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Constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. We classify all surfaces in $\mathbb{H}^2 \times \mathbb{R}$ for which the unit normal makes a constant angle with the $\mathbb{R}$-direction. Here $\mathbb{H}^2$ is the hyperbolic plane.

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1 Introduction

In last years, the study of the geometry of surfaces in 3-dimensional spaces, in particular of product type $\mathbb{M}^2 \times \mathbb{R}$ was developed by a large number of mathematicians. Very recently, in [2] the authors study constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$, namely those surfaces for which the unit normal makes a constant angle with the tangent direction to $\mathbb{R}$. In another recent paper [1] it is proved that if the ambient space is the Euclidean 3-space, the study of surfaces making constant angle with a fixed direction has some important applications to physics, namely it is shown how constant angle surfaces may be used to describe interfaces occurring in special equilibrium configurations of nematic and smectic C liquid crystals. See also [5]. The problem of constant angle surfaces is also studied in the 3-dimensional Heisenberg group [4]. In this article we consider the 3-dimensional Riemannian product $\mathbb{H}^2 \times \mathbb{R}$, and we classify all surfaces making constant angle with the $\mathbb{R}$-direction.

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2 Preliminaries

Let $\widetilde{M} = \mathbb{H}^2 \times \mathbb{R}$ be the Riemannian product of $\left(\mathbb{H}^2(-1), g_H\right)$ and $\mathbb{R}$ with the standard Euclidean metric, where $\mathbb{H}^2(-1)$ denotes the hyperbolic plane of constant curvature -1. Denote by $\widetilde{g} = g_H + dt^2$ the product metric and by $\widetilde{\nabla}$ the Levi Civita connection of $\widetilde{g}$. Denote by t the (global) coordinate on $\mathbb{R}$ and hence $\partial_t = \frac{\partial}{\partial t}$ is the unit vector field in the tangent bundle $T(\mathbb{H}^2 \times \mathbb{R})$ that is tangent to the $\mathbb{R}$-direction.

The Riemann-Christoffel curvature tensor $\widetilde{R}$ of $\mathbb{H}^2 \times \mathbb{R}$ is given by

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) = & - g_H(X_H, W_H) g_H(Y_H, Z_H) \\ & + g_H(X_H, Z_H) g_H(Y_H, W_H) \end{aligned} \tag{1}$$

for any X, Y, Z, W tangent to $\mathbb{H}^2 \times \mathbb{R}$. If X is a tangent vector to $\mathbb{H}^2 \times \mathbb{R}$ we put X_H its projection to the tangent space of $\mathbb{H}^2$.

Let M be a surface in $\widetilde{M} = \mathbb{H}^2 \times \mathbb{R}$. If ξ is a unit normal to M, then the shape operator is denoted by A. We have the formulas of Gauss and Weingarten

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{G}$$

$$\widetilde{\nabla}_X \xi = -AX, \tag{W}$$

for all X and Y tangent to M. Here ∇ is the Levi Civita connection on M and h is the second fundamental form of M. We have $\widetilde{g}(h(X, Y), \xi) = g(X, AY)$ for all X, Y tangent to M, where g is the restriction of $\widetilde{g}$ to M.

Since ∂_t is of unit length, we can decompose ∂_t as

$$\partial_t = T + \cos\theta \, \xi \tag{2}$$

where T is the projection of ∂_t on the tangent space of M and θ is the angle function, defined by

$$\cos\theta = \widetilde{g}(\partial_t, \xi). \tag{3}$$

If X, Y are tangent to M, then we have the following relation

$$g_H(X_H, Y_H) = g(X, Y) - g(X, T) g(Y, T).$$

Thus, if R is the Riemannian curvature on M, then the equation of Gauss can be written as

$$\begin{aligned} R(X, Y, Z, W) = & \; g(AX, W) g(AY, Z) - g(AX, Z) g(AY, W) \\ & - g(X, W) g(Y, Z) + g(X, Z) g(Y, W) \\ & + g(X, W) g(Y, T) g(Z, T) + g(Y, Z) g(X, T) g(W, T) \\ & - g(X, Z) g(Y, T) g(W, T) - g(Y, W) g(X, T) g(Z, T) \end{aligned} \tag{EG}$$

for all $X, Y, Z, W \in T(M)$.

