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On formal Laurent series

Xiao-Xiong Gan and Dariusz Bugajewski

Abstract. Several kinds of formal Laurent series have been introduced with some restrictions so far. This paper systematically sets up a natural definition and structure of formal Laurent series without those restrictions, including introducing a multiplication between formal Laurent series. This paper also provides some results on the algebraic structure of the space of formal Laurent series, denoted by $\mathbb{L}$. By means of the results of the generalized composition of formal power series, we define a composition of a Laurent series with a formal power series and provide a necessary and sufficient condition for the existence of such compositions. The calculus about formal Laurent series is also introduced.

Keywords: analytic function, composition, $\mathbb{C}$-algebra, formal Laurent series, formal power series, integral domain.

Mathematical subject classification: Primary: 40A05; Secondary: 47B33, 40B05, 26E35.

1 Introduction

Let S be a ring and let $l \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. *A formal power series* on S is defined to be a mapping from $\mathbb{N}^l$ to S. A formal power series f in x from $\mathbb{N}$ to S is usually denoted by

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots, \text{ where } a_j \in S \text{ for every } j \in \mathbb{N} \cup \{0\}.$$

In this case a_k is called the kth coefficient of f, for every $k \in \mathbb{N} \cup \{0\}$. We denote the set of all such mappings by $\mathbb{X}(S)$. If $S = \mathbb{C}$, where $\mathbb{C}$ denotes the set of all complex numbers, that is the case considered in this paper, we simply write $\mathbb{X}(\mathbb{C}) = \mathbb{X}$. $\mathbb{X}$ is a commutative $\mathbb{C}$-algebra with 1 and $f \in \mathbb{X}(\mathbb{C})$ is a unit if $a_0 \neq 0$. If $a_0 = 0$, f is called a *nonunit* and we denote the set of all

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nonunits in $\mathbb{X}$ by $m(\mathbb{X})$. It is known that the algebra $\mathbb{X}$ is an integral domain. The structure of the algebra $\mathbb{X}$ has been studied and can be found in many books or papers on complex analysis (see e.g. [5], [8] or [11]).

Laurent series form a natural extension of power series, and therefore formal Laurent series are supposed to be a natural extension of formal power series too. Unfortunately the properties of formal Laurent series are quite different from those of formal power series if they are defined analogously, especially in view of the composition and multiplication.

The composition of formal power series or functional composition has attracted many mathematicians. More than forty years ago, Raney ([10]) investigated the functional composition patterns and provided a proof of the Lagrange inversion formula, that is, if S is a commutative ring with a unit e, then for each formal power series

$$g(x) = \sum_{n=0}^{\infty} b_n x^n \in \mathbb{X}(S) = S^{\mathbb{N}}$$

there exists exactly one nonunit $f \in m(\mathbb{X})$ such that $f(x) = x \cdot g \circ f(x)$. Chaumat and Chollet ([1]) contributed a lot in 2001 when they found that any formal power series A such that $A \circ F$ is analytic, where $F \in m(\mathbb{X})$, is analytic itself.

Gan and Knox ([4]) in 2002 provided a couple of necessary and sufficient conditions for the existence of the composition of formal power series that took away the restriction of nonunitness for the composed formal power series in some formal power series rings. Those results are very useful for establishing the formal Laurent series and the composition of formal Laurent series with formal power series. For convenience of the reader we recall some of those results and definitions below. As we indicated at the beginning of this section, $\mathbb{X} = \mathbb{X}(\mathbb{C})$ unless we define specifically.

Definition 1.1. *Let S be a ring with a metric and let $g \in \mathbb{X}(S)$, say $g(x) = \sum_{k=0}^{\infty} b_k x^k$. We define a subset $\mathbb{X}_g \subset \mathbb{X}(S)$ to be*

$$\mathbb{X}_g = \left\{ f \in \mathbb{X}(S) | f(x) = \sum_{k=0}^{\infty} a_k x^k, \sum_{n=0}^{\infty} b_n a_k^{(n)} \in S, \quad \text{for every } k \in \mathbb{N} \cup \{0\} \right\},$$

where $f^n(x) = \sum_{k=0}^{\infty} a_k^{(n)} x^k$, for all $n \in \mathbb{N}$ ($a_0^{(0)} = 1$, $a_k^{(0)} = 0$, for every $k \in \mathbb{N}$), is created by the product rule. Obviously $\mathbb{X}_g \neq \emptyset$, because $m(\mathbb{X}) \subset \mathbb{X}_g$.

Then the mapping $T_g\colon \mathbb{X}_g \longrightarrow \mathbb{X}(S)$ such that

$$T_g(f)(x) = \sum_{k=0}^{\infty} c_k x^k,$$

where $c_k = \sum_{n=0}^{\infty} b_n a_k^{(n)}$, for $k \in \mathbb{N} \cup \{0\}$, is well-defined. We call $T_g(f)$ the composition of g and f. $T_g(f)$ is also denoted by $g \circ f$ (that is $(g \circ f)(x) = \sum_{n=0}^{\infty} b_n (f(x))^n$).

Definition 1.2. *Let S be a ring and let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series in $\mathbb{X}(S)$. If $a_m \neq 0$ and $a_j = 0$ for all $j > m$, then the degree of f is defined to be the number m and it is denoted by* $\deg(f)$. *If there is no such a number m, then we say that* $\deg(f) = +\infty$.

Definition 1.3. *Let S be a ring and let $g \in \mathbb{X}(S)$, say $g(x) = \sum_{n=0}^{\infty} b_n x^n$. The derivative of g is defined to be the formal power series g' such that*

$$g'(x) = \sum_{n=1}^{\infty} n b_n x^{n-1}.$$

Theorem 1.4 ([4]). *Let S be a field with a metric and let $f, g \in \mathbb{X}(S)$ be given with the forms*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots, \qquad g(x) = b_0 + b_1 x + \cdots + b_n x^n \cdots,$$

and $\deg(f) \neq 0$. *Then the composition $g \circ f$ exists if and only if*

$$\sum_{n=k}^{\infty} \binom{n}{k} b_n a_0^{n-k} \in S \quad \textit{for every } k \in \mathbb{N} \cup \{0\}, \tag{1.1}$$

where

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Let us notice that the condition (1.1) in the Theorem 1.4 is equivalent to the kth derivative of g at a_0, $g^{(k)}(a_0) \in S$ for every $k \in \mathbb{N}$.

Theorem 1.5 ([4]). *Let $f, g \in \mathbb{X}(\mathbb{C})$ be given with the forms as in Theorem* 1.4. *If the series $\sum_{n=0}^{\infty} b_n R^n$ converges for some $R > |a_0|$, then $g \circ f$ exists.*

In his well-known monograph ([5]), Henrici introduced a formal Laurent series as

$$L = c_m z^m + c_{m+1} z^{m+1} + c_{m+2} z^{m+2} + \cdots, \quad c_m \neq 0, \tag{1.2}$$

over $\mathbb{C}$, where m is a fixed integer. In late 1970's, Jones and Thron ([7]) investigated such formal Laurent series. In particular, they examined the relationship between such a formal Laurent series L and a sequence of meromorphic functions $\{R_n(z)\}$.

