BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 45(1), 91-116

ISSN: 1678-7544 (Print) / 1678-7714 (Online)

Minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$ over unbounded domains

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Abstract. We study the existence of minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$ with prescribed boundary data, possibly infinite. We give necessary and sufficient conditions on the "lengths" of the sides of the inscribed polygons in an unbounded domain in $\mathbb{H}^2$, that yield solutions to the minimal surface equation with prescribed boundary data.

Keywords: minimal graphs, unbounded domains, Dirichlet Problem.

Mathematical subject classification: 53A10, 53C42.

1 Introduction

In 1966 H. Jenkins and J. Serrin published the work "The Dirichlet problem for the minimal surface equation", [6], where they show the existence of minimal graphs in $\mathbb{R}^3$ over bounded domains in $\mathbb{R}^2$ with infinite boundary data. J. Spruck in [16] extended the theorem of Jenkins and Serrin to constant mean curvature graphs in $\mathbb{R}^3$ over bounded domains of $\mathbb{R}^2$.

The work of Jenkins and Serrin inspired many extensions to other ambient spaces. For minimal graphs in $\mathbb{S}^2 \times \mathbb{R}$ the existence theorem was proved by H. Rosenberg [13]. In $\mathbb{H}^2 \times \mathbb{R}$, B. Nelli and H. Rosenberg [10] proved the theorem of Jenkins and Serrin over bounded domains in $\mathbb{H}^2$. P. Collin and H. Rosenberg in [1] treated the case in which the domain in $\mathbb{H}^2$ is unbounded and L. Mazet, M. Rodríguez and H. Rosenberg [9] dealt with a more general setting. In [4], J. Gálvez and H. Rosenberg proved this theorem in $\mathbb{M}^2 \times \mathbb{R}$ over bounded domains in $\mathbb{M}^2$ = Hadamard surface. When $\mathbb{M}^2$ is a Riemannian surface we have the work of A.L. Pinheiro [11]. For constant mean curvature graphs in $\mathbb{H}^2 \times \mathbb{R}$, the theorem was proved for bounded domains in $\mathbb{H}^2$ by

Received 28 February 2012.

*Supported by CAPES.

L. Hauswirth, H. Rosenberg and J. Spruck [5] and for unbounded domains by A. Folha and S. Melo in [3]. In Heisenberg space Nil_3, A.L. Pinheiro [12] studied the case of minimal graphs over bounded domains in $\mathbb{R}^2$. In $\widetilde{PSL_2}(\mathbb{R})$, R. Younes in [17] shows the existence of minimal surfaces which are graphs over bounded domains in $\mathbb{H}^2$.

In this paper we study minimal graphs in the space $\widetilde{PSL_2}(\mathbb{R})$. This space is a Riemannian fibration over the hyperbolic plane $\mathbb{H}^2$. Here we are interested in obtaining a Jenkins-Serrin type theorem for minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$ over unbounded domains in $\mathbb{H}^2$.

The $\widetilde{PSL_2}(\mathbb{R})$ space is a simply connected 3-dimensional homogeneous Riemannian manifold with four dimensional isometry group. Denote by $PSL_2(\mathbb{R})$ the quotient Lie group $SL_2(\mathbb{R})/\{\pm Id\}$, where $SL_2(\mathbb{R})$ is the 3-dimensional Lie group of 2×2 real matrices of determinant 1. The universal covering of $PSL_2(\mathbb{R})$ is $\widetilde{PSL_2}(\mathbb{R})$. Let $U\mathbb{H}^2$ be the unit tangent bundle of $\mathbb{H}^2$, i.e. the submanifold of $T\mathbb{H}^2$ consisting of tangent vectors of unit length. We can identify $U\mathbb{H}^2$ with $PSL_2(\mathbb{R})$ and consequently $\widetilde{U\mathbb{H}^2}$ with $\widetilde{PSL_2}(\mathbb{R})$. Furthermore, $\widetilde{PSL_2}(\mathbb{R})$ is diffeomorphic to $\mathbb{H}^2 \times \mathbb{R}$ but $\widetilde{PSL_2}(\mathbb{R})$ is not a product space, see [15] for further information.

In this article we consider a convex domain $\mathcal{D} \subset \mathbb{H}^2$ whose boundary $\partial\mathcal{D}$ is composed of complete geodesic arcs $\{A_i\}$, $\{B_j\}$ and convex arcs $\{C_k\}$ with all the vertices in $\partial_\infty\mathbb{H}^2$. We give necessary and sufficient conditions on the geometry of the domain $\mathcal{D}$ which assure the existence of a function u defined in $\mathcal{D}$, whose graph is minimal and u takes the value $+\infty$ on each A_i, $-\infty$ on each B_j and prescribed continuous data on each C_k. The conditions will be in terms of the lengths of sides of inscribed polygons. Since these quantities are infinite in general, the formulation of the conditions is somewhat delicate.

This paper is organized as follows. In Section 2 we state the main theorems, which will be proved in Section 8. In Sections 3 and 4 we give the necessary background for the study of minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$. Sections 5 and 6 contain a general maximum principle and the flux formula, which are useful tools to prove preliminary results and the necessary conditions of the main theorems. In Section 7, we state results about divergence lines, which are essential to prove the sufficient conditions of the main theorems.

2 Main Theorems

In this section, we establish the theorems that give necessary and sufficient conditions for the existence of minimal graphs in $\widetilde{PSL_2}(\mathbb{R})$ which take, on the boundary of a domain, the values $+\infty$ on $\{A_i\}$, $-\infty$ on $\{B_j\}$ and continuous data on $\{C_k\}$. We begin giving some definitions.

Definition 2.1 (Admissible Domain). *We say that an unbounded domain $\mathcal{D}$ in $\mathbb{H}^2$ is admissible if it is simply connected and $\partial\mathcal{D}$ is a polygon formed by sides $\{A_i\}$, $\left\{B_j\right\}$ and $\{C_k\}$, such that $\{A_i\}$ and $\left\{B_j\right\}$ are geodesics and $\{C_k\}$ are convex curves (with respect to the interior of $\mathcal{D}$). We suppose that two arcs A_i do not have a common end-point; the same for two arcs B_j. All the vertices of $\partial\mathcal{D}$ are supposed at infinity $\partial_\infty\mathbb{H}^2$ ($\partial\mathcal{D}$ is an ideal polygon).*

Remark 2.1. When a convex arc C_k in $\partial\mathcal{D}$ has a vertex point $d_\kappa \in \partial_\infty\mathbb{H}^2$, we assume that the other arc η of $\partial\mathcal{D}$ having d_κ as vertex is asymptotic to C_k at d_κ, that is, for a sequence of points $x_n \in \eta$, converging to d_κ, we have $\mathrm{dist}_{\mathbb{H}^2}(x_n, C_k) \to 0$ as $n \to \infty$.

Definition 2.2 (Dirichlet Problem in $\widetilde{PSL_2(\mathbb{R})}$). *Let $\mathcal{D}$ be an admissible domain. The Dirichlet Problem consists in finding a solution for the minimal surface equation in $\mathcal{D}$ which assumes the value $+\infty$ on each A_i, $-\infty$ on each B_j and prescribed continuous data on each C_k.*

Definition 2.3 (Admissible Inscribed Polygons). *Let $\mathcal{D}$ be an admissible domain. We say that $\mathcal{P}$ is an admissible inscribed polygon if $\mathcal{P} \subset (\mathcal{D} \cup \partial\mathcal{D})$ and its sides are geodesics. Moreover, the vertices of $\mathcal{P}$ are also vertices of $\mathcal{D}$.*

To solve the Dirichlet problem we will find necessary and sufficient conditions in terms of lengths of the sides of $\mathcal{P}$. When the domain is unbounded these quantities can be infinite. Using the ideas as in [1], we proceed as follows.

Let $\mathcal{P}$ be an inscribed polygon in $\mathcal{D} \subset \mathbb{H}^2$ and let $\{d_\kappa\} \in \partial_\infty\mathbb{H}^2$ be the vertices of $\mathcal{P}$. For each κ, we consider a horocycle $\mathcal{H}_\kappa$ at d_κ; such that $\mathcal{H}_i \cap \mathcal{H}_j = \emptyset$, if $i \neq j$. Let F_κ be the convex horodisk with boundary $\mathcal{H}_\kappa$. We suppose that the polygon $\mathcal{P}$ is $\mathcal{P} = \bigcup_l \eta_l$. Each side η_l of $\mathcal{P}$ meets exactly two horodisks. Denote by $\bar{\eta}_l$ the part of η_l outside the two horodisks; and by $|\eta_l|_{\mathbb{H}^2}$ the length of $\bar{\eta}_l$. Note that $\bar{\eta}_l$ is a compact arc of η_l.

So, we define

$$\alpha(\mathcal{P}) = \sum_{A_i \in \mathcal{P}} |A_i|_{\mathbb{H}^2}, \quad \beta(\mathcal{P}) = \sum_{B_j \in \mathcal{P}} \left|B_j\right|_{\mathbb{H}^2} \quad \text{and} \quad \ell(\mathcal{P}) = \sum_l |\eta_l|_{\mathbb{H}^2}$$

where $\mathcal{P} = \bigcup_l \eta_l$.

We are now ready to state the main theorems.

Theorem 2.1. *Let $\mathcal{D} \subset \mathbb{H}^2$ be an admissible domain and suppose that the family $\{C_k\}$ is empty. Then there exists a solution to the Dirichlet Problem in $PSL_2(\mathbb{R})$ if, and only if, for some choice of the horocycles at the vertices,*

$$\alpha(\partial\mathcal{D}) = \beta(\partial\mathcal{D}), \quad \textit{for } \mathcal{P} = \partial\mathcal{D} \tag{1}$$

and

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}) \quad \textit{and} \quad 2\beta(\mathcal{P}) < \ell(\mathcal{P}), \tag{2}$$

for all admissible inscribed polygons $\mathcal{P}$, $\mathcal{P} \neq \partial\mathcal{D}$.

