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Inverse semigroups and combinatorial C*-algebras

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Abstract. We describe a special class of representations of an inverse semigroup S on Hilbert's space which we term *tight*. These representations are supported on a subset of the spectrum of the idempotent semilattice of S, called the *tight spectrum*, which is in turn shown to be precisely the closure of the space of ultra-filters, once filters are identified with semicharacters in a natural way. These representations are moreover shown to correspond to representations of the C*-algebra of the groupoid of germs for the action of S on its tight spectrum. We then treat the case of certain inverse semigroups constructed from semigroupoids, generalizing and inspired by inverse semigroups constructed from ordinary and higher rank graphs. The tight representations of this inverse semigroup are in one-to-one correspondence with representations of the semigroupoid, and consequently the semigroupoid algebra is given a groupoid model. The groupoid which arises from this construction is shown to be the same as the *boundary path* groupoid of Farthing, Muhly and Yeend, at least in the singly aligned, sourceless case.

Keywords: C*-algebras, Cuntz-Krieger algebras, graphs, higher-rank graphs, groupoids, inverse semigroups, semilattices, ultra-filters, boolean algebras, tight Hilbert space representations, crossed products, germs, semigroupoids, categories.

Mathematical subject classification: Primary: 46L05; Secondary: 18B40.

1 Introduction

By a *combinatorial* C*-algebra we loosely mean any C*-algebra which is constructed from a combinatorial object. Among these we include the Cuntz-Krieger algebras built out of 0–1 matrices, first studied in the finite case in [6], and quickly recognized to pertain to the realm of Combinatorics by Enomoto and Watatani [7]. Cuntz-Krieger algebras were subsequently generalized to row-finite matrices in [19], and to general infinite matrices in [12]. Another important class of combinatorial C*-algebras, closely related to the early work of Cuntz and

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Krieger, is formed by the graph C*-algebras [2, 3, 14, 15, 18, 25, 29, 30, 35], including the case of higher rank graphs introduced by Kumjian and Pask in [16], and given its final form by Farthing, Muhly and Yeend in [13]. See also [17, 22, 23, 28]. The monograph [27] is an excellent source of well organized information and references.

Attempting to understand all of these algebras from a single perspective, I have been interested in the notion of semigroupoid C*-algebras [10], which includes the Cuntz-Krieger algebras and the higher rank graph C*-algebras in the general infinite case, provided some technical complications are not present including, but not limited to, *sources*.

The most efficient strategy to study combinatorial C*-algebras has been the construction of a dynamical system which intermediates between Combinatorics and Algebra. In the case of [12], the dynamical system took the form of a partial action of a free group on a topological space, but more often it is represented by an étale, or r-discrete groupoid. In fact, even in the case of [12], the partial action may be encoded by a groupoid [1], [32]. It therefore seemed natural to me that semigroupoid C*-algebras could also be given groupoid models. But, unfortunately, the similarity between the terms *semigroupoid* and *groupoid* has not made the task any easier.

The vast majority of combinatorial C*-algebras may be defined following a standard pattern: the combinatorial object chosen is used to suggest a list of relations, written in the language of C*-algebras, and then one considers the universal C*-algebra generated by partial isometries satisfying such relations.

Partial isometries can behave quite badly from an algebraic point of view, and in particular the product of two such elements needs not be a partial isometry. Should the most general and wild partial isometries be involved in combinatorial algebras, the study of the latter would probably be impossible. Fortunately, though, the partial isometries one usually faces are, without exception, of a tamer nature in the sense that they always generate a *-semigroup consisting exclusively of partial isometries or, equivalently, an inverse semigroup.

In two recent works, namely [25] and [13], this inverse semigroup has been used in an essential way, bridging the combinatorial input object and the groupoid.

$$
\begin{tikzcd}
\boxed{\begin{matrix}\text{Combinatorial}\\ \text{Object}\end{matrix}} \arrow[dr] & & \boxed{\text{Groupoid}} \arrow[r] & \boxed{\begin{matrix}\text{Combinatorial}\\ \text{C*-algebra}\end{matrix}} \\
& \boxed{\begin{matrix}\text{Inverse}\\ \text{Semigroup}\end{matrix}} \arrow[ur] & &
\end{tikzcd}
$$

Diagram 1.1

In both [25] and in [13] the relevant inverse semigroup is made to act on a topological space by means of partial homeomorphisms. The groupoid of germs for this action then turns out to be the appropriate groupoid. However, the above diagram does not describe this strategy quite correctly because the topological space where the inverse semigroup acts is a space of *paths* whose description requires that one looks back at the combinatorial object.

Attempting to adopt this strategy, I stumbled on the fact that it is very difficult to guess the appropriate path space in the case of a semigroupoid. Moreover, earlier experience with partial actions of groups suggested that the path space should be intrinsic to the inverse semigroup.

Searching the literature one indeed finds intrinsic dynamical systems associated to a given inverse semigroup S, such as the natural action of S on the semicharacter space of its idempotent semilattice [24, Proposition 4.3.2]. But, unfortunately, the groupoid of germs for this action turns out not to be the correct one. For example, if one starts with the most basic of all combinatorial algebras, namely the Cuntz algebra $\mathcal{O}_n$, the appropriate combinatorial object is an $n \times n$ matrix of zeros and ones, which in this case consists only of ones, and the inverse semigroup is the Cuntz inverse semigroup, as defined by Renault in [31, III.2.2]. But the groupoid of germs constructed from the above intrinsic action is not the Cuntz groupoid because its C*-algebra is the Toeplitz extension of $\mathcal{O}_n$ [5, Proposition 3.1], rather than $\mathcal{O}_n$ itself. See also [31, III.2.8.i].

If $E = E(S)$ is the idempotent semilattice of an inverse semigroup S, one says that a nonzero map

$$\phi : E \to \{0, 1\}$$

is a *semicharacter* if $\phi(xy) = \phi(x)\phi(y)$, for all x and y in E. The set of all semicharacters, denoted $\hat{E}$, is called the *semicharacter space* of E, and it is a locally compact topological space under the topology of pointwise convergence. The intrinsic action we referred to above is a certain very natural action of S on $\hat{E}$.

If S contains a zero element 0, a quite common situation which can otherwise be easily arranged, then 0 is in E but the above popular definition of semicharacter strangely does not require that $\phi(0) = 0$. In fact the space of all semicharacters is too big, and this is partly the reason why $\hat{E}$ does not yield the correct groupoid in most cases. This is also clearly indicated by the need to reduce the universal groupoid in [25].

One of the main points of this work is that this reduction can be performed in a way that is entirely intrinsic to S, and does not require any more information from the combinatorial object which gave rise to S. In other words, the diagram above can be made to work exactly as indicated.

If ϕ is a semicharacter of a semilattice E then the set

$$\xi = \xi_\phi = \{e \in E : \phi(e) = 1\},$$

which incidentally characterizes ϕ, is a *filter* in the sense that it contains ef, whenever e and f are in ξ, and moreover $f \geqslant e \in \xi$ implies that $f \in \xi$. In case S contains 0, and one chooses to add to the definition of semicharacters the sensible requirement that $\phi(0)$ should be equal to zero then, in addition to the above properties of ξ, one gets $0 \notin \xi$. We then take these simple properties as the definition of a filter.

With the exception of Kellendonk's topological groupoid [20, 9.2], most authors have not paid too much attention to the fact that ultra-filters form an important class of filters, and that these are present in abundance, thanks to Zorn's Lemma. Kellendonk's treatment is however not precisely what we need, perhaps because of the reliance on sequences with only countably many terms.

It then makes sense to pay attention to the set $\hat{E}_\infty$ formed by all semicharacters ϕ for which ξ_ϕ is an ultra-filter. Our apology of ultra-filters notwithstanding, $\hat{E}_\infty$ is not always tractable by the methods of Topology not least because it may fail to be closed in $\hat{E}$. But in what follows we will try to convince the reader that the closure of $\hat{E}_\infty$ within $\hat{E}$, which we denote by $\hat{E}_{tight}$, is the right space to look at.

This explains several instances in the literature where *finite paths* shared the stage with *infinite paths*. Not attempting to compile a comprehensive list, we may cite as examples, in chronological order:

- The description of the spectrum of the Cuntz–Krieger relations for arbitrary 0–1 matrices given at the end of [12, Section 5]. See also [12, 7.3].
- Paterson's description of the unit space of the path groupoid of a graph [25, Proposition 3]. See also [25, Proposition 4].
- The closed invariant space $\partial\Lambda$ within the space of all finite and infinite paths in a higher rank graph Λ, constructed by Farthing, Muhly and Yeend in [13, Definition 5.10]. See also [13, Theorem 6.3].

To fully explain the connection between the groupoid of germs for the natural action of S on $\hat{E}_{tight}$ and the above works would make this paper even longer than it already is, so we have opted instead to restrict attention to semigroupoid C*-algebras. On the one hand these include most of the combinatorial algebras mentioned so far, but on the other hand we have made significant restrictions in order to fend off well known technical complications.

While we do not compromise on *infiniteness*, we assume that our semigroupoid has no *springs*, and *admits least common multiples*. These hypotheses correspond, in the case of a higher rank graph Λ, to the absence of *sources*, and to the fact that Λ is *singly aligned*. Besides allowing for technical simplifications, the existence of least common multiples evokes important connections to Arithmetics, and has an important geometrical interpretation in higher rank graphs.

When Λ is a semigroupoid satisfying all of the these favourable hypotheses, we construct an inverse semigroup $S(\Lambda)$, and then prove in Theorem (18.4) that the semigroupoid C*-algebra is isomorphic to the C*-algebra for the groupoid of germs for the natural action of $S(\Lambda)$ on $\hat{E}_{tight}$, exactly following the strategy outlined in Diagram 1.1.

Although we have not invested all of the necessary energy to study the inverse semigroup constructed from a general higher rank graph, as in [13], we conjecture that the groupoid there denoted by $\mathcal{G}_\Lambda|_{\partial\Lambda}$ is the same as the groupoid $\mathcal{G}_{tight}$ of Theorem (13.3) below, or at least our findings seem to give strong indications that this is so. Should this be confirmed, the assertion made in the introduction of [13] that their groupoid is *fairly far removed from the universal groupoid of* S_Λ might need rectification.

The first part of this work, comprising Sections (3)–(10) is based on Renault's Thesis [31] and Paterson's book [24], and should be considered as a survey of the technical methods we use in the subsequent sections, beginning with a careful study of non-Hausdorff étale groupoids and their C*-algebras. We also discuss actions of inverse semigroups on topological spaces and describe the associated groupoid of germs in detail. Sieben's theory of crossed products by inverse semigroups [33] is included.

We have made a special effort to assume as few hypotheses as possible, and this was of course facilitated by our focus on étale groupoids. We hope this can be used as a guide to the beginner who is primarily interested in the étale case and hence needs not spend much energy on Haar systems.

As a result of our economy of assumptions we have found generalizations of some known results, most notably Theorem (9.9) below, which shows that the C*-algebra of an étale groupoid is a crossed product in Sieben's sense, even in the non-Hausdorff case, with much less stringent hypotheses than the *additivity* assumption of [24, Theorem 3.3.1] or the *fullness* condition of [26, 8.1]. We also present a minor improvement on [24, Proposition 3.3.3], by removing the need for condition (ii) of [24, Definition 3.3.1]. This is presented in Proposition (9.7) below.

Even though we do most of our work based on non-Hausdorff groupoids, we have found an interesting sufficient condition for the groupoid of germs to be

Hausdorff, related to the order structure of inverse semigroups. We show in Theorem (6.2) that if the inverse semigroup S is a semilattice with respect to its natural order

$$s \leqslant t \iff s = ts^*s,$$

then every action of S on a locally compact Hausdorff space, for which the domains of the corresponding partial homeomorphisms are clopen, one has that the associated groupoid of germs is Hausdorff.

A special class of semigroups possessing the above mentioned property (see (6.4)) is formed by the E^*-unitary inverse semigroups, sometimes also called 0-E-unitary, which was defined by Szendrei [34] and has been intensely studied in the semigroup literature. See, for example, [20, Section 9]. Kellendonk's topological groupoid is Hausdorff when S is E^*-unitary [20, 9.2.6], and the related class of E-unitary inverse semigroups have also been shown to provide Hausdorff groupoids [24, Corollary 4.3.2].

It is with section (11) that our original work takes off, where we develop the crucial notion of *tight* representations of a semilattice in a Boolean algebra (11.6). Strangely enough, it is in the realm of these very elementary mathematical constructs that we have found the most important ingredient of this paper. The concept of tight representations may be considered a refinement of an idea which has been dormant in the literature for many years, namely condition (1.3) in [12].

In the following section we study representations of a semilattice into the Boolean algebra $\{0, 1\}$, and its relation to filters and ultra-filters. The central result, Theorem (12.9), is that the space of tight characters is precisely the closure of the set of characters associated to ultra-filters. We also show in (12.11) that tight characters on the idempotent semilattice of an inverse semigroup S are preserved under the natural action of S, thus giving rise to the action of S on $\hat{E}_{tight}$, the dynamical system which occupies our central stage.

In the short section (13) we consider tight Hilbert space representations of a given inverse semigroup S and show in (13.3) that they are in one-to-one correspondence to the representations of $C^*(\mathcal{G}_{tight})$, where $\mathcal{G}_{tight}$ is the groupoid of germs associated to the natural action of S on the tight part of the spectrum of its idempotent semilattice. Perhaps this result could be interpreted as saying that the C*-algebra generated by the range of a universal tight representation of S, which is isomorphic to $C^*(\mathcal{G}_{tight})$ by the result mentioned above, is an important alternative to the classical C*-algebra of an inverse semigroup studied, e.g. in [24, 2.1]. We also believe this addresses the concern expressed by Renault in [31, III.2.8.i].

From section (14) onwards we start our study of the C*-algebra of a semigroupoid Λ and, as in [25] and [13], the first step is to construct an inverse semigroup, which we denote by $S(\Lambda)$. This could be thought of as traversing the leftmost arrow of Diagram 1.1.

In sections (15)–(17) we show that tight representations of $S(\Lambda)$ correspond to certain representations of Λ, which by abuse of language we also call *tight*. This step is crucial because the definition of the semigroupoid C*-algebra naturally emphasizes the semigroupoid, rather than its associated inverse semigroup, so one needs to be able to determine tightness by looking at Λ only.

In the following section we essentially piece together the results so far obtained to arrive at another main result, namely Theorem (18.4), where we show that the semigroupoid C*-algebra is isomorphic to the groupoid C*-algebra of $\mathcal{G}_{tight}$, where $\mathcal{G}_{tight}$ is the groupoid of germs associated to the natural action of S on the tight part of the spectrum of its idempotent semilattice.

Our approach has an aesthetical advantage over [25] or [13] in the sense that our groupoid is constructed based on a very simple idea which can be conveyed in a single sentence, namely that one needs to focus on the set of ultra-filters, and necessarily also on the filters in its boundary. The disadvantage is that it leads to a very abstract picture of our groupoid and one may argue that a more concrete description is desirable.

We believe this concern may be addressed in the most general situation, and a dynamical system much like the one studied in [12] will certainly emerge, although, rather than a partial action of a group, it will be an action of an inverse semigroup. Given the widespread interest in combinatorial objects taking the form of a category, we instead specialize in section (19) to *categorical* semigroupoids, as defined in (19.1). This notion captures an essential property of categories which greatly simplifies the study of $S(\Lambda)$, and hence also of $\mathcal{G}_{tight}$. In Proposition (19.12) we then give a simple characterization of tight characters, resembling very much the description of boundary paths of [13].

In the closing section we focus directly on higher rank graphs and some effort is spent to determine which such objects lead to a semigroupoid admitting least common multiples. Not surprisingly only *singly aligned* higher rank graphs pass the test, in which case we may apply our machinery, arriving at a groupoid model of its C*-algebra.

The literature is rich in very interesting examples of inverse semigroups, such as certain inverse semigroups associated to tilings [20, 9.5], and we believe it might be a very fulfilling task to explore some of these from the point of view of tight representations.

We would like to thank Aidam Sims for bringing to our attention many relevant references on the subject of Higher Rank Graphs.

2 A quick motivation

Let us now briefly discuss the example of the Cuntz inverse semigroup [5], [31, III.2.2], since it is one of the main motivations for this work. The reader is invited to keep this example in mind throughout.

To avoid unnecessary complications we will restrict ourselves to the case $n = 2$.

Consider the semigroup S consisting of an identity 1, a zero element 0, and all the words in four letters, namely p_1, p_2, q_1, q_2, subject to the relations $q_j p_i = \delta_{i,j}$.

It is shown in [5, 1.3] that every element of S may be uniquely written as

$$p_{i_1} \ldots p_{i_k} q_{j_l} \ldots p_{j_1},$$

where $k, l \geqslant 0$, and $i_1, \ldots, i_k, j_1, \ldots, j_l \in \{1, 2\}$. It turns out that S is an inverse semigroup with $1^* = 1$, $0^* = 0$, and $p_i^* = q_i$.

Given a nondegenerated representation σ of S on a Hilbert space H, vanishing on 0, put $S_i = \sigma(p_i)$. It is then elementary to prove that the S_i satisfy

$$S_i^* S_j = \delta_{i,j},$$

which means that the S_i are isometries on H with pairwise orthogonal ranges. Conversely, given any two isometries on H with pairwise orthogonal ranges one may prove that there is a unique representation σ of S such that $\sigma(p_i) = S_i$. In other words the representations of S are in one-to-one correspondence with the pairs of isometries having orthogonal ranges.

If the reader is acquainted with the Cuntz algebra $\mathcal{O}_2$ he or she will likely wonder under which conditions on σ does the relation

$$S_1 S_1^* + S_2 S_2^* = 1 \tag{†}$$

also holds. After fiddling a bit whith this question one will realize that the occurence of the plus sign above is not quite in accordance with the language of semigroups (in which one only has the multiplication operation). In other words it is not immediately clear how to state (†) in the language of semigroups.

In order to approach this problem first notice that the idempotent semillatice $E(S)$ consists of 0, 1, and the idempotents

$$e_{i_1,\ldots,i_k} := p_{i_1} \ldots p_{i_k} q_{i_k} \ldots q_{i_1},$$

where $i_1, \ldots, i_k \in \{1, 2\}$. Clearly 1 is the largest element of $E(S)$, while 0 is the smallest.

Next observe that any nonzero idempotent f *intersects* either e_1 or e_2, in the sense that either fe_1 or fe_2 is nonzero. The set of idempotents $\{e_1, e_2\}$ is therefore what we shall call a *cover* for $E(S)$. One could try to indicate this fact by saying that

$$e_1 \vee e_2 = 1,$$

except that semillatices are only equipped with a meet operation "$\wedge$", rather than a join operation "$\vee$" as we seem to be in need of. However any (meet)-semilattice of projections on a Hilbert space is contained in a Boolean algebra of projections, in which a join operation is fortunately available, namely

$$p \vee q = p + q - pq.$$

Given a representation σ of S on a Hilbert space, one might therefore impose the condition that

$$\sigma(e_1) \vee \sigma(e_2) = 1,$$

which is tantamount to (†). Representations of S obeying this conditions will therefore be in one-to-one correspondence with representations of the Cuntz algebra $\mathcal{O}_2$.

The conclusion is therefore that the missing link between the representation theory of S and the Cuntz algebras lies in the order structure of the semillatice $E(S)$ in relation to Boolean algebras of projections on Hilbert's space.

3 Étale groupoids

In this section we will review the basic facts about étale groupoids which will be needed in the sequel. We follow more or less closely two of the most basic references in the subject, namely [31] and [24]. We will moreover strive to assume as few axioms and hypotheses as possible.

We assume the reader is familiar with the notion of groupoids (in the purely algebraic sense) and in particular with its basic notations: a groupoid is usually denoted by $\mathcal{G}$, its unit space by $\mathcal{G}^{(0)}$, and the set of composable pairs by $\mathcal{G}^{(2)}$. Finally the source and range maps are denoted by $\mathbf{d}$ and $\mathbf{r}$, respectively.

Given our interests, we go straight to the definition of étale groupoids without attempting to first define general locally compact groupoids. We nevertheless begin by recalling from [31, I.2.1] that a *topological groupoid* is a groupoid with a (not necessarily Hausdorff) topology with respect to which both the multiplication and the inversion are continuous.

Definition 3.1. [31, I.2.8] An *étale* (sometimes also called *r-discrete*) groupoid is a topological groupoid $\mathcal{G}$, whose unit space $\mathcal{G}^{(0)}$ is locally compact and Hausdorff in the relative topology, and such that the range map $\mathbf{r} : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

From now on we will assume that we are given an étale groupoid $\mathcal{G}$.

It is well known that $\mathbf{d}(x) = \mathbf{r}(x^{-1})$, for every x in $\mathcal{G}$, and hence $\mathbf{d}$ is a local homeomorphism as well. Like any local homeomorphism, $\mathbf{d}$ and $\mathbf{r}$ are open maps.

Proposition 3.2. *$\mathcal{G}^{(0)}$ is an open subset of $\mathcal{G}$.*

Proof. Let $x_0 \in \mathcal{G}^{(0)}$. By assumption there is an open subset A of $\mathcal{G}$ containing x_0, and an open subset B of $\mathcal{G}^{(0)}$ containing $\mathbf{r}(x_0)$, such that $\mathbf{r}(A) = B$, and $\mathbf{r}|_A$ is a homeomorphism onto B. Set $B' = A \cap B$, and notice that

$$x_0 = \mathbf{r}(x_0) \in A \cap B = B'.$$

Given that A is open in $\mathcal{G}$ we see that B' is open in B, hence $A' := \mathbf{r}^{-1}(B') \cap A$ is open in A, and moreover $\mathbf{r}$ is a homeomorphism from A' to B'.

We next claim that $B' \subseteq A'$. In order to prove it let $x \in B'$. So $x \in B \subseteq \mathcal{G}^{(0)}$, and hence $x = \mathbf{r}(x)$. This implies that $x \in \mathbf{r}^{-1}(B')$, and we already know that $x \in A$, so $x \in r^{-1}(B') \cap A = A'$.

We conclude that $\mathbf{r}$ is a bijective map from A' to B', which restricts to a surjective map (namely the identity) on the subset $B' \subseteq A'$. This implies that $B' = A'$, and since A' is open in $\mathcal{G}$, so is B'. The conclusion then follows from the fact that

$$x_0 \in B' \subseteq \mathcal{G}^{(0)}. \qquad \square$$

Definition 3.3. An open subset $U \subseteq \mathcal{G}$ is said to be a *slice*[1] if the restrictions of $\mathbf{d}$ and $\mathbf{r}$ to U are injective.