Let $\beta\colon \mathbb{Z} \to (0, +\infty)$, where $\mathbb{Z}$ denotes the set of all integers, be a function such that $\beta(0) = 1$. The Banach space $L^p(\beta)$, $1 < p < +\infty$, of all formal Laurent series $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ (cf. Definition 2.1) for which $\|f\|_\beta^p = \sum_{n\in\mathbb{Z}} |a_n|^p \beta(n)^p < +\infty$ has been studied recently and some results about it can be found in [6], [14] and [15]. The formal Laurent series discussed in those papers have their terms running through all integers, differentiating it from and extending (1.2); but they rely on the sequence $\{\beta(n)\}_{n\in\mathbb{Z}}$.

In this paper we are going to set up a space of formal Laurent series analogously to the space of formal power series including its basic operations such as addition, scalar multiplication, product and composition. Moreover, we are going to investigate the calculus of formal Laurent series and provide some results for it.

2 Formal Laurent series and its algebraic structure

Let us begin this section with the definition of a formal Laurent series.

Definition 2.1. *A formal Laurent series on $\mathbb{C}$ is defined to be a mapping from $\mathbb{Z}$ to $\mathbb{C}$. A formal Laurent series g in z from $\mathbb{Z}$ to $\mathbb{C}$ is usually denoted by*

$$\cdots + b_{-n} z^{-n} + \cdots + b_{-1} z^{-1} + b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots \tag{2.1}$$

or just $\sum_{n\in\mathbb{Z}} b_n z^n$, where $b_n \in \mathbb{C}$ for every $n \in \mathbb{Z}$. The zero formal Laurent series is defined to be the series with $b_n = 0$ for every $n \in \mathbb{Z}$ and the identity I is defined to be the series with $b_0 = 1$ and $b_n = 0$ for all other $n \in \mathbb{Z}$.

The series $\sum_{n=0}^{\infty} b_n z^n$ and $\sum_{n=1}^{\infty} b_{-n} z^{-n}$ are called the regular and the principal part of g, and they are denoted by g^+ and g^-, respectively.

The complex conjugate of a Laurent series $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ is defined to be the series $\overline{g}(z) = \sum_{n\in\mathbb{Z}} \overline{b}_n z^n$ where $\overline{b}_n$ is the complex conjugate of b_n, $n \in \mathbb{Z}$. We define the reverse of the formal Laurent series g by

$$\breve{g}(z) = g\left(\frac{1}{z}\right) = \sum_{n\in\mathbb{Z}} \breve{b}_n z^n, \tag{2.2}$$

where $\breve{b}_n = b_{-n}$ *for every* $n \in \mathbb{Z}$. *Finally, we denote by* $\mathbb{L}(\mathbb{C})$, *or simply by* $\mathbb{L}$ *the set of all formal Laurent series over* $\mathbb{C}$ *and define the formal Laurent series addition and scalar multiplication by*

(L-1) $(g + f)(z) = \sum_{n\in\mathbb{Z}}(b_n + a_n)z^n$,

(L-2) $(cg)(z) = \sum_{n\in\mathbb{Z}} cb_n z^n$,

where $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$, $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ *are any two formal Laurent series over* $\mathbb{C}$ *and* $c \in \mathbb{C}$.

It is clear that $\mathbb{L}$ is a linear vector space over $\mathbb{C}$. It is not clear so far whether $\mathbb{L}$ is generally also an algebra. Can we endow a multiplication to formal Laurent series as is done for formal power series?

For the formal Laurent series introduced in [5] or [7], or in (1.2) of this paper, the answer is yes. For the formal Laurent series introduced in Definition 2.1, "*yes*" is not the answer. For example, if $g(z) = \sum_{n=1}^{\infty} z^{-n}$ and $f(z) = \sum_{n=1}^{\infty} z^n$, we can not define any coefficient of $g(z)f(z)$ by the usual multiplication induced from the distribution. However, the product of certain *nonformal* Laurent series, Henrici called them in his monograph ([5], p. 219), or the product of Laurent expansions of the analytic functions g and f on an annulus is well defined. That is, if $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ are Laurent expansions of two analytic functions on the annulus

$$A_{r,s} = \left\{z \in \mathbb{C} \colon r < |z| < s\right\},$$

then $(fg)(z) = \sum_{n\in\mathbb{Z}} c_n z^n$ and

$$c_n = \sum_{m\in\mathbb{Z}} b_m a_{n-m} \quad \text{for every } n \in \mathbb{Z}, \tag{2.3}$$

where each c_n, the coefficient of the Laurent expansion of gf is calculated by

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(t)g(t)}{t^{n+1}}\, dt$$

where $r < \rho < s$ and Γ_ρ is the circle $t = \rho e^{i\theta}$, $0 \le \theta \le 2\pi$.

This result about the product of so-called nonformal Laurent expansions of analytic functions, such as (2.3), indicates a possible product of formal Laurent series defined in Definition 2.1, although we clearly can not introduce the product of formal Laurent series in that way. We simply may not be able to apply the Cauchy integral formula because our formal Laurent series are not necessary the Laurent expansion of analytic functions on some annulus.

Let us begin our consideration with the following example.

Example 2.2. Let

$$g(z) = \sum_{n\in\mathbb{Z}} z^n \quad \text{and} \quad f(z) = \sum_{n\in\mathbb{Z}} 2^{-|n|} z^n.$$

Then g is not analytic on any nonempty annulus and hence we can not apply the product formula (2.3) of nonformal Laurent series to this g. However, it appears that the product gf is a formal Laurent series under the general definition of the product (it will appear in Example 2.14 later).

Now we introduce the *dot product* of formal Laurent series.

Definition 2.3. *Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ be two formal Laurent series over $\mathbb{C}$. We define the dot product of g and f as*

$$g \cdot f = \sum_{n\in\mathbb{Z}} b_n a_n,$$

if $\sum_{n\in\mathbb{Z}} b_n a_n \in \mathbb{C}$; otherwise we say that the dot product $g \cdot f$ does not exist. For our convenience, we also define the p-dot product of g and f as

$$(g \cdot f)^p = \sum_{n\in\mathbb{Z}} b_n^p a_n^p, \quad 0 < p < +\infty,$$

if the series on the right side is convergent.

It is obvious that the dot product is commutative, if it exists. Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n \in \mathbb{L}$ be given. We denote by $DP(g)$ the set of all $f \in \mathbb{L}$ such that $f \cdot g \in \mathbb{C}$. Obviously $DP(g) \neq \emptyset$ for every $g \in \mathbb{L}$ because the zero and the identity formal Laurent series are in the $DP(g)$. Every polynomial is in $DP(g)$, too. We provide a nontrivial example below.

Example 2.4. Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ be given. We define

$$a_n = 2^{-|n|} \frac{1}{|b_n| + |b_{-n}| + 1} \quad \text{for every } n \in \mathbb{Z}.$$

Then $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in DP(g)$ because

$$\sum_{n\in\mathbb{Z}} |a_n b_n| \le \sum_{n\in\mathbb{Z}} 2^{-|n|}.$$

Now, we are going to introduce the so-called *shifting mapping*.