When $\{C_k\}$ is non-empty we are able to prove the result below.

Theorem 2.2. *Let $\mathcal{D} \subset \mathbb{H}^2$ be an admissible domain and suppose that the family $\{C_k\}$ is nonempty. Then there exists a solution to the Dirichlet Problem in $PSL_2(\mathbb{R})$, if and only if for some choice of the horocycles at the vertices,*

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}) \quad \textit{and} \quad 2\beta(\mathcal{P}) < \ell(\mathcal{P}), \tag{3}$$

for all admissible inscribed polygons $\mathcal{P}$, $\mathcal{P} \neq \partial\mathcal{D}$.

Remark 2.2. Let $\mathcal{P}$ be an inscribed polygon in $\mathcal{D} \subset \mathbb{H}^2$ and let $\{d_\kappa\} \in \partial_\infty\mathbb{H}^2$ be the vertices of $\mathcal{P}$. Let $\mathcal{H}_\kappa$ be a horocycle at d_κ. Suppose that the conditions (1), (2) and (3) are satisfied for a family of horocycles $\mathcal{H} = \{\mathcal{H}_\kappa\}$. Take the family $\mathcal{H}' = \{\mathcal{H}_\kappa\}_{\kappa \neq s} \cup \{\mathcal{H}'_s\}$, where $\mathcal{H}'_s$ is contained in the horodisk F_s bounded by $\mathcal{H}_s$. Conditions (1), (2) and (3) also hold for the family $\mathcal{H}'$. In the following, the subindices T and T' are used to clarify the dependence of the terms with respect to $\mathcal{H}$ and $\mathcal{H}'$ respectively.

Indeed, if $\mathcal{P} = \partial\mathcal{D}$

$$\alpha(\partial\mathcal{D}_{T'}) - \beta(\partial\mathcal{D}_{T'}) = \alpha(\partial\mathcal{D}_T) - \beta(\partial\mathcal{D}_T) = 0,$$

then condition (1) holds for this family $\mathcal{H}'$.

Now, consider $\mathcal{P} \neq \partial\mathcal{D}$. Suppose that the horocycle $\mathcal{H}_s$ meets sides A_i and B_j of the polygon $\mathcal{P}$, for some i, j. The part that is added in $\alpha(\mathcal{P}_{T'})$ is also added in $\beta(\mathcal{P}_{T'})$ and $\ell(\mathcal{P}_{T'})$, and this preserves the inequalities. See Figure 1. When $\mathcal{H}_s$ meets sides A_i and C_k, for some i, k (or A_i and E, where E is interior arc in $\mathcal{D}$) the part that is added in $\alpha(\mathcal{P}_{T'})$ is also added in $\ell(\mathcal{P}_{T'})$ and $\beta(\mathcal{P}_{T'})$ remains the same. The case where $\mathcal{H}_s$ meets B_j and C_k, for some j, k (or B_j and E) is similar as above. If $\mathcal{H}_s$ meets C_k and E, for some k, then the part added in $\ell(\mathcal{P}_{T'})$ does not change the inequalities. So conditions (2) and (3) hold for this family $\mathcal{H}'$.

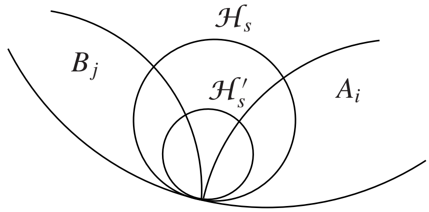

Figure 1: $\mathcal{H}_s$ and $\mathcal{H}'_s$ in the disk model for $\mathbb{H}^2$.

3 The space $\widetilde{PSL_2}(\mathbb{R})$

The space $\widetilde{PSL_2}(\mathbb{R})$ is a simply connected 3-dimensional homogeneous manifold whose isometry group has dimension 4. This manifold is a Riemannian fibration over the hyperbolic plane $\mathbb{H}^2$ and its fibers are geodesics tangent to a unitary Killing field ξ. The bundle curvature is the number τ such that $\widetilde{\nabla}_X \xi = \tau X \times \xi$ for any vector field X, where $\widetilde{\nabla}_X$ is the Riemannian connection of $\widetilde{PSL_2}(\mathbb{R})$. The bundle projection $\pi : \widetilde{PSL_2}(\mathbb{R}) \to \mathbb{H}^2$ is a Riemannian submersion, see [17].

We take the half-plane model of the hyperbolic plane $\mathbb{H}^2$,

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\},$$

with metric

$$ds_{\mathbb{H}^2} = \lambda^2(dx^2 + dy^2), \quad \lambda = \frac{1}{y}.$$

The metric in $\widetilde{PSL_2}(\mathbb{R})$ is

$$ds^2 = \lambda^2(dx^2 + dy^2) + \left(2\tau\frac{\lambda_y}{\lambda}dx - 2\tau\frac{\lambda_x}{\lambda}dy + dz\right)^2,$$

where τ is the bundle curvature. For more information, see [2].

Now we will build an orthonormal frame $\{E_1, E_2, E_3\}$ on $\widetilde{PSL_2}(\mathbb{R})$. Denote by E_3 the vector field ξ. Let $\{e_1, e_2\}$ be the orthonormal frame of $\mathbb{H}^2$ given by $e_1 = \frac{1}{\lambda}\partial_x$ and $e_2 = \frac{1}{\lambda}\partial_y$. We denote E_1, E_2 the horizontal lifts to $\widetilde{PSL_2}(\mathbb{R})$ of e_1 and e_2, respectively. Thus

$$d\pi(E_i) = e_i \quad \text{and} \quad \langle E_i, E_3\rangle = 0, \; i = 1, 2.$$

Here $\langle\ ,\ \rangle$ is the scalar product in $\widetilde{PSL_2}(\mathbb{R})$.

Using this information, we can give expression of E_i in local coordinates

$$E_1 = \frac{1}{\lambda}\partial_x - 2\tau\frac{\lambda_y}{\lambda^2}\partial_z, \quad E_2 = \frac{1}{\lambda}\partial_y + 2\tau\frac{\lambda_x}{\lambda^2}\partial_z \quad \text{and} \quad E_3 = \partial_z.$$

So $\{E_1, E_2, E_3\}$ defined as above is an orthonormal frame on $\widetilde{PSL_2(\mathbb{R})}$.

4 Graphs

In the space $\widetilde{PSL_2(\mathbb{R})}$ a graph is the image of a section of the Riemannian submersion. Let $s : D \to \widetilde{PSL_2(\mathbb{R})}$ be a section of the Riemannian submersion $\pi : \widetilde{PSL_2(\mathbb{R})} \to \mathbb{H}^2$, i.e., s is a map that satisfies $\pi \circ s = Id_D$. We define by Σ_0 the surface given by $\{z = 0\}$ and we identify the domain $D \subset \mathbb{H}^2$ with its lift to Σ_0, i.e., if $(x, y) \in D$ then its lift is the point of $\widetilde{PSL_2(\mathbb{R})}$ whose coordinates are $(x, y, 0)$.

Given a function $u \in C^2(D)$ we define the graph $G(u)$ of u on D as

$$G(u) = \left\{(x, y, u(x, y)) \in \widetilde{PSL_2(\mathbb{R})};\ (x, y) \in D\right\},$$

where the value $u(x, y)$ is the distance from the lift of $(x, y) \in \mathbb{H}^2$ to $s(x, y) \in \pi^{-1}(x, y)$ along the geodesic fiber through $(x, y, 0)$.

We take the function $F(x, y, z) = z - u(x, y)$ in $\widetilde{PSL_2(\mathbb{R})}$. Clearly $G(u) = F^{-1}(0)$ and let $N = \frac{\overline{\nabla} F}{\|\overline{\nabla} F\|}$ be the unitary vector normal field to $G(u)$ in $\widetilde{PSL_2(\mathbb{R})}$ pointing up. Here the gradient $\overline{\nabla}$ is calculated in the metric of $\widetilde{PSL_2(\mathbb{R})}$ and $\|.\|$ is the norm in $\widetilde{PSL_2(\mathbb{R})}$. To calculate the mean curvature H of $G(u)$, with respect to the normal N, choose the vectors $v_1, v_2 \in \widetilde{PSL_2(\mathbb{R})}$ such that $\{v_1, v_2, N\}$ is an orthonormal basis of $T\left(\widetilde{PSL_2(\mathbb{R})}\right)$. As N is a unitary field we have $\left\langle \widetilde{\nabla}_N N, N\right\rangle = 0$. Then

$$\begin{aligned}
-2H &= \sum_{i=1}^{2} \left\langle \widetilde{\nabla}_{v_i} N, v_i\right\rangle \\
&= \sum_{i=1}^{2} \left\langle \widetilde{\nabla}_{v_i} N, v_i\right\rangle + \left\langle \widetilde{\nabla}_N N, N\right\rangle \\
&= div(N),
\end{aligned}$$

where div denotes the divergence in $\widetilde{PSL_2(\mathbb{R})}$.

Hence $div\left(\frac{\overline{\nabla}F}{\|\overline{\nabla}F\|}\right) = -2H$.