Since $\mathbf{d}$ and $\mathbf{r}$ are local homeomorphisms, for every slice U one has that $\mathbf{d}$ and $\mathbf{r}$ are homeomorphisms from U onto $\mathbf{d}(U)$ and $\mathbf{r}(U)$, respectively. For the same reason $\mathbf{d}(U)$ and $\mathbf{r}(U)$ are open subsets of $\mathcal{G}^{(0)}$, and hence also open in $\mathcal{G}$. It is obvious that every open subset of a slice is also a slice.

Proposition 3.4. *$\mathcal{G}^{(0)}$ is a slice.*

[1] Slices are sometimes referred to as *open $\mathcal{G}$-sets*. Some authors use the notation $\mathcal{G}^{op}$ for the set of all slices.

Proof. By (3.2) $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$. Since $\mathbf{r}$ and $\mathbf{d}$ coincide with the identity on $\mathcal{G}^{(0)}$, they are injective. □

We next present a crucial property of slices, sometimes used as the definition of étale groupoids [24, Definition 2.2.3]:

Proposition 3.5. *The collection of all slices forms a basis for the topology of $\mathcal{G}$.*

Proof. Let V be an open subset of $\mathcal{G}$ and let $x_0 \in V$. We must prove that there exists a slice U such that $x_0 \in U \subseteq V$.

Since $\mathbf{r}$ is a local homeomorphism there is an open subset A_1 of $\mathcal{G}$ containing x_0, and an open subset B_1 of $\mathcal{G}^{(0)}$ containing $\mathbf{r}(x_0)$, such that $\mathbf{r}(A_1) = B_1$, and $\mathbf{r}|_{A_1}$ is a homeomorphism onto B_1. Since $\mathbf{d}$ is also a local homeomorphism, we may choose an open subset A_2 of $\mathcal{G}$ containing x_0, and an open subset B_2 of $\mathcal{G}^{(0)}$ containing $\mathbf{d}(x_0)$, such that $\mathbf{d}(A_2) = B_2$, and $\mathbf{d}|_{A_2}$ is a homeomorphism onto B_2. Therefore $U := A_1 \cap A_2 \cap V$ is a slice containing x_0, and contained in V. □

If U is a slice then $\mathbf{r}(U)$ is an open subset of the locally compact Hausdorff space $\mathcal{G}^{(0)}$, and hence $\mathbf{r}(U)$ also possesses these properties. Since U is homeomorphic to $\mathbf{r}(U)$ we have:

Proposition 3.6. *Every slice is a locally compact Hausdorff space in the relative topology.*

If V is a subset of $\mathcal{G}$ which is open and Hausdorff, observe that the set of all intersections $V \cap U$, where U is a slice, forms a basis for the relative topology of V by (3.5). Notice that $V \cap U$ is locally compact because it is an open subset of the locally compact space U. From this it is easy to see that V itself is locally compact. This proves:

Proposition 3.7. *Every open Hausdorff subset of $\mathcal{G}$ is locally compact.*

The following result is proved in Proposition (2.2.4) of [24], and although we are not assuming the exact same set of hypotheses, the proof given there works under our conditions:

Proposition 3.8. *If U and V are slices then*

(i) $U^{-1} = \{u^{-1} : u \in U\}$ *is a slice, and*

(ii) $UV = \{uv : u \in U,\ v \in V,\ (u, v) \in \mathcal{G}^{(2)}\}$ *is a (possibly empty) slice.*

The theory of continuous functions on locally compact Hausdorff spaces has many rich features which one wishes to retain in the study of non necessarily Hausdorff groupoids. The definition of $C_c(\mathcal{G})$, first used in [4], and also given in [24], takes advantage of the abundance of Hausdorff subspaces of $\mathcal{G}$.

Definition 3.9. We shall denote by $C_c^0(\mathcal{G})$ the set of all complex valued functions f on $\mathcal{G}$ for which there exists a subset $V \subseteq \mathcal{G}$, such that

(i) V is open and Hausdorff in the relative topology,

(ii) f vanishes outside V, and

(iii) the restriction of f to V is continuous and compactly supported[2].

We finally define $C_c(\mathcal{G})$ as the linear span of $C_c^0(\mathcal{G})$ within the space of all complex valued functions on $\mathcal{G}$.

We would like to stress that functions in $C_c(\mathcal{G})$ might not be continuous relative to the global topology of $\mathcal{G}$.

Suppose that V is an open Hausdorff subset of $\mathcal{G}$ and let $f \in C_c(V)$. Considering f as a function on $\mathcal{G}$ by extending it to be zero outside V, it is immediate that $f \in C_c^0(\mathcal{G})$, and hence also $f \in C_c(\mathcal{G})$. This said we will henceforth view $C_c(V)$ as a subset of $C_c(\mathcal{G})$.

Proposition 3.10. *Let $\mathscr{C}$ be a covering of $\mathcal{G}$ consisting of slices. Then $C_c(\mathcal{G})$ is linearly spanned by the collection of all subspaces of the form $C_c(U)$, where $U \in \mathscr{C}$.*

Proof. Given $f \in C_c^0(\mathcal{G})$ pick V as in (3.9). Observe that V is locally compact Hausdorff by (3.7), and that $\{U \cap V : U \in \mathscr{C}\}$ is a covering for V. We may then use a standard partition of unit argument to prove that f may be written as a finite sum of functions $f_i \in C_c(V \cap U_i) \subseteq C_c(U_i)$, where each U_i is a slice in $\mathscr{C}$. This concludes the proof. □

We are now about to introduce the operations that will eventually lead to the C*-algebra of $\mathcal{G}$. Normally this is done by first introducing a Haar system on $\mathcal{G}$. In étale groupoids a Haar system is just a collection of counting measures, so the whole issue becomes a lot simpler. So much so that we can get away without even mentioning Haar systems.

[2]That is $f|_V \in C_c(V)$, where the latter has the usual meaning.

Proposition 3.11. *Given $f, g \in C_c(\mathcal{G})$ define, for every $x \in \mathcal{G}$,*

$$(f \star g)(x) = \sum_{\substack{(y,z)\in\mathcal{G}^{(2)} \\ x=yz}} f(y)g(z), \qquad \text{and} \qquad f^*(x) = \overline{f(x^{-1})}.$$

Then

(i) *$f \star g$ and f^* are well defined complex functions on $\mathcal{G}$ belonging to $C_c(\mathcal{G})$,*

(ii) *if $f \in C_c(U)$ and $g \in C_c(V)$, where U and V are slices, then $f \star g \in C_c(UV)$.*

(iii) *if $f \in C_c(U)$, where U is a slice, then $f^* \in C_c(U^{-1})$.*

Proof. The parts of the statement concerning f^* are trivial, so we leave them as exercises. We begin by addressing the finiteness of the sum above. For this we use (3.10) to write $f = \sum_{i=1}^n f_i$, where each $f_i \in C_c(U_i)$, and U_i is a slice.

If $x = yz$, and $f(y)g(z)$ is nonzero, then $f_i(y)$ is nonzero for some $i = 1, \ldots, n$, and hence $y \in U_i$. Observing that $\mathbf{r}(y) = \mathbf{r}(x)$, and that there exists at most one $y \in U_i$ with that property, we see that there exists at most n pairs (y, z) such that $yz = x$, and $f(y) \neq 0$. This proves that the above sum is finite, and hence that $f \star g$ is a well defined complex valued function on $\mathcal{G}$.

Since "$\star$" is clearly a bilinear operation, in order to prove that $f \star g \in C_c(\mathcal{G})$, one may again use (3.10) in order to assume that $f \in C_c(U)$ and $g \in C_c(V)$, where U and V are slices. That is, it suffices to prove (ii), which we do next.

So let us be given f and g as in (ii). If $(f \star g)(x) \neq 0$, then there exists at least one pair $(y, z) \in \mathcal{G}^{(2)}$, such that $x = yz$, $y \in U$, and $z \in V$, but since $\mathbf{r}(y) = \mathbf{r}(x)$, and $\mathbf{d}(z) = \mathbf{d}(x)$, we necessarily have that

$$y = \mathbf{r}_U^{-1}\mathbf{r}(x), \quad \text{and} \quad z = \mathbf{d}_V^{-1}\mathbf{d}(x),$$

where we are denoting by $\mathbf{r}_U$ the restriction of $\mathbf{r}$ to U, and by $\mathbf{d}_V$ the restriction of $\mathbf{d}$ to V. It is then easy to see that

$$(f \star g)(x) = \begin{cases} f\big(\mathbf{r}_U^{-1}\mathbf{r}(x)\big)\, g\big(\mathbf{d}_V^{-1}\mathbf{d}(x)\big), & \text{if } x \in UV, \\ 0 \qquad\qquad\qquad\qquad , & \text{otherwise.} \end{cases} \tag{3.11.1}$$

In addition the above formula for $f \star g$ proves that it is continuous on UV, so we must only show that $f \star g$ is compactly supported on UV. If $A \subseteq U$ and

$B \subseteq V$ are the compact supports of f and g in U and V, respectively, we claim that AB is compact. In fact, since $\mathcal{G}^{(0)}$ is Hausdorff, we have that

$$\mathcal{G}^{(2)} = \{(x, y) \in \mathcal{G} \times \mathcal{G} : \mathbf{d}(x) = \mathbf{r}(y)\}$$

is closed in $\mathcal{G} \times \mathcal{G}$. So $(A \times B) \cap \mathcal{G}^{(2)}$ is closed in $A \times B$, and hence compact. Since AB is the image of $(A \times B) \cap \mathcal{G}^{(2)}$ under the continuous multiplication operator, we conclude that AB is compact, as claimed. Observing that $f \star g$ vanishes outside AB, we deduce that $f \star g \in C_c(UV)$. □

It is now routine to show that $C_c(\mathcal{G})$ is an associative complex *-algebra with the operations defined above.

We have already commented on the fact that $C_c(V) \subseteq C_c(\mathcal{G})$, for every open Hausdorff subset V of $\mathcal{G}$, and hence $C_c(\mathcal{G}^{(0)}) \subseteq C_c(\mathcal{G})$. A quick glance at the definitions of the operations will convince the reader that $C_c(\mathcal{G}^{(0)})$ is also a *-subalgebra of $C_c(\mathcal{G})$, the induced multiplication and adjoint operations corresponding to the usual pointwise operations on $C_c(\mathcal{G}^{(0)})$.

Proposition 3.12. *Let U be a slice and let $f \in C_c(U)$. Then $f \star f^*$ lies in $C_c(\mathcal{G}^{(0)})$.*

Proof. Given that U is a slice, we have that $UU^{-1} = \mathbf{r}(U)$. Moreover, by (3.11.iii) we have that $f^* \in C_c(U^{-1})$, and hence by (3.11.ii),

$$f \star f^* \in C_c(UU^{-1}) \subseteq C_c\big(\mathbf{r}(U)\big) \subseteq C_c(\mathcal{G}^{(0)}). \qquad \square$$

We would now like to discuss representations of $C_c(\mathcal{G})$. So let H be a Hilbert space and let

$$\pi : C_c(\mathcal{G}) \to B(H)$$

be a *-representation. Obviously the restriction of π to $C_c(\mathcal{G}^{(0)})$ is a *-representation of the latter. Since $C_c(\mathcal{G}^{(0)})$ is the union of C*-algebras[3], π is necessarily contractive with respect to the norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{x \in \mathcal{G}^{(0)}} |f(x)|, \quad \forall f \in C_c(\mathcal{G}^{(0)}).$$

Therefore, if U is a slice and $f \in C_c(U)$, we have

$$\|\pi(f)\|^2 = \|\pi(f)\pi(f)^*\| = \|\pi(f \star f^*)\| \leqslant \|f \star f^*\|_\infty, \tag{3.13}$$

[3] Namely the subalgebras of $C_c(\mathcal{G}^{(0)})$ formed by all continuous functions that vanish outside a fixed compact subset $K \subseteq \mathcal{G}^{(0)}$.

because $f \star f^* \in C_c(\mathcal{G}^{(0)})$, by (3.12). Notice that for every $x \in \mathcal{G}^{(0)}$ we have

$$(f \star f^*)(x) = \sum_{x=yz} f(y)\overline{f(z^{-1})},$$

where any nonzero summand must correspond to a pair (y, z) such that $\mathbf{d}(y) = \mathbf{r}(z) = \mathbf{d}(z^{-1})$, with both $y, z^{-1} \in U$. Since U is a slice this implies that $y = z^{-1}$, but since $\mathbf{r}(y) = \mathbf{r}(x)$, we have that $y = \mathbf{r}_U^{-1}\big(\mathbf{r}(x)\big)$, so, provided $(f \star f^*)(x)$ is nonzero one has that $(f \star f^*)(x) = |f(\mathbf{r}_U^{-1}\big(\mathbf{r}(x)\big)|^2$, therefore

$$\|f \star f^*\|_\infty = \sup_{x\in\mathcal{G}^{(0)}} |(f \star f^*)(x)| = \sup_{u\in U} |f(u)|^2 = \|f\|_\infty^2,$$

where we are also denoting by $\|\cdot\|_\infty$ the sup norm on $C_c(U)$. Combining this with (3.13) we have proven:

Proposition 3.14. *If π is any *-representation of $C_c(\mathcal{G})$ on a Hilbert space H then for every slice U and for every $f \in C_c(U)$ one has that $\|\pi(f)\| \leqslant \|f\|_\infty$.*

By (3.10) any $f \in C_c(\mathcal{G})$ may be written a finite fum $f = \sum_{i=1}^n f_i$, where $f_i \in C_c(U_i)$, and U_i is a slice. So for every representation π of $C_c(\mathcal{G})$ we have

$$\|\pi(f)\| \leqslant \sum_{i=1}^n \|\pi(f_i)\| \leqslant \sum_{i=1}^n \|f_i\|_\infty, \tag{3.15}$$

by (3.14). Regardless of its exact significance,[4] the right-hand side of (3.15) depends only on f and not on π. This means that

$$|||f||| := \sup_\pi \|\pi(f)\| < \infty, \tag{3.16}$$

for all $f \in C_c(\mathcal{G})$. It is then easy to see that $|||\cdot|||$ is a C*-seminorm on $C_c(\mathcal{G})$ and hence its Hausdorff completion is a C*-algebra.

Definition 3.17. The C*-algebra of $\mathcal{G}$, denoted $C^*(\mathcal{G})$, is defined to be the completion of $C_c(\mathcal{G})$ under the norm $|||\cdot|||$ defined above. We will moreover denote by

$$i : C_c(\mathcal{G}) \to C^*(\mathcal{G}) \tag{3.17.1}$$

the natural inclusion given by the completion process, which is injective by [31, 4.2.i].

[4] It is related to the so called I-norm of f [31, II.1.3].

Let us now study approximate units in $C^*(\mathcal{G})$. For this recall that $C_c(\mathcal{G}^{(0)})$ is a subalgebra of $C_c(\mathcal{G})$.

Proposition 3.18. *Let $\{u_i\}_{i\in I}$ be a bounded selfadjoint approximate unit for $C_c(\mathcal{G}^{(0)})$ relative to the norm $\|\cdot\|_\infty$. Then $\{i(u_i)\}_{i\in I}$ is an approximate unit for $C^*(\mathcal{G})$.*

Proof. It is obviously enough to prove that $\{u_i\}_{i\in I}$ is a bounded approximate unit for $C_c(\mathcal{G})$ relative to $|||\cdot|||$. In view of (3.10) it in fact suffices to verify that for every slice U and for every $f \in C_c(U)$ one has that

$$\lim_i |||f \star u_i - f||| = 0.$$

By (3.4) we have that $\mathcal{G}^{(0)}$ is a slice, and it is easy to see that $U\mathcal{G}^{(0)} = U$, so by (3.11) we have that $f \star u_i \in C_c(U)$. Moreover, for every $x \in U$ one has that

$$(f \star u_i)(x) = f(x)u_i(\mathbf{d}(x)).$$

By (3.14) we conclude that

$$|||f \star u_i - f||| \leqslant \sup_{x\in U} \big|f(x)u_i(\mathbf{d}(x)) - f(x)\big|,$$

which converges to zero because u_i converges uniformly to 1 on every compact subset of $\mathcal{G}^{(0)}$, such as the image under $\mathbf{d}$ of the compact support of f. $\square$

4 Inverse semigroup actions

Recall that a semigroup S is said to be an *inverse semigroup* if for every $s \in S$, there exists a unique $s^* \in S$ such that

$$ss^*s = s, \quad \text{and} \quad s^*ss^* = s^*. \tag{4.1}$$

It is well known that the correspondence $s \mapsto s^*$ is then an involutive anti-homomorphism. One usually denotes by $E(S)$ the set of all idempotent elements of S, such as s^*s, for every $s \in S$. For a thorough treatment of this subject the reader is referred to [20], and [24].

We next recall the definition of one of the most important examples of inverse semigroups:

Definition 4.2. If X is any set we denote by $\mathcal{I}(X)$ the inverse semigroup formed by all bijections between subsets of X, under the operation given by composition of functions in the largest domain in which the composition may be defined.

The following is a crucial concept to be studied throughout the remaining of this work.

Definition 4.3. Let S be an inverse semigroup and let X be a locally compact Hausdorff topological space. An *action* of S on X is a semigroup homomorphism

$$\theta : S \to \mathcal{I}(X)$$

such that,

(i) for every $s \in S$ one has that θ_s is continuous and its domain is open in X,

(ii) the union of the domains of all the θ_s coincides with X.

Fix for the duration of this section an action θ of S on X.

Observe that if $s \in S$ then from (4.1) we get

$$\theta_s\theta_{s^*}\theta_s = \theta_s, \quad \text{and} \quad \theta_{s^*}\theta_s\theta_{s^*} = \theta_{s^*},$$

which implies that $\theta_{s^*} = \theta_s^{-1}$.

Notice that the range of each θ_s coincides with the domain of $\theta_s^{-1} = \theta_{s^*}$, and hence it is open as well. This also says that θ_s^{-1} is continuous, so θ_s is necessarily a homeomorphism onto its range.

In the absence of the property expressed in the last sentence of the above definition one may replace X by the open subspace X_0 formed by the union of the domains of all the θ_s. It is then apparent that θ gives an action of S on X_0 with all of the desired properties. In other words, the restriction imposed by that requirement is not so severe.

It is well known that if e is an idempotent, that is, if $e^2 = e$, then θ_e is the identity map on its domain.

Notation 4.4. For every idempotent $e \in E(S)$ we will denote[5] by D_e the domain (and range) of θ_e.

It is easy to see that θ_s and θ_{s^*s} share domains, and hence the domain of θ_s is D_{s^*s}. Likewise the range of θ_s is given by D_{ss^*}. Thus θ_s is a homeomorphism between the open sets

$$\theta_s : D_{s^*s} \to D_{ss^*}.$$

[5]Some authors adopt the notation D_s, even if s is not idempotent, to mean the *range* of θ_s, which therefore coincides with our D_{ss^*}. We shall however not do so in order to avoid introducing an unnecessary convention: one could alternatively choose to use D_s to denote the *domain* of θ_s. Once one is accustomed to the idea that the source and range projections of a partial isometry u are u^*u and uu^*, respectively, the notations D_{s^*s} and D_{ss^*} require no convention to convey the idea of domain and range.

If e and f are idempotents it is easy to conclude from the identity $\theta_e\theta_f = \theta_{ef}$, that $D_e \cap D_f = D_{ef}$. The next result appears in [33, 4.2].

Proposition 4.5. *For each $s \in S$ and $e \in E(S)$ one has that*

$$\theta_s(D_e \cap D_{s^*s}) = D_{ses^*}.$$

Proof. By the observation above we have

$$\theta_s(D_e \cap D_{s^*s}) = \theta_s(D_{es^*s}),$$

which coincides with the range of $\theta_s\theta_{es^*s} = \theta_{ses^*s}$. The conclusion then follows from the following calculation:

$$ses^*s(ses^*s)^* = ses^*s\, s^*ses^* = ses^*. \qquad \square$$

Our next short term goal is to construct a *groupoid of germs* from θ. However, given the examples of inverse semigroups that we have in mind, we would rather not assume that θ is given in terms of a *localization*, as in [24, Theorem 3.3.2]. Nor do we want to assume that S is *additive*, as in [24, Corollary 3.3.2]. We also want to avoid using the condition of *fullness* [26, 5.2], which is used to prove a result [26, 8.1] similar to what we are looking for in the Hausdorff case.

Definition 4.6. [24, page 140] We will denote by Ω the subset of $S \times X$ given by

$$\Omega = \big\{(s, x) \in S \times X : x \in D_{s^*s}\big\},$$

and for every (s, x) and (t, y) in Ω we will say that $(s, x) \sim (t, y)$, if $x = y$, and there exists an idempotent e in $E(S)$ such that $x \in D_e$, and $se = te$. The equivalence class of (s, x) will be called the *germ of s at x*, and will be denoted by $[s, x]$.

Given (s, x) and (t, y) in Ω such that $(s, x) \sim (t, y)$, and letting e be the idempotent mentioned in (4.6), observe that

$$x \in D_e \cap D_{s^*s} \cap D_{t^*t} = D_{es^*st^*t}.$$

If we set $e_0 = es^*st^*t$, it then follows that $se_0 = te_0$. So, upon replacing e by e_0, we may always assume that the idempotent e in (4.6) satisfies $e \leqslant s^*s, t^*t$.