Definition 2.5. *Let $k \in \mathbb{Z}$ be given. The kth-shifting mapping $S_k : \mathbb{L} \longrightarrow \mathbb{L}$ is defined by*

$$S_k(f)(z) = \sum_{n\in\mathbb{Z}} a_{n-k} z^n \quad \textit{for every} \;\; f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathbb{L}.$$

Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n \in \mathbb{L}$ be given. We define the reverse-shifting set of g as

$$\mathbb{L}(g) = \left\{ f \in \mathbb{L} | S_k(\breve{f}) \in DP(g) \quad \textit{for every} \;\; k \in \mathbb{Z} \right\}.$$

Let us mention that the multiplication operator M_z considered in ([6] and [14]) on some particular formal Laurent series can be understood as some particular kth-shifting mapping $S_k : \mathbb{L} \longrightarrow \mathbb{L}$ because

$$S_1(f)(z) = \sum_{n\in\mathbb{Z}} a_{n-1} z^n = \sum_{n\in\mathbb{Z}} a_n z^{n+1} = zf(z) = (M_z f)(z)$$

for every $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathbb{L}$. Similarly, $S_k(f)(z) = z^k f(z)$ for every $k \in \mathbb{Z}$.

By Definition 2.5, $f \in \mathbb{L}(g)$ if $\sum_{m\in\mathbb{Z}} b_m a_{k-m} \in \mathbb{C}$ for every $k \in \mathbb{Z}$, because it is equivalent to $g \cdot S_k(\breve{f}) \in \mathbb{C}$, for every $k \in \mathbb{Z}$. One checks that $g \cdot S_k(\breve{f})$ is just the c_k in (2.3), the kth coefficient of the product of so-called nonformal Laurent series which require that both f and g are analytic on an annulus. We introduce the following proposition.

Proposition 2.6. *Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ be two formal Laurent series on $\mathbb{C}$. Then $f \in \mathbb{L}(g)$ if and only if $g \in \mathbb{L}(f)$.*

Proof. In fact,

$$g \cdot S_k(\breve{f}) = \sum_{m\in\mathbb{Z}} b_m a_{k-m} = \sum_{n\in\mathbb{Z}} b_{k-n} a_n = f \cdot S_k(\breve{g}),$$

for every $k \in \mathbb{Z}$, what completes the proof. □

Now we are going to define the multiplication of two formal Laurent series.

Definition 2.7. *Let*

$$g(z) = \sum_{n\in\mathbb{Z}} b_n z^n \quad \textit{and} \quad f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

be two formal Laurent series on $\mathbb{C}$. We have

$$b_m z^m f(z) = \sum_{n\in\mathbb{Z}} b_m a_n z^{n+m} = \sum_{k\in\mathbb{Z}} b_m a_{k-m} z^k \quad \textit{for every } m \in \mathbb{Z}.$$

We define the product of f and g, denoted by fg, to be

(L-3) $fg(z) = \sum_{k\in\mathbb{Z}} d_k z^k$, *if* $d_k = \sum_{m\in\mathbb{Z}} b_m a_{k-m} \in \mathbb{C}$ *for every* $k \in \mathbb{Z}$.

One can notice that $d_k \in \mathbb{C}$ for every $k \in \mathbb{Z}$ if $f \in \mathbb{L}(g)$. Actually d_k is the series formed by all coefficients of k-th term determined by $g(z)f(z)$ through the distributive law and $\mathbb{L}(g)$ is the set of all $f \in \mathbb{L}$ such that $fg \in \mathbb{L}$. By Proposition 2.6, $fg = gf$ for every $f \in \mathbb{L}(g)$, and therefore the product of formal Laurent series is commutative if it exists.

Remark 1. Let f and g be as those in Definition 2.1. If $b_{-n} = 0 = a_{-n}$ for every $n \in \mathbb{N}$, then the formal Laurent series g and f become the formal power series and each d_k is reduced to be the coefficient of the Cauchy product of these two formal power series. One can also check that $g(z)f(z) = f(z)$ if $g = I$.

Remark 2. If g and f are analytic on an annulus, then the coefficients of the Laurent expansion of fg have the same forms as in (L-3) (see [5], p. 219).

The following two examples show that the existence of the dot product of two formal Laurent series does not have relationship with the existence of the product defined in (L-3). It is an open question whether these two conditions are related.

Example 2.8. Let $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ and $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ such that

$a_{-n} = n$, $b_{-n} = n^{-3}$ and $a_n = 1$, $b_n = n^{-2}$ for every $n \in \mathbb{N}$, $a_0 = b_0 = 1$.

Then $f \cdot g = 1 + \sum_{n=1}^{\infty} \frac{n}{n^3} + \sum_{n=1}^{\infty} \frac{1}{n^2} \in \mathbb{R}$, so $f \in DP(g)$. However,

$$d_0 = \sum_{m\in\mathbb{Z}} b_m a_{0-m} = 1 + \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} n^{-3}$$

does not exists, so $f \notin \mathbb{L}(g)$.

Example 2.9. Let $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ and $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ such that

$a_{-n} = n$, $b_{-n} = n^{-2}$ and $a_n = 1$, $b_n = n^{-3}$, for every $n \in \mathbb{N}$, $a_0 = b_0 = 1$.

Then

$$\sum_{m=-|k|-1}^{-\infty} b_m a_{k-m} = \sum_{m=-|k|-1}^{-\infty} m^{-2} \text{ and } \sum_{m=|k|+1}^{\infty} b_m a_{k-m} = \sum_{m=|k|+1}^{\infty} m^{-3}(m-k),$$

and hence $d_k = \sum_{m\in\mathbb{Z}} b_m a_{k-m} \in \mathbb{R}$ for every $k \in \mathbb{Z}$ or $f \in \mathbb{L}(g)$. However

$$f \cdot g = 1 + \sum_{n=1}^{\infty} a_{-n} b_{-n} + \sum_{n=1}^{\infty} a_n b_n = 1 + \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n^3}$$

does not exist, so $f \notin DP(g)$.

Proposition 2.10. *Let $f, g \in \mathbb{L}$. Then*

(i) $\mathbb{L}(g) \neq \emptyset$;

(ii) $f \in \mathbb{L}(g) \Longrightarrow \alpha f \in \mathbb{L}(g)$ *for every* $\alpha \in \mathbb{C}$;

(iii) $f, h \in \mathbb{L}(g) \Longrightarrow f + h \in \mathbb{L}(g)$.

Proof. If we denote by $E(P)$ the set of all polynomials over $\mathbb{C}$, then $E(P) \subset \mathbb{L}(g)$, what proves (i). Items (ii) and (iii) are also obvious. □

Now, let us provide a nontrivial example of the product of formal Laurent series.

Example 2.11. Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ be given. Let us define

$$a_n = 2^{-|n|} \frac{1}{1 + \sum_{i=-2|n|}^{2|n|} |b_i|} \quad \text{for every } n \in \mathbb{Z},$$

and let $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. We show that $S_k(\check{f}) \in DP(g)$ for every $k \in \mathbb{Z}$.