Using the expression of the curvature $\widetilde{R}$ of $\mathbb{H}^2 \times \mathbb{R}$, after a straightforward computation, we can write the equation of Codazzi as

$$\nabla_X AY - \nabla_Y AX - A[X, Y] = \cos\theta\big(g(X, T)Y - g(Y, T)X\big) \qquad \textbf{(EC)}$$

for all $X, Y \in T(M)$.

Proposition 2.1. *Let X be a tangent vector to M. We have*

$$\begin{cases} \nabla_X T = \cos\theta\, AX \\ X(\cos\theta) = -g(AX, T). \end{cases} \tag{4}$$

Proof. For any X tangent to M we can write

$$X = X_H + g(X, T)\, \partial_t\ .$$

We have

$$\widetilde{\nabla}_X \partial_t = \widetilde{\nabla}_{X_H} \partial_t + g(X, T)\widetilde{\nabla}_{\partial_t}\partial_t = 0.$$

On the other hand,

$$\begin{aligned} \widetilde{\nabla}_X \partial_t &= \widetilde{\nabla}_X T + \widetilde{\nabla}_X(\cos\theta\ \xi) \\ &= \nabla_X T + h(X, T) + X(\cos\theta)\xi - (\cos\theta)AX. \end{aligned}$$

Identifying the tangent and the normal part respectively, one gets

$$\nabla_X T = \cos\theta\, AX \quad \text{and} \quad X(\cos\theta)\xi = -h(X, T).$$

Hence the conclusion. $\square$

From now on consider that θ is constant; for a given orientation of $\mathbb{R}$, suppose that $\theta \in [0, \pi)$. Then, from the previous proposition we have $g(AX, T) = 0$ for every X tangent to M (at p), which is equivalent to

$$g(AT, X) = 0\ , \quad \forall X \in T_p(M). \tag{5}$$

This means that, if $T \neq 0$, T is a principal direction with principal curvature 0.

Remark 2.2. If $T = 0$ on M, then ∂_t is always normal so, $M \subseteq \mathbb{H}^2 \times \{t_0\}$, for $t_0 \in \mathbb{R}$.

If $T \neq 0$, we consider

$$e_1 = \frac{1}{||T||}\, T, \tag{6}$$

where $||T|| = \sin\theta$.

Let e_2 be a unit vector tangent to M and perpendicular to e_1. Then the shape operator A takes the following form

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$$

for a certain function λ on M. Hence we have

$$h(e_1, e_1) = 0, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \lambda\, \xi. \tag{7}$$

Proposition 2.3. *If M is a constant angle surface in $\mathbb{H}^2 \times \mathbb{R}$ with constant angle $\theta \neq 0$, then M has constant Gaussian curvature $K = -\cos^2\theta$ and the projection T of $\dfrac{\partial}{\partial t}$ is a principal direction with principal curvature* 0.

Proof. We have to prove only the first part of this statement. To do this, we decompose $e_1, e_2 \in T(M)$ as

$$e_1 = E_1 + \sin\theta\, \partial_t\,, \quad e_2 = E_2 \tag{8}$$

with $E_1, E_2 \in T(\mathbb{H}^2)$ (since e_2 is perpendicular to ∂_t). We immediately have

$$g_H(E_1, E_1) = \cos^2\theta, \quad g_H(E_1, E_2) = 0, \quad g_H(E_2, E_2) = 1.$$

Putting $X = W = e_1$ and $Y = Z = e_2$ in the Gauss equation **(EG)** and combining (1) and (7), we find that the Gaussian curvature of M satisfies

$$K = -\cos^2\theta. \tag{9}$$

□

We conclude this section with the following

Proposition 2.4. *The Levi Civita connection of g on M is given by*

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, \quad \nabla_{e_2} e_1 = \lambda \cot\theta\; e_2, \\ \nabla_{e_1} e_2 &= 0, \quad \nabla_{e_2} e_2 = -\lambda \cot\theta\; e_1. \end{aligned} \tag{10}$$

Proof. Using (4) and (6) we obtain

$$\nabla_X e_1 = \frac{1}{\sin\theta}\,\nabla_X T = \cot\theta\; AX.$$

From (7) we then obtain the first two formulas. The other two formulas then follow immediately. □

3 Characterization of constant angle surfaces

In this section we classify the constant angle surfaces M in $\mathbb{H}^2 \times \mathbb{R}$. There exist two trivial cases, namely $\theta = 0$ and $\theta = \frac{\pi}{2}$. As we have already seen, in the first case one has that $\frac{\partial}{\partial t}$ is always normal and hence M is an open part of $\mathbb{H}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$. In the second case $\frac{\partial}{\partial t}$ is always tangent. This corresponds to the Riemannian product of a curve in $\mathbb{H}^2$ and $\mathbb{R}$.