Proposition 4.7. *Given (s, x) and (t, y) in Ω such that $x = \theta_t(y)$, one has that*

(i) $(st, y) \in \Omega$, *and*

(ii) *the germ $[st, y]$ depends only on the germs $[s, x]$ and $[t, y]$.*

Proof. Initially observe that

$$y = \theta_{t^*}(x) \in \theta_{t^*}(D_{s^*s} \cap D_{tt^*}) \overset{(4.5)}{=} D_{t^*s^*st} = D_{(st)^*st},$$

so (st, y) indeed belongs to Ω. Next let (s', x) and (t', y) be elements of Ω such that $(s', x) \sim (s, x)$ and $(t', y) \sim (t, y)$. Therefore there are idempotents e and f such that $x \in D_e$, $y \in D_f$, $se = s'e$, and $tf = t'f$. We then have that

$$\theta_{t'}(y) = \theta_{t'}\big(\theta_f(y)\big) = \theta_{t'f}(y) = \theta_{tf}(y) = \theta_t\big(\theta_f(y)\big) = \theta_t(y) = x.$$

In other words, the fact that $\theta_t(y) = x$ does not depend on representatives. By (i) it then follows that $(s't', y) \in \Omega$ and we will be finished once we prove that $(st, y) \sim (s't', y)$. For this let d be the idempotent given by $d = ft^*et$, and we claim that $y \in D_d$. To see this notice that since $x \in D_e \cap D_{tt^*}$, it follows that

$$y = \theta_{t^*}(x) \in \theta_{t^*}(D_e \cap D_{tt^*}) = D_{t^*et},$$

and since $y \in D_f$ by assumption, we deduce that

$$y \in D_f \cap D_{t^*et} = D_{ft^*et} = D_d.$$

This proves our claim. In addition we have

$$s't'd = s't'ft^*et = s'tft^*et = s'etft^*t = setft^*t = stft^*et = std,$$

proving that $(st, y) \sim (s't', y)$, as required. ∎

Let

$$\mathcal{G} = \Omega/\sim$$

be the set of all germs, and put

$$\mathcal{G}^{(2)} = \big\{\big([s, x], [t, y]\big) \in \mathcal{G} \times \mathcal{G} : x = \theta_t(y)\big\}. \tag{4.8}$$

For $\big([s, x], [t, y]\big) \in \mathcal{G}^{(2)}$ define

$$[s, x] \cdot [t, y] = [st, y], \tag{4.9}$$

and

$$[s, x]^{-1} = \big[s^*, \theta_s(x)\big]. \tag{4.10}$$

We leave it for the reader to prove:

Proposition 4.11. *$\mathcal{G}$ is a groupoid with the operations defined above, and the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ naturally identifies with X under the correspondence*

$$[e, x] \in \mathcal{G}^{(0)} \mapsto x \in X,$$

where e is any idempotent such that $x \in D_e$.

Although we are not providing a proof of the result above, we observe that the last part of the statement depends upon the assumption made in the last sentence of Definition (4.3).

The *source*[6] map of $\mathcal{G}$ is clearly given for every $[t, x] \in \mathcal{G}$ by

$$\mathbf{d}[t, x] = [t, x]^{-1}[t, x] = [t^*, \theta_t(x)]\,[t, x] = [t^*t, x].$$

Enforcing the identification referred to in (4.11) we will write

$$\mathbf{d}[t, x] = x.$$

With respect to the *range* map, a similar reasoning gives

$$\mathbf{r}[t, x] = \theta_t(x).$$

We would now like to give $\mathcal{G}$ a topology. For this, given any $s \in S$, and any open subset $U \subseteq D_{s^*s}$, let

$$\Theta(s, U) = \{[s, x] \in \mathcal{G} : x \in U\}. \tag{4.12}$$

Proposition 4.13. *Let s and t be elements of S and let U and V be open sets with $U \subseteq D_{s^*s}$, and $V \subseteq D_{t^*t}$. If $[r, z] \in \Theta(s, U) \cap \Theta(t, V)$ then there exists an idempotent e and an open set $W \subseteq D_{(re)^*re}$ such that*

$$[r, z] \in \Theta(re, W) \subseteq \Theta(s, U) \cap \Theta(t, V).$$

Proof. By assumption $[r, z] = [s, x] = [t, y]$, for some $x \in U$ and $y \in V$. But this implies that $z = x = y$, so $z \in U \cap V$. In addition there are idempotents e and f such that $z \in D_e$, $z \in D_f$, $re = se$, and $rf = tf$. Replacing e and f by ef, we may assume without loss of generality that $e = f$, hence $re = se = te$.

[6] Given the several uses of the letter "s" in the setting of semigroups, we have decided to allow the idea of "domain" to determine the letter to denote the source map, a convention that is not rare in the literature.

Set $W = U \cap V \cap D_{(re)^*re}$. Since $z \in D_{r^*r} \cap D_e = D_{r^*re} = D_{(re)^*re}$, we see that $z \in W$, and hence

$$[r, z] = [re, z] \in \Theta(re, W).$$

In order to prove that $\Theta(re, W) \subseteq \Theta(s, U) \cap \Theta(t, V)$, let $[re, x]$ be a generic element of $\Theta(re, W)$, so that $x \in W$. Noticing that $x \in U$, and that

$$[re, x] = [se, x] = [s, x],$$

we see that $[re, x] \in \Theta(s, U)$, and a similar reasoning gives $[re, x] \in \Theta(t, V)$. □

By the result above we see that the collection of all $\Theta(s, U)$ forms the basis of a topology on $\mathcal{G}$. From now on $\mathcal{G}$ will be considered to be equipped with this topology, and hence $\mathcal{G}$ is a topological space.

Proposition 4.14. *With the above topology $\mathcal{G}$ is a topological groupoid.*

Proof. Our task is to prove that the multiplication and inversion operations on $\mathcal{G}$ are continuous. For this let $[s, x]$ and $[t, y]$ be elements of $\mathcal{G}$ such that $\big([s, x], [t, y]\big) \in \mathcal{G}^{(2)}$. Moreover suppose that the product of these elements lie in a given open set $W \subseteq \mathcal{G}$. Therefore, there exists some $r \in S$ and an open set $V \subseteq D_{r^*r}$, such that

$$[s, x][t, y] = [st, y] \in \Theta(r, V) \subseteq W.$$

This implies that $y \in V$ and that there exists some idempotent e such that $y \in D_e$, and $ste = re$.

Setting $U = V \cap D_e \cap D_{t^*t}$, we will prove that the product of any pair of elements

$$\big([s, x'], [t, y']\big) \in \big(\Theta(s, D_{s^*s}) \times \Theta(t, U)\big) \cap \mathcal{G}^{(2)} \tag{4.14.1}$$

lies in W. The product referred to is clearly given by $[st, y']$, and since $y' \in U \subseteq D_e$, we have

$$[st, y'] = [r, y'] \in \Theta(r, V) \subseteq W.$$

Observing that $x \in D_{s^*s}$, and $y \in V \subseteq U$, we see that the set appearing in (4.14.1) is a neighborhood of $\big([s, x], [t, y]\big)$ in the relative topology of $\mathcal{G}^{(2)}$. This proves that multiplication is continuous.

With respect to inversion let $s \in S$ and let $U \subseteq D_{s^*s}$ be an open set. From the definition of the inversion in (4.10) it is clear that

$$\Theta(s, U)^* = \Theta(s^*, \theta_s(U)),$$

from which the continuity of the inversion follows immediately. □

We shall now begin to work towards proving that $\mathcal{G}$ is an étale groupoid.

Proposition 4.15. *Given $s \in S$, let $U \subseteq D_{s^*s}$ be an open set. Then the map*

$$\phi : x \in U \mapsto [s, x] \in \Theta(s, U)$$

is a homeomorphism, where $\Theta(s, U)$ of course carries the topology induced from $\mathcal{G}$.

Proof. By the definition of the equivalence relation in (4.6) it is obvious that ϕ is a bijective map. Let $V \subseteq U$ be an open subset. Then clearly $\phi(V) = \Theta(s, V)$, which is open in $\mathcal{G}$, proving that ϕ is an open mapping. To prove that ϕ is continuous at any given $x \in U$, let W be a neighborhood of $\phi(x)$ in $\Theta(s, U)$, so there exists some $t \in S$ and an open set $V \subseteq D_{t^*t}$, such that

$$[s, x] = \phi(x) \in \Theta(t, V) \subseteq W \subseteq \Theta(s, U).$$

Clearly this implies that $x \in V \subseteq U \subseteq D_{s^*s}$. In addition there exists some idempotent e such that $x \in D_e$, and $se = te$. For every $y \in D_e \cap V$ observe that

$$\phi(y) = [s, y] = [t, y] \in \Theta(t, V) \subseteq W,$$

which means that $\phi(D_e \cap V) \subseteq W$. Since $D_e \cap V$ is a neighborhood of x in U, we see that ϕ is continuous. □

We have already seen that $\mathcal{G}^{(0)}$, the unit space of $\mathcal{G}$, corresponds to X. The result above helps to complete that picture by showing that the correspondence is topological:

Corollary 4.16. *The identification of $\mathcal{G}^{(0)}$ with X given by* (4.11) *is a homeomorphism.*

Proof. Given $[e, x] \in \mathcal{G}^{(0)}$ we have that D_e is an open subset of X containing x and $\Theta(e, D_e)$ is an open subset of $\mathcal{G}^{(0)}$ containing $[e, x]$. The result then follows from the fact that

$$\phi : y \in D_e \mapsto [e, y] \in \Theta(e, D_e)$$

is a homeomorphism by (4.15). □

Having assumed that X is locally compact and Hausdorff, it follows from the above result that $\mathcal{G}^{(0)}$ shares these properties. The source map on every basic open set $\Theta(s, U)$ is a homeomorphism onto U because it is the inverse of the map ϕ of (4.15). This implies that $\mathbf{d}$ is a local homeomorphism, and hence so is $\mathbf{r}$. This implies that:

Proposition 4.17. *The groupoid $\mathcal{G} = \mathcal{G}(\theta, S, X)$ constructed above, henceforth called the groupoid of germs of the system (θ, S, X), is an étale groupoid.*

The following identifies important slices in $\mathcal{G}$.

Proposition 4.18. *For every $s \in S$ and every open subset $U \subseteq D_{s^*s}$, one has that $\Theta(s, U)$ is a slice.*

Proof. By definition of the topology on $\mathcal{G}$ we have that $\Theta(s, U)$ is open in S. Recall that source map is given by $\mathbf{d} : [s, x] \mapsto x$, whose restriction to $\Theta(s, U)$ is the inverse of the map ϕ of (4.15), so it is injective. With respect to the restriction of the range map $\mathbf{r}$ on $\Theta(s, U)$, notice that $\mathbf{r} = \theta_s \circ \mathbf{d}$, which is injective. □

5 Example: Action of the inverse semigroup of slices

The main goal of this section is to present an example of inverse semigroup actions which is intrinsic to every étale groupoid. We therefore fix an étale groupoid $\mathcal{G}$ from now on. Denote[7] by $S(\mathcal{G})$ the set of all slices in $\mathcal{G}$. It is well known [24, Proposition 2.2.4] that $S(\mathcal{G})$ is an inverse semigroup under the operations

$$UV = \big\{uv : u \in U,\ v \in V,\ (u, v) \in \mathcal{G}^{(2)}\big\}, \quad \text{and} \quad U^* = \big\{u^{-1} : u \in U\big\},$$

for all slices U and V in $S(\mathcal{G})$. The idempotent semilattice of $S(\mathcal{G})$ is easily seen to consist precisely of the open subsets of $\mathcal{G}^{(0)}$.

Henceforth denoting by

$$X := \mathcal{G}^{(0)},$$

we wish to define an action θ of $S(\mathcal{G})$ on X. Given a slice U we have already mentioned that $\mathbf{d}(U)$ and $\mathbf{r}(U)$ are open subsets of X, and moreover that the maps

$$\mathbf{d}_U : U \to \mathbf{d}(U), \quad \text{and} \quad \mathbf{r}_U : U \to \mathbf{d}(U),$$

obtained by restricting $\mathbf{d}$ and $\mathbf{r}$, respectively, are homeomorphisms. Given $x \in \mathbf{d}(U)$ we let

$$\theta_U(x) = \mathbf{r}_U\big(\mathbf{d}_U^{-1}(x)\big). \tag{5.1}$$

Clearly θ_U is a homeomorphism from $\mathbf{d}(U)$ to $\mathbf{r}(U)$. It is interesting to observe that $\theta_U(x) = y$, if and only if there exists some $u \in U$ such that $\mathbf{d}(u) = x$

[7] As already observed some authors denote this set by $\mathcal{G}^{op}$.

and $\mathbf{r}(u) = y$. Thus, if we view θ_U as a set of ordered pairs, according to the technical definition of functions, we have

$$\theta_U = \big\{\big(\mathbf{d}(u), \mathbf{r}(u)\big) : u \in U\big\}. \tag{5.2}$$

We would like to show that $\theta_U\theta_V = \theta_{UV}$, for all $U, V \in S(\mathcal{G})$. Assuming that $\theta_U\theta_V(x) = z$, or equivalently that $(x, z) \in \theta_U\theta_V$, there exists $y \in X$, such that $(y, z) \in \theta_U$ and $(x, y) \in \theta_V$, so we may pick $u \in U$ and $v \in V$ such that $\mathbf{d}(v) = x$, $\mathbf{r}(v) = y = \mathbf{d}(u)$, and $\mathbf{r}(u) = z$.

$$\begin{tikzcd} x \arrow[r, bend left, "v"] & y \arrow[r, bend left, "u"] & z \end{tikzcd}$$

Therefore we have that $uv \in UV$, and since

$$(x, z) = \big(\mathbf{d}(v), \mathbf{r}(u)\big) = \big(\mathbf{d}(uv), \mathbf{r}(uv)\big) \in \theta_{UV},$$

we see that $\theta_{UV}(x) = z$. Conversely, if we are given that $\theta_{UV}(x) = z$, there exists some $w \in UV$ such that $\mathbf{d}(w) = x$, and $\mathbf{r}(w) = z$. Writing $w = uv$, with $u \in U$ and $v \in V$, set $y = \mathbf{r}(v) = \mathbf{d}(u)$. Then

$$(x, y) = \big(\mathbf{d}(w), \mathbf{r}(v)\big) = \big(\mathbf{d}(v), \mathbf{r}(v)\big) \in \theta_V,$$

and similarly $(y, z) = \big(\mathbf{d}(u), \mathbf{r}(u)\big) \in \theta_U$, and we see that $(x, z) \in \theta_U\theta_V$, thus proving that $\theta_U\theta_V = \theta_{UV}$.

The last condition to be checked in order to prove that θ is an action is (4.3.ii), but this is obvious because $X = \mathcal{G}^{(0)}$ is a slice by (3.4), and θ_X is clearly the identity map defined on the whole of X. With this we have proven:

Proposition 5.3. *The correspondence* $U \mapsto \theta_U$, *defined by* (5.1), *gives an action of* $S(\mathcal{G})$ *on the unit space of* $\mathcal{G}$.

Given any *-subsemigroup[8] $S \subseteq S(\mathcal{G})$, one may restrict θ to S, thus obtaining a semigroup homomorphism

$$\theta|_S : S \to \mathcal{I}(X)$$

which is an action of S on X, provided (4.3.ii) may be verified. The next result gives sufficient conditions for the groupoid of germs for such an action to be equal to $\mathcal{G}$.

[8] A subsemigroup of an inverse semigroup is said to be a *-subsemigroup if it is closed under the * operation, in which case it is clearly an inverse semigroup in itself.

Proposition 5.4. *Let $\mathcal{G}$ be an étale groupoid and let S be a *-subsemigroup of $S(\mathcal{G})$ such that*

(i) $\mathcal{G} = \bigcup_{U \in S} U$, *and*

(ii) *for every $U, V \in S$, and every $u \in U \cap V$, there exists $W \in S$, such that $u \in W \subseteq U \cap V$.*

Then $\theta|_S$ is an action of S on $X = \mathcal{G}^{(0)}$, and the groupoid of germs for $\theta|_S$ is isomorphic to $\mathcal{G}$.

Proof. Given $x \in X$, there exists some $U \in S$ such that $x \in U$, by (i), and so $(x, x) = \big(\mathbf{d}(x), \mathbf{r}(x)\big) \in \theta_U$, and in particular x is in the domain of U. This proves (4.3.ii) and hence $\theta|_S$ is indeed an action of S on X.

Let us temporarily denote the groupoid of germs for $\theta|_S$ by $\mathcal{H}$. Observe that the domain of θ_U is $\mathbf{d}(U)$, so $\mathcal{H}$ is given by

$$\mathcal{H} = \big\{[U, x] : U \in S,\ x \in \mathbf{d}(U)\big\}.$$

Given a germ $[U, x] \in \mathcal{H}$ we therefore have that there exists a unique $u_0 \in U$ such that $\mathbf{d}(u_0) = x$, because $\mathbf{d}|_U$ is injective.

We claim that u_0 depends only on the germ $[U, x]$. For this suppose that $[U, x] = [V, x]$, for some $V \in S$, which means that there is an idempotent $E \in S$ such that $x \in \mathbf{d}(E)$ and $UE = VE$. As observed earlier, E is necessarily a subspace of X and hence $E = \mathbf{d}(E)$. Applying the definition of the product one gets

$$UE = \{u \in U : \mathbf{d}(u) \in E\},$$

and since $\mathbf{d}(u_0) = x \in \mathbf{d}(E) = E$, we conclude that $u_0 \in UE$. Therefore also $u_0 \in VE$, and in particular $u_0 \in V$. This is to say that the unique element $v \in V$, with $\mathbf{d}(v) = x$, is u_0, so the claim is proved. We may then set $\phi([U, x]) = u$, thus obtaining a well defined map

$$\phi : \mathcal{H} \to \mathcal{G}.$$

Employing the homeomorphisms $\mathbf{d}_U = \mathbf{d}|_U : U \to \mathbf{d}(U)$, for every slice U, one may concretely describe ϕ by

$$\phi\big([U, x]\big) = \mathbf{d}_U^{-1}(x). \tag{5.4.1}$$

Another interesting characterization of ϕ is

$$\phi\big([U, x]\big) = u \iff u \in U, \text{ and } \mathbf{d}(u) = x \tag{5.4.2}$$

for every $U \in S$, and every $x \in \mathbf{d}(U)$. To see that ϕ is surjective let $u \in \mathcal{G}$. We may invoke (i) to find some $U \in S$ such that $u \in U$, and hence $[U, \mathbf{d}(u)]$ is in $\mathcal{H}$ and

$$\phi\big([U, \mathbf{d}(u)]\big) = u. \tag{5.4.3}$$

We will next prove that ϕ is injective, and for this we let $[U_1, x_1]$ and $[U_2, x_2]$ be germs in $\mathcal{H}$ such that

$$\phi\big([U_1, x_1]\big) = \phi\big([U_2, x_2]\big).$$

Denoting by w the common value of the terms above we have by (5.4.2) that

$$w \in U_i, \quad \text{and} \quad \mathbf{d}(w) = x_i, \quad \forall\, i = 1, 2.$$

In particular $w \in U_1 \cap U_2$, so (ii) applies providing some $W \in S$ such that $w \in W \subseteq U_1 \cap U_2$. The fact that $W \subseteq U_i$ may be described in terms of the semigroup structure of $S(\mathcal{G})$ by saying that $W = U_i W^* W$, (compare (6.1)), which in particular implies that

$$U_1 W^* W = U_2 W^* W.$$

Moreover

$$x_1 = x_2 = \mathbf{d}(w) \in \mathbf{d}(W) = \mathbf{d}(W^* W),$$

thus proving that $[U_1, x_1] = [U_2, x_2]$.

Let us now prove that ϕ is a homeomorphism. For this pick a germ $[U, x] \in \mathcal{H}$ and recall from (4.18) that

$$\Theta_U := \Theta\big(U, \mathbf{d}(U)\big) = \big\{[U, y] : y \in \mathbf{d}(U)\big\}$$

is a slice in $\mathcal{H}$, which clearly contains $[U, x]$. The image of Θ_U under ϕ is obviously U, and the restriction of ϕ to Θ_U is certainly continuous on Θ_U by (5.4.1). On the other hand, for each $u \in U$, we have that

$$\phi^{-1}(u) = [U, \mathbf{d}(u)],$$

by (5.4.3). Since ϕ^{-1} sends U into the slice Θ_U, in order to prove that ϕ^{-1} is continuous on U, it is enough to prove that $\delta \circ \phi^{-1}$ is continuous, where we are denoting by δ the source map for the groupoid $\mathcal{H}$. That composition is clearly given by

$$\delta \circ \phi^{-1}(u) = \delta\big([U, \mathbf{d}(u)]\big) = \mathbf{d}(u), \quad \forall\, u \in U,$$

which is well known to be continuous.

It remains to prove that ϕ is an isomorphism of groupoids. For this let $[U, x]$ and $[V, y]$ be germs is $\mathcal{H}$ and let $u = \phi([U, x])$, and $v = \phi([V, y])$, so that $u \in U$, $\mathbf{d}(u) = x$, $v \in V$, and $\mathbf{d}(v) = y$, according to (5.4.2). It is useful to remark that

$$\big(y, \mathbf{r}(v)\big) = \big(\mathbf{d}(v), \mathbf{r}(v)\big) \in \theta_V,$$

by (5.2), so $\theta_V(y) = \mathbf{r}(v)$. By (4.8) we have that $\big([U, x][V, y]\big) \in \mathcal{H}^{(2)}$ if and only $x = \theta_V(y)$, which is equivalent to saying that $\mathbf{d}(u) = \mathbf{r}(v)$, or that

$$\Big(\phi\big([U, x]\big), \phi\big([V, y]\big)\Big) = (u, v) \in \mathcal{G}^{(2)}.$$

This says that two elements in $\mathcal{H}$ may be multiplied if and only if their images under ϕ in $\mathcal{G}$ may be multiplied. In this case we have by (4.9) that

$$[U, x][V, y] = [UV, y].$$

On the other hand, notice that $uv \in UV$, and that $\mathbf{d}(uv) = \mathbf{d}(v) = y$, so

$$\phi\big([UV, y]\big) = uv = \phi\big([U, x]\big)\phi\big([V, y]\big),$$

thus proving that ϕ is a homomorphism of groupoids. $\square$

Conditions (5.4.i-ii) look very much like the definition of a topological base for $\mathcal{G}$. Therefore if S is a *full* *-subsemigroup of $S(\mathcal{G})$, in the sense of [26, 5.2], then S clearly satisfies (5.4.i-ii). However the latter conditions are clearly much weaker than to require that S be a base for the topology of $\mathcal{G}$. For example, if $\mathcal{G}$ is a groupoid consisting only of units, that is, if $\mathcal{G}$ is a topological space, then $\mathcal{G}$ itself is a slice and the singleton $\{\mathcal{G}\}$ is a *-subsemigroup of $S(\mathcal{G})$ which is not full, but satisfies (5.4.i-ii).

6 The Hausdorff property for the groupoid of germs

Quoting Paterson [24], the theory of non Hausdorff groupoids presented in section (3), and employed throughout this paper, already has enough of the Hausdorff property to allow for the efficient use of standard topological methods. However should a groupoid be Hausdorff in the true sense of the word it is definitely good to be aware of it.

It is not easy to determine conditions on an inverse semigroup S to ensure that the groupoid $\mathcal{G}(\theta, S, X)$ of (4.17) be Hausdorff for any action θ of S on any space X, especially because even groups may present difficulties. However the actions we are interested in have a special property which may be exploited in order to obtain such a characterization. In what follows we would like to describe this result.