Calculating $\overline{\nabla}F$ and $\|\overline{\nabla}F\|$ we find

$$\begin{aligned}
\overline{\nabla}F &= \left(-2\tau\frac{\lambda_y}{\lambda^3} - \frac{u_x}{\lambda^2}\right)\partial_x + \left(2\tau\frac{\lambda_x}{\lambda^3} - \frac{u_y}{\lambda^2}\right)\partial_y \\
&\quad + \left(1 + 4\tau^2\frac{\lambda_y^2}{\lambda^4} + 2\tau\frac{\lambda_y u_x}{\lambda^3} - 2\tau\frac{\lambda_x u_y}{\lambda^3} + 4\tau^2\frac{\lambda_x^2}{\lambda^4}\right)\partial_z \\
&= \left(-2\tau\frac{\lambda_y}{\lambda^2} - \frac{u_x}{\lambda}\right)E_1 + \left(2\tau\frac{\lambda_x}{\lambda^2} - \frac{u_y}{\lambda}\right)E_2 + E_3
\end{aligned}$$

and

$$\|\overline{\nabla}F\| = \left(-2\tau\frac{\lambda_y}{\lambda^2} - \frac{u_x}{\lambda}\right)^2 + \left(2\tau\frac{\lambda_x}{\lambda^2} - \frac{u_y}{\lambda}\right)^2 + 1.$$

So

$$\frac{\overline{\nabla}F}{\|\overline{\nabla}F\|} = \frac{\rho}{W}E_1 + \frac{\omega}{W}E_2 + \frac{1}{W}E_3.$$

Here $\rho = \left(-2\tau\frac{\lambda_y}{\lambda^2} - \frac{u_x}{\lambda}\right)$, $\omega = \left(2\tau\frac{\lambda_x}{\lambda^2} - \frac{u_y}{\lambda}\right)$ and $W^2 = \rho^2 + \omega^2 + 1$.

Therefore

$$div\left(\frac{\overline{\nabla}F}{\|\overline{\nabla}F\|}\right) = div\left(\frac{\rho}{W}E_1 + \frac{\omega}{W}E_2 + \frac{1}{W}E_3\right).$$

Since E_3 is a Killing field, $div(E_3) = 0$ and $div\left(\frac{1}{W}E_3\right) = 0$. Then

$$\begin{aligned}
&div\left(\frac{\rho}{W}E_1 + \frac{\omega}{W}E_2\right) + div\left(\frac{1}{W}E_3\right) \\
&= div\left(\frac{\rho}{W}E_1 + \frac{\omega}{W}E_2\right) \\
&= \sum_{i=1}^{2}\left\langle\widetilde{\nabla}_{E_i}\left(\frac{\rho}{W}E_1 + \frac{\omega}{W}E_2\right), E_i\right\rangle \\
&= \sum_{i=1}^{2}\left\langle\nabla_{e_i}d\pi\left(\frac{\rho}{W}E_1 + \frac{\omega}{W}E_2\right), e_i\right\rangle_{\mathbb{H}^2} \\
&= div_{\mathbb{H}^2}\left(\frac{\rho}{\lambda W}\partial_x + \frac{\omega}{\lambda W}\partial_y\right).
\end{aligned}$$

Thus

$$-2H = div_{\mathbb{H}^2}\left(\frac{\rho}{\lambda W}\partial_x + \frac{\omega}{\lambda W}\partial_y\right).$$

The equation above is called the H-graph equation. So u is a solution for the minimal surface equation if u satisfies

$$Mu := div_{\mathbb{H}^2}\left(\frac{\rho}{\lambda W}\partial_x + \frac{\omega}{\lambda W}\partial_y\right) = 0. \tag{4}$$

5 Maximum principle

The next result is a General Maximum Principle proved in [12]. We fix some notations. Let $\pi : \overline{M}^3 \to M^2$ be a Riemannian submersion and N_u the upwards unit normal vector to the graph $G(u)$ of u in $\overline{M}^3$. We write this vector as $N_u = -\frac{\mathcal{G}^u}{W_u} + \frac{1}{W_u}$, where $-\frac{\mathcal{G}^u}{W_u}$ is the horizontal part of N_u and $W_u^2 = 1 + \|\mathcal{G}^u\|^2_{\overline{M}}$. Since $\mathcal{G}^u$ is horizontal, we can identify $\mathcal{G}^u$ with its projection $d\pi(\mathcal{G}^u)$. So for the functions u, v in Σ_0 we have, as proved in [7], that

$$\mathcal{G}^u - \mathcal{G}^v = \nabla u - \nabla v,$$

here ∇ denotes the gradient on M^2.

Now we state a lemma proved in [7].

Lemma 5.1. *Let u and v be functions in $C^2(D)$, D is a domain in M^2. Then*

$$\left\langle \mathcal{G}^u - \mathcal{G}^v, \frac{\mathcal{G}^u}{W_u} - \frac{\mathcal{G}^v}{W_v} \right\rangle_{M^2} = \frac{W_u + W_v}{2}\|N_u - N_v\|_{\overline{M}^3} \geq 0,$$

with equality at a point if and only if $\nabla u = \nabla v$.

Theorem 5.1. General Maximum Principle. *Consider $D \subset M^2$ a bounded domain. Let $u, v \in C^2(D)$ be such that their graphs are minimal surfaces in $\overline{M}^3$. Let $I \subset \partial D$ be a finite set of points such that $\partial D - I$ consists of smooth arcs and suppose that u and v extend continuously to each smooth arc of $\partial D - I$. If $u \leq v$ on $\partial D - I$, then $u \leq v$ on D.*

6 Flux formula

Let u be a solution of the minimal surface equation (4) on a domain $D \subset \mathbb{H}^2$. We consider the half-space model of $\mathbb{H}^2$ with $\lambda = \frac{1}{y}$. We denote by $-X_u$

the horizontal projection of the upwards normal vector to the graph of u in $\widetilde{PSL_2}(\mathbb{R})$. So

$$-X_u = d\pi\left(\frac{\rho}{W}E_1 + \frac{\omega}{W}E_2\right) = \left(\frac{y\rho}{W}\partial_x + \frac{y\omega}{W}\partial_y\right),$$

where $\rho = 2\tau - yu_x$, $\omega = -yu_y$ and $W^2 = 1 + \rho^2 + \omega^2$.

Definition 6.1. *Let η be an arc in D and ν the unit normal to η. We define the flux of u across η to be*

$$F_u(\eta) = \int_\eta \langle X_u, \nu\rangle_{\mathbb{H}^2} ds,$$

ds is the arclength of η in $\mathbb{H}^2$.

Since u is a solution of the minimal surface equation, we know that X_u has null divergence, see equation (4). Moreover, $|X_u|_{\mathbb{H}^2} \leq 1$. So we can extend the definition of flux to an arc η in $\partial\mathcal{D}$. In this case, we consider the vector ν as the outer conormal vector of ∂D.

Lemma 6.1. *Let u be a solution of the minimal surface equation in D. Then the following conditions hold:*

(i) *if η is a piecewise smooth arc in $\overline{D}$, then $|F_u(\eta)| \leq |\eta|_{\mathbb{H}^2}$;*

(ii) *if $D' \subset D$ is a compact bounded domain, then $F_u(\partial D') = 0$;*

(iii) *if $\eta \subset D$, then $|F_u(\eta)| < |\eta|_{\mathbb{H}^2}$.*

For the proof of (i) and (ii) of the lemma above is sufficient to observe that $|X_u|^2_{\mathbb{H}^2} \leq 1$ and $div_{\mathbb{H}^2} X_u = 0$ and to use Stokes Theorem. To prove (iii) we use the fact that $|X_u|^2_{\mathbb{H}^2} < 1$ in the interior of D.

Lemma 6.2. *Let η be a convex arc in ∂D. Let u be a solution which is continuous on η. Then*

$$|F_u(\eta)| < |\eta|_{\mathbb{H}^2}.$$

For the proof of this lemma, see [17].

Lemma 6.3. *Let u be a solution of the minimal surface equation in D. If $u \to +\infty$ on an arc η in ∂D (respectively, $-\infty$), then η is a geodesic of $\mathbb{H}^2$ and $F_u(\eta) = |\eta|_{\mathbb{H}^2}$ (respectively, $-|\eta|_{\mathbb{H}^2}$).*

Proof. We take $p \in \eta$ such that $\mathrm{dist}_{\mathbb{H}^2}(p, \partial\eta) > 0$ and a sequence of points $\{p_m\}$ in D such that $\{p_m\} \to p$. So, we have $u(p_m) \to +\infty$. We denote $v_m = u - u(p_m)$ and $G(v_m)$ the graph of v_m in $\widetilde{PSL}_2(\mathbb{R})$ over D. By Curvature estimates, from [14], we get a $\delta > 0$ independent of m, such that a neighborhood of each $P_m = (p_m, v_m(p_m)) = (p_m, 0)$ in $G(v_m)$ is a graph $G_{P_m}(v_m, \delta)$, of bounded geometry (C^2- bounded), over the disk $\mathbb{D}(P_m, \delta)$ of radius δ centered at the origin of $T_{P_m}G(v_m)$. Here $T_{P_m}G(v_m)$ is the tangent plane of $G(v_m)$ at P_m.

For m large enough, the tangent planes $T_{P_m}G_{P_m}(v_m, \delta)$ are almost vertical, otherwise the vertical projection by π of $G_{P_m}(v_m, \delta)$ would have points outside D. So a subsequence of $T_{P_m}G_{P_m}(v_m, \delta)$ converges to a vertical plane Π when $\{p_m\} \to p$. This implies that the disks $\mathbb{D}(P_m, \delta)$ converge to a disk $\mathbb{D}(P, \delta)$ in Π, where $P = (p, 0)$, and the graphs $G_{P_m}(v_m, \delta)$ converge to a minimal graph G_P over $\mathbb{D}(P, \delta) \subset \Pi$. We know that the plane Π is tangent to $\pi^{-1}(\eta)$ in the point P of $\widetilde{PSL}_2(\mathbb{R})$, where $\pi^{-1}(\eta) \subset \widetilde{PSL}_2(\mathbb{R})$ is the vertical cylinder on η.

