ (A \otimes h^{-1}) \circ \rho_A = (i_A \otimes \varepsilon_C) \circ \omega'_A. \tag{75}$$

Indeed, by the same arguments used in the proof of (64) we have

$$\begin{aligned}
& \mu_A \circ (A \otimes h^{-1}) \circ \rho_A \\
= \;& m_{\omega'_A} \circ \varphi_{A\otimes_{A_C} A} \circ (A \otimes (\gamma_A^{-1} \circ p_{A\otimes C} \circ (\eta_A \otimes C))) \circ \rho_A \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ \varphi_{A\square C} \circ (A \otimes (p_{A\otimes C} \circ (\eta_A \otimes C))) \circ \rho_A \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (\mu_A \otimes C) \circ (A \otimes (\nabla_{A\otimes C} \circ (\eta_A \otimes C))) \circ \rho_A \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ \rho_A
\end{aligned}$$

and by Lemma 2.9,

$$\begin{aligned}
& m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ \rho_A \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ p_{A\otimes C} \circ (\mu_A \otimes C) \circ (\eta_A \otimes \rho_A) \\
= \;& m_{\omega'_A} \circ \gamma_A^{-1} \circ r_A \circ (\eta_A \otimes A) \\
= \;& m_{\omega'_A} \circ q_{A,A} \circ (\eta_A \otimes A) \\
= \;& \mu_A \circ (\eta_A \otimes ((i_A \otimes \varepsilon_C) \circ \omega'_A)) \\
= \;& (i_A \otimes \varepsilon_C) \circ \omega'_A.
\end{aligned}$$

Therefore (75) holds.

On the other hand, the identity

$$\mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \rho_A = id_A \tag{76}$$

also holds. To prove it compute:

$$\begin{aligned}
& \mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \rho_A \\
= \;& \mu_A \circ ((\mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C}) \otimes h) \circ (A \otimes \delta_C) \circ \rho_A \\
= \;& \mu_A \circ ((\mu_A \circ (A \otimes h^{-1}) \circ \rho_A) \otimes h) \circ \rho_A \\
= \;& \mu_A \circ \circ(((i_A \otimes \varepsilon_C) \circ \omega'_A) \otimes h) \circ \rho_A \\
= \;& \omega_A \circ (((A_C \otimes \varepsilon_C) \circ \omega'_A) \otimes C) \circ \rho_A \\
= \;& \omega_A \circ \Omega_A \circ \omega'_A \\
= \;& id_A,
\end{aligned}$$

where the first equality follows by the associativity of μ_A and (64), the second one by (38) and the third one by (75). In the fourth one we used the definition of h and the left A_C-module condition for the morphism ω_A. The fifth one follows by (55) and the last one by the properties of the idempotent Ω_A.

Then we have,

$$\mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \rho_A = id_A = \mu_A \circ (A \otimes e) \circ \rho_A$$

and as a consequence

$$\mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ i_{A\otimes C} \circ \gamma_A \circ q_{A,A} = \mu_A \circ (A \otimes e) \circ i_{A\otimes C} \circ \gamma_A \circ q_{A,A}$$

or, equivalently,

$$\mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ i_{A\otimes C} = \mu_A \circ (A \otimes e) \circ i_{A\otimes C}.$$

Therefore,

$$\mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \nabla_{A\otimes C} = \mu_A \circ (A \otimes e) \circ \nabla_{A\otimes C}$$

and composing with $\eta_A \otimes C$ we obtain

$$\mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \psi \circ (C \otimes \eta_A) = \mu_A \circ (A \otimes e) \circ \psi \circ (C \otimes \eta_A).$$

Thus, by (57) we prove that $h^{-1} \wedge h = e$, i.e. $h \in Reg^{WR}(C, A)$, because

$$\mu_A \circ (A \otimes e) \circ \psi \circ (C \otimes \eta_A) = e.$$

To finish the proof it remains to prove (65). First, note that by (58) and the equality $h^{-1} \wedge h = e$ it is possible to obtain an explicit formula for ψ as in (60). Also by (75), (58) and (64) we have

$$(i_A \otimes \varepsilon_C) \circ \Omega_A \circ (\eta_{A_C} \otimes C) = \mu_A \circ (A \otimes h^{-1}) \circ \rho_A \circ h = h \wedge h^{-1}. \quad (77)$$