Recall e.g. from [20, 1.4.6] that an inverse semigroup S is naturally equipped with a partial order defined by

$$s \leqslant t \iff s = ts^*s, \quad \forall s, t \in S. \tag{6.1}$$

Proposition 6.2. *Suppose that S is an inverse semigroup which is a semilattice*[9] *with respect to its natural order. Let θ be an action of S on a locally compact Hausdorff space X, such that for each $s \in S$, the domain D_{s^*s} of θ_s is closed (besides being open). Then $\mathcal{G}(\theta, S, X)$ is Hausdorff.*

Proof. Let $[s, x]$ and $[t, y]$ be two distinct elements of $\mathcal{G}(\theta, S, X)$. We need to find disjoint open subsets U and V of $\mathcal{G}(\theta, S, X)$, such that $[s, x] \in U$, and $[t, y] \in V$. If $x \neq y$ this is quite easy: separate x and y within X using disjoint open sets $A, B \subseteq X$, and take $U = \Theta(s, A \cap D_{s^*s})$ and $V = \Theta(t, B \cap D_{t^*t})$.

Let us then treat the less immediate case in which $x = y$. For this let $u = s \wedge t$ and notice that

$$su^*u = u = tu^*u,$$

and hence $x \notin D_{u^*u}$, or else $[s, x] = [t, x]$, by (4.6). As we are assuming that D_{u^*u} is closed we deduce that $V = X \setminus D_{u^*u}$ is an open neighborhood of x in X. Setting $W = V \cap D_{s^*s} \cap D_{t^*t}$, it is clear that

$$[s, x] \in \Theta(s, W), \quad \text{and} \quad [t, x] \in \Theta(t, W).$$

It therefore suffices to prove that $\Theta(s, W)$ and $\Theta(t, W)$ are disjoint sets. Arguing by contradiction suppose that $[r, z] \in \Theta(s, W) \cap \Theta(t, W)$. It follows that $[r, z] = [s, z] = [t, z]$, and hence there are idempotents e and f such that z lies in D_e and in D_f, and moreover such that $re = se$, and $rf = tf$. By replacing e and f with ef, we may assume that $e = f$, in which case $re = se = te$. Then

$$s(re)^*(re) = ser^*re = rer^*re = rr^*ree = re,$$

so $re \leqslant s$, and similarly $re \leqslant t$, so $re \leqslant u$. This implies that $re = reu^*u$, whence $r^*re = r^*reu^*u \leqslant u^*u$, and therefore

$$z \in D_{r^*r} \cap D_e = D_{r^*re} \subseteq D_{u^*u},$$

[9] Not to be confused with the semilattice of idempotents of S, this means that for every $s, t \in S$, there is a maximum among the elements of S which are smaller than both s and t. Tradition suggests that this element be denoted by $s \wedge t$. It is convenient to observe that if e and f are idempotents in S then the product ef coincides with $e \wedge f$. However if s and t are not idempotents then the product st is not always the same as $s \wedge t$.

which contradicts the fact that $z \in W$. ◻

In view of this result it is interesting to find examples of inverse semigroups which are semilattices. Recall that a *zero* in an inverse semigroup S is an element $0 \in S$ such that

$$0s = s0 = 0, \quad \forall s \in S.$$

An inverse semigroup S with zero is said to be E^*-unitary if for every $e, s \in S$, one has that

$$0 \neq e^2 = e \leqslant s \implies s^2 = s.$$

In other words, if an element dominates a nonzero idempotent then that element itself is an idempotent. The E^*-unitary inverse semigroups have been intensely studied in the semigroup literature. See, for example, [20, Section 9].

The following result resembles the fact that two analytic functions on a common connected domain, and agreeing on an open subset, must be equal.

Lemma 6.3. *Let S be an E^*-unitary inverse semigroup and let $s, t \in S$ be such that $s^*s = t^*t$, and $se = te$, for some nonzero idempotent $e \leqslant s^*s$. Then $s = t$.*

Proof. Notice that the idempotent $f = ses^*$ is nonzero because $e = s^*ses^*s$. We have that

$$ts^*f = ts^*ses^* = tt^*tes^* = tes^* = ses^* = f,$$

so $f \leqslant ts^*$, which implies that ts^* is idempotent. In particular it follows that $ts^* = (ts^*)^* = st^*$, so st^* is idempotent as well. We next claim that $ss^* = tt^*$. In fact

$$tt^* = tt^*tt^* = ts^*st^* = st^*ts^* = ss^*ss^* = ss^*.$$

Setting $u = ts^*t$, we have that

$$u^*u = t^*st^*ts^*t = t^*ss^*ss^*t = t^*ss^*t = t^*tt^*t = t^*t.$$

Therefore also $u^*u = s^*s$, while

$$t = tt^*t = tu^*u, \quad \text{and} \quad s = ss^*s = su^*u,$$

so it is enough to prove that $tu^* = su^*$. We have

$$su^* = st^*st^* = st^* = ts^* = tt^*ts^* = tt^*st^* = tu^*.$$ ◻

The following result is probably well known to semigroup theorists:

Proposition 6.4. *If S is an E^*-unitary inverse semigroup with zero, then S is a semilattice with respect to its usual order.*

Proof. We must prove that $s \wedge t$ exists for every $s, t \in S$. If there exists no nonzero $u \in S$, such that $u \leqslant s, t$, it is clear that $s \wedge t = 0$. So suppose the contrary and fix any such nonzero u. We then claim that

$$st^*t = ts^*s = tt^*s = ss^*t. \tag{6.4.1}$$

Let $f = s^*st^*t$. Since $u^*u \leqslant s^*s$, and $u^*u \leqslant t^*t$, we have that $u^*u \leqslant f$. Setting

$$\tilde{s} = sf, \quad \text{and} \quad \tilde{t} = tf,$$

notice that $\tilde{s}$ and $\tilde{t}$ share initial projections because

$$\tilde{s}^*\tilde{s} = fs^*sf = f = ft^*tf = \tilde{t}^*\tilde{t}.$$

Also notice that

$$\tilde{s}u^*u = sfu^*u = su^*u = u = tu^*u = tfu^*u = \tilde{t}u^*u.$$

Employing (6.3) we then deduce that $\tilde{s} = \tilde{t}$. So

$$st^*t = ss^*st^*t = sf = \tilde{s} = \tilde{t} = tf = ts^*s.$$

This shows the equality between the first and second terms in (6.4.1). Since $0 \neq u^* \leqslant s^*, t^*$, we may apply the above argument to s^*, t^*, u^* in order to prove that $s^*tt^* = t^*ss^*$, which implies that $tt^*s = ss^*t$, so the third and fourth terms in (6.4.1) agree.

The fact that $u \leqslant s, t$ implies that $su^*u = u = tu^*u$. Left multiplying this by t^* we have that

$$t^*su^*u = t^*tu^*u = u^*u,$$

so t^*s is idempotent by the fact that S is E^*-unitary. Applying the same reasoning to s^*, t^* and u^*, we have that ts^* is idempotent as well. Thus both t^*s and ts^* are selfadjoint, and hence

$$st^*t = ts^*t = tt^*s,$$

proving the equality between the first and third terms in (6.4.1), hence concluding the proof of our claim. We shall next prove that the element $m(s,t) := st^*t$, satisfies

$$u \leqslant m(s,t) \leqslant s, t.$$

It is obvious that $m(s,t) \leqslant s, t$. Recalling that $u^*u \leqslant f$, notice that

$$u = su^*u = sfu^*u = ss^*st^*tu^*u = st^*tu^*u = m(s,t)u^*u,$$

so $u \leqslant m(s,t)$. Therefore $m(s,t)$ is the infimum of s and t. ◻

7 Pre-grading structure of $C^*(\mathcal{G})$

In this section we return to our earlier standing hypotheses, namely that θ is an action of the inverse semigroup S on the locally compact Hausdorff space X. We will again be dealing with the groupoid of germs of the system (θ, S, X), denoted here simply by $\mathcal{G}$.

Our aim is to show that $C^*(\mathcal{G})$ admits a *pre-grading* over S, as explained below:

Definition 7.1. Let A be any C*-algebra and let S be an inverse semigroup. A *pre-grading*[10] of A over S is a family of closed linear subspaces $\{A_s\}_{s\in S}$ of A, such that for every $s, t \in S$ on has that

(i) $A_s A_t \subseteq A_{st}$,

(ii) $A_s^* = A_{s^*}$,

(iii) if $s \leqslant t$ (see (6.1)), then $A_s \subseteq A_t$,

(iv) A is the closed linear span of the union of all A_s.

The pre-grading is said to be *full* if in addition $A_s A_t$ is dense in A_{st}.

We begin by introducing some terminology:

Notations 7.2.

(i) For each $s \in S$ and each $f \in C_0(D_{s^*s})$ we will denote by $\alpha_s(f)$ the element of $C_0(D_{ss^*})$ given by

$$\alpha_s(f)\big|_x = f\big(\theta_{s^*}(x)\big), \quad \forall\, x \in D_{ss^*}.$$

(ii) Given $s \in S$ we will use the shorthand notation Θ_s for the slice $\Theta(s, D_{s^*s})$.

(iii) The restriction of the source and range maps to Θ_s will be denoted by $\mathbf{d}_s$ and $\mathbf{r}_s$, respectively.

(iv) If f is any complex valued function on D_{s^*s} we will denote the composition $f \circ \mathbf{d}_s$ by $\delta_s f$. This is by definition a function on Θ_s which we shall also view as a function on $\mathcal{G}$ by extending it to be zero outside Θ_s.

(v) If f is any complex valued function on D_{ss^*} we will denote the composition $f \circ \mathbf{r}_s$ by $f\delta_s$, with the same convention making $f\delta_s$ a function supported on Θ_s.

[10] We use the term pre-grading to suggest that we are not requiring any sort of linear independence of the subspaces A_s, as is usually required for gradings over groups.

Since Θ_s is a slice we have that $\mathbf{d}_s$ is a homeomorphism, with domain Θ_s, onto $\mathbf{d}(\Theta_s) = D_{s^*s}$. The inverse of $\mathbf{d}_s$ is then given by

$$\mathbf{d}_s^{-1} : x \in D_{s^*s} \mapsto [s, x] \in \Theta_s.$$

Compare (4.15).

It is important not to mistake $\delta_s f$ by $f \circ \mathbf{d}$, since the latter does not necessarily vanish outside Θ_s. Also notice that because $\mathbf{d}_s$ is a homeomorphism one has that $\delta_s f \in C_c\big(\Theta_s\big)$ if and only if $f \in C_c(D_{s^*s})$. In this case we obviously have that $\delta_s f \in C_c(\mathcal{G})$. Similar observations apply to $f\delta_s$.

The reader will be able to tell between the notations of (7.2.iv) and (7.2.v) by taking note of which side of f does δ_s appear. The following is intended to conciliate these points of view.

Proposition 7.3. *Given $f \in C_c(D_{s^*s})$ one has that $\delta_s f = \alpha_s(f)\delta_s$.*

Proof. Clearly both $\delta_s f$ and $\alpha_s(f)\delta_s$ are functions supported on Θ_s. Thus, given any $x \in D_{s^*s}$ we have

$$\begin{aligned}(\alpha_s(f)\delta_s)([s,x]) &= \alpha_s(f)\big(\mathbf{r}([s,x])\big) = \alpha_s(f)\big(\theta_s(x)\big) = f(\theta_{s^*}(\theta_s(x))) \\ &= f(x) = f\big(\mathbf{d}([s,x])\big) = (\delta_s f)([s,x]).\end{aligned}$$

□

The two notations are therefore completely interchangeable. We shall however prefer to use $\delta_s f$, perhaps because our notation for $[s, x]$ already favours sources over ranges. After all when one speaks of the "germ of a function f at a point x", the emphasis is on the point x in the domain of f, rather that the point $f(x)$ in the range of f.

Proposition 7.4. *If $s, t \in S$ then*

(i) $\Theta_s\Theta_t = \Theta_{st}$,

(ii) $\Theta_s^{-1} = \Theta_{s^*}$.

Proof. Given $[st, y] \in \Theta_{st}$ we have that $y \in D_{(st)^*st}$. By (4.5) if follows that

$$D_{(st)^*st} = D_{t^*s^*st} = \theta_{t^*}(D_{s^*s} \cap D_{tt^*}).$$

Therefore, there exists $x \in D_{s^*s} \cap D_{tt^*}$ such that $y = \theta_{t^*}(x)$ and hence $x = \theta_t(y)$. After verifying that $y \in D_{t^*t}$ we then conclude that

$$\big([s,x],[t,y]\big) \in (\Theta_s \times \Theta_t) \cap \mathcal{G}^{(2)},$$

and hence $[st, y] = [s, x]\,[t, y] \in \Theta_s\Theta_t$. This proves that $\Theta_{st} \subseteq \Theta_s\Theta_t$. The converse inclusion is trivial, so (i) is proved. The proof of (ii) follows by inspection. □

Proposition 7.5. *Given $s, t \in S$, let $f \in C_c(D_{s^*s})$, and $g \in C_c(D_{t^*t})$. Then*

(i) $(\delta_s f) \star (\delta_t g) = \delta_{st} h$, *where* $h = \alpha_{t^*}(f\alpha_t(g))$,

(ii) $(\delta_s f)^* = \delta_{s^*}\, \alpha_s(\bar{f})$.

On the other hand, if $f \in C_c(D_{ss^*})$, and $g \in C_c(D_{tt^*})$, then

(iii) $(f\delta_s) \star (g\delta_t) = h\delta_{st}$, *where* $h = \alpha_s(\alpha_{s^*}(f)g)$,

(iv) $(f\delta_s)^* = \alpha_{s^*}(\bar{f})\delta_{s^*}$.

Proof. Since $\delta_s f \in C_c(\Theta_s)$ and $\delta_t g \in C_c(\Theta_t)$ we have by (3.11.ii) that

$$(\delta_s f) \star (\delta_t g) \in C_c(\Theta_s\Theta_t) \overset{(7.4)}{=} C_c(\Theta_{st}).$$

Given $[st, y] \in \Theta_{st}$ recall from the proof of (7.4) that $[st, y] = [s, x]\,[t, y]$, where $x = \theta_t(y)$. Therefore

$$\begin{aligned}(\delta_s f) \star (\delta_t g)([st, y]) &= (\delta_s f)([s, x])\, (\delta_t g)([t, y]) = f(x)g(y)\\ &= f\big(\theta_t(y)\big)g(y) = h(y),\end{aligned}$$

where the last equality is to be taken as the definition of h. It is tempting to write $h = \alpha_{t^*}(f)g$, except that we are reserving the expression $\alpha_{t^*}(f)$, defined in (7.2.i), for functions $f \in C_c(D_{tt^*})$, and all we know about f is that it lies in $C_c(D_{s^*s})$. The reader will find that the expression given in the statement is an alternative way to describe h which respects the domains of α_t and α_{t^*}, the fundamental point being that $\alpha_t(g)$ is in $C_c(D_{tt^*})$, and the latter is an ideal in the space of continuous functions. This proves (i).

With respect to (ii), for every $\gamma \in \mathcal{G}$ we have that

$$(\delta_s f)^*(\gamma) = \overline{(\delta_s f)(\gamma^{-1})},$$

so the support of $(\delta_s f)^*$ is contained in $\Theta_s^{-1} = \Theta_{s^*}$. Given $[s^*, x] \in \Theta_{s^*}$ we than compute

$$\begin{aligned}(\delta_s f)^*([s^*, x]) &= \overline{(\delta_s f)([s^*, x]^{-1})} = \overline{(\delta_s f)([s, \theta_{s^*}(x)])} = \overline{f(\theta_{s^*}(x))}\\ &= \alpha_s(\bar{f})(x) = \big(\delta_{s^*}\, \alpha_s(\bar{f})\big)([s^*, x]).\end{aligned}$$

Points (iii) and (iv) follow respectively from (i) and (ii), with the aid of (7.3). □

The expression for h in (7.5.iii) is a fundamental formula underlying the algebrization of partially defined maps. It's first appearance in the literature dates back at least to [8, 3.4], and may be found also in [33, Section 5] and in [9, Section 2], the latter being its twisted version.

Proposition 7.6. *If $s, t \in S$ are such that $s \leqslant t$, then*

(i) $D_{s^*s} \subseteq D_{t^*t}$,

(ii) *for every $f \in C_c(D_{s^*s})$ one has that $\delta_s f = \delta_t f$,*

(iii) *for every $f \in C_c(D_{ss^*})$ one has that $f\delta_s = f\delta_t$,*

(iv) $\Theta_s \subseteq \Theta_t$,

(v) $C_c(\Theta_s) \subseteq C_c(\Theta_t)$.

Proof. Given that $s = ts^*s$ we have

$$D_{s^*s} = D_{(ts^*s)^*ts^*s} = D_{s^*st^*t} = D_{s^*s} \cap D_{t^*t},$$

from where (i) follows. To prove (iv) let $[s, x] \in \Theta_s$, so we have that $x \in D_{s^*s} \subseteq D_{t^*t}$ and hence $[t, x]$ belongs to Θ_t. In addition, setting $e = s^*s$, the fact that $x \in D_e$, and $se = te$ implies that

$$[s, x] = [t, x] \in \Theta_t, \tag{7.6.1}$$

proving (iv), and consequently also proving (v).

To prove (ii) let $f \in C_c(D_{s^*s})$, so also $f \in C_c(D_{t^*t})$ by (i). Using (v) we may view both $\delta_s f$ and $\delta_t f$ as elements of $C_c(\Theta_t)$. Given $[t, x] \in \Theta_t$, where $x \in D_{t^*t}$, we either have that $x \notin D_{s^*s}$, in which case $[t, x] \notin \Theta_s$ and hence, recalling that $\delta_s f$ is supported in Θ_s, we have

$$\delta_s f([t, x]) = 0 = f(x) = \delta_t f([t, x]).$$

On the other hand, if $x \in D_{s^*s}$, we have that

$$\delta_t f([t, x]) = f(x) = \delta_s f([s, x]) \overset{7.6.1}{=} \delta_s f([t, x]).$$

This concludes the proof of (ii), and (iii) follows as well in view of (7.3). ◻

In what follows we give the result promised at the beginning of this section:

Proposition 7.7. *Let $i : C_c(\mathcal{G}) \to C^*(\mathcal{G})$ be the natural map defined in* (3.17.1). *For each $s \in S$, let A_s denote the closure of $i\big(C_c(\Theta_s)\big)$ within $C^*(\mathcal{G})$. Then the collection $\{A_s\}_{s \in S}$ is a full pre-grading of $C^*(\mathcal{G})$.*

Proof. It is obvious that $\{\Theta_s\}_{s\in S}$ is a covering of $\mathcal{G}$, so (7.1.iv) follows immediately from (3.10) and the fact that $i(C_c(\mathcal{G}))$ is dense in $C^*(\mathcal{G})$.

Given that $\mathbf{d}_s$ is a homeomorphism it is clear that $C_c(\Theta_s)$ consists precisely of the elements of the form $\delta_s f$, where f runs in $C_c(D_{s^*s})$. Therefore (7.1.i–ii) follow respectively from (7.5.i–ii). The third axiom of pre-gradings is an obvious consequence of (7.6.v), so we are left with proving that our pre-grading is full. For this let $s, t \in S$ and pick any element in $C_c(\Theta_{st})$, which is necessarily of the form $\delta_{st}h$, where $h \in C_c(D_{(st)^*st})$. Recall e.g. from the proof of (7.4.i) that $D_{(st)^*st} = \theta_{t^*}(D_{s^*s} \cap D_{tt^*})$, so

$$\alpha_t(h) = h \circ \theta_{t^*} \in C_c(D_{s^*s} \cap D_{tt^*}).$$

We may then write $\alpha_t(h) = fk$, where both f and k are in $C_c(D_{s^*s} \cap D_{tt^*})$. Observing that

$$k \in C_c(D_{s^*s} \cap D_{tt^*}) \subseteq C_c(D_{tt^*}) = C_c\big(\theta_t(D_{t^*t})\big),$$

the function $g = k \circ \theta_t = \alpha_{t^*}(k)$ lies in $C_c(D_{t^*t})$, and hence $\delta_t g \in C_c(\Theta_t)$. In addition we have that $\delta_s f \in \Theta_s$, so

$$C_c(\Theta_s)C_c(\Theta_t) \ni (\delta_s f) \star (\delta_t g) = \delta_{st}\Big(\alpha_{t^*}\big(f\alpha_t(g)\big)\Big) = \delta_{st}\big(\alpha_{t^*}(fk)\big) = \delta_{st}h.$$

This shows that $C_c(\Theta_{st}) \subseteq C_c(\Theta_s)C_c(\Theta_t)$, from where one sees that our pre-grading is in fact full. □

8 Universal property of $C^*(\mathcal{G})$

As before we fix an action θ of an inverse semigroup S on a locally compact Hausdorff topological space X. We will assume in addition that S is countable and that X is second countable,[11] due to the use of measure theory methods. We shall retain the notation $\mathcal{G}$ for the groupoid of germs of the system (θ, S, X).

Recall from (7.2.i) that for $s \in S$ we denote by α_s the isomorphism from $C_c(D_{s^*s})$ to $C_c(D_{ss^*})$ given by $\alpha_s(f) = f \circ \theta_{s^*}$.

Definition 8.1. A *covariant representation* of the system (θ, S, X) on a Hilbert space H is a pair (π, σ), where π is a nondegenerate *-representation of $C_0(X)$ on H, and $\sigma : S \to B(H)$ satisfies

[11] In case of absolute necessity one may perhaps dispense with the second countability assumption at the expense of working with the σ-algebra of Baire (instead of Borel) measurable sets, assuming in addition that every D_e is Baire measurable.

(i) $\sigma_{st} = \sigma_s \sigma_t$,

(ii) $\sigma_{s^*} = \sigma_s^*$,

(iii) $\pi(\alpha_s(f)) = \sigma_s \pi(f) \sigma_{s^*}$,

(iv) $\overline{\pi\big(C_0(D_e)\big)H} = \sigma_e(H)$,

for every $s, t \in S$, $f \in C_0(D_{s^*s})$, and $e \in E(S)$.

From now on we fix a covariant representation (π, σ) of (θ, S, X) on H.

We will write $\tilde{\pi}$ for the canonical weakly continuous extension of π to the algebra $\mathscr{B}(X)$ of all bounded Borel measurable functions on X. It is well known that for each open subset $U \subseteq X$, one has that the range of $\tilde{\pi}(1_U)$ coincides with $\overline{\pi\big(C_0(U)\big)H}$, where $1_U \in \mathscr{B}(X)$ denotes the characteristic function of U. Therefore (8.1.iv) may be expressed by saying that

$$\sigma_e = \tilde{\pi}(1_{D_e}), \quad \forall\, e \in E(S). \tag{8.2}$$

In particular it follows that

$$\sigma_e \pi(f) = \pi(f) \sigma_e, \quad \forall\, f \in C_0(X). \tag{8.3}$$

Our next main goal will be to show that there exists a *-representation $\sigma \times \pi : C_c(\mathcal{G}) \to B(H)$, such that for every $s \in S$, and $f \in C_c(D_{s^*s})$, one has that $(\sigma \times \pi)(\delta_s f) = \sigma_s \pi(f)$.