Let $k \in \mathbb{Z}$ be given. We have

$$g \cdot S_k(\check{f}) = \sum_{m\in\mathbb{Z}} b_m a_{k-m} = \sum_{m\in\mathbb{Z}} b_m 2^{-|k-m|} \frac{1}{1 + \sum_{i=-2|k-m|}^{2|k-m|} |b_i|}.$$

If $m > |2k|$, we have $0 < m < 2(m-k) = 2|k-m|$. Then

$$|b_m| 2^{-|k-m|} \frac{1}{1 + \sum_{i=-2|k-m|}^{2|k-m|} |b_i|} < 2^{-|k-m|},$$

so

$$\sum_{m=|2k|}^{\infty} |b_m a_{k-m}| < \sum_{m=|2k|}^{\infty} 2^{-|k-m|} < +\infty.$$

If $m < -|2k|$, we have $m < 0$ and $m + |2k| < 0$, so

$$-2|k-m| = -2(k-m) = m + m - 2k < m + m + 2|k| < m.$$

Hence

$$|b_m| 2^{-|k-m|} \frac{1}{1 + \sum_{i=-2|k-m|}^{2|k-m|} |b_i|} < 2^{-|k-m|},$$

and then

$$\sum_{m=-|2k|}^{-\infty} |b_m a_{k-m}| < \sum_{m=-|2k|}^{-\infty} 2^{-|k-m|} < +\infty.$$

Therefore

$$\sum_{m\in\mathbb{Z}} |b_m a_{k-m}| < +\infty.$$

Hence $g \cdot S_k(\check{f}) \in \mathbb{C}$, so $S_k(\check{f}) \in DP(g)$ for every $k \in \mathbb{Z}$. Thus $f \in \mathbb{L}(g)$.

Banach spaces l^p for $p \geq 1$ play a very important role in functional analysis. In this paper, by l^p for $p \geq 1$ we will understand the set of all sequences $(b_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$ such that $\sum_{n\in\mathbb{Z}} |b_n|^p < +\infty$ ($\sum_{n\in\mathbb{Z}} |b_n|^p$ is said to be convergent if and only if both $\sum_{n=0}^{\infty} |b_n|^p$ and $\sum_{n=1}^{\infty} |b_{-n}|^p$ are convergent). Obviously the Hölder Inequality is true after this extension. By l^∞ we will denote the collection of all bounded sequences $(b_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$.

Let $1 \leq p \leq +\infty$ and let us define the mapping $A: l^p \longrightarrow \mathbb{L}$ by $A((b_n)) = \sum_{n\in\mathbb{Z}} b_n z^n = g(z)$ for every $(b_n)_{n\in\mathbb{Z}} \in l^p$. It is clear that A is an isomorphism onto its image with the operations (L-1) and (L-2), and therefore we may consider l^p as embedded into $\mathbb{L}$ by A. Let us define $\mathbb{L}_p$ as

$$\mathbb{L}_p = \left\{ f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathbb{L} \colon (a_n)_{n\in\mathbb{Z}} \in l^p \right\}.$$

As usual, we define $\|\cdot\|_p$ as

$$\|f\|_p = \left(\sum_{n\in\mathbb{Z}} |a_n|^p \right)^{1/p},$$

if $f \in \mathbb{L}_p$ for $1 \leq p < +\infty$, and $\|f\|_\infty = \sup\{|a_n| \colon n \in \mathbb{Z}\}$, if $f \in \mathbb{L}_\infty$.

Basing on the above considerations, we prove the following theorem.

Theorem 2.12. *Let* $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n \in \mathbb{L}_p$, *where* $1 \leq p < +\infty$. *Then*

(i) $\mathbb{L}_q \subset \mathbb{L}(g)$ *if* $p > 1$ *and* $1/p + 1/q = 1$;

(ii) $\mathbb{L}_\infty \subset \mathbb{L}(g)$ *if* $p = 1$;

(iii) $\mathbb{L}_1 \subset \mathbb{L}(g)$ *if* $p = +\infty$;

(iv) *if* $\phi \in \mathbb{L}_p$ *and* $\psi \in \mathbb{L}_q$ *where* $1/p + 1/q = 1$, *then* $\phi\psi \in \mathbb{L}(g)$ *for every* $g \in \mathbb{L}_1$.

Proof. Suppose that $1 < p < +\infty$. Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence in l^q and let us write $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. Then

$$\left(\sum_{m\in\mathbb{Z}} |a_m|^q\right)^{1/q} < +\infty \quad \text{and} \quad \left(\sum_{m\in\mathbb{Z}} |b_m|^p\right)^{1/p} < +\infty.$$

Applying Hölder's Inequality we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |b_m a_{k-m}| &\leq \left(\sum_{m\in\mathbb{Z}} |b_m|^p\right)^{1/p} \left(\sum_{m\in\mathbb{Z}} |a_{k-m}|^q\right)^{1/q} \\ &= \left(\sum_{m\in\mathbb{Z}} |b_m|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} |a_n|^q\right)^{1/q}\end{aligned}$$

which proves (i).

Now suppose that $p = 1$. Let $(a_n) \in l^\infty$ be given and let $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. Then $|a_n| < M$ for every $n \in \mathbb{Z}$ for some $M > 0$ so we have

$$\sum_{m\in\mathbb{Z}} |b_m a_{k-m}| \leq \sum_{m\in\mathbb{Z}} M|b_m| = M\left(\sum_{m\in\mathbb{Z}} |b_m|\right) < +\infty$$

for every $k \in \mathbb{Z}$, and then $f \in \mathbb{L}(g)$. This proves (ii).

The proof of (iii) is similar to (ii).

Now, let $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathbb{L}_p$ and $\psi(z) = \sum_{n\in\mathbb{Z}} c_n z^n \in \mathbb{L}_q$. We show that $\phi\psi \in \mathbb{L}(g)$, where $g \in \mathbb{L}_1$.

By (i) and (ii), it is clear that $\phi\psi \in \mathbb{L}$. Put $\phi\psi(z) = \sum_{k\in\mathbb{Z}} d_k z^k$, where $d_k = \sum_{m\in\mathbb{Z}} a_m c_{k-m}$ for every $k \in \mathbb{Z}$. Applying Hölder's Inequality we get

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |d_k b_{n-k}| &= \sum_{k\in\mathbb{Z}} |\sum_{m\in\mathbb{Z}} a_m c_{k-m}| \cdot |b_{n-k}| \\
&\le \sum_{k\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} |a_m c_{k-m}| \right) \cdot |b_{n-k}| \\
&\le \sum_{k\in\mathbb{Z}} \left[\left(\sum_{m\in\mathbb{Z}} |a_m|^p \right)^{1/p} \left(\sum_{m\in\mathbb{Z}} |c_{k-m}|^q \right)^{1/q} \right] \cdot |b_{n-k}| \\
&= \sum_{k\in\mathbb{Z}} \left(\|\phi\|_p \|\psi\|_q \right) \cdot |b_{n-k}| \\
&= \|\phi\|_p \|\psi\|_q \|g\|_1 < +\infty.
\end{aligned}$$

It means that $\phi\psi \in \mathbb{L}(g)$. □

Theorem 2.13. *$\mathbb{L}_1$ is a linear algebra. In particular, $f^k \in \mathbb{L}_1$ for every $f \in \mathbb{L}_1$ and $k \in \mathbb{N}$.*

Proof. It is obvious that $\mathbb{L}_1$ is a linear space over $\mathbb{C}$. We show that $\mathbb{L}_1$ is closed in view of the multiplication of formal Laurent series. Let $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ and $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ be any two formal Laurent series in $\mathbb{L}_1$. By Theorem 2.12 (ii), $fg \in \mathbb{L}$. We show that $fg \in \mathbb{L}_1$.