We can take coordinates (u, v) on M such that the metric g on M has the form

$$g = du^2 + \beta^2(u, v)\, dv^2 \tag{11}$$

with $\partial_u := \frac{\partial}{\partial u} = e_1$ and $\partial_v := \frac{\partial}{\partial v} = \beta\, e_2$, where β is a smooth function on M. This can be done since $[e_1, e_2]$ is collinear with e_2. We have

$$0 = \left[\partial_u, \partial_v\right] = \left[\partial_u, \beta e_2\right] = \beta_u e_2 + \beta\left[e_1, e_2\right] = (\beta_u - \beta\lambda\cot\theta)\, e_2$$

and hence β satisfies the following PDE

$$\beta_u = \beta\lambda\cot\theta. \tag{12}$$

Using Proposition 2.4 one can now write the Levi Civita connection of g on M in terms of the coordinates u and v, namely

$$\nabla_{\partial_u}\partial_u = 0, \quad \nabla_{\partial_u}\partial_v = \nabla_{\partial_v}\partial_u = \lambda\cot\theta\;\partial_v, \quad \nabla_{\partial_v}\partial_v = -\beta\beta_u\partial_u + \frac{\beta_v}{\beta}\,\partial_v. \tag{13}$$

Proposition 3.1. *The two functions λ and β are given by*

$$\lambda(u, v) = \sin\theta\tanh\left(u\cos\theta + C(v)\right) \tag{14}$$

$$\beta(u, v) = D(v)\cosh\left(u\cos\theta + C(v)\right), \tag{15}$$

or

$$\lambda(u, v) = \pm\sin\theta \tag{16}$$

$$\beta(u, v) = D(v)e^{\pm u\cos\theta} \tag{17}$$

where C and D are smooth functions depending on v, $D(v) \neq 0$ for any v.

Proof. From the equation of Codazzi (**EC**), if we put $X = e_1$ and $Y = e_2$ one obtains that λ must satisfy the following PDE

$$\lambda_u = \sin\theta\cos\theta - \lambda^2\cot\theta. \tag{18}$$

By integration, one gets (14) or (16). Now, solving (12) we obtain β. □

There are many models for the hyperbolic plane (e.g. the Klein model, the Poincaré disk, the upper half plane H^+, the Minkowski model $\mathcal{H}$), cf. [7]. The study of the constant angle surfaces was done by the authors in [3] by using the upper half plane model of the hyperbolic plane. In the following we will deal with the Minkowski model or the hyperboloid model for $\mathbb{H}^2$.

We denote by $\mathbb{R}^3_1$ the Minkowski 3-space with coordinates x, y and z, endowed with the Lorentzian metric tensor

$$< \cdot, \cdot >= dx^2 + dy^2 - dz^2.$$

Then $\mathbb{H}^2$ can be considered as the upper sheet ($z > 0$) of the hyperboloid

$$\left\{(x, y, z) \in \mathbb{R}^3_1 \ : \ x^2 + y^2 - z^2 = -1\right\}.$$

The external unit normal to $\mathcal{H}$ in a point $p \in \mathcal{H} \subset \mathbb{R}^3_1$ is $N = p$ and we have $\langle N, N\rangle = -1$.