Lemma 8.4. *Let J be a finite subset of S and suppose that for each $s \in J$ we are given $f_s \in C_c(D_{s^*s})$ such that $\sum_{s \in J} \delta_s f_s = 0$, in $C_c(\mathcal{G})$. Then $\sum_{s \in J} \sigma_s \pi(f_s) = 0$, in $B(H)$.*

Proof. Fix, for the time being, two elements $\xi, \eta \in H$. For each $s \in S$, let $\mu_s = \mu_{s,\xi,\eta}$ be the finite Borel measure on Θ_s given by

$$\mu_s(A) = \Big\langle \sigma_s \tilde{\pi}(1_{\mathbf{d}(A)}) \xi, \eta \Big\rangle,$$

for every Borel measurable $A \subseteq \Theta_s$, where $1_{\mathbf{d}(A)}$ stands for the characteristic function on $\mathbf{d}(A)$.

Since $\mathbf{d}$ is a homeomorphism from Θ_s to D_{s^*s}, one has that $\mathbf{d}(A)$ is a measurable subset of D_{s^*s}, and hence also of X. Therefore $1_{\mathbf{d}(A)} \in \mathscr{B}(X)$, so that $\tilde{\pi}(1_{\mathbf{d}(A)})$ is well defined. That μ_s is indeed a countably additive measure follows from the corresponding well known property of $\tilde{\pi}$.

If $B \subseteq D_{s^*s}$ is a measurable set, let $A = \mathbf{d}_s^{-1}(B)$, so that A is a measurable subset of Θ_s and $B = \mathbf{d}(A)$. Notice that

$$\delta_s 1_B = 1_B \circ \mathbf{d}_s = 1_A,$$

so

$$\int_{\Theta_s} \delta_s 1_B \, d\mu_s = \int_{\Theta_s} 1_A \, d\mu_s = \mu_s(A) = \left\langle \sigma_s \tilde{\pi}(1_{\mathbf{d}(A)})\xi, \eta \right\rangle = \left\langle \sigma_s \tilde{\pi}(1_B))\xi, \eta \right\rangle,$$

from where one easily deduces that

$$\int_{\Theta_s} \delta_s f \, d\mu_s = \left\langle \sigma_s \tilde{\pi}(f)\xi, \eta \right\rangle, \quad \forall f \in \mathscr{B}(X). \tag{8.4.1}$$

We next claim that for every $s, t \in S$, and every measurable set $A \subseteq \Theta_s \cap \Theta_t$, one has that

$$\mu_s(A) = \mu_t(A). \tag{8.4.2}$$

In order to prove it observe that $B := \mathbf{d}(A) = \mathbf{d}_s(A) = \mathbf{d}_t(A)$ is a Borel subset of $D_{s^*s} \cap D_{t^*t}$ and

$$A = \left\{[s, x] : x \in B\right\} = \left\{[t, x] : x \in B\right\}.$$

For every $x \in B$ we moreover have that $[s, x] = [t, x]$, so there exists $e \in E(S)$ such that $x \in D_e$, and $se = te$. It therefore follows that

$$B \subseteq \bigcup_{\substack{e \in E(S) \\ se = te}} D_e.$$

Since we are assuming that S is countable, so is $E(S)$ and we may decompose B as a disjoint union of measurable subsets $\{B_n\}_{n \in \mathbf{N}}$, such that each B_n is a subset of some D_{e_n}, and $se_n = te_n$. Obviously A is then the disjoint union of the sets

$$A_n = \mathbf{d}_s^{-1}(B_n) = \mathbf{d}_t^{-1}(B_n).$$

Notice that for each $n \in \mathbf{N}$ we have

$$\begin{aligned} \sigma_s \tilde{\pi}(1_{\mathbf{d}(A_n)}) &= \sigma_s \tilde{\pi}(1_{B_n}) = \sigma_s \tilde{\pi}(1_{D_{e_n}} 1_{B_n}) \\ &= \sigma_s \tilde{\pi}(1_{D_{e_n}}) \tilde{\pi}(1_{B_n}) \overset{(8.2)}{=} \sigma_s \sigma_{e_n} \tilde{\pi}(1_{B_n}) = \sigma_{se_n} \tilde{\pi}(1_{B_n}), \end{aligned}$$

and similarly for t. Since $se_n = te_n$ we have that

$$\sigma_s \tilde{\pi}(1_{\mathbf{d}(A_n)}) = \sigma_t \tilde{\pi}(1_{\mathbf{d}(A_n)}),$$

whence $\mu_s(A_n) = \mu_t(A_n)$. The countable additivity of μ_s and μ_t then take care of (8.4.2).

Let M be the measurable subset of $\mathcal{G}$ given by $M = \bigcup_{s\in J}\Theta_s$, where J is as in the statement. It is an easy exercise in measure theory to prove that there exists a measure μ on M such that $\mu(A) = \mu_s(A)$, for every $s \in J$, and $A \subseteq \Theta_s$. We then have that

$$\Big\langle \sum_{s\in J} \sigma_s\pi(f_s)\xi, \eta\Big\rangle \overset{(8.4.1)}{=} \sum_{s\in J} \int_{\Theta_s} \delta_s f_s \, d\mu_s = \sum_{s\in J} \int_M \delta_s f_s \, d\mu$$
$$= \int_M \sum_{s\in J} \delta_s f_s \, d\mu = 0.$$

Since ξ and η are arbitrary we conclude that $\sum\limits_{s\in J} \sigma_s\pi(f_s) = 0$, as stated. □

We thus arrive at the main result of this section.

Theorem 8.5. *Let S be a countable inverse semigroup, let θ be an action of S on the second countable locally compact Hausdorff space X, and let $\mathcal{G}$ be the corresponding groupoid of germs* (4.17)*. Given any covariant representation (π, σ) of (θ, S, X) on a Hilbert space H there exists a unique *-representation $\sigma \times \pi$ of $C^*(\mathcal{G})$ on H such that*

$$(\sigma \times \pi)\big(i(\delta_s f)\big) = \sigma_s\pi(f), \quad \textit{and} \quad (\sigma \times \pi)\big(i(g\delta_s)\big) = \pi(g)\sigma_s,$$

for every $s \in S$, every $f \in C_c(D_{s^*s})$, and every $g \in C_c(D_{ss^*})$, where $i : C_c(\mathcal{G}) \to C^*(\mathcal{G})$ is the canonical map.

Proof. Given any $f \in C_c(\mathcal{G})$ use (4.18) and (3.10) to write

$$f = \sum_{k=1}^{n} \delta_{s_k} f_k,$$

where $s_1, \ldots, s_n \in S$, and $f_k \in C_c(D_{s_k^* s_k})$, for all $k = 1, \ldots, n$. Define

$$(\sigma \times \pi)(f) = \sum_{k=1}^{n} \sigma_{s_k}\pi(f_k).$$

That $\sigma \times \pi$ is well defined is a consequence of (8.4). It is obviously also linear, and we claim that it is a *-homomorphism. In order to prove the preservation of multiplication, we may use linearity to reduce our task to proving only that

$$(\sigma \times \pi)(\delta_s f \star \delta_t g) = (\sigma \times \pi)(\delta_s f)\ (\sigma \times \pi)(\delta_t g),$$

for every $f \in C_c(D_{s^*s})$ and $g \in C_c(D_{t^*t})$. By (7.5.i) the left-hand side equals

$$\begin{aligned}(\sigma \times \pi)\big(\delta_{st}\alpha_{t^*}\big(f\alpha_t(g)\big)\big) &= \sigma_{st}\pi\big(\alpha_{t^*}\big(f\alpha_t(g)\big)\big) = \sigma_{st}\sigma_{t^*}\pi\big(f\alpha_t(g)\big)\sigma_t \\ &= \sigma_{st}\sigma_{t^*}\pi(f)\pi\big(\alpha_t(g)\big)\sigma_t = \sigma_s\sigma_t\sigma_{t^*}\pi(f)\sigma_t\pi(g)\sigma_{t^*}\sigma_t \\ &\overset{(8.3)}{=} \sigma_s\pi(f)\sigma_t\sigma_{t^*}\sigma_t\sigma_{t^*}\sigma_t\pi(g) = \sigma_s\pi(f)\sigma_t\pi(g) \\ &= (\sigma \times \pi)(\delta_s f)\ (\sigma \times \pi)(\delta_t g).\end{aligned}$$

Our claim will then be proved once we show that

$$(\sigma \times \pi)\big((\delta_s f)^*\big) = \big((\sigma \times \pi)(\delta_s f)\big)^*.$$

By (7.5.ii) the left-hand side equals

$$\begin{aligned}(\sigma \times \pi)\big(\delta_{s^*}\,\alpha_s(\bar{f})\big) &= \sigma_{s^*}\pi\big(\alpha_s(\bar{f})\big) = \sigma_{s^*}\sigma_s\pi(\bar{f})\sigma_{s^*} = \pi(\bar{f})\sigma_{s^*}\sigma_s\sigma_{s^*} \\ &= \pi(f)^*\sigma_{s^*} = \big(\sigma_s\pi(f)\big)^* = \big((\sigma \times \pi)(\delta_s f)\big)^*.\end{aligned}$$

This proves our claim. By (3.16) we then conclude that

$$\|(\sigma \times \pi)(f)\| \leqslant |||f|||, \quad \forall\, f \in C_c(\mathcal{G}),$$

which implies that $\sigma \times \pi$ factors through i, producing a *-representation of $C^*(\mathcal{G})$, by abuse of language also denoted by $\sigma \times \pi$, clearly satisfying the first identity in the statement.

In order to prove the second identity let $s \in S$ and $g \in C_c(D_{ss^*})$. Set $f = \alpha_{s^*}(g)$, so that $g = \alpha_s(f)$, and $f \in C_c(D_{s^*s})$. Therefore

$$g\delta_s = \alpha_s(f)\delta_s \overset{(7.3)}{=} \delta_s f,$$

so

$$\begin{aligned}(\sigma \times \pi)\big(i(g\delta_s)\big) &= (\sigma \times \pi)\big(i(\delta_s f)\big) = \sigma_s\pi(f) = \sigma_s\sigma_s^*\sigma_s\pi(f) \\ &\overset{(8.3)}{=} \sigma_s\pi(f)\sigma_s^*\sigma_s = \pi\big(\alpha_s(f)\big)\sigma_s = \pi(g)\sigma_s.\end{aligned}$$

□

9 Inverse semigroup crossed products

The main goal of this section is to show that, in the context of the previous section, $C^*(\mathcal{G})$ is naturally isomorphic to the inverse semigroup crossed product $C_0(X) \rtimes_\alpha S$.

In the first part of this section we shall therefore briefly review the theory of inverse semigroup crossed products based on [33] and [24], not only for the convenience of the reader, but also because we will present a few improvements.

Definition 9.1. An *action* of an inverse semigroup[12] S on a C*-algebra A is a semigroup homomorphism

$$\alpha : S \to \mathcal{I}(A),$$

(see (4.2) for a definition of $\mathcal{I}(A)$) such that

(i) for every $s \in S$, the domain (and hence also the range) of α_s is a closed two sided ideal of A, and α_s is a *-homomorphism,

(ii) the linear span of the union of the domains of all the α_s is dense in A.

As in the case of actions on locally compact spaces, defined in (4.3), for every $e \in E(S)$, we denote by J_e the domain of α_e. For each $s \in S$ one therefore has that α_s is a *-isomorphism from J_{ss^*} to J_{ss^*}. See also footnote (5).

Given an action of S on a locally compact space X in the sense of (4.3), it is easy to produce an action of S on $A = C_0(X)$, this time in the sense of (9.1): observing that $J_e := C_0(D_e)$ is an ideal in $C_0(X)$, for each $s \in S$, one takes $\alpha_s : J_{s^*s} \to J_{ss^*}$ to be given by (7.2.i). To check that (9.1.ii) holds one uses (4.3.ii) and the Stone–Weierstrass Theorem.

From now on we fix an action of S on a C*-algebra A.

One then considers the linear space

$$L = \bigoplus_{s\in S} J_{ss^*}. \tag{9.2}$$

If e is an idempotent notice that J_e appears in the above direct sum as many times as there are elements $s \in S$ with $ss^* = e$.

Any element x in L is of the form $x = (a_s)_{s\in S}$, where $a_s \in J_{ss^*}$, and $a_s = 0$ for all but finitely many s. Given $s \in S$ and $a \in J_{ss^*}$, we shall denote by $a\delta_s$ the element of L which is identically zero except for its s^{th} component which is equal to a. Any element of L, say $x = (a_s)_{s\in S}$, is therefore given by

$$x = \sum_{s\in S} a_s\delta_s, \tag{9.3}$$

[12] Sieben assumes that S is unital [33, 4.1] and that α_e is the identity map on A. Attempting to avoid units Paterson instead assumes that the family of the domains of the α_s forms an upward directed chain [24, Definition 3.3.1.ii]. These assumptions are designed to be used in proving the equivalence between covariant representations of the system and *-representations of the covariance algebra. See [33, 5.6] and [24, Proposition 3.3.3]. As we will see below there is a way to get around this problem without assuming either of this extra conditions.

where the sum has finitely many nonzero terms. Based on [8] and [21], Sieben defines a *-algebra structure on L according to which

$$(a\delta_s)(b\delta_t) = \alpha_s\big(\alpha_{s^*}(a)b\big)\delta_{ts}, \quad \text{and} \qquad (a\delta_s)^* = \alpha_{s^*}(a^*)\delta_{s^*},$$

for every $s, t \in S$, $a \in J_{ss^*}$, and $b \in J_{tt^*}$.

The crossed product $A \rtimes_\alpha L$ is then defined (see below) as a certain completion of L. However, contrary to what happens with similar constructions, one does not expect L to survive the completion process intact: if e and f are idempotents and $a \in J_e \cap J_f$, so that $a\delta_e$ and $a\delta_f$ are elements of L, the construction is such that $a\delta_e - a\delta_f = 0$, when passing to the crossed product.

Let us now review the construction of the crossed product. Sieben first defines [33, 4.5] a covariant representation of the system (α, S, A) on a Hilbert space H (up to the fact that our S needs not have a unit) precisely as in (8.1), except that $C_0(X)$ is replaced by A, and $C_0(D_e)$ is replaced by J_e. Risking being a bit monotonous the definition is:

Definition 9.4. A *covariant representation* of the system (α, S, A) on a Hilbert space H is a pair (π, σ), where π is a nondegenerate *-representation of A on H, and $\sigma : S \to B(H)$ satisfies

(i) $\sigma_{st} = \sigma_s\sigma_t$,

(ii) $\sigma_{s^*} = \sigma_s^*$,

(iii) $\pi(\alpha_s(a)) = \sigma_s\pi(a)\sigma_{s^*}$,

(iv) $\overline{\pi\big(J_e\big)H} = \sigma_e(H)$,

for every $s, t \in S$, $a \in J_{ss^*}$, and $e \in E(S)$.

It is then easy to see [33, 5.3] that for every covariant representation (π, σ) the formula

$$(\pi \times \sigma)\left(\sum_{s\in S} a_s\delta_s\right) = \sum_{s\in S} \pi(a_s)\sigma_s$$

defines a nondegenerate *-representation of L on H.

If $e, f \in E(S)$ are such that $e \leqslant f$ (meaning that $ef = e$), then $\alpha_e\alpha_f = \alpha_e$, which gives $J_e \subseteq J_f$. If moreover and $a \in J_e$ we may speak of two different elements of L, namely $a\delta_e$ and $a\delta_f$. Moreover notice that

$$\begin{aligned}(\pi \times \sigma)(a\delta_e) &= \pi(a)\sigma(e) = \pi(a)\sigma(ef) = \pi(a)\sigma(e)\sigma(f)\\ &= \pi(a)\sigma(f) = (\pi \times \sigma)(a\delta_f),\end{aligned}$$

where our use of the identity $\pi(a)\sigma(e) = \pi(a)$ is justified by (9.4.iv).

Restricting one's attention to representations of L which behave as $\pi \times \sigma$ in the above respect is an important insight due to Paterson.

Definition 9.5. [24, 3.87] A *-homomorphism ϕ from L into another *-algebra will be called *admissible* if for every $e, f \in E(S)$, with $e \leqslant f$, and every $a \in J_e$, one has that $\phi(a\delta_e) = \phi(a\delta_f)$.

Following Sieben [33, 5.6], Patterson [24, Proposition 3.3.3] proves that every admissible nondegenerate *-representation Π of L on a Hilbert space H is given as above for a covariant representation (π, σ) of (α, S, A). Under the assumption that S is unital the construction of the first component of the covariant representation, namely π, is a breeze: for every a in A one simply defines $\pi(a) = \Pi(a\delta_1)$. Paterson avoids units by requiring that the J_e be upward directed. However it is possible to get around this problem with bare hands:

Lemma 9.6. *Given an admissible nondegenerate *-representation Π of L on a Hilbert space H, there exists a *-representation π of A on H such that*

$$\pi(a) = \Pi(a\delta_e), \quad \forall\, e \in E(S), \quad \forall\, a \in J_e.$$

Proof. We first claim that for every $s \in S$, $e \in E(S)$, and $a \in J_e \cap J_{ss^*}$ one has that

$$\Pi(a\delta_{es}) = \Pi(a\delta_s). \tag{9.6.1}$$

Since $J_{es(es)^*} = J_{ess^*} = J_e \cap J_{ss^*}$, both elements appearing as arguments to Π in (9.6.1) are indeed in L. We have

$$\begin{aligned}(a\delta_{es} - a\delta_s)(a\delta_{es} - a\delta_s)^* &= (a\delta_{es} - a\delta_s)\big(\alpha_{s^*e}(a^*)\delta_{s^*e} - \alpha_{s^*}(a^*)\delta_{s^*}\big)\\ &= -aa^*\delta_{ss^*e} + aa^*\delta_{ss^*},\end{aligned}$$

so admissibility implies that $\Pi(a\delta_{es} - a\delta_s)\Pi(a\delta_{es} - a\delta_s)^* = 0$, from which (9.6.1) follows. Let

$$A_0 = \sum_{e\in E(S)} J_e,$$

so that A_0 is a dense *-subalgebra of A. Given a in A_0, write it as a finite sum $a = \sum_{e\in E(S)} a_e$, with $a_e \in J_e$, and define

$$\pi(a) = \sum_{e\in E(S)} \Pi(a_e\delta_e).$$

We claim that $\pi(a)$ does not depend on the choice of the a_e's. Proving this claim is tantamount to proving that when a vanishes, so does the right-hand side above. Since Π is nondegenerate it is in fact enough to prove that

$$\sum_{e\in E(S)} \Pi(a_e\delta_e)\Pi(b\delta_s) = 0,$$

for every $s \in S$ and $b \in J_{ss^*}$. The left-hand side above equals

$$\sum_{e\in E(S)} \Pi\big((a_e\delta_e)(b\delta_s)\big) = \sum_{e\in E(S)} \Pi(a_e b\delta_{es}) \overset{(9.6.1)}{=} \sum_{e\in E(S)} \Pi(a_e b\delta_s)$$
$$= \Pi\Big(\sum_{e\in E(S)} a_e b\delta_s\Big) = \Pi(ab\delta_s) = 0.$$

This proves that π is a well defined map on A_0. To prove that π is a *-representation let $e, f \in E(S)$, $a \in J_e$, and $b \in J_f$. Then

$$\pi(a)\pi(b) = \Pi(a\delta_e)\Pi(b\delta_f) = \Pi(ab\delta_{ef}) = \pi(ab).$$

We leave it for the reader the easy proof that π preserves the star operation. Summarizing, π is a *-representation of the dense subalgebra $A_0 \subseteq A$ on H. Any finite sum of ideals among the J_e gives a closed *-subalgebra of A. This implies that π is norm-decreasing on A_0 and hence extends to a *-representation of A, which clearly satisfies the required conditions. ◻

Inserting the result above into Sieben's proof of [33, 5.6], or Paterson's proof of [24, Proposition 3.3.3], we arrive at the following:

Proposition 9.7. *Let S be a (not necessarily unital) inverse semigroup and let α be an action of S on a C*-algebra A. Then the association*

$$(\pi, \sigma) \longmapsto \Pi = \pi \times \sigma$$

is a one-to-one correspondence between covariant representations (π, σ) *of* (α, S, A) *and admissible nondegenerate *-representations* Π *of* L.

Recall from [33, 5.4] that the *crossed product of A by S relative to the action* α, denoted $A \rtimes_\alpha S$, is defined to be the Hausdorff completion of L in the norm

$$|||x||| = \sup_{\Pi} \|\Pi(x)\|,$$

where the supremum is taken over all representations of L of the form $\Pi = \pi \times \sigma$ (equivalently over all admissible nondegenerate representations). As such, it is evident that to the classes of objects put in correspondence by (9.7), one can add the nondegenerate *-representations of $A \rtimes_\alpha S$.

This concludes our review of inverse semigroup crossed products, so we will now return to considering actions of inverse semigroups on topological spaces.

Theorem 9.8. *Let S be a countable inverse semigroup, let X be a second countable locally compact Hausdorff space, and let θ be an action of S on X in the sense of* (4.3). *Denoting by $\mathcal{G}$ the groupoid of germs of* (θ, S, X) *one has that $C^*(\mathcal{G})$ is isomorphic to $C_0(X) \rtimes_\alpha S$, where α is the action of S on $C_0(X)$ given by* (7.2.i).

Proof. Choose a faithful nondegenerate *-representation

$$\Psi : C_0(X) \rtimes_\alpha S \to B(H),$$

where H is a Hilbert space. That representation, once composed with the natural map

$$j : L \to C_0(X) \rtimes_\alpha S,$$

yields a *-representation $\Pi = \Psi \circ j$, of L on H which is clearly admissible and nondegenerate. By (9.7) there exists a covariant representation (π, σ) of $\big(\alpha, S, C_0(X)\big)$ on H such that

$$\Pi(f\delta_s) = \pi(f)\sigma_s,$$

for every $s \in S$, and $f \in J_{ss^*} = C_0(D_{ss^*})$. Invoking (8.5) we deduce that there exists a *-representation $\Phi = \pi \times \sigma$ of $C^*(\mathcal{G})$ on H such that

$$\Phi\big(i(f\delta_s)\big) = \pi(f)\sigma_s = \Pi(f\delta_s) = \Psi\big(j(f\delta_s)\big),$$

for every $s \in S$, and every $f \in C_c(D_{ss^*})$.