Let $fg(z) = \sum_{n\in\mathbb{Z}} d_k z^k$, where $d_k = \sum_{m\in\mathbb{Z}} a_m b_{k-m}$ for every $k \in \mathbb{Z}$. Then

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |d_k| &= \sum_{k\in\mathbb{Z}} |\sum_{m\in\mathbb{Z}} a_m b_{k-m}| \\
&\le \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |a_m b_{k-m}| \\
&= \sum_{m\in\mathbb{Z}} |a_m| \sum_{k\in\mathbb{Z}} |b_{k-m}| \\
&= \|f\|_1 \|g\|_1 < +\infty.
\end{aligned}$$

Thus, $fg \in \mathbb{L}_1$ and hence $\mathbb{L}_1$ is a linear algebra. Taking $f = g$ and repeating the above process k-times, we get

$$f^k \in \mathbb{L}_1 \quad \text{for every} \quad f \in \mathbb{L}_1 \quad \text{and} \quad k \in \mathbb{N}. \qquad \square$$

Remark 3. Obviously every sequence $(b_n)_{n\in\mathbb{Z}}$ with $\lim_{n\to\pm\infty} b_n \in \mathbb{C}$ must be bounded and thus $g = \sum_{n\in\mathbb{Z}} b_n z^n \in \mathbb{L}_\infty$. We conclude by Theorem 2.12 (iii) that $\mathbb{L}_1 \subset \mathbb{L}(g)$, which provides a wide range of formal Laurent series that have well defined products. If we notice that many sequences converge in some spaces but their corresponding formal Laurent series converge nowhere such as $g(z) = \sum_{n\in\mathbb{Z}} z^n$ or $f(z) = \sum_{n\in\mathbb{Z}} \frac{1}{n} z^n$, we may realize how far away we have been from the analyticity.

We met a difficulty in Example 2.2, where the formal Laurent series g is nowhere analytic and therefore the product of g and any Laurent series, formal or nonformal, did not yet exist. However, with the definition (L-3), we have the following particular example.

Example 2.14. Let

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n = \sum_{n\in\mathbb{Z}} 2^{-|n|} z^n \quad \text{and} \quad g(z) = \sum_{n\in\mathbb{Z}} b_n z^n = \sum_{n\in\mathbb{Z}} z^n$$

(that is, $a_n = 2^{-|n|}$ and $b_n = 1$ for every $n \in \mathbb{Z}$). The formal Laurent series g is not analytic on any nonempty annulus. However, $gf \in \mathbb{L}$ by Theorem 2.12 (iii), because $g \in \mathbb{L}_\infty$ and $f \in \mathbb{L}_1$. If we write $gf(z) = \sum_{k\in\mathbb{Z}} d_k z^k$, we have

$$\begin{aligned} d_k &= \sum_{m\in\mathbb{Z}} a_m b_{k-m} = \sum_{-\infty}^{m=-1} a_m b_{k-m} + \sum_{m=0}^{\infty} a_m b_{k-m} \\ &= \sum_{m=1}^{\infty} 2^{-m} + \sum_{m=0}^{\infty} 2^{-m} = 1 + 2 = 3. \end{aligned}$$

Then we have

$$g(z)f(z) = \sum_{k\in\mathbb{Z}} d_k z^k = \sum_{n\in\mathbb{Z}} 3z^n = 3g(z).$$

Proposition 2.15. *For any $f, g \in \mathbb{L}$,*

(i) $f \in \mathbb{L}(g)$ *if and only if* $\check{f} \in \mathbb{L}(\check{g})$;

(ii) $f \in \mathbb{L}(\check{g})$ *if and only if* $\check{f} \in \mathbb{L}(g)$;

(iii) $f \in \mathbb{L}(\overline{g})$ *if and only if* $\overline{f} \in \mathbb{L}(g)$.

Proof. Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ and $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. By (L-3), $\breve{f} \in \mathbb{L}(\breve{g})$ if and only if

$$\sum_{m\in\mathbb{Z}} \breve{b}_m \breve{a}_{k-m} = \sum_{m\in\mathbb{Z}} b_{-m} a_{-(k-m)} = \sum_{m\in\mathbb{Z}} b_m a_{-k-m} \in \mathbb{C} \quad \text{for every } k \in \mathbb{Z}.$$

This is equivalent to $\sum_{m\in\mathbb{Z}} a_m b_{j-m} \in \mathbb{C}$ for every $j \in \mathbb{Z}$, or $f \in \mathbb{L}(g)$. This proves (i).

Since $\breve{\breve{f}} = f$ for all $f \in \mathbb{L}$, (ii) is a consequence of (i).

Similarly we can prove (iii). □

It is well known that the space of the formal power series over $\mathbb{C}$ is an integral domain. We have already seen in the Example 2.14 that $(f(z) - 3)g(z) = 0$ but neither $(f - 3)$ nor g is the zero formal Laurent series. The next proposition will provide more details about $\mathbb{L}(g)$, if we pursue an integral domain.

Proposition 2.16. *Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ be a formal Laurent series. Then*

(i) *$zg(z) = g(z)$ if and only if $b_n = b$ for every $n \in \mathbb{Z}$ for some $b \in \mathbb{C}$;*

(ii) *if $P_n(z) = \sum_{k=0}^{n} p_k z^k$ is a polynomial of degree n and $g = \sum_{n\in\mathbb{Z}} bz^n$, $b \in \mathbb{C}$, then*

$$\left(P_n(z) - \sum_{k=0}^{n} p_k\right) g(z) = 0;$$

(iii) *if $g(z) = \sum_{n\in\mathbb{Z}} bz^n$ and the series $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ is such that $\sum_{n\in\mathbb{Z}} a_n = A \in \mathbb{C}$ converges absolutely, then*

$$(f(z) - A)g(z) = 0.$$

Proof. Suppose that $zg(z) = g(z)$, it follows that $b_n = b_{n-1}$ for every $n \in \mathbb{Z}$. Let us write $b = b_0$, we have $b = b_n$ for every $n \in \mathbb{Z}$.

Conversely, suppose that $g(z) = \sum_{n\in\mathbb{Z}} bz^n$. Then

$$zg(z) = \sum_{n\in\mathbb{Z}} bz^{n+1} = \sum_{n\in\mathbb{Z}} bz^n = g(z).$$

This proves (i).

Now by (i), we can easily obtain that $z^k g(z) = g(z)$ for every $k \in \mathbb{N}$ and for g defined in (ii). Then $(p_k z^k - p_k)g(z) = 0$ for $0 \le k \le n$, and we get

$$\left(P_n(z) - \sum_{k=0}^{n} p_k\right) g(z) = 0.$$

Finally, $fg \in \mathbb{L}$ by Theorem 2.12 (iii). Writing $fg(z) = \sum_{k\in\mathbb{Z}} d_k z^k$, we have

$$d_k = \sum_{n\in\mathbb{Z}} b_n a_{k-n} = \sum_{n\in\mathbb{Z}} b a_{k-n} = bA.$$

Hence $f(z)g(z) = \sum_{k\in\mathbb{Z}} d_k z^k = \sum_{k\in\mathbb{Z}} bAz^k = A\sum_{n\in\mathbb{Z}} bz^n = Ag(z)$. □

Proposition 2.17. *Let* $g = \sum_{n\in\mathbb{Z}} bz^n$, $b \in \mathbb{C}\setminus\{0\}$ *and let* $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathbb{L}(g)$. *Then*

$$f(z)g(z) = 0 \text{ if and only if } \sum_{n\in\mathbb{Z}} a_n = 0.$$

Proof. It is clear that $f(z)g(z) = 0$ if and only if $\sum_{m\in\mathbb{Z}} b_m a_{n-m} = 0$ for every $n \in \mathbb{Z}$. Then $\sum_{m\in\mathbb{Z}} b_m a_{n-m} = 0$ is equivalent to $\sum_{m\in\mathbb{Z}} ba_{n-m} = 0$ or $\sum_{n\in\mathbb{Z}} a_n = 0$. □

Remark 4. If $f \in \mathbb{L}_1$, then $f \in \mathbb{L}(f)$ by Theorem 2.13. Since $\mathbb{L}_\infty \subset \mathbb{L}(f)$ by Theorem 2.12 (ii), the above two propositions show that $\mathbb{L}(f)$ is not an integral domain for every $f \in \mathbb{L}_1$. What kind of $\mathbb{L}(g)$ can be an integral domain? Is there any $\mathbb{L}(g)$ that is an integral domain? Those problems are still open.