We suppose that the curve η is not a geodesic. Then there exists a geodesic Γ in $\mathbb{H}^2$ tangent to η at p. We know that G_P and $\pi^{-1}(\Gamma)$ are tangent at P. If G_P is on one side of $\pi^{-1}(\Gamma)$, by the maximum principle, we have $G_P \subset \pi^{-1}(\Gamma)$ and η would be a geodesic. If G_P is on both sides of $\pi^{-1}(\Gamma)$ we have that $G_P \cap \pi^{-1}(\Gamma)$ is composed of $k \geq 2$ curves passing through P meeting transversally at P. So in a neighborhood of P these curves separate G_P in $2k$ components and the adjacent components lie in alternate sides of $\pi^{-1}(\Gamma)$. Moreover the normal vector to G_P alternates from pointing down to pointing up when one goes from one component to the another. This cannot occur. Therefore η is a geodesic in $\mathbb{H}^2$.

Now we prove the second part of the lemma. Suppose that $u \to +\infty$ on η. We know that the tangent planes $T_P G(u)$ are almost vertical at points sufficiently close to η and the normal vector $N(P)$ is almost horizontal. So, using the definition of X_u, we have that close to η, $\langle X_u, \nu\rangle_{\mathbb{H}^2}$ approaches one. Thus, for $\varepsilon > 0$ small

$$\langle X_u, \nu\rangle_{\mathbb{H}^2} \geq 1 - \varepsilon.$$

Therefore, for all small $\varepsilon > 0$

$$\int_\eta \langle X_u, \nu\rangle_{\mathbb{H}^2}\, ds \geq \int_\eta (1 - \varepsilon) ds.$$

As ε tends to zero

$$\int_\eta \langle X_u, \nu\rangle_{\mathbb{H}^2} \geq |\eta|_{\mathbb{H}^2}\, .$$

However, by Lemma 6.1, $\int_\eta \langle X_u, \nu\rangle_{\mathbb{H}^2} \leq |\eta|_{\mathbb{H}^2}$. Then

$$\int_\eta \langle X_u, \nu\rangle_{\mathbb{H}^2} = |\eta|_{\mathbb{H}^2}\,.$$

The case $u \to -\infty$ is similar, just observe that $\langle X_u, \nu\rangle_{\mathbb{H}^2}$ approaches -1 at points sufficiently near to η, so $X_u = -\nu$ along η. $\square$

Remark 6.1. By the second part of the lemma above we conclude that if a solution u tends to $+\infty$ on geodesic arcs η_1, η_2 of ∂D, then, using the triangle inequality, these arcs cannot meet at a strictly convex corner in ∂D.

We can write a generalization of the second part of Lemma 6.3.

Lemma 6.4. *Let $\{u_n\}$ be a sequence of continuous solutions on $\overline{D}$. If $\{u_n\}$ diverges uniformly to $+\infty$ (respectively, $-\infty$) on compact sets of $\eta \subset \partial D$ while remains uniformly bounded on compact sets of Ω, then $F_{u_n}(\eta) \to |\eta|_{\mathbb{H}^2}$ (respectively, $F_{u_n}(\eta) \to -|\eta|_{\mathbb{H}^2}$).*

The next lemma is almost a converse of Lemma 6.3. We follow the ideas in [9].

Lemma 6.5. *Let u be a solution in D and $\widetilde{\eta} \subset \partial D$ a geodesic arc such that $F_u(\eta) = |\eta|_{\mathbb{H}^2}$ $(F_u(\eta) = -|\eta|_{\mathbb{H}^2})$, for every compact arc $\eta \subset \widetilde{\eta}$. Then u assumes the boundary value $+\infty$ $(-\infty)$ on $\widetilde{\eta}$.*

Proof. Let η be a compact arc as in the statement of the lemma, small enough, such that there exists a geodesic triangle $\mathcal{T}$ contained in D whose sides are η, S_1 and S_2 satisfying length(S_1)=length(S_2). We denote by Δ the region of D bounded by $\mathcal{T}$. We consider the solution v which takes values $+\infty$ on η and $v = u$ on S_1, S_2. This solution thank to Lemma 5.3 in [17].

We need to show that $u = v$ in $\Delta \cup \mathcal{T}$. If this is not the case, the set $O = \{u - v < \epsilon\}$ is different from empty, for $\epsilon > 0$ a regular value of $u - v$. Let D' be the connected component of the complement of O in Δ which has $\mathcal{T} - \eta$ in its boundary and let O' be the complement of D' in Δ. In particular we see that $O \subset O'$ and $\partial O' \subset \partial O$. Let q be a point in $\partial O' - \eta$. For $\mu > 0$, let $O'(\mu) = \{p \in O'; \mathrm{dist}_{\mathbb{H}^2}(p, \eta) > \mu\}$. Let q_1, q_2 be the end-points of the connected component of $\partial O' \cap \partial O'(\mu)$ which contains q. Let p_i be the projection of q_i on η, $i = 1, 2$. Let $\widetilde{O}(\mu)$ be the domain bounded by the segments $[p_1, q_1]$, $[p_2, q_2]$, the arc $[p_1, p_2] \subset \eta$ and the boundary component of $O'(\mu)$ between q_1, q_2, which is denoted by $\Gamma(\mu)$. As $F_u(\partial O') = 0 = F_v(\partial O')$ we obtain

$$
\begin{aligned}
0 &= F_u(\partial O') - F_v(\partial O') \\
&= \int_{\Gamma(\mu)} \langle X_u - X_v, \nu\rangle_{\mathbb{H}^2} + \int_{[p_1,q_1]\cup[p_2,q_2]} \langle X_u - X_v, \nu\rangle_{\mathbb{H}^2} \\
&\quad + \int_{[p_1,p_2]} \langle X_u - X_v, \nu\rangle_{\mathbb{H}^2}.
\end{aligned}
$$

On $\Gamma(\mu)$ the vector $X_u - X_v$ points outside $\widetilde{O}(\mu)$, because $X_u - X_v = \nabla u - \nabla v$. Then, we can write

$$
\begin{aligned}
0 &< \int_{\Gamma(\mu)} \langle X_u - X_v, \nu\rangle_{\mathbb{H}^2} \\
&= -\int_{[p_1,q_1]\cup[p_2,q_2]} \langle X_u - X_v, \nu\rangle_{\mathbb{H}^2} - \int_{[p_1,p_2]} \langle X_u - X_v, \nu\rangle_{\mathbb{H}^2} \\
&\leq 4\mu,
\end{aligned}
$$

because the last term in the second line vanishes using the hypothesis $F_u(\eta) = |\eta|_{\mathbb{H}^2}$ and Lemma 6.3 applied to v. Note that the integral on $\Gamma(\mu)$ increases when $\mu \to 0$. That contradicts previous inequality. Therefore, $u = v$ and $u \to +\infty$ on η.

The proof for $u \to -\infty$ in the geodesic arc η when $F_u(\eta) = -|\eta|_{\mathbb{H}^2}$, is analogous as above taking $v = -\infty$ on η and $v = u$ on S_1 and S_2. $\square$

Lemma 6.6. *Let D be a domain whose boundary ∂D contains an arc η and let $\{u_n\}$ be a sequence of solutions in D with each u_n continuous on η. Then if the sequence diverges to $+\infty$ uniformly on compact subsets of D while remaining uniformly bounded on compact subsets of η, we have*

$$\lim_{n\to\infty} F_{u_n}(\eta) = -|\eta|.$$

7 Divergence lines

In this section we will use a technique developed in [8] which allows to describe the properties of the sets where a sequence (not necessarily monotone) of solutions of the minimal surface equation converges or diverges without the aid of a maximum principle. Many ideas found here were inspired by [9].

We denote by D a domain in $\mathbb{H}^2$ with piecewise smooth boundary and solutions are always solutions of the minimal surface equation (4).

Definition 7.1. *Let D be a domain with piecewise smooth boundary, and $\{u_n\}$ a sequence of solutions in D. We define the convergence set as*

$$\mathcal{U} = \{p \in D;\ |\nabla u_n(p)| \ \text{is bounded independently of}\ n\}$$

and the divergence set as

$$\mathcal{V} = D - \mathcal{U}.$$

Lemma 7.1. *Let $p \in D$ and $\{u_n\}$ be a sequence of solutions in D. If $p \in \mathcal{U}$, there is a subsequence of $\{v_n = u_n - u_n(p)\}$ converging uniformly to a solution in a neighborhood of p in D. In particular $\mathcal{U}$ is open. If $p \in \mathcal{V}$, then there is a compact geodesic arc $L_p(\delta) \in D$ of length 2δ centered at p such that, after passing to a subsequence, $\{N_{v_n}(q, v_n(q))\}$ converges to a horizontal vector N_q. The vector N_q is such that $d\pi(N_q)$ is orthogonal to $L_p(\delta)$ at every point $q \in L_p(\delta)$. The constant $\delta > 0$ only depends on* $\mathrm{dist}_{\mathbb{H}^2}(p, \partial D)$.

Remark 7.1. For any $q \in D$ and $\{v_n = u_n - u_n(q)\}$, we note that

$$N_{u_n}(q, u_n(q)) = N_{v_n}(q, v_n(q))$$

after vertical translation, and the convergence and divergence sets are the same for $\{u_n\}$ and $\{v_n\}$.

Proof of Lemma 7.1. We denote by $G(v_n)$ the graph of v_n in $\widetilde{PSL_2}(\mathbb{R})$ over D. We fix p in D such that the distance from $P = (p, v_n(p)) = (p, 0)$ to the boundary of $G(v_n)$ is bigger than or equal to $\mathrm{dist}_{\mathbb{H}^2}(p, \partial D)$. So the curvature estimates, see [14], give us a $\delta_1 > 0$ (independent of n) such that a neighborhood of P in $G(v_n)$ is a graph, in geodesic coordinates, with bounded geometry, over the disk of radius δ_1, $\mathbb{D}_n(P, \delta_1)$, centered at the origin of $T_P G(v_n)$. We call this graph $G_P(v_n, \delta_1)$.