We recall the notion of the Lorentzian cross-product (see e.g. [7]):

$$\begin{aligned} \boxtimes : \mathbb{R}^3_1 \times \mathbb{R}^3_1 \ &\longrightarrow \mathbb{R}^3_1, \\ ((a_1, a_2, a_3), (b_1, b_2, b_3)) \ &\mapsto (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_2b_1 - a_1b_2). \end{aligned}$$

As analogue to the vector cross product in the Euclidean space, it has similar algebraic and geometric properties:

(i) $a \boxtimes b$ is perpendicular to a and b, i.e. $\langle a \boxtimes b, a\rangle = \langle a \boxtimes b, b\rangle = 0$;

(ii) $b \boxtimes a = -a \boxtimes b$;

(iii) $\langle a \boxtimes b, a \boxtimes b\rangle = -\langle a, a\rangle\langle b, b\rangle + \langle a, b\rangle^2$ for all $a, b \in \mathbb{R}^3_1$.

Let M be a 2-dimensional surface in $\mathcal{H} \times \mathbb{R} \subset \mathbb{R}^3_1 \times \mathbb{R}$. On the ambient space we consider the product metric:

$$g_o = dx^2 + dy^2 - dz^2 + dt^2.$$

Denote by $\overset{\mathrm{o}}{\nabla}$ the Levi Civita connection on $\mathbb{R}^3_1 \times \mathbb{R}$ and let $D^\perp$ be the normal connection of M in $\mathbb{R}^3_1 \times \mathbb{R}$. If $\widetilde{\xi}$ is the unit normal to $\widetilde{M}$, then $\widetilde{\xi}(p_1, p_2, p_3, p_4) = (p_1, p_2, p_3, 0)$. The shape operator on M w.r.t. $\widetilde{\xi}$ is denoted by $\widetilde{A}$ and will be computed below.

Theorem 3.2. *A surface M in $\mathcal{H} \times \mathbb{R}$ is a constant angle surface if and only if the position vector F is, up to isometries of $\mathcal{H} \times \mathbb{R}$, locally given by*

$$F(u,v) = \big(\cosh(u\cos\theta) f(v) + \sinh(u\cos\theta) f(v) \boxtimes f'(v), u \sin\theta\big), \quad (19)$$

where f is a unit speed curve on $\mathcal{H}$.

Remark 3.3. This result is similar to that given in Theorem 2 of [2].

Proof of the Theorem. First we have to prove that the given immersion (19) is a constant angle surface in $\mathcal{H} \times \mathbb{R}$. To do this we compute the tangent vectors (in an arbitrary point on M)

$$\begin{aligned} F_u(u,v) &= \big(\cos\theta\big[\sinh(u\cos\theta) f(v) + \cosh(u\cos\theta) f(v) \boxtimes f'(v)\big], \sin\theta\big) \\ F_v(u,v) &= \big(\cosh(u\cos\theta) f'(v) + \sinh(u\cos\theta) f(v) \boxtimes f''(v), 0\big) = \\ &= \big(\big[\cosh(u\cos\theta) - \kappa(v)\sinh(u\cos\theta)\big]\, f'(v), 0\big), \end{aligned}$$

where κ is the geodesic curvature of the curve f. Let us give some details. Since $f(v)$ lies on the hyperboloid it follows that $f'(v)$ is spacelike. But the curve f has unit speed so, $\langle f'(v), f'(v)\rangle = 1$. In each point of the curve f one has an orthonormal basis, namely $\{f(v), f'(v), f(v) \boxtimes f'(v)\}$. Taking into account that $\langle f'(v), f''(v)\rangle = 0$ for all v, one can express $f''(v)$ as linear combination of $f(v)$ and $f(v) \boxtimes f'(v)$. From the theory of curves, the curvature $\kappa(v) = |f''(v)|$ ($f''(v)$ is not timelike) and hence the following identity holds $f''(v) = f(v) + \kappa(v) f(v) \boxtimes f'(v)$. As consequence $f(v) \boxtimes f''(v) = -\kappa(v) f'(v)$.