Observe that the notation "$f\delta_s$" means different things here: an element of L as in (9.3), or an element of $C_c(\mathcal{G})$ as in (7.2.v). However the context should suffice to distinguish between these uses.

It follows that Φ maps $C^*(\mathcal{G})$ into the image of $C_0(X) \rtimes_\alpha S$ through Ψ in $B(H)$, and since Ψ is faithful we can produce a *-homomorphism

$$\phi : C^*(\mathcal{G}) \to C_0(X) \rtimes_\alpha S,$$

such that

$$\phi\big(i(f\delta_s)\big) = j(f\delta_s), \quad \forall s \in S, \quad \forall f \in C_0(D_{s^*s}).$$

Leaving this aside for a moment consider the map

$$\gamma : \sum_{s \in S} f_s\delta_s \in L \longmapsto \sum_{s \in S} f_s\delta_s \in C_c(\mathcal{G}),$$

where again the double meaning of $f_s\delta_s$ should bring no confusion. Using (7.5.iii–iv) it is immediate that γ is a *-homomorphism, and by (7.6.iii) one sees that it is admissible. Therefore the composition $i \circ \gamma$ extends to give a *-homomorphism

$$\psi : C_0(X) \rtimes_\alpha S \to C^*(\mathcal{G}),$$

satisfying

$$\psi(j(f\delta_s)) = i(f\delta_s), \quad \forall s \in S, \quad \forall f \in C_0(D_{s^*s}).$$

This proves that ψ and ϕ are each other's inverse, and hence isomorphisms. □

We may use our methods to obtain the following generalization of [24, Theorem 3.3.1] and [26, 8.1].

Proposition 9.8. *Let $\mathcal{G}$ be a étale groupoid with second countable unit space and let S be a countable*[13] **-subsemigroup of $S(\mathcal{G})$ satisfying* (5.4.i–ii). *Let moreover θ be the restriction to S of the action of $S(\mathcal{G})$ on $\mathcal{G}^{(0)}$ given by* (5.2)*, and denote by α the induced action of S on $C_0(\mathcal{G}^{(0)})$, as in* (7.2.i). *Then*

$$C^*(\mathcal{G}) \simeq C_0(\mathcal{G}^{(0)}) \rtimes_\alpha S.$$

Proof. Let $\mathcal{H}$ be the groupoid of germs for the given action of S on $\mathcal{G}^{(0)}$. Applying (9.8) we conclude that

$$C^*(\mathcal{H}) \simeq C_0(\mathcal{G}^{(0)}) \rtimes_\alpha S,$$

but we also have that

$$\mathcal{H} \simeq \mathcal{G},$$

by (5.4), so the statement follows. □

[13] If such an S exists then $\mathcal{G}$ itself is second countable.

10 Action on the spectrum

As before we will let S be an inverse semigroup, but we will no longer postulate the existence of actions of S on exogenous topological spaces. Instead we will construct actions on spaces which are intrinsic to S. These spaces will actually be constructed from the idempotent semilattice of S, which we will denoted simply by E.

Definition 10.1. Let E be any semilattice. A *semicharacter* of E is a nonzero map

$$\phi : E \to \{0, 1\},$$

such that $\phi(ef) = \phi(e)\phi(f)$, for all $e, f \in E$. The set of all semicharacters equipped with the topology of pointwise convergence (equivalently the relative topology from the product space $\{0, 1\}^E$) is called the *spectrum* of E and is denoted $\hat{E}$.

It is easy to see that $\hat{E}$ is a locally compact Hausdorff topological space.

Definition 10.2. For every $e \in E$ we will denote by D_e the subset of $\hat{E}$ formed by all semicharacters ϕ such that $\phi(e) = 1$.

Given that the correspondence $\phi \mapsto \phi(e)$ is continuous in the topology of pointwise convergence, we see that D_e is a clopen subset of $\hat{E}$.

Notice that $\hat{E}$ may fail to be compact since there may exist a net of semicharacters converging pointwise to the identically zero map (which is not a character by definition). No such net may exist inside D_e because its semicharacters take the value 1 at e. So D_e is actually closed in $\{0, 1\}^E$, hence compact.

Proposition 10.3. *Let $s \in S$ and $\phi \in D_{s^*s}$.*

(i) *The map $\theta_s(\phi) : e \in E \mapsto \phi(s^*es) \in \{0, 1\}$ is a semicharacter in D_{ss^*}.*

(ii) *The map $\theta_s : \phi \in D_{s^*s} \mapsto \theta_s(\phi) \in D_{ss^*}$ is a homeomorphism.*

(iii) *The map $\theta : s \in S \mapsto \theta_s \in \mathcal{I}(\hat{E})$ is a semigroup homomorphism.*

(iv) *θ is an action of S on $\hat{E}$, as defined in* (4.3).

Proof. For $e, f \in E$ we have

$$\begin{aligned}\theta_s(\phi)(ef) &= \phi(s^*efs) = \phi(s^*ess^*fs) = \phi(s^*es)\phi(s^*fs) \\ &= \theta_s(\phi)(e)\, \theta_s(\phi)(f),\end{aligned}$$

so $\theta_s(\phi)$ is multiplicative. In addition

$$\theta_s(\phi)(ss^*) = \phi(s^*ss^*s) = \phi(s^*s) = 1,$$

so $\theta_s(\phi) \in D_{ss^*}$. For every net $\{\phi_i\}_i$ converging to ϕ in D_{s^*s}, and for every $e \in E$, one has that

$$\lim_i \theta_s(\phi_i)(e) = \lim_i \phi_i(s^*es) = \phi(s^*es) = \theta_s(\phi)(e),$$

so we see that $\{\theta_s(\phi_i)\}_i$ converges to $\theta_s(\phi)$, proving that θ_s is continuous. We next claim that θ_s is bijective and $\theta_s^{-1} = \theta_{s^*}$. In fact, for all $\phi \in D_{s^*s}$ and all $e \in E$, we have

$$\theta_{s^*}\big(\theta_s(\phi)\big)(e) = \theta_s(\phi)(ses^*) = \phi(s^*ses^*s) = \phi(s^*s)\phi(e)\phi(s^*s) = \phi(e),$$

so $\theta_{s^*} \circ \theta_s$ is the identity on D_{s^*s}. By exchanging s and s^* we have that $\theta_s \circ \theta_{s^*}$ is also the identity on D_{ss^*}, verifying our claim, and also giving

$$\theta_{s^*} = \theta_s^{-1}.$$

This proves also that θ_s^{-1} is continuous, so θ_s is a homeomorphism as required by (ii).

Before we tackle (iii) observe that for every $e, f \in E$ one has that $D_e \cap D_f = D_{ef}$. In addition we claim that

$$\theta_s(D_{s^*s} \cap D_e) = D_{ses^*}. \tag{10.3.1}$$

In fact, a semicharacter ϕ lies in the set displayed on the left-hand side above if and only if

$$\begin{aligned} \theta_s^{-1}(\phi) \in D_{s^*s} \cap D_e = D_{s^*se} &\iff \theta_{s^*}(\phi)(s^*se) = 1 \\ &\iff \phi(ss^*ses^*) = 1 \\ &\iff \phi(ses^*) = 1 \\ &\iff \phi \in D_{ses^*}. \end{aligned}$$

In particular, given $s, t \in S$, the domain of $\theta_t \circ \theta_s$ is given by

$$\theta_s^{-1}(D_{ss^*} \cap D_{t^*t}) = \theta_{s^*}(D_{ss^*} \cap D_{t^*t}) = D_{s^*t^*ts} = D_{(ts)^*ts},$$

which is precisely the domain of θ_{ts}. Moreover for every $\phi \in D_{(ts)^*ts}$, and every $e \in E$, we have

$$\theta_s\big(\theta_t(\phi)\big)(e) = \theta_t(\phi)(s^*es) = \phi(t^*s^*est) = \theta_{st}(\phi)(e),$$

proving that $\theta_s \circ \theta_t = \theta_{st}$, and (iii) follows. To prove (iv) it is now enough to check (4.3.ii). For this it suffices to observe that if $\phi \in \hat{E}$ then ϕ is nonzero by definition, and hence there exists $e \in E$ such that $\phi(e) = 1$. Thus ϕ lies in the domain of any θ_s for which $s^*s = e$, for example $s = e$. ☐

Definition 10.4. Let H be a Hilbert space. A map $\sigma : S \to B(H)$ will be called a *representation of S on H* if for every $s, t \in S$ one has

(i) $\sigma_{st} = \sigma_s \sigma_t$,

(ii) $\sigma_{s^*} = \sigma_s^*$.

We have already encountered such objects when we studied covariant representations, as defined in (8.1). The difference is that here there is no representation π of $C_0(X)$ to go along with σ.

For every $e \in E$, denote by 1_e the characteristic function of $D_e \subseteq \hat{E}$. A concrete description of 1_e may be given e.g. by

$$1_e(\phi) = \phi(e), \quad \forall \phi \in \hat{E}. \tag{10.5}$$

Since D_e is clopen we have that 1_e is continuous. Moreover D_e is compact so $1_e \in C_c(\hat{E}) \subseteq C_0(\hat{E})$.

Proposition 10.6. *Let σ be a representation of S on a Hilbert space H. Then there exists a unique *-representation π_σ of $C_0(\hat{E})$ on H such that $\pi_\sigma(1_e) = \sigma_e$, for every $e \in E$. In addition the pair (π_σ, σ) is a covariant representation of the system $(\theta, S, \hat{E})$*

Proof. The Stone–Weierstrass theorem readily implies that the set of all 1_e's span a dense subalgebra of $C_0(\hat{E})$, from where uniqueness follows. To prove existence, let A be the closed *-subalgebra of $B(H)$ generated by $\{\sigma_e : e \in E\}$. It is immediate that A is commutative, so let us denote the spectrum of A by $\hat{A}$. Given $\psi \in \hat{A}$, observe that the map

$$\phi : e \in E \mapsto \psi(\sigma_e) \in \{0, 1\}$$

is a semicharacter of E (it is nonzero because ψ is nonzero). This allows us to define a map

$$j : \psi \in \hat{A} \longmapsto \phi = \psi \circ \sigma \in \hat{E},$$

which is obviously continuous and injective. If we temporarily (and heretically) alter the definition of both $\hat{A}$ and $\hat{E}$ by dropping the requirement that characters (in the case of $\hat{A}$) and semicharacters (in the case of $\hat{E}$) be nonzero, then the map j above will satisfy $j(0) = 0$. This means that, returning to the usual (and sacrosanct) notion of spectrum, j is a proper map. It follows that

$$\pi_\sigma : f \in C_0(\hat{E}) \longmapsto f \circ j \in C_0(\hat{A}) = A$$

is a well defined surjective *-homomorphism. Since $A \subseteq B(H)$, we may view π_σ as a representation of A on H. Let us next prove that

$$\pi_\sigma(1_e) = \sigma_e. \tag{10.6.1}$$

To prove it observe that for every $\psi \in \hat{A}$ we have

$$\psi(\pi_\sigma(1_e)) = \widehat{\pi_\sigma(1_e)}(\psi) = 1_e(j(\psi)) = 1_e(\psi \circ \sigma) = (\psi \circ \sigma)(e) = \psi(\sigma_e),$$

proving (10.6.1). In order to prove (8.1.iii) let $s \in S$ and $f \in C_0(\hat{E})$. Since the algebra generated by the 1_e is dense in $C_0(\hat{E})$, we may assume that $f = 1_e$, for some $e \in E$. Denoting by $\alpha_s(f) = f \circ \theta_{s^*}$, for $f \in C_0(D_{s^*s})$, as in (7.2.i), notice that $\alpha_s(1_e) = 1_e \circ \theta_{s^*}$ is the characteristic function of

$$\{\phi \in \hat{E} : \theta_{s^*}(\phi) \in D_e\} = \{\phi \in \hat{E} : \phi(s^*es) = 1\} = D_{s^*es},$$

that is, $\alpha_s(1_e) = 1_{s^*es}$. Therefore

$$\sigma_s \pi_\sigma(1_e)\sigma_{s^*} \overset{(10.6.1)}{=} \sigma_s\sigma_e\sigma_{s^*} = \sigma_{ses^*} = \pi_\sigma(1_{s^*es}) = \pi_\sigma(\alpha_s(1_e)).$$

Addressing (8.1.iv) notice that the compacity of D_e implies that $C_0(D_e) = C(D_e)$ is a unital algebra with unit 1_e. It follows that

$$\overline{\pi_\sigma\big(C_0(D_e)\big)(H)} = \pi_\sigma(1_e)(H) \overset{(10.6.1)}{=} \sigma_e(H),$$

and the proof is complete. ◻

We do not want to be restricted to studying only the action of S on $\hat{E}$. In fact the most interesting intrinsic actions take place on subsets of $\hat{E}$. But of course only invariant subsets matter.

Definition 10.7. We say that a subset $X \subseteq \hat{E}$ is invariant if for every $s \in S$ one has that

$$\theta_s(D_{s^*s} \cap X) \subseteq X.$$

In that case, for every $e \in E$ we denote

$$D_e^X = D_e \cap X,$$

and for every $s \in S$ we let

$$\theta_s^X : D_{s^*s}^X \to D_{ss^*}^X$$

be given by restricting θ_s.

It is then elementary to prove that the correspondence

$$\theta^X : s \in S \mapsto \mathcal{I}(X) \tag{10.8}$$

is an action of S on X.

These actions, for suitably chosen subsets $X \subseteq \hat{E}$, will dominate our attention throughout this work. It is therefore interesting that we can sometimes guarantee that its groupoid of germs is Hausdorff:

Corollary 10.9. *Let S be an inverse semigroup which is a semilattice with respect to its natural order (such as an E^*-unitary inverse semigroup). If E denotes the idempotent semilattice of S, and if $X \subseteq \hat{E}$ is a closed invariant subspace, then the groupoid of germs for the action θ^X of S on X, as defined in* (10.7)*, is a Hausdorff groupoid.*

Proof. As pointed out shortly after (10.5), we have that D_e is a compact subset of $\hat{E}$, for every $e \in E$. Hence $D_e^X = D_e \cap X$ is closed. The statement then follows from (6.2). $\square$

The following result shows that invariant subsets may be found underlying Hilbert space representations of S.

Proposition 10.10. *Given a representation σ of S on a Hilbert space H, write the kernel of π_σ as $C_0(U)$, where U is an open subset of $\hat{E}$. Then $X := \hat{E} \setminus U$ is a closed invariant subset.*

Proof. Given $s \in S$ and $\phi \in D_{s^*s} \cap X$, suppose by contradiction that $\theta_s(\phi) \notin X$. Then $\theta_s(\phi) \in U \cap D_{ss^*}$, and by Urysohn's Theorem, there exists

$$f \in C_0(U \cap D_{ss^*})$$

such that $f(\theta_s(\phi)) = 1$. Then

$$0 = \sigma_{s^*}\pi_\sigma(f)\sigma_s = \pi_\sigma\big(\alpha_{s^*}(f)\big),$$

which implies that $\alpha_{s^*}(f) \in C_0(U)$. Since

$$\alpha_{s^*}(f)(\phi) = f\big(\theta_s(\phi)\big) = 1,$$

we conclude that $\phi \in U$, but we have taken ϕ in X. This is a contradiction and hence X is indeed invariant. $\square$

Definition 10.11. Let σ be a representation of S on a Hilbert space H. We will say that σ is *supported* on a given subset $X \subseteq \hat{E}$ if the representation π_σ of (10.6) vanishes on $C_0(\hat{E} \setminus X)$.

Fix for the time being a closed invariant set $X \subseteq \hat{E}$ and let

$$\mathcal{G}^X = \mathcal{G}(\theta^X, S, X), \tag{10.12}$$

be the groupoid of germs associated to the system (θ^X, S, X). Observe that for every $e \in E$ one has that D_e^X (defined in (10.7)) is a compact open subset of X. Denoting by 1_e^X the characteristic function of $D_e^X \subseteq X$, we then have that $1_e^X \in C_c(X)$. Employing the notation introduced in (7.2.v) we see that for every $s \in S$,

$$1_{ss^*}^X \delta_s \in C_c(\Theta_s) \subseteq C_c(\mathcal{G}^X),$$

where Θ_s was defined in (7.2.ii).

Proposition 10.13. *Let $X \subseteq \hat{E}$ be a closed invariant set. Then the correspondence*

$$\sigma^X : s \in S \mapsto i\big(1_{ss^*}^X \delta_s\big) \in C^*(\mathcal{G}^X),$$

(recall that i was defined in (3.17.1)*) is a representation of S (where we imagine $C^*(\mathcal{G})$ as an operator algebra via any faithful *-representation) which is supported on X. In fact, the set U referred to in* (10.10) *is precisely equal to $\hat{E} \setminus X$.*

Proof. For simplicity in this proof we will occasionally drop the superscripts "X", as it will cause no confusion. We will moreover identify $C_c(\mathcal{G}^X)$ with its copy within $C^*(\mathcal{G}^X)$, hence dropping "i" as well. For $s, t \in S$ we have by (7.5.iii)

$$\sigma_s \sigma_t = (1_{ss^*}\delta_s) \star (1_{tt^*}\delta_t) = \alpha_s\big(\alpha_{s^*}(1_{ss^*})1_{tt^*}\big)\delta_{st} = \alpha_s(1_{s^*s}1_{tt^*})\delta_{st} = 1_{st(st)^*}\delta_{st},$$

where the last step follows easily from (10.3.1). This proves (10.4.i), and (10.4.ii) may easily be proved with the aid of (7.5.iv). Thus σ^X is indeed a representation of S in $C^*(\mathcal{G}^X)$, but we must still identify the set U of (10.10). The relevant representation of $C_0(\hat{E})$ should really be denoted π_{σ^X}, but we will simply denote it by π. By (10.6.1) we have

$$\pi(1_e) = \sigma_e = 1_e^X \delta_e. \tag{†}$$

Identifying the unit space of $\mathcal{G}^X$ with X as in (4.16), and hence identifying $C_0(X)$ as a subalgebra of $C^*(\mathcal{G}^X)$, we may write (†) as

$$\pi(1_e) = 1_e^X = 1_e|_X,$$

so we conclude that $\pi(f) = f|_X$, for all $f \in C_0(\hat{E})$. The kernel of π is therefore seen to be $C_0(\hat{E} \setminus X)$. ◻

The above representation is universal in the following sense:

Theorem 10.14. *Let S be a countable inverse semigroup and let $\sigma : S \to B(H)$ be a representation which is supported on a closed invariant subset $X \subseteq \hat{E}$. Then there exists a *-representation ρ of $C^*(\mathcal{G}^X)$ on H such that $\rho\big(i(1^X_{ss^*}\delta_s)\big) = \sigma_s$, for all $s \in S$, and hence the diagram*

$$\begin{tikzcd} S \arrow[rr, "\sigma"] \arrow[dr, "\sigma^X"'] & & B(H) \\ & C^*(\mathcal{G}^X) \arrow[ur, "\rho"'] & \end{tikzcd}$$

commutes.

Proof. Let π_σ be as in (10.6). Since π_σ vanishes on $C_0(\hat{E} \setminus X)$ we may factor π_σ through $C_0(X)$ obtaining a representation π of $C_0(X)$ on H such that the diagram

$$\begin{tikzcd} C_0(\hat{E}) \arrow[rr, "\pi_\sigma"] \arrow[dr] & & B(H) \\ & C_0(X) \arrow[ur, "\pi"'] & \end{tikzcd}$$

commutes, where the southeast arrow is given by restriction. We then claim that (π, σ) is a covariant representation of the system (θ^X, S, X). In order to prove it let $f \in C_0(X)$ and choose $g \in C_0(\hat{E})$ whose restriction to X gives f. Then for every $s \in S$ we have

$$\begin{aligned} \sigma_s \pi(f) \sigma_{s^*} &= \sigma_s \pi_\sigma(g) \sigma_{s^*} = \pi_\sigma\big(\alpha_s(g)\big) = \pi\left(\alpha_s(g)|_X\right) \\ &= \pi\left(\alpha^X_s(g|_X)\right) = \pi\left(\alpha^X_s(f)\right) \end{aligned}$$

where we are denoting by α^X the action of S on $C_0(X)$ associated to θ^X, as in (7.2.i). This proves (8.1.iii). To check (8.1.iv) observe that for every $e \in E$ we have

$$\pi\big(C_0(D^X_e)\big)H = \pi(1^X_e)(H) = \pi_\sigma(1_e)(H) = \sigma_e(H),$$

concluding the proof that (π, u) is covariant.

We next wish to apply Theorem (8.5) to this covariant representation, so we must address the countability restrictions: since E is countable, the product space

$\{0, 1\}^E$ is metrizable and hence second countable. We are then given the green light to apply the said Theorem and hence there exists a *-representation ρ of $C^*(\mathcal{G}^X)$ on H such that

$$\rho\big(i(f\delta_s)\big) = \pi(f)\sigma_s,$$

for every $s \in S$, and every $f \in C_c(D_{ss^*})$. We then conclude that

$$\begin{aligned}\rho(\sigma_s^X) &= \rho\big(i(1_{ss^*}^X\delta_s)\big) = \pi(1_{ss^*}^X)\sigma_s \\ &= \pi_\sigma(1_{ss^*})\sigma_s \overset{(10.6.1)}{=} \sigma_{ss^*}\sigma_s = \sigma_{ss^*s} = \sigma_s.\end{aligned}$$ □

The following is a main result:

Corollary 10.15. *Let S be a countable inverse semigroup and let X be a closed invariant subset of $\hat{E}$. Then there is a one-to-one correspondence between representations σ of S supported on X and representations ρ of the C*-algebra of the groupoid of germs for the action θ^X of S on X. If σ and ρ correspond to each other then*

$$\rho\big(i(1_{ss^*}^X\delta_s)\big) = \sigma_s,$$

for all $s \in S$.

Proof. Follows immediately from (10.13) and (10.14). □

One should notice that any representation of S is supported on $\hat{E}$, so we obtain the following version of [24, Theorem 4.4.1]:

Corollary 10.16. *If S is a countable inverse semigroup then there is a one-to-one correspondence between representations of S and representations of the C*-algebra of the groupoid of germs for the natural action θ of S on $\hat{E}$.*

11 Representations of semilattices

We have intentionally postponed until now a very delicate and subtle conceptual problem. If S contains a zero element 0, and σ is a Hilbert space representation of S, is it not natural to expect that $\sigma_0 = 0$? However, including our development so far, most treatments of inverse semigroups completely ignore this issue. In fact some of the better known examples of inverse semigroup representations, such as Wordingham's Theorem [24, Theorem 2.2.2], do send zero to a nonzero element!