If $p \in \mathcal{U}$ the sequence $\{|\nabla u_n|\}$ is bounded, then there is a subsequence of $\{N_{v_n}(P)\}$, still called $\{N_{v_n}(P)\}$, which converges to a non horizontal vector and consequently the tangent planes associated to this subsequence converges to a non vertical plane Π. The disks $\mathbb{D}_n(P, \delta_1)$ converge to a disk $\mathbb{D}_{\delta_1}$ of radius δ_1 centered at the origin of Π and the graphs $G_P(v_n, \delta_1)$ converge to a minimal graph $G_P(\delta_1)$ over $\mathbb{D}_{\delta_1}$. Since this plane Π is a non vertical, there is $\widetilde{\delta}$, $0 < \widetilde{\delta} \leq \delta_1$, such that $G_P(\delta_1)$ is a graph over a disk in D centered at p of radius $\widetilde{\delta}$. We conclude that there is a neighborhood of $p \in D$ such that a subsequence of $\{v_n\}$ converges to a solution in this neighborhood.

Now suppose that $p \in \mathcal{V}$. Since $\{|\nabla u_n|\}$ is unbounded there is a subsequence of $\{N_{v_n}(P)\}$ that converges to a horizontal vector N_P. So (for this subsequence) the tangent planes $T_P G(v_n)$ converge to a vertical plane Π and the graphs $G_P(v_n, \delta_1)$ converge to a minimal graph $G_P(\delta')$ over a disk of radius $\delta' \leq \delta_1$ centered at the origin of Π.

We take the geodesic $L_p \subset D$ passing through p orthogonal to $d\pi(N_P)$ and let $\widetilde{L}_p$ be its horizontal lift. We know that $G_P(\delta') \subset \pi^{-1}(L_p)$, i.e., $G_P(\delta')$ is contained in the minimal surface $\pi^{-1}(L_p)$ (see the proof of Lemma 6.3).

Now, let $\delta \leq \delta'$. We take $\widetilde{L}_p(\delta) \subset \widetilde{L}_p$ the geodesic arc contained in $G_P(\delta') \cap \widetilde{L}_p$ which contains P and has length 2δ. Since $G_P(\delta') \subset \pi^{-1}(L_p)$ we have that, for all $q \in L_p(\delta)$, $\{N_{v_n}(q, v_n(q))\}$ converges to a horizontal vector N_q at $(q, v_n(q))$ orthogonal to $\widetilde{L}_p(\delta)$. Therefore, $d\pi(N_q)$ is orthogonal to $L_p(\delta)$ at q, $\forall q \in L_p(\delta)$. $\square$

Remark 7.2. The lemma above shows that the convergence set is a domain.

Lemma 7.2. *Given $p \in \mathcal{V}$, there is a geodesic $L \subset D$, which passes by p such that, after passing to a subsequence, $\{N_{v_n}(q, v_n(q))\}$ converges to a horizontal vector N_q, $\forall q \in L$, with the property that $d\pi(N_q)$ is orthogonal to L. This geodesic L contains $L_p(\delta)$ and $L \subset \mathcal{V}$.*

Proof. Let L be the geodesic in D which contains $L_p(\delta)$ joining points of ∂D ($L_p(\delta)$ is given in Lemma 7.1). We denote by $\overline{pq}$, $q \in L$, the compact arc in L between p, q, and by $N_{v_n}|_{\overline{pq}}$ the normal vector to $G(v_n)$ at the points $(q_1, v_n(q_1))$ where $q_1 \in \overline{pq}$. We define

$$\Lambda = \left\{ q \in L \;\middle|\; \begin{array}{l} \text{there is a subsequence of } \{v_n\} \text{ such that } N_{v_n}|_{\overline{pq}} \text{ (associated} \\ \text{with this subsequence) becomes horizontal with its vertical} \\ \text{projection orthogonal to } \overline{pq}. \end{array} \right\}$$

We want to prove that $\Lambda = L$. Since $p \in \Lambda$, $\Lambda \neq \emptyset$. We will prove that Λ is open and closed. First, we will prove that Λ is open. Let q be a point in Λ. We denote $\{v_{\Lambda(n)}\}$ the subsequence associated to Λ. Since $\Lambda \subset \mathcal{V}$, Lemma 7.1 ensures the existence a geodesic arc $L_q(\delta)$ with center q such that, after passing to a subsequence, $\{N_{v_{\Lambda(n)}}|_{L_q(\delta)}\}$ converges to horizontal vector N with $d\pi(N)$ orthogonal to $L_q(\delta)$. We note that $d\pi(N)$ is orthogonal to $L_q(\delta)$ and to $\overline{pq}$ simultaneously. From which we deduce that $L_q(\delta) \subset L$ and thus Λ is open.

Now we will prove that Λ is closed. We take a sequence of points $\{q_m\}$ in Λ such that $q_m \to q \in L$. We will show that $q \in \Lambda$. For each m, there is a subsequence of $\{v_n\}$ such that $\{N_{v_n}|_{\overline{pq}_m}\}$ becomes horizontal with its vertical projection orthogonal to $\overline{pq}_m$. By the diagonal process we obtain a subsequence of $\{v_n\}$ such that (for this subsequence) $\{N_{v_n}|_{\overline{pq}_m}\}$ converges to a horizontal vector with its vertical projection orthogonal to L in $\overline{pq}_m$, $\forall m$. Then by Lemma 7.1 we can find a geodesic $L_{q_m}(\delta)$ through q_m, (for m large, δ depends only on the distance from q to ∂D) such that $\{N_{v_n}|_{\overline{pq}_m}\}$ converges to a horizontal vector N such that $d\pi(N)$ is orthogonal to $L_{q_m}(\delta)$. Thus $L_{q_m}(\delta) \subset L$. Since $q_m \to q$, we have that, for all m large enough, $q \in L_{q_m}(\delta)$. Consequently $q \in \Lambda$. □

An important conclusion of the lemma above is that the divergence set $\mathcal{V}$ is the union of geodesics L_i, which are called *divergence lines*, $\mathcal{V} = \bigcup_{i \in I} L_i$.

Lemma 7.3. *Let $\{u_n\}$ be a sequence of solutions in D. Suppose that the divergence set $\mathcal{V}$ of $\{u_n\}$ consists of a countable number of divergence lines. Then there is a subsequence of $\{u_n\}$, again denoted by $\{u_n\}$, such that:*

1. *The divergence set of $\{u_n\}$ consists of a countable number of pairwise disjoint divergence lines.*
2. *For any connected component $\mathcal{U}'$ of $\mathcal{U} = D - \mathcal{V}$ and for any $p \in \mathcal{U}'$, the sequence $\{u_n - u_n(p)\}$ converges uniformly on compact subsets of $\mathcal{U}'$ to a solution in $\mathcal{U}'$.*

Proof. Let L_1 be a divergence line of $\{u_n\}$. We take $p_1 \in L_1$ and $\{v_n^1 = u_n - u_n(p_1)\}$. Lemma 7.2 guarantees that, after passing to a subsequence, $\{N_{v_n^1}(q_1, v_n^1(q_1))\}$ converges to a horizontal vector N_{q_1} such that $d\pi(N_{q_1})$ is orthogonal to L_1 at q_1, for all q_1 in L_1. The divergence set of this subsequence is contained in the divergence set of the original sequence, so the divergence set associated to this subsequence has only a countable number of lines. This subsequence is still denoted by $\{u_n\}$ and its divergence set by $\mathcal{V}$.

Suppose that there exists a divergence line $L_2 \neq L_1$ in $\mathcal{V}$. We take $p_2 \in L_2$ and $\{v_n^2 = u_n - u_n(p_2)\}$. Then, we can find a subsequence such that $\{N_{v_n^2}(q_2, v_n^2)\}$ converges to a horizontal vector N_{q_2} such that $d\pi(N_{q_2})$ is orthogonal to L_2 at q_2, for each $q_2 \in L_2$. If $L_1 \cap L_2 \neq \emptyset$ we take a point $q \in L_1 \cap L_2$. Then a subsequence of $\{N_{v_n^2}(q, v_n(q))\}$ converges to a horizontal vector N_q where $d\pi(N_q)$ is orthogonal to L_1 and L_2 simultaneously at q. So L_1 and L_2 are tangent at q. Since L_1, L_2 are geodesics, we obtain $L_1 = L_2$. We continue this process to get a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, whose divergence set is composed of a countable number of pairwise disjoint divergence lines. This proves 1.

Now we prove part 2 of Lemma 7.3. Lemma 7.1 shows that there is a subsequence of $\{u_n\}$ and a neighborhood of each point $p \in \mathcal{U}$ such that $\{u_n - u_n(p)\}$ (associated with this subsequence) converges to a minimal graph, and this convergence is uniform on compact subsets of this neighborhood. Then taking a countable dense sequence $\{p_i\}$ in $\mathcal{U}'$, by the diagonal process we obtain a subsequence of $\{u_n\}$ such that $\{u_n - u_n(p)\}$ converges uniformly on compact subsets of $\mathcal{U}'$, $\forall p \in \mathcal{U}'$. $\square$

Lemma 7.4. *Let $\{u_n\}$ be a sequence of solutions in D, such that its divergence set is composed of a countable number of pairwise disjoint divergence lines. Suppose that $\{u_n\}$ converges to a solution u in a connected set $\mathcal{U}' \subset D$. Let η be a compact arc in $\partial\mathcal{U}'$ included in a divergence line of $\{u_n\}$ such that $X_{u_n} \to \nu$ along η, where ν is the outer conormal to η with respect to $\mathcal{U}'$. Then if $p \in \mathcal{U}'$ and $q \in \eta$, we have*

$$\lim_{n\to\infty} (u_n(q) - u_n(p)) = +\infty.$$

Proof. We choose p, q as in the hypothesis of the lemma. Since $X_{u_n} \to \nu$ along η, we have that $F_{u_n}(\eta) \to |\eta|$, where $F_{u_n}(\eta)$ is the flux of u_n across η. So Lemma 6.5 ensures that $u \to +\infty$ on η.