3 Composition of a formal Laurent series with a formal power series

Composition of formal power series is a very interesting issue which also distinguishes formal power series from the regular sequences. Some applications of the composition have been developed rapidly in many fields including the differential equations recently (see [2], [3], [9], [12]). Some kinds of composition of formal Laurent series with formal power series were discussed in [5], where the formal Laurent series has a finite principal part and the composed formal power series must be nonunit.

Except for the analytic functions on an annulus, the composition of formal Laurent series has not been established so far.

We have seen that the multiplication for formal Laurent series is not a simple issue, Henrich had similar comments in his book ([5], p. 219), too. However, the possible composition involving formal Laurent series could be more complicated. For example, composition of formal Laurent series must face the existence of the inverse or reciprocal of the formal Laurent series and the uniqueness of such existence. Such an existence and uniqueness are guaranteed in the space of formal power series, as we discussed in Section 1. Do we have similar results in $\mathbb{L}$? First let us consider the following examples.

Example 3.1. Let $f(z) = \sum_{n=1}^{\infty} z^n$ and $g(z) = \sum_{n\in\mathbb{Z}} z^n$. We know that $g^+(f(z))$ is a formal power series but we do not know what $g^-(f(z))$ is.

Example 3.2. Let $f(z) = 1 - z$ and $g(z) = \sum_{n\in\mathbb{Z}} z^n$. Obviously $(1-z)^k$ is a polynomial for all $k \in \mathbb{N}$ and $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ is a well-defined formal power series. We can easily find out that

$$g^+(f(z)) = \sum_{n=0}^{\infty} (f(z))^n$$

is not defined.

Example 3.1 shows that we must have the inverse of some kind of formal Laurent series before we establish a composition of Laurent series. On the other hand, Example 3.2 shows that we may not have the composition even if the composed formal power series is a unit or a polynomial. These two examples tell us that generally it is impossible to have a composition of a formal Laurent series with a nonunit and also may not be possible to have a composition of a formal Laurent series with a unit. This is a significant difference contrasting with the composition of formal power series.

By means of the generalized composition established in [4], however, we are able to establish the composition of a formal Laurent series with a formal power series. The existence of the composition of formal Laurent series with formal Laurent series is open.

In next definition, we keep the notations $\mathbb{X}_g$ and T_g defined in Definition 1.1 and extend the composition to formal Laurent series g.

Definition 3.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a unit formal power series on $\mathbb{C}$. If $n \in \mathbb{N} \cup \{0\}$, we define $f^n(z) = \sum_{k=0}^{\infty} a_k^{(n)} z^k$ as we did in Definition 1.1. Write $f^{-1}(z) = \sum_{k=0}^{\infty} a_k^{(-1)} z^k$ and then write*

$$f^{-n}(z) = \left[f^{-1}(z)\right]^n = \sum_{k=0}^{\infty} a_k^{(-n)} z^k \quad \text{for every } n \in \mathbb{N},$$

where f^{-1} is the inverse formal power series of f such that the Cauchy product $f^{-1}(z) f(z) = 1$. Since f is a unit, f^{-n} is well-defined for all $n \in \mathbb{N}$.

Similarly as we did in Definition 1.1, let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ be a formal Laurent series and extend $\mathbb{X}_g$ to be

$$\mathbb{X}_g = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{X}(\mathbb{C})\colon \sum_{n\in\mathbb{Z}} b_n a_k^{(n)} \in \mathbb{C} \quad \text{for every } k \in \mathbb{N} \cup \{0\} \right\}.$$

If $\mathbb{X}_g \neq \emptyset$, *for example if g is a formal power series or g is a Laurent expansion of an analytic functions, we define* $T_g \colon \mathbb{X}_g \longrightarrow \mathbb{X}$ *by*

$$T_g(f)(z) = \sum_{k=0}^{\infty} c_k z^k$$

where $c_k = \sum_{n\in\mathbb{Z}} b_n a_k^{(n)} \in \mathbb{C}$ *for every* $k \in \mathbb{N} \cup \{0\}$. $T_g(f)$ *is called the composition of the formal Laurent series g with the unit formal power series f on* $\mathbb{C}$, *or the composition of g with f.* $T_g(f)$ *is also denoted by* $g \circ f$ *(that is* $(g \circ f)(z) = \sum_{k\in\mathbb{Z}} b_k[f(z)]^k$*).*

Now, we prove the following theorem.

Theorem 3.4. *Let* $g(z) = \sum_{k\in\mathbb{Z}} b_k z^k$ *be a Laurent series over* $\mathbb{C}$, *and let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be a formal power series on* $\mathbb{C}$ *such that* $a_0 \neq 0$. *Then* $g(f(z))$ *is a formal power series, that is,* $g(f(z)) = \sum_{n=0}^{\infty} c_n z^n$ *with* $c_n \in \mathbb{C}$ *for every* $n \in \mathbb{N} \cup \{0\}$, *if and only if both*

$$\sum_{n=k}^{\infty} \binom{n}{k} b_{-n} a_0^{k-n} \in \mathbb{C} \quad \text{and} \quad \sum_{n=k}^{\infty} \binom{n}{k} b_n a_0^{n-k} \in \mathbb{C}$$

are true for every $k \in \mathbb{N} \cup \{0\}$.

Proof. Write $g = g^- + g^+$, where g^- is the principal part of g and g^+ is the regular part of g. By Theorem 1.4, $g^+ \circ f$ exists if and only if

$$\sum_{n=k}^{\infty} \binom{n}{k} b_n a_0^{n-k} \in \mathbb{C}$$

for every $k \in \mathbb{N} \cup \{0\}$.

Since $a_0 \neq 0$, it follows that $f^{-1}(z) = \frac{1}{f(z)}$ is a well-defined formal power series. If we write

$$f^{-1}(z) = \sum_{n=0}^{\infty} a_n^{(-1)} z^n,$$

then

$$a_0^{(-1)} = \frac{1}{a_0},\ a_1^{(-1)} = -\frac{a_1}{a_0^2},\ a_2^{(-1)} = \frac{a_1^2 - a_0 a_2}{a_0^3}, \cdots .$$

Then $g^-(f(z)) = \sum_{n=1}^{\infty} b_{-n}(f(z))^{-n} = \sum_{n=1}^{\infty} b_{-n}\big(f^{-1}(z)\big)^n$.