We will calculate now both ξ and $\widetilde{\xi}$. The second normal vector is nothing but the position vector where we take the last component to be 0, namely we have

$$\widetilde{\xi}(u,v) = \big(\cosh(u\cos\theta) f(v) + \sinh(u\cos\theta) f(v) \boxtimes f'(v), 0\big).$$

Looking for the expression of the unitary normal ξ as linear combination of f, f', $f \boxtimes f'$ and ∂_t we find

$$\xi(u,v) = \big(-\sin\theta\,\big[\sinh(u\cos\theta) f(v) + \cosh(u\cos\theta) f(v) \boxtimes f'(v)\big], \cos\theta\big).$$

This is direct consequence of the following conditions:

$$\tilde{g}(\xi, F_u) = 0, \quad \tilde{g}(\xi, F_v) = 0, \quad \tilde{g}(\xi,\xi) = 1 \quad \text{and} \quad \langle p_1(\xi), p_1(F(u,v))\rangle = 0$$

(due the fact that ξ is tangent to $\mathcal{H} \times \mathbb{R}$), where $p_1 : \mathbb{R}^3_1 \times \mathbb{R} \longrightarrow \mathbb{R}^3_1$ is the natural projection. It follows $\langle \xi, \partial_t\rangle = \cos\theta$ (which is a constant).

Conversely, consider a surface M in $\mathcal{H} \times \mathbb{R}$ given by the following isometric immersion

$$F : M \longrightarrow \mathcal{H} \times \mathbb{R} \hookrightarrow \mathbb{R}^3_1 \times \mathbb{R}, \quad F = (F_1, F_2, F_3, F_4).$$

Suppose the constancy of the angle function θ. If M is one of the trivial cases (see page 89), then it can be parameterized by (19). From now on we consider $\theta \notin \{0, \frac{\pi}{2}\}$.

We have

$$(F_4)_u = \widetilde{g}(F_u, \partial_t) = \widetilde{g}(F_u, T + \cos\theta\xi) = g(\partial_u, T) = \sin\theta$$

and

$$(F_4)_v = \widetilde{g}(F_v, \partial_t) = g(\partial_v, T) = 0.$$

These relations and the initial condition $F_4(0, 0) = 0$ yield

$$F_4 = u \sin\theta. \tag{20}$$

If $X = (X_1, X_2, X_3, X_4)$ is tangent to M, then $\overset{\circ}{\nabla}_X \widetilde{\xi} = (X_1, X_2, X_3, 0)$. It follows

- $D^\perp_X \widetilde{\xi} = \langle (X_1, X_2, X_3, 0), \xi \rangle \, \xi = -\cos\theta \langle X, T \rangle \xi$
- $D^\perp_X \xi = \cos\theta \langle X, T \rangle \widetilde{\xi}$.

Since $\overset{\circ}{\nabla}_X \widetilde{\xi} = -\widetilde{A}\, X + D^\perp_X \widetilde{\xi}$ for every X tangent to M, we are able to give $\widetilde{A}$ in terms of the basis $\{\partial_u, \partial_v\}$

$$\widetilde{A} = \begin{pmatrix} -\cos^2\theta & 0 \\ 0 & -1 \end{pmatrix}.$$

From (2), taking the j^{th} component, and because $T = \sin\theta F_u$, one has

$$\xi_j = -\tan\theta (F_j)_u \tag{21}$$

for all $j = 1, 2, 3$. Here $\xi = (\xi_1, \xi_2, \xi_3, \cos\theta)$.

Applying now the formula of Gauss, using the expressions of the shape operators A and $\widetilde{A}$ and (13) we find:

$$(F_j)_{uu} = \cos^2\theta F_j \tag{22}$$

$$(F_j)_{uv} = \lambda \cot\theta (F_j)_v \tag{23}$$

$$(F_j)_{vv} = -\beta\beta_u (F_j)_u + \frac{\beta_v}{\beta} (F_j)_v - \lambda\beta^2 \tan\theta (F_j)_u + \beta^2 F_j. \tag{24}$$

Let us sketch the proof of (22). If $\widetilde{h}$ is the second fundamental form of the immersion $M \hookrightarrow \mathbb{R}^3_1 \times \mathbb{R}$ then one can prove that $\widetilde{h}(\partial_u, \partial_u) = \cos^2\theta\, \widetilde{\xi}$ (since $\langle \widetilde{\xi}, \widetilde{\xi} \rangle = -1$). We have

$$F_{uu} = \overset{\circ}{\nabla}_{\partial_u} \partial_u = \nabla_{\partial_u}\partial_u + \widetilde{h}(\partial_u, \partial_u) = \cos^2\theta\ p_1(F(u, v))$$

and now take the j^{th} component ($j = 1, 2, 3$). In the same manner we can show (23) and (24).