Claim 7.1. For n large, there is an $\epsilon > 0$ such that $\frac{\partial u_n}{\partial t} \geq 0$ on $\{\widetilde{\Upsilon(t)}; -\epsilon < t \leq 0\}$, where $\widetilde{\Upsilon(t)}$ is the horizontal lift of the geodesic arc $\{\Upsilon(t); -\epsilon < t \leq 0\} \subset \mathcal{U}'$ with $-\theta < t \leq 0$, $\theta \geq \epsilon$, $\Upsilon(0) = q$ and $\Upsilon'(0) = \nu$. The inequality is strict on $\{\Upsilon(t); -\epsilon < t < 0\}$.

Indeed, using Lemma 7.1 and the fact that $u\big|_\eta = +\infty$, we obtain an $\epsilon > 0$ such that, $\frac{\partial u}{\partial t} \geq 1$ in $\{\widetilde{\Upsilon(t)}; -\epsilon < t < 0\}$. The convergence $u_n \to u$ implies that $\frac{\partial u_n}{\partial t} > 0$ in $\{\widetilde{\Upsilon(t)}; -\epsilon < t < -\theta_0\}$, for every $0 < \theta_0 < \epsilon$ and $n \geq n_0(\theta_0)$.

Let Q be a point in $\widetilde{\eta}$, where $\widetilde{\eta}$ is the horizontal lift of η and $\pi(Q) = q$. If the claim is not true, considering a subsequence if necessary, there is a sequence $\{q_m\}$ in $\{\Upsilon(t); -\theta_0 \leq t \leq 0\}$, such that, $q_m \to q$ and $\frac{\partial u_n}{\partial t}(Q_m) = 0$ for the points in the sequence $\{Q_m\}$.

If the sequence $\{|\nabla u_n(q_m)|\}$ is bounded we have, from the curvature estimates, that $\{|\nabla u_n|\}$ is uniformly bounded over a disk $D(q_m)$ of radius independent of n, centered at q_m. Since $q_m \to q$ the sequence $\{|\nabla u_n(q)|\}$ is bounded, because for m large enough, $q \in D(q_m)$. This contradicts the fact that q is contained in the divergence set.

If the sequence $\{|\nabla u_n(q_m)|\}$ is unbounded, we consider the sequence $\{u_n - u_n(q_m)\}$ and $\mathbb{D}_n^1$ the disk of radius δ in the graph of $\{u_n - u_n(q_m)\}$ centered at $(q_m, 0)$ given by the curvature estimates, with δ independent of n. Since $\frac{\partial u_n}{\partial t}(Q_m) = 0$, the disks $\mathbb{D}_n^1$ converge to a vertical disk centered at $(q, 0)$ of radius δ in $\pi^{-1}(\Upsilon)$, here Υ is a geodesic through q orthogonal to η. Let $\mathbb{D}_n^2$ be a disk of radius δ centered at $(q, 0)$ in the graph of $\{u_n - u_n(q)\}$. Since η

is contained in a divergence line, $\{\mathbb{D}_n^2\}$ converges to a vertical disk centered at $(q,0)$ in $\pi^{-1}(\eta)$. Then, for n large enough, these disks $\mathbb{D}_n^1$ and $\mathbb{D}_n^2$ intersect transversally. But this is impossible because the normal vectors to $\mathbb{D}_n^1$ and $\mathbb{D}_n^2$ only depend on the gradient of u_n, so they are the same vector (on domains where both sequences are defined) for the two sequences. This proves Claim 7.1.

Let $q_t \in \mathcal{U}'$ be the point $q_t = \Upsilon(t)$, $t < 0$, for t small enough. Claim 7.1 assures, for n large, that

$$u_n(q) - u_n(p) \geq u_n(q_t) - u_n(p) \geq u(q_t) - u(p) - 1.$$

The second inequality came from the convergence of $\{u_n\}$ to u. The third term is as large as we want, because $u\,\big|_\eta = +\infty$. $\square$

For completeness we state an important Remark made in [9].

Remark 7.3. Let L be a divergence line and suppose there exists two components Ω_1 and Ω_2 of $\mathcal{U}$ such that $L \subset \partial\Omega_i$, $i = 1, 2$. Consider the points $p_1 \in \Omega_1$, $p_2 \in \Omega_2$. Passing to a subsequence, $\{u_n - u_n(p_i)\}$ converges uniformly on compact sets of Ω_i to a minimal graph $u_i : \Omega_i \to \mathbb{R}$. Assume $F_{u_1}(T) = |T|_{\mathbb{H}^2}$ for each bounded arc $T \subset L$, when L is oriented as $\partial\Omega_1$. Then $F_{u_2}(T) = -|T|_{\mathbb{H}^2}$, when L is oriented as $\partial\Omega_2$. We deduce from Lemma 7.4 that $\{(u_n - u_n(p_1))\,|_L\}$ diverges to $+\infty$ and $\{(u_n - u_n(p_2))\,|_L\}$ diverges to $-\infty$. In particular, we can deduce that $\{u_n - u_n(p_1)\}$ diverges to $+\infty$ uniformly on compacts sets of Ω_2 and $\{u_n - u_n(p_2)\}$ diverges to $-\infty$ uniformly on compacts sets of Ω_1. See Figure 2.

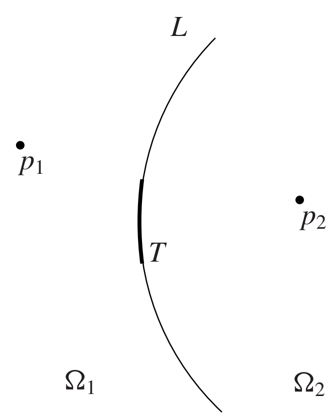

Figure 2: Domains Ω_1 and Ω_2.

Lemma 7.5. *Let $\eta \subset \partial D$ be a smooth arc. We consider a sequence of solutions $\{u_n\}$ in D, such that, $\lim_{n\to\infty} u_n|_\eta = f$, f a continuous function. Then an end-point of the divergence line cannot be an interior point of η.*

Proof. We consider p a point of η and η' a neighborhood of p in η such that $\overline{\eta'} \subset \eta$.

First we suppose that η is a strictly convex arc. Let Γ be the geodesic arc in $\mathbb{H}^2$ joining the end-points of η'. We consider η' small enough such that the domain Δ, bounded by η' and Γ, is contained in D.

We define $M = \max_{\eta'} |f|$. So we can take $|u_n| < M + 1$ on η', for n large enough. Using Theorem 1 of [17], we have that there are minimal graphs $v^+, v^- : \Delta \to \mathbb{R}$ given by

$$v^+ = \begin{cases} M+1 & \text{on } \eta' \\ +\infty & \text{on } \Gamma \end{cases} \quad \text{and} \quad v^- = \begin{cases} -M-1 & \text{on } \eta' \\ -\infty & \text{on } \Gamma. \end{cases}$$

By the general maximum principle we have

$$v^- \leq u_n \leq v^+,$$

for all n.

Thus, by the compactness principle, see [17], we have $\Delta \subset \mathcal{U}$. Therefore, p is not an end-point for any divergence line.

Now consider η a geodesic curve and L a divergence line with an end-point in $p \in \eta$. Then there is a subset $V \subset D$ which contains a subarc (containing p) of η in its boundary, and the sequence diverges to $\pm\infty$ on V. Assume that the sequence diverges to $+\infty$. Taking a point $q \in \eta \cap \partial V$, denote by $\overline{pq}$ the arc contained in η joining the points p and q. Let s be a point in L and $\overline{ps}$ the arc in L joining p and s. Denote by $\overline{sq}$ the geodesic arc joining s and q. Suppose that q is as close to s as necessary, in order to guarantee $\overline{sq} \subset V$. We choose this "triangle" T so that the sequence $\{u_n\}$ diverges to $+\infty$ in the domain $\Delta_T \subset V$ bounded by T. By the flux formula,

$$0 = F_{u_n}(\overline{ps}) + F_{u_n}(\overline{pq}) + F_{u_n}(\overline{sq}).$$

By Lemma 6.6 we have,

$$\lim_{n\to+\infty} F_{u_n}(\overline{pq}) = -|\overline{pq}|.$$

Since $\overline{ps} \subset L$, using Remark 7.3, either

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = |\overline{ps}| \quad \text{or} \quad \lim_{n\to+\infty} F_{u_n}(\overline{ps}) = -|\overline{ps}|.$$

First, suppose that

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = -|\overline{ps}|.$$

Then,

$$\begin{aligned} 0 &= \lim_{n\to+\infty} F_{u_n}(\overline{ps}) + \lim_{n\to+\infty} F_{u_n}(\overline{pq}) + \lim_{n\to+\infty} F_{u_n}(\overline{sq}) \\ &\leq -|\overline{ps}| - |\overline{pq}| + |\overline{sq}|. \end{aligned}$$

But this contradicts the triangular inequality.

Now we consider the case where

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = |\overline{ps}|.$$

By Lemma 7.4 we have that $\{u_n\}$ diverges to $+\infty$ on a subset of $D - V$ which has L and a subarc of η in its boundary. Then applying the same argument as above, we get a contradiction.

Now, suppose that there are two, or more, divergence lines with end-points in p. We fix two divergence lines, L_1, L_2. The point $p \in \eta$ divides η in two curves η_1, η_2. We orient L_1, L_2, η_1, η_2, such that, W_1 is the domain bounded in part by $L_1 \cup \eta_1$ and not containing L_2; W_2 is the domain bounded in part by $\eta_2 \cup L_2$ and not containing L_1; and W_3 is the domain bounded in part by $L_1 \cup L_2$ and not containing $\eta_1 \cup \eta_2$. Let $q \in L_1$, $s \in L_2$, $p_1 \in \eta$, $p_2 \in \eta$ be points. Denote by $\overline{pq}$ the arc in L_1 joining p and q, by $\overline{ps}$ the arc in L_2 joining p and s, by $\overline{qs} \subset W_3$ the geodesic arc joining q to s, by $\overline{qp_1} \subset W_1$ the geodesic arc joining q and p_1, and by $\overline{sp_2} \subset W_2$ the geodesic arc joining s and p_2. In some of these subsets W_i, $i = 1, 2, 3$ the sequence $\{u_n\}$ diverges to $+\infty$. Suppose that in W_3 the sequence diverges to $+\infty$.