The problem with zero is but the tip of an iceberg which we will now explore. The issue apparently only concerns the idempotent semilattice of S.

So we will now fix an abstract semilattice[14] E**, which will always be assumed to contain a smallest element** 0**.**

Definition 11.1. Given a partially ordered set X with smallest element 0, we shall say that two elements x and y in X *intersect*, in symbols $x \Cap y$, if there is a nonzero $z \in X$ such that $z \leqslant x, y$. Otherwise we will say that x and y are *disjoint*, in symbols $x \perp y$.

If E is a semilattice it is easy to see that x and y intersect if and only if $x \wedge y \neq 0$.

Definition 11.2. Let E be a semilattice and let $\mathscr{B} = (\mathscr{B}, 0, 1, \wedge, \vee, \neg)$ be a Boolean algebra. By a *representation* of E in $\mathscr{B}$ we mean a map $\beta : E \to \mathscr{B}$, such that

(i) $\beta(0) = 0$, and

(ii) $\beta(x \wedge y) = \beta(x) \wedge \beta(y)$, for every $x, y \in E$.

Recall that a Boolean algebra $\mathscr{B}$ is also a semilattice under the standard order relation given by

$$x \leqslant y \iff x = x \wedge y, \quad \forall x, y \in \mathscr{B}.$$

Fix for the time being a representation β of a semilattice E in a Boolean algebra $\mathscr{B}$. For every $x, y \in E$, such that $x \leqslant y$, one has that $x = x \wedge y$, and hence

$$\beta(x) = \beta(x \wedge y) = \beta(x) \wedge \beta(y),$$

which means that $\beta(x) \leqslant \beta(y)$. In other words, β preserves the respective order relations. On the other hand if $x, y \in E$ are such that $x \perp y$, one has that $\beta(x) \perp \beta(y)$, which may also be expressed in $\mathscr{B}$ by saying that

$$\beta(x) \leqslant \neg\, \beta(y).$$

More generally, if X and Y are finite subsets of E, and one is given an element $z \in E$ such that $z \leqslant x$ for every $x \in X$, and $z \perp y$ for every $y \in Y$, it follows that

$$\beta(z) \leqslant \bigwedge_{x \in X} \beta(x) \wedge \bigwedge_{y \in Y} \neg\, \beta(y). \tag{11.3}$$

[14] A *semilattice* is by definition a partially ordered set E such that for every $x, y \in E$, there exists a maximum among the elements which are smaller than x and y. Such an element is said to be the *infimum* of x and y, and is denoted $x \wedge y$.

The set of all such z's will acquire an increasing importance, so we make the following:

Definition 11.4. Given finite subsets $X, Y \subseteq E$, we shall denote by $E^{X,Y}$ the subset of E given by

$$E^{X,Y} = \big\{z \in E : z \leqslant x,\ \forall x \in X,\ \text{and}\ z \perp y,\ \forall y \in Y\big\}.$$

Notice that if X is nonempty and $x_{\min} = \bigwedge_{x \in X} x$, one may replace X in (11.4) by the singleton $\{x_{\min}\}$, without altering $E^{X,Y}$. However there does not seem to be a similar way to replace Y by a smaller set.

Definition 11.5. Given any subset F of the semilattice E, we shall say that a subset $Z \subseteq F$ is a *cover for* F, if for every nonzero $x \in F$, there exists $z \in Z$ such that $z \Cap x$. If $y \in E$ and Z is a cover for $F = \{x \in E : x \leqslant y\}$, we will say that Z is a *cover for* y.

The notion of covers is relevant to the introduction of the following central concept (compare [12, 1.3]):

Definition 11.6. Let β be a representation of the semilattice E in the Boolean algebra $\mathscr{B}$. We shall say that β is *tight* if for every finite subsets $X, Y \subseteq E$, and for every finite cover Z for $E^{X,Y}$, one has that

$$\bigvee_{z \in Z} \beta(z) \geqslant \bigwedge_{x \in X} \beta(x) \wedge \bigwedge_{y \in Y} \neg\, \beta(y).$$

Notice that the reverse inequality "$\leqslant$" always holds by (11.3). Thus, when β is tight, we actually get an equality above. We should also remark that in the absence of any finite cover Z for any $E^{X,Y}$, every representation is considered to be tight by default.

It should be stressed that the definition above is meant to include situations in which X, Y, or Z are empty, and in fact this will often be employed in the sequel. It might therefore be convenient to reinforce the convention according to which the supremum of the empty subset of a Boolean algebra is zero, and that its infimum is 1.

For example, if $X = Y = \varnothing$, then $E^{X,Y} = E$, and hence a cover Z for $E^{X,Y}$ must contain quite a lot of elements. If a representation β is tight then the supremum of $\beta(z)$ over such a cover is required to coincide with 1. This may be considered as a *nondegeneracy* condition for tight representations (applicable only when E admits a finite cover).

In certain cases the verification of tightness may be simplified by assuming that $X \neq \varnothing$:

Lemma 11.7. *Let $\beta : E \to \mathscr{B}$ be a representation of the semilattice E in the Boolean algebra $\mathscr{B}$ and suppose that β is known to satisfy the tightness condition* (11.6) *only when X is nonempty. If moreover*

(i) *E contains a finite set X such that $\bigvee_{x\in X}\beta(x)=1$, or*

(ii) *E does not admit any finite cover,*

then β satisfies (11.6) *in full, i.e., β is tight.*

Proof. Our task is therefore to prove the tightness condition even when $X=\varnothing$. So, let $Y\subseteq E$ be a finite set and let Z be a finite cover for $E^{\varnothing,Y}$. Notice that for every $u\in E$, either $u \Cap y$, for some $y\in Y$, or $u\in E^{\varnothing,Y}$, in which case $u\Cap z$, for some $z\in Z$. Therefore $Y\cup Z$ is a finite cover for E. Under hypothesis (ii) this is impossible, meaning that there are no finite covers for $E^{\varnothing,Y}$, so there is nothing to be done. We therefore assume (i), and we must show that

$$\bigvee_{z\in Z}\beta(z)\geqslant\bigwedge_{y\in Y}\neg\,\beta(y). \tag{11.7.1}$$

Let X be as in (i). We claim that for each $x\in X$, the set $x\wedge Z:=\{x\wedge z : z\in Z\}$ is a cover for $E^{\{x\},Y}$. In fact, given a nonzero

$$w\in E^{\{x\},Y}\subseteq E^{\varnothing,Y},$$

there exists some $z\in Z$ such that $z\Cap w$. Since $w\leqslant x$, we have

$$w\wedge x\wedge z=w\wedge z\neq 0,$$

so $w\Cap(x\wedge z)$, concluding the proof of our claim. By hypothesis β satisfies the tightness condition with respect to the cover $x\wedge Z$ for $E^{\{x\},Y}$, and hence

$$\bigvee_{z\in Z}\beta(x\wedge z)\geqslant\beta(x)\wedge\bigwedge_{y\in Y}\neg\,\beta(y). \tag{11.7.2}$$

We therefore have

$$\bigvee_{z\in Z}\beta(z)\overset{\text{(i)}}{=}\bigvee_{z\in Z}\Big(\bigvee_{x\in X}\beta(x)\Big)\wedge\beta(z)=\bigvee_{x\in X}\bigvee_{z\in Z}\beta(x\wedge z)\overset{(11.7.2)}{\geqslant}\bigvee_{x\in X}\Big(\beta(x)\wedge\bigwedge_{y\in Y}\neg\,\beta(y)\Big)$$

$$=\Big(\bigvee_{x\in X}\beta(x)\Big)\wedge\bigwedge_{y\in Y}\neg\,\beta(y)=\bigwedge_{y\in Y}\neg\,\beta(y),$$

proving (11.7.1). ☐

The following alternative characterization of tightness is apparently even weaker than the above:

Proposition 11.8. *Let β be a representation of the semilattice E in the Boolean algebra $\mathscr{B}$, satisfying either* (i) *or* (ii) *of (11.7). Then β is tight if and only if for every $x \in E$ and for every finite cover Z for x, one has that*

$$\bigvee_{z \in Z} \beta(z) \geqslant \beta(x).$$

Proof. The only if part is immediate since $\{u \in E : u \leqslant x\} = E^{\{x\},\varnothing}$. To prove the converse implication let $X, Y \subseteq E$ be finite subsets and let Z be a cover for $E^{X,Y}$. Using (11.7) we may assume that X is nonempty, so let $x_{\min} = \bigwedge_{x \in X} x$. We claim that $Y \cup Z$ is a cover for $E^{\{x_{\min}\},\varnothing}$. In order to prove it pick $u \leqslant x_{\min}$. Then clearly $u \leqslant x$, for every $x \in X$.

Suppose first that $u \notin E^{X,Y}$. Then u necessarily fails to be disjoint from some $y \in Y$, meaning that $x \Cap y$, and thus proving that u intersects some element of $Y \cup Z$. On the other hand, if $u \in E^{X,Y}$, then our assumption guarantees that there exists some element z in Z, and hence also in $Y \cup Z$, which intersects x. This proves our claim, and so the hypothesis gives

$$\beta(x_{\min}) \leqslant \bigvee_{u \in Y \cup Z} \beta(u),$$

and hence also

$$\begin{aligned}\beta(x_{\min}) \wedge \Big(\bigwedge_{y \in Y} \neg\,\beta(y)\Big) &\leqslant \Big(\bigvee_{u \in Y \cup Z} \beta(u)\Big) \wedge \Big(\bigwedge_{y \in Y} \neg\,\beta(y)\Big) \\ &= \bigvee_{u \in Y \cup Z} \Big(\beta(u) \wedge \bigwedge_{y \in Y} \neg\,\beta(y)\Big).\end{aligned}$$

Referring to the term $\beta(u) \wedge \bigwedge_{y \in Y} \neg\,\beta(y)$, appearing above, notice that it is zero for every $u \in Y$. In case $u \in Z$, then because $Z \subseteq E^{X,Y}$, we see that $\beta(u) \leqslant \neg\,\beta(y)$, for all $y \in Y$, and hence the alluded term coincides with $\beta(u)$. The right-hand side of the expression displayed above thus becomes simply $\bigvee_{u \in Z} \beta(u)$, and since $\beta(x_{\min}) = \bigwedge_{x \in X} \beta(x)$, the left-hand side is

$$\Big(\bigwedge_{x \in X} \beta(x)\Big) \wedge \Big(\bigwedge_{y \in Y} \neg\,\beta(y)\Big). \qquad \square$$

When E happens to be a Boolean algebra there is a very elementary characterization of tight representations:

Proposition 11.9. *Suppose that E is a semilattice admitting the structure of a Boolean algebra which induces the same order relation as that of E, and let $\beta : E \to \mathscr{B}$ be a representation of E in some Boolean algebra $\mathscr{B}$. Then β is tight if and only if it is a Boolean algebra homomorphism.*

Proof. Supposing that β is tight, notice that $\{1\}$ is a cover for $E^{\varnothing,\{0\}}$, so

$$\beta(1) = \neg\, \beta(0) = \neg\, 0 = 1.$$

Given $x \in E$ notice that $\{\neg\, x\}$ is a cover for $E^{\varnothing,\{x\}}$, therefore

$$\beta(\neg\, x) = \neg\, \beta(x).$$

Since $x \vee y = \neg\,(\neg\, x \wedge \neg\, y)$, for all $x, y \in E$, we may easily prove that $\beta(x \vee y) = \beta(x) \vee \beta(y)$. Thus β is a Boolean algebra homomorphism, as required.

In order to prove the converse implication let $X, Y \subseteq E$ be finite sets and let Z be a finite cover for $E^{X,Y}$. Let

$$z_0 = \bigvee_{z\in Z} z, \quad x_0 = \bigwedge_{x\in X} x, \quad \text{and} \quad \bar{y}_0 = \bigwedge_{y\in Y} \neg\, y.$$

It is obvious that $z_0 \leqslant x_0 \wedge \bar{y}_0$, and we claim that in fact $z_0 = x_0 \wedge \bar{y}_0$. We will prove it by checking that

$$\neg\, z_0 \wedge x_0 \wedge \bar{y}_0 = 0.$$

Let $u = \neg\, z_0 \wedge x_0 \wedge \bar{y}_0$, and notice that the fact that $u \leqslant x_0 \wedge \bar{y}_0$ implies that $u \in E^{X,Y}$. Arguing by contradiction, and hence supposing that u is nonzero, we deduce that $u \Cap z$, for some $z \in Z$, but this contradicts the fact that $u \leqslant \neg\, z_0$. This proves our claim so, assuming that β is a Boolean algebra homomorphism, we have

$$\bigvee_{z\in Z} \beta(z) = \beta\Big(\bigvee_{z\in Z} z\Big) = \beta(z_0) = \beta(x_0 \wedge \bar{y}_0) = \bigwedge_{x\in X} \beta(x) \wedge \bigwedge_{y\in Y} \neg\, \beta(y),$$

showing that β is tight. ◻

Not all semilattices admit tight injective representations. In order to study this issue in detail it is convenient to introduce the following:

Definition 11.10. Let E be a semilattice and let $x, y \in E$ be such that $y \leqslant x$. We shall say that y is *dense* in x if there is no nonzero $z \in E$ such that $z \perp y$ and $z \leqslant x$. Equivalently, if $E^{\{x\},\{y\}} = \{0\}$.

Obviously each $x \in E$ is dense in itself but it is conceivable that some $y \neq x$ is dense in x. For a concrete example notice that in the semilattice $E = \{0, 1/2, 1\}$, where $0 \leqslant 1/2 \leqslant 1$, one has that $1/2$ is dense in 1.

In the general case, whenever y is dense in x we have that $E^{\{x\},\{y\}} = \{0\}$, and hence the empty set is a cover for $E^{\{x\},\{y\}}$. Therefore for every tight representation β of E one has that

$$0 = \beta(x) \wedge \neg\, \beta(y),$$

which means that $\beta(x) \leqslant \beta(y)$. Since the opposite inequality also holds, we have that $\beta(x) = \beta(y)$. Thus no tight representation of E can possibly separate x and y. The reader is referred to [11] for a thorough study of this and related problems. For future reference we record this conclusion in the next:

Proposition 11.11. *If $y \leqslant x$ are elements in the semilattice E, such that y is dense in x, then $\beta(y) = \beta(x)$ for every tight representation β of E.*

We will have a lot more to say about tight representations in the following sections.

12 Filters and characters

As in the previous section we fix a semilattice E with smallest element 0. A fundamental tool for the study of tight representations of E is the notion of filters, which we shall discuss in this section.

Definition 12.1. Let X be any partially ordered set with minimum element 0. A *filter* in X is a nonempty subset $\xi \subseteq X$, such that

(i) $0 \notin \xi$,

(ii) if $x \in \xi$ and $y \geqslant x$, then $y \in \xi$,

(iii) if $x, y \in \xi$, there exists $z \in \xi$, such that $x, y \geqslant z$.

An *ultra-filter* is a filter which is not properly contained in any filter.

Given a partially ordered set X and any nonzero element $x \in X$, it is elementary to prove that

$$\xi = \{y \in X : y \geqslant x\}$$

is a filter containing x. By Zorn's Lemma there exists an ultra-filter containing ξ, thus every nonzero element in X belongs to some ultra-filter.

When E is a semilattice, given the existence of $x \wedge y$ for every $x, y \in E$, condition (12.1.iii) may be replaced by

$$x, y \in \xi \;\Rightarrow\; x \wedge y \in \xi. \tag{12.2}$$

The following is an important fact about filters in semilattices which also benefits from the existence of $x \wedge y$.

Lemma 12.3. *Let E be a semilattice and let ξ be a filter in E. Then ξ is an ultra-filter if and only if ξ contains every element $y \in E$ such that $y \Cap x$, for every $x \in \xi$.*

Proof. In order to prove the "if" part let η be a filter such that $\xi \subseteq \eta$. Given $y \in \eta$ one has that for every $x \in \xi$, both y and x lie in η, and hence (12.2) implies that $y \wedge x \in \eta$, so $y \wedge x \neq 0$, and hence $y \Cap x$. By hypothesis $y \in \xi$, proving that $\eta = \xi$, and hence that ξ is an ultra-filter.

Conversely let ξ be an ultra-filter and suppose that $y \in E$ is such that $y \Cap x$, for every $x \in \xi$. Defining

$$\eta = \big\{u \in E : u \geqslant y \wedge x, \text{ for some } x \in \xi\big\},$$

we claim that η is a filter. By hypothesis $0 \notin \eta$. Also if $u_1, u_2 \in \eta$, choose for every $i = 1, 2$ some $x_i \in \xi$ such that $u_i \geqslant y \wedge x_i$. Then

$$u_1 \wedge u_2 \geqslant (y \wedge x_1) \wedge (y \wedge x_2) = y \wedge (x_1 \wedge x_2),$$

so $u \in \eta$. Given that (12.1.ii) is obvious we see that η is indeed a filter, as claimed. Noticing that $\xi \subseteq \eta$ we have that $\eta = \xi$, because ξ is an ultra-filter. Since $y \in \eta$, we deduce that $y \in \xi$. □

The study of representations of our semilattice E in the most elementary Boolean algebra of all, namely $\{0, 1\}$, leads us to the following specialization of the notion of semicharacters:

Definition 12.4. By a *character* of E we shall mean any nonzero representation of E in the Boolean algebra $\{0, 1\}$. The set of all characters will be denoted by $\hat{E}_0$.

Thus, a character is nothing but a semicharacter which vanishes at 0. Perhaps the widespread use of the term *semicharacter* is motivated by the fact that it shares prefix with the term *semilattice*. If this is really the case then our choice of the term *character* may not be such a good idea but alas, we cannot think of a better term.

It is easy to see that $\hat{E}_0$ is a closed subset of $\hat{E}$, and hence that $\hat{E}_0$ is locally compact.

Given a character ϕ, observe that

$$\xi_\phi = \big\{x \in E : \phi(x) = 1\big\}, \tag{12.5}$$

is a filter in E (it is nonempty because ϕ is assumed not to be identically zero). Conversely, given a filter ξ, define for every $x \in E$,

$$\phi_\xi(x) = \begin{cases} 1, & \text{if } x \in \xi, \\ 0, & \text{otherwise.} \end{cases} \tag{12.6}$$

It is then easy to see that ϕ_ξ is a character. Therefore we see that (12.5) and (12.6) give one-to-one correspondences between $\hat{E}_0$ and the set of all filters.

Proposition 12.7. *If ξ is an ultra-filter then ϕ_ξ is a tight representation of E in* $\{0, 1\}$.

Proof. Let $X, Y \subset E$ be finite subsets and let Z be a cover for $E^{X,Y}$. In order to prove that

$$\bigvee_{z\in Z} \phi(z) \geqslant \prod_{x\in X} \phi(x) \prod_{y\in Y} (1 - \phi(y)),$$

it is enough to show that if the right-hand side equals 1, then so do the left-hand side. This is to say that if $x \in \xi$ for every $x \in X$, and $y \notin \xi$ for every $y \in Y$, then there is some $z \in Z$, such that $z \in \xi$.

By (12.3), for each $y \in Y$ there exists some $x_y \in \xi$ such that $y \perp x_y$. Supposing by contradiction that $Z \cap \xi = \varnothing$, then for every $z \in Z$ there exists, again by (12.3), some $x_z \in \xi$, such that $z \perp x_z$. Set

$$w = \bigwedge_{x\in X} x \wedge \bigwedge_{y\in Y} x_y \wedge \bigwedge_{z\in Z} x_z.$$

Since $w \in \xi$ we have that $w \neq 0$. Obviously $w \leqslant x$ for every $x \in X$, and $w \perp y$ for every $y \in Y$, and hence $w \in E^{X,Y}$. Since Z is a cover there exists some $z_1 \in Z$ such that $w \Cap z_1$. However, since $w \leqslant x_{z_1} \perp z_1$, we have that $w \perp z_1$, a contradiction. ◻

Definition 12.8. We shall denote by $\hat{E}_\infty$ the set of all characters $\phi \in \hat{E}_0$ such that ξ_ϕ is an ultra-filter. Also we will denote by $\hat{E}_{tight}$ the set of all tight characters.

Employing the terminology just introduced we may rephrase (12.7) by saying that $\hat{E}_\infty \subseteq \hat{E}_{tight}$. The following main result further describes the relationship between $\hat{E}_\infty$ and $\hat{E}_{tight}$.

Theorem 12.9. *Let E be a semilattice with smallest element 0, and let $\hat{E}_\infty$ and $\hat{E}_{\rm tight}$ be as defined in (12.8). Then the closure of $\hat{E}_\infty$ in $\hat{E}_0$ coincides with $\hat{E}_{\rm tight}$.*

Proof. Since the condition for any given ϕ in $\hat{E}_0$ to belong to $\hat{E}_{tight}$ is given by equations it is easy to prove that $\hat{E}_{tight}$ is closed within $\hat{E}_0$, and since $\hat{E}_\infty \subseteq \hat{E}_{tight}$ by (12.7), we deduce that

$$\overline{\hat{E}_\infty} \subseteq \hat{E}_{tight}.$$

To prove the reverse inclusion let us be given $\phi \in \hat{E}_{tight}$. We must therefore show that ϕ can be arbitrarily approximated by elements from $\hat{E}_\infty$. Let U be a neighborhood of ϕ within $\hat{E}_0$. By definition of the product topology, U contains a neighborhood of ϕ of the form

$$\begin{aligned} V = V_{X,Y} = \Big\{ \psi \in \hat{E}_0 : \psi(x) = 1, \text{ for all } x \in X, \quad \text{and} \\ \psi(y) = 0, \text{ for all } y \in Y \Big\}, \end{aligned}$$

where X and Y are finite subsets of E. We next claim that $E^{X,Y} \neq \{0\}$. In order to prove this suppose the contrary, and hence $Z = \varnothing$ is a cover for $E^{X,Y}$. Since ϕ is tight we conclude that

$$0 = \bigvee_{z \in Z} \phi(z) = \prod_{x \in X} \phi(x) \prod_{y \in Y} (1 - \phi(y)).$$

However, since ϕ is supposed to be in V, we have that $\phi(x) = 1$ for all $x \in X$, and $\phi(y) = 0$ for all $y \in Y$, which means that the right-hand side of the expression displayed above equals 1. This is a contradiction and hence our claim is proved.

We are therefore allowed to choose a nonzero $z \in E^{X,Y}$, and further to pick an ultra-filter ξ such that $z \in \xi$. Observe that $\phi_\xi \in \hat{E}_\infty$, and the proof will be concluded once we show that $\phi_\xi \in U$.