Then, (65) holds because

$$\begin{aligned}
& \psi \circ (C \otimes h^{-1}) \circ \delta_C \\
= \; & (\mu_A \otimes C) \circ (h^{-1} \otimes (\rho_A \circ (h \wedge h^{-1}))) \circ \delta_C \\
= \; & (\mu_A \otimes C) \circ (h^{-1} \otimes (((\rho_A \circ i_A) \otimes \varepsilon_C) \circ \Omega_A \circ (\eta_{A_C} \otimes C))) \circ \delta_C \\
= \; & (\mu_A \otimes C) \circ (h^{-1} \otimes (((\zeta_A \circ i_A) \otimes \varepsilon_C) \circ \Omega_A \circ (\eta_{A_C} \otimes C))) \circ \delta_C \\
= \; & \zeta_A \circ (h^{-1} \wedge (\mu_A \circ (A \otimes h^{-1}) \circ \rho_A \circ h)) \\
= \; & \zeta_A \circ (h^{-1} \wedge (\mu_A \circ (A \otimes h^{-1}) \circ \nabla_{A\otimes C} \circ (h \otimes C) \circ \delta_C)) \\
= \; & \zeta_A \circ (h^{-1} \wedge (h \wedge h^{-1})) \\
= \; & \zeta_A \circ ((h^{-1} \wedge h) \wedge h^{-1}) \\
= \; & \zeta_A \circ (e \wedge h^{-1}) \\
= \; & \zeta_A \circ h^{-1}.
\end{aligned}$$

Finally, a trivial consequence of the previous Theorem is the following Corollary.

Corollary 3.12. *Let* (A, C, ψ) *be a partial entwining structure such that there exists a morphism* $\rho_A : A \to A \otimes C$ *such that* (A, μ_A, ρ_A) *belongs to* $\mathcal{M}_A^C(\psi)$. *Consider the following assertions:*

(i) $A_C \hookrightarrow A$ *is a partial* C*-cleft extension.*

(ii) $A_C \hookrightarrow A$ *is a partial* C*-Galois extension with normal basis.*

Then (i) $\Rightarrow$ (ii). *If* $A \otimes -$ *preserves coequalizers,* (ii) $\Rightarrow$ (i).

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References

[1] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez and A.B. Rodríguez Raposo. *Weak C-cleft extensions, weak entwining structures and weak Hopf algebras*. J. Algebra, **284** (2005), 679–704.

[2] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez and A.B. Rodríguez Raposo. *Weak C-cleft extensions and weak Galois extensions*. J. Algebra, **299** (2006), 276–293.

[3] J.N. Alonso Álvarez, J.M. Fernández Vilaboa and R. González Rodríguez. *Weak braided Hopf algebras.* Indiana Univ. Math. J., **57** (2008), 2423–2458.

[4] G. Böhm, F. Nill and K. Szlachányi. *Weak Hopf algebras I: Integral theory and* C^**-structures*. J. Algebra, **221** (1999), 385–438.

[5] G. Böhm and J. Vercruysse. *Morita theory for coring extensions and cleft bicomodules*. Adv. Math., **209** (2007), 611–648.

[6] T. Brzeziński and P.M. Hajac. *Coalgebra extensions and algebra coextensions of Galois type*. Commun. Algebra, **27** (1999), 1347–1367.

[7] T. Brzeziński. *On modules associated to coalgebra Galois extensions*. J. Algebra, **215** (1999), 290–317.

[8] S. Caenepeel and E. De Groot. *Modules over weak entwining structures*. Contemp. Math., **267** (2000), 31–54.

[9] S. Caenepeel and C. Janssen. *Partial entwining structures*, arXiv:math/0610524 (2007).

[10] S. Caenepeel and C. Janssen. *Partial (co)actions of Hopf algebras and partial Hopf-Galois theory*. Comm. Algebra, **36** (2008), 2923–2946.

[11] S.U. Chase and M.E. Sweedler. *Hopf algebras and Galois theory*. Springer-Verlag, Berlin-Heilderberg-New York (1969).

[12] I.G. Connell. *On the group ring*. Can. J. Math., **15** (1965), 650–685.

[13] M. Dokuchaev, R. Exel and P. Piccione. *Partial representations and partial group algebras*. J. Algebra, **226** (2000), 505–532.

[14] M. Dokuchaev and R. Exel. *Associativity of crossed products by partial actions, enveloping actions and partial representations*. Trans. Amer. Math. Soc., **357** (2005), 1931–1952.

[15] M. Dokuchaev, M. Ferrero and A. Paques. *Partial actions and Galois theory*. J. Pure Appl. Algebra, **208** (2007), 77–87.

[16] Y. Doi and M. Takeuchi. *Cleft comodule algebras for a bialgebra*. Comm. Algebra, **14** (1986), 801–817.

[17] R. Exel. *Twisted partial actions: a classification of regular C^*-algebraic bundles*. Proc. London Math. Soc., **74** (1997), 417–443.

[18] J.M. Fernández Vilaboa and E. Villanueva Novoa. *A characterization of the cleft comodule triples*. Comm. Algebra, **16** (1988), 613–622.

[19] H.F. Kreimer and M. Takeuchi. *Hopf algebras and Galois extensions of an algebra*. Indiana Univ. Math. J., **30** (1981), 675–691.

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