If,

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = -|\overline{ps}| \quad \text{and} \quad \lim_{n\to+\infty} F_{u_n}(\overline{pq}) = -|\overline{pq}|,$$

when the arcs are oriented as ∂W_3, then applying the flux formula to the triangle formed by $\overline{ps}$, $\overline{pq}$ and $\overline{qs}$, we obtain a contradiction as before.

If, when the arcs are oriented as ∂W_3, either

$$\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = |\overline{ps}| \quad \text{or} \quad \lim_{n\to+\infty} F_{u_n}(\overline{pq}) = |\overline{pq}|,$$

then doing as we have done before to the triangle formed by $\overline{qp_1}$, $\overline{pq}$ and $\overline{p_1 p}$, if $\lim_{n\to+\infty} F_{u_n}(\overline{pq}) = |\overline{pq}|$, or to the triangle formed by $\overline{ps}$, $\overline{pp_2}$ and $\overline{sp_2}$, if $\lim_{n\to+\infty} F_{u_n}(\overline{ps}) = |\overline{ps}|$, we obtain a contradiction. $\square$

8 Proof of the Theorems

We fix some notations. Let $\{\mathcal{H}_\kappa(m)\}$ be a sequence of nested horocycles at d_κ such that $\mathcal{H}_\kappa(m)$ converges to d_κ as $m \to +\infty$. We choose $\mathcal{H}_\kappa(m)$, such that $\mathcal{H}_\kappa(m+1) \subset F_\kappa(m)$, $F_\kappa(m)$ is the convex horodisk bounded by $\mathcal{H}_\kappa(m)$. Here $\bar{\gamma}_\kappa(m) = \mathcal{H}_\kappa(m) \cap (\partial\mathcal{D} \cup \mathcal{D})$.

Let $\mathcal{D}(m)$ be the domain bounded by

$$\partial\mathcal{D}(m) = \big(\partial\mathcal{D} - \{\cup_\kappa F_\kappa(m)\}\big) \bigcup \big(\cup_\kappa \gamma_\kappa(m)\big),$$

where $\gamma_\kappa(m)$ is the geodesic arc in $\mathcal{D}$ having the same end-points as $\bar{\gamma}_\kappa(m)$. Let $\mathcal{P}$ be the boundary of a domain Ω. Similarly, we define $\Omega(m)$ the domain whose boundary is

$$\mathcal{P}(m) = \big(\mathcal{P} - \{\cup_\kappa F_\kappa(m)\}\big) \bigcup \big(\cup_\kappa \gamma'_\kappa(m)\big),$$

where $\gamma'_\kappa(m)$ are the geodesic arcs contained in $\Omega \cap \{\cup_\kappa F_\kappa(m)\}$ joining the points $\mathcal{P} \cap \{\cup_\kappa \mathcal{H}_\kappa(m)\}$.

Proof of Theorem 2.1. We suppose that the conditions (1) and (2) are true for polygons $\mathcal{P}$ in $\mathcal{D}$.

Claim 8.1. There is a solution in $\mathcal{D}$ with boundary values

$$u_n = \begin{cases} n & \text{on } \cup_i A_i \\ -n & \text{on } \cup_j B_j. \end{cases} \tag{5}$$

Proof of Claim 8.1. For each m, the existence Theorem (Proposition 5.4 in [17]) says, there is a solution of the minimal surface equation u_m in $\mathcal{D}(m)$ with boundary values

$$u_m = \begin{cases} n & \text{on } \cup_i A_i(m) \\ -n & \text{on } \cup_j B_j(m) \\ 0 & \text{on } \cup_\kappa \gamma_\kappa(m). \end{cases}$$

We fix m_0. For all $m > m_0$, we have $\{u_m\}$ is a sequence of solutions of the minimal surface equation in $\mathcal{D}(m_0)$ bounded above by n and below by $-n$. By the Compactness Principle, see [17], there is a subsequence of $\{u_m\}$ that converges in $\mathcal{D}(m_0)$ to a solution. By the boundary values of the $\{u_m\}$, we have $u_m|_{A_i(m_0)} = n$ and $u_m|_{B_j(m_0)} = -n$. By the diagonal process, we have in $\mathcal{D}$ a solution u_n, given by

$$u_n = \begin{cases} n & \text{on } \cup_i A_i \\ -n & \text{on } \cup_j B_j. \end{cases}$$

□

So using Claim 8.1 we have that there exists a solution of the minimal surface equation given by (5).

We consider the sequence $\{u_n\}$ formed by the solutions above. We will prove that the sequence $\{u_n\}$ does not have divergence lines. For this we assume that there exists some divergence line and we obtain a contradiction.

We know, by Lemma 7.5, that the end-points of these lines are in the vertices of $\mathcal{D}$. Since $\partial\mathcal{D}$ has only a finite number of vertices, we can suppose that the divergence set consists of a finite number of disjoint divergence lines. These lines separate the domain $\mathcal{D}$ in at least two connected components, and the interior of these components belong to the convergence domain $\mathcal{U}$. By Lemma 7.4, in some connected component of the convergence set, the sequence $\{u_n\}$ diverges to $+\infty$ or $-\infty$. We suppose that in some connected component of the convergent set $\mathcal{U}'$ the sequence diverges to $+\infty$ (the case $-\infty$ is similar).

Since $\mathcal{U}' \subset \mathcal{U}$, we have that the sequence $\{u_n - u_n(p)\},\ p \in \mathcal{U}'$, converges uniformly on compact subsets of $\mathcal{U}'$ to a solution of the minimal surface equation u in $\mathcal{U}'$. On the other hand, by the choice of $\mathcal{U}'$ we have $u_n(p) \to +\infty,\ p \in \mathcal{U}'$. Moreover, we note that $\partial\mathcal{U}' = \mathcal{P}$ is an admissible polygon. Taking the family of horocycles $\{\mathcal{H}_\kappa(m)\}$ at d_κ, we can choose $\mathcal{P}$ satisfying the next result.

Claim 8.2. We can choose $\mathcal{P}$ so that

$$\begin{aligned} & F_u\left(\mathcal{P}(m) - \left[\{\cup_i A_i(m)\} \cup \{\cup_\kappa(\gamma_\kappa'(m))\}\right]\right) \\ = \quad & -\left|\mathcal{P}(m) - \left[\{\cup_i A_i(m)\} \cup \{\cup_\kappa(\gamma_\kappa'(m))\}\right]\right|_{\mathbb{H}^2} \end{aligned}$$

where $\partial\mathcal{U}' = \mathcal{P}$. See [9] for a proof.

We are supposing that there is some divergence line, so $\mathcal{P} \neq \partial\mathcal{D}$. Then the hypothesis are the inequalities (2). Using the flux formula and Claim 8.2 we have

$$\begin{aligned} 0 &= F_u\left(\mathcal{P}(m)\right) \\ &= F_u\left(\mathcal{P}(m) - \left[\{\cup_i A_i(m)\} \cup \{\cup_\kappa(\gamma_\kappa'(m))\}\right]\right) \\ &\quad + F_u\left(\{\cup_i A_i(m)\} \cup \{\cup_\kappa(\gamma_\kappa'(m))\}\right) \\ &\leq -\left|\mathcal{P}(m) - \left[\{\cup_i A_i(m)\} \cup \{\cup_\kappa(\gamma_\kappa'(m))\}\right]\right|_{\mathbb{H}^2} \\ &\quad + \left|\{\cup_i A_i(m)\} \cup \{\cup_\kappa(\gamma_\kappa'(m))\}\right|_{\mathbb{H}^2} \\ &= 2\alpha(\mathcal{P}) - \ell(\mathcal{P}) + \left|\cup_\kappa\left(\gamma_\kappa'(m)\right)\right|_{\mathbb{H}^2}. \end{aligned}$$

As $m \to \infty$, $|\cup_\kappa (\gamma'_\kappa(m))|_{\mathbb{H}^2}$ tends to zero, because we have only sides A_i and B_i. So

$$0 \leq 2\alpha(\mathcal{P}) - \ell(\mathcal{P}),$$

which contradicts the hypothesis. Then, the sequence $\{u_n\}$ does not have any divergence lines.

Thus, $\mathcal{D}$ is the convergence domain and this implies that there is a subsequence of $\{u_n - u_n(p)\}$, $p \in \mathcal{D}$, which converges to a solution u on $\mathcal{D}$. Here we use the hypothesis (1) in $\mathcal{P} = \partial\mathcal{D}$.

If the sequence $\{u_n\}$ is bounded at the point $p \in \mathcal{D}$, u has the desired boundary values, that is, $u|_{A_i} = +\infty$ and $u|_{B_i} = -\infty$. We will show that even if the sequence $\{u_n\}$ is unbounded the solution u has the boundary values as prescribed.