By Theorem 1.4, $g^-(f(z))$ exists if and only if

$$\sum_{n=k}^{\infty} \binom{n}{k} b_{-n} \left(a_0^{(-1)}\right)^{n-k} \in \mathbb{C} \quad \text{for every } \ k \in \mathbb{N} \cup \{0\}$$

which is equivalent to

$$\sum_{n=k}^{\infty} \binom{n}{k} b_{-n} a_0^{k-n} \in \mathbb{C} \quad \text{for every } \ k \in \mathbb{N} \cup \{0\}.$$ □

Example 3.5. Let $g(z) = \sum_{n=1}^{\infty} (2z)^{-n} + \sum_{n=0}^{\infty} z^n$ and $f(z) = \frac{3}{4} + \sum_{n=1}^{\infty} n!z^n$. Then

$$g^-(z) = \sum_{n=1}^{\infty} (2z)^{-n} \quad \text{and} \quad g^+(z) = \sum_{n=0}^{\infty} z^n.$$

It is clear that f diverges everywhere except $z = 0$ and then $g \circ f$ does not make any sense in the classical complex analysis. However

$$(g^- \circ f)(z) \in \mathbb{L} \quad \text{and} \quad (g^+ \circ f)(z) \in \mathbb{L}$$

by Theorem 1.5 because g^- converges for $|z| > 1/2$ and g^+ converges for $|z| < 1$. Then $(g \circ f) \in \mathbb{L}$.

The above example is a particular case of the following proposition.

Proposition 3.6. *Let $r, s \in \mathbb{R} \cup \{+\infty\}$ with $0 \le r < s$ and let $g(z) = \sum_{k\in\mathbb{Z}} b_k z^k$ be a Laurent series which converges on the annulus*

$$A_{r,s} = \{z \in \mathbb{C}\colon r < |z| < s\}.$$

If $f \in \mathbb{X}(\mathbb{C})$ with $f(0) = a_0 \in A_{r,s}$, then $g \circ f \in \mathbb{L}$.

Proof. Let us write $g = g^- + g^+$. Then $g^-(z) = \sum_{n=1}^{\infty} b_{-n} z^{-n}$ converges for all z such that $|z| > r$, or $|1/z| < 1/r$, and $g^+(z) = \sum_{n=0}^{\infty} b_n z^n$ converges for all z such that $|z| < s$.

Since $a_0 \in A_{r,s}$, it follows that $|1/a_0| < 1/r$. Applying Theorem 1.5 to g^- and f^{-1} we have that $(g^- \circ f) \in \mathbb{L}$. Similarly, because $|a_0| < s$, applying Theorem 1.5 to g^+ and f we obtain that $(g^+ \circ f) \in \mathbb{L}$.

Thus, $(g \circ f)(z) \in \mathbb{L}$. □

Proposition 3.7. *Let $g(z) = \sum_{k\in\mathbb{Z}} b_k z^k$ be a Laurent series over $\mathbb{C}$, and let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a formal power series on $\mathbb{C}$ such that $a_0 \neq 0$. If there exists some $m \in \mathbb{N}$ such that $b_k = 0$ for every $k \le -m$, then $g \circ f \in \mathbb{X}$ if and only if $g^+ \circ f \in \mathbb{X}$.*

Proof. Since $b_k = 0$ for every $k \le -m$, it follows that $g^-(z) = \sum_{i=1}^{m-1} b_{-i} z^{-i}$. Then

$$(g^- \circ f)(z) = \sum_{i=1}^{m-1} b_{-i} f^{-i}(z) \in \mathbb{X}$$

is always true because $f(0) = a_0 \neq 0$.

Thus, $g \circ f \in \mathbb{X}$ if and only if $g^+ \circ f \in \mathbb{X}$ because $g \circ f = g^- \circ f + g^+ \circ f$. □

Definition 3.8. *Let $g(z) = \sum_{n\in\mathbb{Z}} b_n z^n$ be a formal Laurent series over $\mathbb{C}$. Then the derivative of g is a formal Laurent series defined by*

$$g'(z) = \sum_{n\in\mathbb{Z}} (b_n z^n)' = \sum_{n\in\mathbb{Z}} n b_n z^{n-1}.$$

The higher order derivatives of g are defined recursively.

It is obvious that $g' = (g^-)' + (g^+)'$.

Corollary 3.9. *Let $g(z) = \sum_{k\in\mathbb{Z}} b_k z^k$ be a Laurent series over $\mathbb{C}$, and let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a formal power series on $\mathbb{C}$ such that $a_0 \neq 0$. Then $(g \circ f)(z)$ is a formal power series, that is, $(g \circ f)(z) = \sum_{n=0}^{\infty} c_n z^n$ with $c_n \in \mathbb{C}$ for every $n \in \mathbb{N} \cup \{0\}$, if and only if*

$$(g^+)^{(k)}(a_0) \in \mathbb{C} \ \text{ and } \ (\breve{g}^+)^{(k)}(a_0^{-1}) \in \mathbb{C} \quad \text{for every } \ k \in \mathbb{N} \cup \{0\}.$$

Proof. By [4], $(g^+)^{(k)}(a_0) \in \mathbb{C}$ for every $k \in \mathbb{N} \cup \{0\}$ is equivalent to

$$\sum_{n=k}^{\infty} \binom{n}{k} b_n a_0^{n-k} \in \mathbb{C} \quad \text{for every } \ k \in \mathbb{N} \cup \{0\}.$$

Further, let us note that $\breve{g}^+ = \sum_{n=1}^{\infty} b_{-n} z^n$. Thus

$$\left(\breve{g}^+\right)^{(k)}\left(a_0^{-1}\right) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) b_{-n} \left(a_0^{-1}\right)^{n-k} \in \mathbb{C}$$

$$\text{for every } \ k \in \mathbb{N} \cup \{0\}$$

is equivalent to

$$\sum_{n=k}^{\infty} \binom{n}{k} b_{-n} a_0^{k-n} \in \mathbb{C} \quad \text{for every } \ k \in \mathbb{N} \cup \{0\}.$$

It is enough to apply Theorem 3.4 to complete the proof. □

4 Calculus on formal Laurent series

Let us begin this section with commonly known lemma which will be very useful for our purposes.

Lemma 4.1. *Suppose $\sum_{n=1}^{\infty} na_n$, where $a_n \in \mathbb{C}$ for $n \in \mathbb{N}$ is convergent. Then $\sum_{n=1}^{\infty} a_n$ is also convergent.*

Proof. Let us consider the power series $\sum_{n=1}^{\infty} na_n x^{n-1}$, $x \in \mathbb{R}$. Obviously the radius of convergence of this series is not less than 1. However, by the Abel Limit Theorem [13] (more precisely, by the proof of this theorem) we know that $\sum_{n=1}^{\infty} na_n x^{n-1}$ is uniformly convergent on the interval [0,1] and thus we may integrate termwise, getting

$$\int_0^1 \left(\sum_{n=1}^{\infty} na_n x^{n-1} \right) dx = \sum_{n=1}^{\infty} na_n \int_0^1 x^{n-1} dx = \sum_{n=1}^{\infty} a_n.$$

Thus $\sum_{n=1}^{\infty} a_n$ is convergent which ends the proof. □

Remark 5. Lemma 4.1 also holds for $\sum_{n \in \mathbb{Z}} na_n$ if we write

$$\sum_{n \in \mathbb{Z}} na_n = \sum_{m=1}^{\infty} (-m) a_{-m} + \sum_{n=0}^{\infty} na_n$$

and apply the above proof to $\sum_{m=1}^{\infty} m(-a_{-m})$ and $\sum_{n=0}^{\infty} na_n$ respectively.