Case 1: λ satisfies (14). Integrating (23) one gets

$$(F_j)_v = H_j(v) \cosh(u \cos\theta + C(v)),$$

where H_j is an arbitrary function. Hence

$$F_j = \int_0^v \cosh(u\cos\theta + C(\tau)) H_j(\tau) d\tau + I_j(u),$$

where I_j is an arbitrary function. Substituting in (22) we obtain

$$I_j = K_j \cosh(u\cos\theta) + L_j \sinh(u\cos\theta), \tag{25}$$

where K_j and L_j are real constants.

We define the following functions

$$f_j = K_j + \int_0^v \cosh C(\tau) H_j(\tau) d\tau, \quad (j = 1, 2, 3)$$

$$g_j = L_j + \int_0^v \sinh C(\tau) H_j(\tau) d\tau, \quad (j = 1, 2, 3).$$

Case 2: λ satisfies (16). One gets

$$F_j = e^{\pm u\cos\theta} \int_0^v H_j(\tau) d\tau + I_j(u)$$

with I_j having the same form as in (25).

In this case we put

$$f_j = K_j + \int_0^v H_j(\tau)d\tau, \quad (j = 1, 2, 3)$$

$$g_j = L_j \pm \int_0^v H_j(\tau)d\tau, \quad (j = 1, 2, 3).$$

Let $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3)$.

Summarizing, in both cases F is of the following form:

$$F = \big(\cosh(u\cos\theta) f + \sinh(u\cos\theta) g, u\sin\theta\big). \tag{26}$$

Let $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$. Then we have

$$\sum_{j=1}^{3} \epsilon_j F_j^2 = -1, \tag{27}$$

$$\sum_{j=1}^{3} \epsilon_j (F_j)_u^2 = \cos^2\theta, \tag{28}$$

$$\sum_{j=1}^{3} \epsilon_j (F_j)_u (F_j)_v = 0, \tag{29}$$

$$\sum_{j=1}^{3} \epsilon_j (F_j)_v^2 = \beta^2. \tag{30}$$

From (27) and (28) one obtains

$$\sum \epsilon_j f_j^2 - \sum \epsilon_j g_j^2 = -2. \tag{i}$$

Now, relations (27) and (29) can be written as

$$\begin{aligned} &\sum \epsilon_j f_j^2 \cosh^2(u\cos\theta) + \sum \epsilon_j g_j^2 \sinh^2(u\cos\theta) \\ &\quad + 2\sum \epsilon_j f_j g_j \sinh(u\cos\theta)\cosh(u\cos\theta) = -1 \end{aligned} \tag{ii}$$

and

$$\begin{aligned}&\Big(\sum \epsilon_j f_j f_j' + \sum \epsilon_j g_j g_j'\Big) \sinh(u\cos\theta)\cosh(u\cos\theta)\\&\quad + \sum \epsilon_j f_j' g_j \cosh^2(u\cos\theta) + \sum \epsilon_j f_j g_j' \sinh^2(u\cos\theta) = 0.\end{aligned} \tag{iii}$$

By a derivation in (27) one has

$$\begin{aligned}&\sum \epsilon_j f_j f_j' \cosh^2(u\cos\theta) + \sum \epsilon_j g_j g_j' \sinh^2(u\cos\theta)\\&\quad + \Big(\sum \epsilon_j f_j g_j' + \sum \epsilon_j f_j' g_j\Big) \sinh(u\cos\theta)\cosh(u\cos\theta) = 0\end{aligned} \tag{iv}$$

$$\begin{aligned}&\Big(\sum \epsilon_j f_j^2 + \sum \epsilon_j g_j^2\Big) \sinh(u\cos\theta)\cosh(u\cos\theta)\\&\quad + \sum \epsilon_j f_j g_j (\cosh^2(u\cos\theta) + \sinh^2(u\cos\theta)) = 0.\end{aligned} \tag{v}$$

Finally (i), (ii) and (v) yield

$$\sum \epsilon_j f_j^2 = -1, \quad \sum \epsilon_j g_j^2 = 1, \quad \sum \epsilon_j f_j g_j = 0.$$

Moreover, it follows

$$\sum \epsilon_j f_j f_j' = 0, \quad \sum \epsilon_j g_j g_j' = 0, \quad \sum \epsilon_j f_j g_j' + \sum \epsilon_j f_j' g_j = 0.$$

From (iii) we get

$$\sum \epsilon_j f_j' g_j = 0 \quad \text{and} \quad \sum \epsilon_j f_j g_j' = 0. \tag{31}$$

Hence, the relation (iv) is identically satisfied.