We suppose the sequence $\{u_n(p)\}$ tends to $-\infty$. So, on each A_i, we have that $\{u_n - u_n(p)\} \to +\infty$ when $n \to +\infty$. By the flux formula we can show that on B_i the sequence $\{u_n - u_n(p)\}$ tends to $-\infty$. Indeed, applying the flux formula in $\mathcal{P}(m)$,

$$\begin{aligned}
0 &= \lim_{n\to\infty} F_{u_n}(\mathcal{P}(m)) \\
&= \sum_i \lim_{n\to\infty} F_{u_n}(A_i(m)) + \sum_i \lim_{n\to\infty} F_{u_n}(B_i(m)) + \sum_\kappa \lim_{n\to\infty} F_{u_n}(\gamma_\kappa(m)) \\
&\geq \alpha(\mathcal{P}) - \beta(\mathcal{P}) - \sum_\kappa |\gamma'_\kappa(m)|_{\mathbb{H}^2}.
\end{aligned}$$

As $m \to +\infty$, $\left|\gamma'_\kappa(m)\right| \to 0$, then $0 \geq \alpha(\mathcal{P}) - \beta(\mathcal{P})$. But by the hypothesis, $\alpha(\mathcal{P}) = \beta(\mathcal{P})$ and this implies that $\lim_{n\to\infty} F_{u_n}(B_i(m)) = -|B_i(m)|$. So, by Lemma 6.5, $\{u_n - u_n(p)\} \to -\infty$ on B_i, $\forall i$.

We suppose the sequence $\{u_n(p)\}$ tends to $+\infty$. So $\{u_n - u_n(p)\} \to -\infty$ on each B_i. Using the flux formula we can show that on A_i the sequence $\{u_n - u_n(p)\}$ tends to $+\infty$. By the flux formula,

$$\begin{aligned}
0 &= \lim_{n\to\infty} F_{u_n}(\mathcal{P}(m)) \\
&= \sum_i \lim_{n\to\infty} F_{u_n}(A_i(m)) + \sum_i \lim_{n\to\infty} F_{u_n}(B_i(m)) + \sum_\kappa \lim_{n\to\infty} F_{u_n}\left(\gamma'_\kappa(m)\right) \\
&\leq \alpha(\mathcal{P}) - \beta(\mathcal{P}) + \sum_\kappa |\gamma'_\kappa(m)|_{\mathbb{H}^2}.
\end{aligned}$$

As $m \to \infty$ we have $\beta(\mathcal{P}) \leq \alpha(\mathcal{P})$. Since we cannot have $\alpha(\mathcal{P}) > \beta(\mathcal{P})$, by the hypothesis, we have $\lim_{n\to\infty} F_{u_n}(A_i(m)) = |A_i(m)|_{\mathbb{H}^2}$. Then $\{u_n - u_n(p)\}$ tends to $+\infty$ on A_i, $\forall i$.

To show the conditions (1) and (2) we suppose that there is a solution u in $\mathcal{D}$ to the Dirichlet problem. First, we prove the equation (1) Applying the flux formula to $\mathcal{P}(m) = \partial\mathcal{D}(m)$, we have

$$0 = F_u\left(\mathcal{P}(m)\right) = \sum_i F_u(A_i(m)) + \sum_i F_u(B_i(m)) + \sum_\kappa F_u(\gamma'_\kappa(m)).$$

Therefore, by the flux formula, we have the inequalities

$$\begin{aligned}\sum_i |A_i(m)|_{\mathbb{H}^2} - \sum_i |B_i(m)|_{\mathbb{H}^2} - \sum_\kappa \left|\gamma'_\kappa(m)\right|_{\mathbb{H}^2} &= \\ \alpha(\partial\mathcal{D}) - \beta(\partial\mathcal{D}) - \sum_\kappa |\gamma'_\kappa(m)|_{\mathbb{H}^2} &\leq 0\end{aligned}$$

$$\begin{aligned}\sum_i |A_i(m)|_{\mathbb{H}^2} - \sum_i |B_i(m)|_{\mathbb{H}^2} + \sum_\kappa \left|\gamma'_\kappa(m)\right|_{\mathbb{H}^2} &= \\ \alpha(\partial\mathcal{D}) - \beta(\partial\mathcal{D}) + \sum_\kappa |\gamma'_\kappa(m)|_{\mathbb{H}^2} &\geq 0\end{aligned}$$

So, as $m \to \infty$, we have $|\gamma_\kappa(m)|_{\mathbb{H}^2} \to 0$. Thus

$$\alpha(\mathcal{D}) = \beta(\mathcal{D}).$$

Now, we prove the inequalities (2). Let $\mathcal{P}$ be a polygon in $\mathcal{D}$ whose sides are A_i, B_j and E_l. By the flux formula, we know that $F_u(\mathcal{P}(m)) = 0$. So, we can write

$$\begin{aligned}F_u\left(\cup_i A_i(m)\right) &= -F_u\left(\mathcal{P}(m) - \cup_i A_i(m)\right) \\ &= -F_u\left(\{\cup_j B_j(m)\} \cup \{\cup_l E_l(m)\} \cup \{\cup_\kappa \gamma'_\kappa(m)\}\right)\end{aligned}$$

Then, using the equality above and the flux formula

$$\begin{aligned}\alpha(\mathcal{P}) &= \sum_i |A_i(m)|_{\mathbb{H}^2} \\ &= F_u\left(\cup_i A_i(m)\right) = -F_u\left(\{\cup_j B_j(m)\} \cup \{\cup_l E_l(m)\} \cup \{\cup_\kappa \gamma'_\kappa(m)\}\right) \\ &\leq \left|F_u\left(\{\cup_j B_j(m)\} \cup \{\cup_l E_l(m)\} \cup \{\cup_\kappa \gamma'_\kappa(m)\}\right)\right| \\ &\leq \left|F_u\left(\cup_j B_j(m)\right)\right| + |F_u(\cup_l E_l(m))| + |F_u(\cup_\kappa \gamma'_\kappa(m))| \\ &\leq \beta(\mathcal{P}) + |F_u\left(\cup_l E_l(m)\right)| + \sum_\kappa \left|\gamma'_\kappa(m)\right|_{\mathbb{H}^2} \\ &= \ell(\mathcal{P}) - \alpha(\mathcal{P}) - \sum_l |E_l(m)|_{\mathbb{H}^2} + |F_u\left(\cup_l E_l(m)\right)| + \sum_\kappa \left|\gamma'_\kappa(m)\right|_{\mathbb{H}^2}.\end{aligned}$$

Since $|F_u(\cup_l E_l(m))| < \sum_l |E_l(m)|_{\mathbb{H}^2}$ and $\left|\gamma'_\kappa(m)\right|_{\mathbb{H}^2} \to 0$, $\forall\kappa$ when $m \to \infty$, then $2\alpha(\mathcal{P}) < l(\mathcal{P})$.

Similarly, by the equation $F_u(\mathcal{P}(m)) = 0$ we have

$$F_u\left(\cup_j B_j(m)\right) = -F_u\left(\mathcal{P}(m) - \cup_j B_j(m)\right).$$

Using this equation, the inequality $2\beta < \ell(\mathcal{P})$ follows analogously as above. Therefore Theorem 2.1 is proved. □

Proof of Theorem 2.2. Let us assume that there exists a solution u to the Dirichlet problem. The proof is similar to Theorem 2.1 applied in the polygons $\mathcal{P}$ in $\mathcal{D}$ formed by arcs A_i, B_j, $C_k \subset \partial\mathcal{D}$ and arcs $E_l \subset (\mathcal{D} \cup \partial\mathcal{D})$.

To show that there exists a solution u to the Dirichlet Problem we suppose that the inequalities (3) are true for all polygons $\mathcal{P}$ in $\mathcal{D}$.

Claim 8.3. *There exists a solution u_n in $\mathcal{D}$ such that*

$$u_n = \begin{cases} n, & \text{on } A_i \\ -n, & \text{on } B_j \\ f_n, & \text{on } C_k, \end{cases}$$

where $f_n = \varphi \circ f$, $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) = \begin{cases} x, & -n \leq x \leq n \\ -n, & x < -n \\ n, & x > n. \end{cases}$$

We assume that the claim is true. We take the sequence of solutions $\{u_n\}$ in $\mathcal{D}$ defined by

$$u_n = \begin{cases} n, & \text{on } A_i \\ -n, & \text{on } B_j \\ f_n, & \text{on } C_k. \end{cases}$$

The convergence set of $\{u_n\}$ is $\mathcal{D}$, as in the proof of Theorem 2.1. So there is a subsequence which converges to a solution u on $\mathcal{D}$. Using barrier functions given by [17], we have that the limit of this subsequence on the boundary is the limit of the boundary values and also the limit solution extends continuously to the boundary, so we deduce that u takes the desired boundary values.

Now we prove Claim 8.3. We will proceed as in the proof of Claim 8.1.

Let $\{d_l\}$ be the vertices of domain $\mathcal{D}$. We can assume that each vertex d_l is in $\{(x, y) \in \mathbb{R}^2; y = 0\}$. For each l let $\sigma_l[m]$ be the geodesics which are semicircles centered at d_l of radius $\frac{1}{m}$. So, for m big enough, each $\sigma_l[m]$ divides the domain $\mathcal{D}$ in exactly two components (Δ_1^l e Δ_2^l), one of them, say Δ_1^l, having d_l in its asymptotic boundary. Now, let $\varrho_l[m]$ be the geodesic arc contained in $\sigma_l[m]$ joining the boundary points of $\mathcal{D}$.

We can find a solution with prescribed boundary values using the Existence Theorem in [17]. Let $A_i[m]$ be the compact arcs of A_i that lie outside Δ_1^l, $B_j[m]$ the compact arcs of B_j outside Δ_1^l and $C_k[m]$ the compact arcs in C_k outside Δ_1^l. So, there exists

$$u_n = \begin{cases} n & \text{on } A_i[m] \\ -n & \text{on } B_j[m] \\ f_n & \text{on } C_k[m] \\ 0 & \text{on } \varrho_l[m], \end{cases}$$

where $f_n = \varphi \circ f$, $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) = \begin{cases} x, & -n \leq x \leq n \\ -n, & x < -n \\ n, & x > n. \end{cases}$$

From now on, the proof is analogous to the one of Claim 8.1. Hence, Claim 8.3 is proved. □

Acknowledgments. The author would like to thank Harold Rosenberg for introducing her to the problem and for the many fruitful conversations. Also the author would like to thank the referee for making many useful suggestions.

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