It is clear that if $g(z) = \sum_{n \in \mathbb{Z}} b_n z^n$ and $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ are two formal Laurent series over $\mathbb{C}$, then we have

(i) $(f+g)' \in \mathbb{L}$ and $(f+g)' = f' + g'$, and

(ii) $(cf)' \in \mathbb{L}$ and $(cf)' = cf'$.

These two properties are obvious consequences of Definition 3.8.

The following theorem provides further interesting properties of the derivative of a formal Laurent series.

Theorem 4.2. *Let $g(z) = \sum_{n \in \mathbb{Z}} b_n z^n$, $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ be two formal Laurent series. Then*

(iii) $f' \in \mathbb{L}(g)$ *implies that* $f \in \mathbb{L}(g)$;

(iv) $f' \in \mathbb{L}(g)$ *if and only if* $g' \in \mathbb{L}(f)$;

(v) $(fg)' \in \mathbb{L}$ *and* $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$, *provided that* $f' \in \mathbb{L}(g)$;

Proof.

(iii) Let $f' \in \mathbb{L}(g)$, that is $f'g(z) = \sum_{n\in\mathbb{Z}} w_n z^n \in \mathbb{L}$ where

$$w_n = \sum_{m\in\mathbb{Z}} (m+1)a_{m+1}b_{n-m-1} \in \mathbb{C} \quad \text{for every } n \in \mathbb{Z}.$$

By Lemma 4.1 (and Remark 5), $\sum_{m\in\mathbb{Z}} a_{m+1}b_{n-m-1} \in \mathbb{C}$ for $n \in \mathbb{Z}$, so $\sum_{m\in\mathbb{Z}} a_m b_{n-m} \in \mathbb{C}$ for $n \in \mathbb{Z}$, and therefore $f \in \mathbb{L}(g)$.

(iv) Suppose $f' \in \mathbb{L}(g)$ and write $gf'(z) = \sum_{n\in\mathbb{Z}} r_n z^n \in \mathbb{L}$, where $r_n = \sum_{m\in\mathbb{Z}} b_m(n+1-m)a_{n+1-m} \in \mathbb{C}$ for $n \in \mathbb{Z}$. Since

$$r_n = (n+1)\sum_{m\in\mathbb{Z}} b_m a_{n+1-m} - \sum_{m\in\mathbb{Z}} m b_m a_{n+1-m}$$

and, by Lemma 4.1, $\sum_{m\in\mathbb{Z}} b_m a_{n+1-m} \in \mathbb{C}$, we deduce that

$$\sum_{m\in\mathbb{Z}} m b_m a_{n+1-m} \in \mathbb{C}$$

for $n \in \mathbb{Z}$ which means that $fg'(z) \in \mathbb{L}$ or $g' \in \mathbb{L}(f)$.

Conversely we suppose that $g' \in \mathbb{L}(f)$, we can use similar approach to prove that $f' \in \mathbb{L}(g)$. This is (iv).

(v) Finally, applying (iii) and (iv) we have that $f'g, g'f \in \mathbb{L}$. If we write $(fg)(z) = \sum_{n\in\mathbb{Z}} d_n z^n$, then

$$(fg)'(z) = \sum_{n\in\mathbb{Z}} d_n' z^n = \sum_{n\in\mathbb{Z}} (n+1)d_{n+1}z^n,$$

or $d_n' = (n+1)d_{n+1}$ for every $n \in \mathbb{Z}$. We also write

$$gf'(z) = \sum_{n\in\mathbb{Z}} r_n z^n \;\text{ where }\; r_n = \sum_{m\in\mathbb{Z}} b_m(n+1-m)a_{n+1-m},$$

$$g'f(z) = \sum_{n\in\mathbb{Z}} s_n z^n \;\text{ where }\; s_n = \sum_{m\in\mathbb{Z}} (m+1)b_{m+1}a_{n-m} = \sum_{m\in\mathbb{Z}} m b_m a_{n-m+1}.$$

Then
$$\begin{aligned} r_n + s_n &= \textstyle\sum_{m\in\mathbb{Z}} \big(b_m(n+1-m)a_{n+1-m} + m b_m a_{n-m+1}\big) \\ &= \textstyle\sum_{m\in\mathbb{Z}} (n+1)b_m a_{n-m+1} \\ &= (n+1)d_{n+1} = d_n'. \end{aligned}$$

We complete the proof. □

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References

[1] J. Chaumat and A.M. Chollet. *On composite formal power series.* Trans. Amer. Math. Soc., **353**(4) (2001),1691–1703.

[2] R. Enoch. *Formal power series solutions of Schroder's equation.* Aequationes Math., **74**(1-2) (2007), 26–61.

[3] X. Gan. *A generalized chain rule for formal power series.* Comm. Math. Anal., **2**(1) (2007), 37–44.

[4] X. Gan and N. Knox. *On composition of formal power series.* Int. J. Math. and Math. Sci., **30**(12) (2002), 761–770.

[5] P. Henrici. *Applied and Computational Complex Analysis.* John Wiley and Sons, Wiley Classics Library Edition, New York, London, Sydney, Toronto (1988).

[6] K. Hedayatian. *On the reflexivity of the multiplication operator on Banach spaces of formal Laurent series.* Int. J. Math., **18**(3) (2007), 231–234.

[7] W. Jones and W. Thron. *Sequences of meromorphic functions corresponding to a formal Laurent series.* SIAM J. Math. Anal., **10**(1) (1979), 1–17.

[8] I. Niven. *Formal power series.* Amer. Math. Monthly, **76**(8) (1969), 871–889.

[9] D. Parvica and M. Spurr. *Unique summing of formal power series solutions to advanced and delayed differential equations.* Discrete Cont. Dyn. Syst., Suppl. 730–737 (2005).

[10] G. Ranney. *Functional composition patterns and power series reversion.* Trans. Amer. Math. Soc., **94**(3) (1960), 441–451.

[11] R. Remmert. *Theory of Complex Functions*. Fouth corrected printing, Springer-Verlag, New York, Berlin, Heidelberg (1998).

[12] Y. Siuya. *Formal power series solutions in a parameter.* J. Diff. Eq., **190**(2) (2003), 559–578.

[13] K.R. Stromberg. *An Introduction to Classic Real Analysis.* Wadsworth International Group, Belmont, California (1981).

[14] K. Seddighi and B. Yousefi. *On the reflexivity of operators on function spaces.* Proc. Amer. Math. Soc., **116** (1992), 45–52.

[15] B. Yousefi and R. Soltani. *Hypercyclicity on the Banach space of formal Laurent series.* Int. J. Appl. Math., **12**(3) (2003), 251–256.

Xiao-Xiong Gan
Department of Mathematics
Morgan State University
Baltimore, MD 21251
USA

E-mail: xiao-xiong.gan@morgan.edu

Dariusz Bugajewski
Department of Mathematics and Computer Science
Adam Mickiewicz University
Umultowska 87
61-614 Poznań
POLAND

E-mail: ddbb@amu.edu.pl