We can write these last equations in another way:

$$\begin{array}{rcl}\langle f, f\rangle &=& -1\\ \langle g, g\rangle &=& 1\\ \langle f, g\rangle &=& 0\\ \langle f', g\rangle &=& 0\\ \langle f, g'\rangle &=& 0\end{array} \quad \text{and} \quad \begin{array}{rcl}\langle f, f'\rangle &=& 0\\ \langle g, g'\rangle &=& 0.\end{array} \tag{32}$$

We still have to develop the relation (30). This yields

$$\langle H(v), H(v)\rangle = \langle f', f'\rangle - \langle g', g'\rangle = D^2(v), \quad H = (H_1, H_2, H_3).$$

Remark that f can be thought as a curve on $\mathcal{H}$ (while g is not).

Since $\langle f', f'\rangle \geq 0$ (which can be easily proved), one can change the v-coordinate such that f becomes a unit speed curve in $\mathcal{H}$; this corresponds to $D(v)^2 \cosh^2 C(v) = 1$ or $D(v)^2 = 1$ (depending on the value of λ).

We have $g \perp f$ and $g \perp f'$. Due the geometric properties of the Lorentzian cross product, g is collinear to $f \boxtimes f'$. We have $\langle g, g\rangle = 1$ and $\langle f \boxtimes f', f \boxtimes f'\rangle = 1$ and hence $g = \pm f \boxtimes f'$. We can assume that $g = f \boxtimes f'$. Then F is given by (19) as we wanted to prove. $\square$

Remark 3.4. Looking for all minimal constant angle surfaces in $\mathcal{H} \times \mathbb{R}$, these must be totally geodesic in $\mathcal{H} \times \mathbb{R}$. Hence we obtain the following surfaces:

(1) $\mathcal{H} \times \{t_0\}$, $t_0 \in \mathbb{R}$

(2) $f \times \mathbb{R}$ with f a geodesic line in $\mathcal{H}$.

Remark 3.5. A surface M in $\mathcal{H} \times \mathbb{R}$ is a non-minimal constant mean curvature constant angle surface if and only if it is parameterized by (19) where f is the parabola explicitly given by

$$f(v) = \left(1 + \frac{v^2}{2}\right) K - \varepsilon \frac{v^2}{2}\, L + vK \boxtimes L.$$

Here K and L are orthogonal unitary timelike, respectively spacelike vectors in $\mathbb{R}^3_1$ and $\varepsilon = \pm 1$.

Proof. First, since M is CMC surface, due to (7), λ must be a constant and hence λ is given by $\lambda = \varepsilon \sin\theta$, with $\varepsilon = \pm 1$. In this case we have

$$f_j = K_j + \int_0^v H_j(\tau)d\tau \quad \text{and} \quad g_j = L_j + \varepsilon \int_0^v H_j(\tau)d\tau,$$

where K_j and L_j are real constants ($j = 1, 2, 3$). See Case 2 in the proof of the main theorem.

Denoting by $K = (K_1, K_2, K_3)$ and $L = (L_1, L_2, L_3)$ it immediately follows that $\langle K, K\rangle = -1$, $\langle L, L\rangle = 1$ and $\langle K, L\rangle = 0$. Moreover, $V = K - \varepsilon L$ is a lightlike vector and $f(v) - \varepsilon g(v) = V$, for all v and $H(v) = (H_1, H_2, H_3)$ lies in a plane orthogonal to V.

Considering the base $\{K, L, K \boxtimes L\}$ one obtains the expression of f and g, namely

$$f(v) = K + \frac{1}{2}\, A^2(v)\big(K - \varepsilon L\big) + A(v)K \boxtimes L$$

and $g(v) = f(v) \boxtimes f'(v)$, where $A(v)$ is a smooth function. After a change of the parameter v (we can do this since $|A'(v)| = |f'(v)| = 1$), we get the statement. $\square$

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