For every $x \in X$ and $y \in Y$, we have that $z \leqslant x$ and $z \perp y$, hence $x \in \xi$ and $y \notin \xi$. This entails $\phi_\xi(x) = 1$ and $\phi_\xi(y) = 0$, so $\phi_\xi \in V \subseteq U$, as required. □

Before we close this section let us discuss the issue of tight filters in the idempotent semilattice of an inverse semigroup. We specifically want to prove that the correspondence described by (10.3.ii) preserves tight characters. For this we need an auxiliary result:

Lemma 12.10. *Let S be an inverse semigroup with zero and let E be the idempotent semilattice of S. Given finite subsets X and Y of E, with X nonempty, let Z be a finite cover for $E^{X,Y}$. Then for every $s \in S$ one has that sZs^* is a cover for E^{sXs^*,sYs^*}.*

Proof. Let w be a nonzero element of E such that $w \leqslant sxs^*$ for every $x \in X$, and $w \perp sys^*$ for every $y \in Y$. Then

$$(s^*ws)y = s^*wss^*sy = s^*wsys^*s = 0,$$

so $s^*ws \perp y$, for every $y \in Y$. For every $x \in X$ we have that

$$(s^*ws)x = s^*wss^*sx = s^*wsxs^*s = s^*ws,$$

so $s^*ws \leqslant x$. This shows that $s^*ws \in E^{X,Y}$, and we claim that $s^*ws \neq 0$. For this choose $x \in X$ (allowed because X is nonempty) and observe that $w \leqslant sxs^* \leqslant ss^*$. So

$$0 \neq w = ss^*wss^*,$$

which implies our claim. By hypothesis there exists some $z \in Z$ such that $s^*ws \Cap z$. Noticing that

$$0 \neq (s^*ws)z = s^*wss^*sz = s^*wszs^*s,$$

we deduce that $wszs^* \neq 0$, so $w \Cap szs^*$. □

The promised preservation of tightness is in order:

Proposition 12.8. *Let S be an inverse semigroup with zero and let E be the idempotent semilattice of S. Given $s \in S$ and a tight character ϕ on E such that $\phi(s^*s) = 1$, one has that the character $\theta_s(\phi)$ defined in* (10.3.ii) *is also tight.*

Proof. In view of the requirement that X be nonempty in (12.10) we will use (11.7) for the characterization of tight characters. We may do so for $\theta_s(\phi)$ because $\theta_s(\phi)(ss^*) = 1$.

So let X and Y be finite subsets of E, with X nonempty, and let Z be a cover for $E^{X,Y}$. Then

$$\begin{aligned}\bigvee_{z\in Z}\theta_s(\phi)(z) &= \bigvee_{z\in Z}\phi(s^*zs) = \bigvee_{z'\in s^*Zs}\phi(z')\\ &= \prod_{x'\in s^*Xs}\phi(x')\prod_{y'\in s^*Ys}1-\phi(y')\\ &= \prod_{x\in X}\phi(s^*xs)\prod_{y\in Y}1-\phi(s^*ys)\\ &= \prod_{x\in X}\theta_s(\phi)(x)\prod_{y\in Y}1-\theta_s(\phi)(y),\end{aligned}$$

where we have used (12.10) and the hypothesis that ϕ is tight in walking through the third equal sign above. This concludes the proof. □

If the content of this work is to be subsumed in a single idea, than that idea is that the most natural intrinsic action of S on a topological space is the restriction of the action θ to $\hat{E}_{tight}$, as defined by (10.8). In the following sections we hope to convince the reader of its relevance.

13 Tight representations of inverse semigroups

Throughout this section we will fix an inverse semigroup S with 0. Suppose we are given a representation σ of S on a Hilbert space H and denote by A the closed unital *-subalgebra of $B(H)$ generated by the identity operator and $\{\sigma_e : e \in E(S)\}$. Since A is abelian we see that the set

$$\mathscr{B}_A = \{e \in A : e^2 = e\}$$

is a Boolean algebra relative to the operations

$$e \wedge f = ef, \quad e \vee f = e + f - ef, \quad \text{and} \quad \neg e = 1 - e,$$

for all $e, f \in \mathscr{B}_A$. Provided we assume that $\sigma_0 = 0$, it is clear that the restriction of σ to $E(S)$ is a representation of $E(S)$ in $\mathscr{B}_A$, in the sense of Definition (11.2).

Definition 13.1. A representation σ of S on a Hilbert space H is said to be *tight* if the restriction of σ to $E(S)$ is a tight representation of $E(S)$ in the Boolean algebra $\mathscr{B}_A$, in the sense of (11.6).

Notice that, at the very least, tight representations are required to satisfy $\sigma_0 = 0$.

Theorem 13.2. *A representation σ of S on a Hilbert space H is tight if and only if it is supported in $\hat{E}_{tight}$.*

Proof. Let π_σ be the *-representation of $C_0(\hat{E})$ on H given by (10.6), and write $\mathrm{Ker}(\pi_u) = C_0(U)$, for a suitable open subset $U \subseteq \hat{E}$. Fix finite subsets $X, Y \subseteq E$ and a finite cover Z for $E(S)^{X,Y}$. The condition for tightness of σ is that

$$\bigvee_{z\in Z} \sigma_z = \prod_{x\in X} \sigma_x \prod_{y\in Y} (1 - \sigma_y), \tag{†}$$

which, in view of (10.6.1), is equivalent to

$$\bigvee_{z\in Z} \pi_\sigma(1_z) = \prod_{x\in X} \pi_\sigma(1_x) \prod_{y\in Y} \big(1 - \pi_\sigma(1_y)\big),$$

or to

$$\bigvee_{z\in Z} 1_z - \prod_{x\in X} 1_x \prod_{y\in Y} (1 - 1_y) \in C_0(U).$$

If $f = f_{X,Y,Z}$ is the function on the left-hand side of the expression displayed above then to say that $f \in C_0(U)$ means that $f(\phi) = 0$, for every $\phi \notin U$.

Using (10.5) notice that for every $\phi \in \hat{E}$, to say that $f(\phi) = 0$, is the same as saying that

$$\bigvee_{z\in Z} \phi(z) = \prod_{x\in X} \phi(x) \prod_{y\in Y} (1 - \phi(y)). \tag{‡}$$

Summarizing, σ is tight if and only if for every X, Y and Z, as above, one has that (‡) holds for every $\phi \in \hat{E} \setminus U$. But this is precisely expressing that

$$\hat{E} \setminus U \subseteq \hat{E}_{tight},$$

which is equivalent to $\hat{E} \setminus \hat{E}_{tight} \subseteq U$, or to saying that π_σ vanishes on $C_0(\hat{E} \setminus \hat{E}_{tight})$. The last condition means, by definition, that σ is supported in $\hat{E}_{tight}$. □

The following result largely subsumes our main point so far:

Theorem 13.3 *Let S be a countable inverse semigroup with zero and let $\mathcal{G}_{tight}$ be the groupoid of germs associated to the restriction of the action θ of* (10.3.iv) *to the closed invariant space $\hat{E}_{tight} \subseteq \hat{E}$. Then there is a one-to-one correspondence between tight Hilbert space representations of S and *-representations of $C^*(\mathcal{G}_{tight})$. An explicit form of this correspondence is given by the formula at the end of* (10.15).

Proof. Follows immediately from (10.15) and (13.2). ◻

14 The inverse semigroup associated to a semigroupoid

With this section we start to discuss an application of our methods to semigroupoid C*-algebras, as defined in [10]. Our task here will be to construct an inverse semigroup $S(\Lambda)$ from a given semigroupoid Λ.

We begin by recalling a few basic concepts from the theory of semigroupoids. See [10] for more details. A *semigroupoid* is a triple $(\Lambda, \Lambda^{(2)}, \cdot)$ such that Λ is a set, $\Lambda^{(2)}$ is a subset of $\Lambda \times \Lambda$, and

$$\cdot : \Lambda^{(2)} \to \Lambda$$

is an operation which is associative in the following sense: if $f, g, h \in \Lambda$ are such that either

- $(f, g) \in \Lambda^{(2)}$ and $(g, h) \in \Lambda^{(2)}$, or
- $(f, g) \in \Lambda^{(2)}$ and $(fg, h) \in \Lambda^{(2)}$, or
- $(g, h) \in \Lambda^{(2)}$ and $(f, gh) \in \Lambda^{(2)}$,

then all of (f, g), (g, h), (fg, h) and (f, gh) lie in $\Lambda^{(2)}$, and

$$(fg)h = f(gh).$$

Moreover, for every $f \in \Lambda$, we will let

$$\Lambda^f = \left\{g \in \Lambda : (f, g) \in \Lambda^{(2)}\right\}.$$

From now on we fix a semigroupoid Λ.

If $f, g \in \Lambda$ we will say that f *divides* g, or that g is a *multiple* of f, in symbols $f \mid g$, if either

- $f = g$, or
- there exists $h \in \Lambda$ such that $fh = g$.

We recall from [10] that division is reflexive, transitive and invariant under multiplication on the left.

A useful artifice is to introduce a unit for Λ, that is, pick some element in the universe outside Λ, call it 1, set $\tilde{\Lambda} = \Lambda \dot{\cup} \{1\}$, and for every $f \in \tilde{\Lambda}$ put

$$1f = f1 = f.$$

Then, whenever $f \mid g$, regardless of whether $f = g$ or not, there always exists $x \in \tilde{\Lambda}$ such that $g = fx$.

We will find it useful to extend the definition of Λ^f, for $f \in \tilde{\Lambda}$, by putting

$$\Lambda^1 = \Lambda.$$

Nonetheless, even if $f1$ is a meaningful product for every $f \in \Lambda$, we *will not* include 1 in Λ^f. In the few occasions that we need to refer to the set of *all* elements x in $\tilde{\Lambda}$ for which fx makes sense we shall use $\Lambda^f \cup \{1\}$.

It is interesting to notice that, as a consequence of the associative axiom, for every $f \in \tilde{\Lambda}$, and $g \in \Lambda^f$, one has

$$\Lambda^{fg} = \Lambda^g. \tag{14.1}$$

Note that condition above does not allow for $g = 1$, since 1 is never in Λ^f. Besides, if $g = 1$ then the above equality will most likely fail. It is also easy to see that if $g \in \Lambda$, and $h \in \Lambda^g \cup \{1\}$, then

$$g \in \Lambda^f \iff gh \in \Lambda^f, \tag{14.2}$$

for every $f \in \tilde{\Lambda}$.

Recall from [10, Section 3] that a *spring* is an element $f \in \Lambda$ such that

$$\Lambda^f = \varnothing.$$

If f is a spring one is therefore not allowed to right-multiply it by any element, that is, fg is never a legal multiplication, unless $g = 1$. In some key places below we will suppose that Λ has no springs.

We should be aware that $\tilde{\Lambda}$ is *not* a semigroupoid. Otherwise, since $f1$ and $1g$ are meaningful products, the associativity axiom would imply that $(f1)g$ is also a meaningful product, but this is clearly not always the case. Nevertheless it is interesting to understand precisely which one of the three clauses of the associativity property is responsible for this problem. As already observed, the first clause does fail irremediably when $g = 1$. However it is easy to see that all other clauses do generalize to $\tilde{\Lambda}$. This is quite useful, since when we are developing a computation, having arrived at an expression of the form $(fg)h$, and therefore having already checked that all products involved are meaningful, we most often want to proceed by writing

$$\ldots = (fg)h = f(gh),$$

and this is fortunately meaningful and correct for all $f, g, h \in \tilde{\Lambda}$, because it does not rely on the delicate first clause of associativity.

Given $f, g \in \Lambda$, we say that f and g *intersect* if they admit a *common multiple*, that is, an element $m \in \Lambda$ such that $f \mid m$ and $g \mid m$. Otherwise we will say that f and g are *disjoint*. We will write

$$f \Cap g \tag{14.3}$$

when f and g intersect, and

$$f \perp g \tag{14.4}$$

when f and g are disjoint. Incidentally this notation employs the same symbols "$\Cap$" and "$\perp$", defined in (11.1) in connection to semilattices, with different (although deeply related) meanings, and we will rely on the context to determine the correct interpretation of our notation. Employing the unitization $\tilde{\Lambda}$ notice that $f \Cap g$ if and only if there are $x, y \in \tilde{\Lambda}$ such that $fx = gy$.

Definition 14.5. We shall say that an element $f \in \tilde{\Lambda}$ is *monic* if for every $g, h \in \tilde{\Lambda}$ we have

$$fg = fh \Rightarrow g = h.$$

Observe that the above includes the implication $fg = f \Rightarrow g = 1$. Obviously 1 is monic. Moreover notice that if Λ has a right identity, that is, an element e such that $fe = f$, for all $f \in \Lambda$, then there are no monic elements since $fe = f$, but $e \neq 1$.

Definition 14.6. Let $f, g \in \Lambda$ be such that $f \Cap g$. We shall say that an element $m \in \Lambda$ is a *least common multiple* of f and g, if m is a common multiple of f and g and for every other common multiple h, one has that $m \mid h$.

From now on we shall assume the following:

Standing Hypothesis 14.7. Λ is a semigroupoid in which every element is monic, and moreover every intersecting pair of elements admits a least common multiple.

Observe that if $f, g, h \in \tilde{\Lambda}$ then

$$f \mid g \text{ and } g \mid f \Rightarrow f = g. \tag{14.8}$$

In fact, writing $f = gx$, and $g = fy$, for $x, y \in \tilde{\Lambda}$, we deduce that

$$g = fy = (gx)y = g(xy),$$

which implies that $xy = 1$, but this can only happen if $x = y = 1$, and hence $f = g$.

If f and g are intersecting elements in Λ and if m_1 and m_2 are both least common multiples of f and g, then $m_1 \mid m_2$ and $m_2 \mid m_1$, so $m_1 = m_2$ by (14.8). Therefore there is exactly one least common multiple for f and g, which we denote as

$$\operatorname{lcm}(f, g).$$

We next relate the notion of least common multiples to the categorical notion of *pull-backs*.

Proposition 14.9. *Let f and g be intersecting elements in Λ, and write* $\operatorname{lcm}(f, g) = fp = gq$. *Then (p, q) is the unique pair of elements in $\tilde{\Lambda}$ such that*

(i) *$fp = gq$, and*

(ii) *for every other pair of elements $p', q' \in \tilde{\Lambda}$ such that $fp' = gq'$, there exists a unique $r \in \tilde{\Lambda}$ such that $p' = pr$, and $q' = qr$.*

$$\begin{tikzcd}
 & & \bullet \arrow[dr, "f"] & \\
\bullet \arrow[urr, "p'"] \arrow[r, "r"] \arrow[drr, "q'"'] & \bullet \arrow[ur, "p"'] \arrow[dr, "q"] & & \bullet \\
 & & \bullet \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. We initially notice that the occurrence of black dots in our diagram is not intended to give the idea of *source* or *range*, as we are not assuming that our semigroupoid is a category.

Given (p', q') as in (ii) notice that $m' := fp'$ is a common multiple of f and g. Therefore $m \mid m'$, so there exists $r \in \tilde{\Lambda}$ such that $m' = mr$. It follows that

$$fp' = m' = mr = fpr.$$

Since f is monic we deduce that $p' = pr$, and a similar reasoning gives $q' = qr$. The uniqueness of r follows from the fact that p is monic. Next let us address the uniqueness of (p, q), by assuming that (p_1, q_1) and (p_2, q_2) are two pairs satisfying (i) and (ii).

Applying (ii) twice we conclude that there are r and s in $\tilde{\Lambda}$ such that $p_2 = p_1 r$, and $q_2 = q_1 r$ on the one hand, and $p_1 = p_2 s$, and $q_1 = q_2 s$, on the other. Since

$$p_1 = p_2 s = p_1 r s,$$

we deduce that $rs = 1$, whence $r = s = 1$ and uniqueness follows. □

We will now begin the actual construction of the inverse semigroup $S(\Lambda)$. The first step is to consider a certain collection of subsets of Λ:

Definition 14.10. We shall let $\mathcal{Q}$ denote the collection of all subsets of Λ of the form

$$Q^F = \bigcap_{f \in F} \Lambda^f,$$

where F is a nonempty finite subset of Λ. By default the empty set will also be included in $\mathcal{Q}$.

Since we have prohibited 1 to be in any Λ^f, no member of $\mathcal{Q}$ is allowed to contain 1. In addition we have prohibited 1 to be in the set F above (recall that $F \subseteq \Lambda$), so Λ^1 is never involved in the intersection of sets making up Q^F, above. Therefore Λ is only a member of $\mathcal{Q}$ if there exists some $f \in \Lambda$ for which $\Lambda^f = \Lambda$, which is not always the case, and rarely true in the examples we wish to consider.

It is noteworthy that $\mathcal{Q}$ is closed under intersections and hence it is a semilattice with smallest element $\varnothing$. As already noticed it may or may not contain a largest element.

The underlying set of the inverse semigroup we wish to construct may already be introduced:

Definition 14.11. We will let $S(\Lambda)$ denote the set

$$S(\Lambda) = \big\{(f, A, g) \in \tilde{\Lambda} \times \mathcal{Q} \times \tilde{\Lambda} \ : \ A \subseteq \Lambda^f \cap \Lambda^g\big\}.$$

We will tacitly assume that all elements of $S(\Lambda)$ of the form (f, A, g), with $A = \varnothing$, are identified with each other, forming an *equivalence class* which we will call *zero* and denote by 0.

Apart from the identification referred to above, no other identifications will be implicitly or explicitly made.

We will now work towards defining the multiplication operation on $S(\Lambda)$. The following rudimentary notation will be extremely useful:

Definition 14.12. Given $f \in \tilde{\Lambda}$ and $A \in \mathcal{Q}$ we shall let

$$f^{-1}(A) = \big\{g \in \Lambda^f : fg \in A\big\}.$$

The true meaning of $f^{-1}(A)$ is revealed next:

Proposition 14.13. *Given $A \in \mathcal{Q}$ one has*

(i) $1^{-1}(A) = A$,

(ii) *if* $f \in A$, *then* $f^{-1}(A) = \Lambda^f$,

(iii) *if* $f \in \Lambda \setminus A$, *then* $f^{-1}(A) = \varnothing$.

Proof. Skipping the obvious first statement write $A = \bigcap_{h \in H} \Lambda^h$, where $H \subseteq \Lambda$ is a finite subset. We begin by proving (iii) by contradiction. So, supposing that $f \in \Lambda$ and $f^{-1}(A)$ is nonempty pick $g \in f^{-1}(A)$. Then $fg \in A$, which means that $(h, fg) \in \Lambda^{(2)}$ for all $h \in H$. By the associativity property (and the fact that $f \neq 1$) we deduce that $(h, f) \in \Lambda^{(2)}$, and hence that $f \in A$, proving (iii).

As for (ii) if $f \in A$, then again by the associativity property we have that $(h, fg) \in \Lambda^{(2)}$ for all $h \in H$ and $g \in \Lambda^f$, so $fg \in A$, of $g \in f^{-1}(A)$. □

A couple of elementary facts related to the above notation are:

Proposition 14.14. *Let* $f, g \in \tilde{\Lambda}$, *and* $A \in \mathscr{Q}$.

(i) *If* $g^{-1}\big(f^{-1}(A)\big)$ *is nonempty, then* $g \in \Lambda^f \cup \{1\}$,

(ii) *If* $g \in \Lambda^f \cup \{1\}$, *then* $(fg)^{-1}(A) = g^{-1}\big(f^{-1}(A)\big)$.

Proof.

(i) The result is obvious if either $f = 1$ or $g = 1$, so we suppose $f, g \in \Lambda$. Clearly $f^{-1}(A)$ is nonempty, so $f \in A$ by (14.13.iii), in which case $f^{-1}(A) = \Lambda^f$ by (14.13.ii). The hypothesis is then that $g^{-1}(\Lambda^f)$ is nonempty and, again by (14.13.iii), we conclude that $g \in \Lambda^f$.

(ii) Left to the reader. □

We are now ready to describe the multiplication operation on $S(\Lambda)$.

Definition 14.15. Given (f, A, g) and (h, B, k) in $S(\Lambda)$ we will let

$$(f, A, g)(h, B, k) = \begin{cases} \big(fu\ ,\ u^{-1}(A) \cap v^{-1}(B)\ ,\ kv\big)\ , & \text{if } \operatorname{lcm}(g, h) = gu = hv, \\ \qquad\qquad 0 \qquad\qquad\qquad , & \text{if } g \perp h. \end{cases}$$

There is a slight hitch in the above definition in the sense that nothing guarantees that fu and kv are legal products. However, if u is not in $\Lambda^f \cup \{1\}$, then u is not in A either, because $A \subseteq \Lambda^f$. By (14.13) we deduce that $u^{-1}(A) = \varnothing$, and hence we define the product to be zero by the rule that any triple with the empty set in the middle represents zero in $S(\Lambda)$, regardless of the fact that fu is not defined. The same argument applies if v is not in $\Lambda^g \cup \{1\}$.

Rather than include a third clause in the definition above we shall accept illegal products fu and kv, only as long as the empty set rule applies.

There is a diagrammatic interpretation for the product: in case $(f, A, g)(h, B, k)$ is nonzero, we have that $g \Cap h$, so we may write a *pull-back diagram* for (g, h), as displayed in the diamond at the center of the diagram below.

$$\begin{tikzcd}
 & & u^{-1}(A)\cap v^{-1}(B) \arrow[dl, "u"'] \arrow[dr, "v"] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
\bullet & & \bullet & & \bullet
\end{tikzcd}$$

Imagining that the element (f, A, g) is represented by the triangle in the lower left corner, including the decoration "A" at its top vertex, and similarly for (h, B, k), the product is then represented by the big triangle encompassing the whole diagram, with $u^{-1}(A) \cap v^{-1}(B)$ as decoration. This idea is used to prove the following:

Theorem 14.16. *Let S be an inverse semigroup satisfying* (14.7). *Then the multiplication on $S(\Lambda)$ introduced above is well defined and associative, and hence $S(\Lambda)$ is a semigroup. It is moreover an inverse semigroup with zero, where the adjoint operation is given by*

$$(f, A, g)^* = (g, A, f).$$

Proof. Since the middle coordinate of 0 is the empty set, it is clear that

$$0s = s0 = 0, \quad \forall s \in S(\Lambda).$$

Next, given (f, A, g) and (h, B, k) in $S(\Lambda)$ we must show that their product in fact lies in $S(\Lambda)$. Clearly this is so if the product comes out to be zero, so we suppose otherwise, and hence $\mathrm{lcm}(g, h) = gu = hv$, for suitable $u, v \in \Lambda$. We claim that $u^{-1}(A) \subseteq \Lambda^{fu}$. This is obvious if $u = 1$, and if $u \neq 1$, (14.13) applies to give

$$u^{-1}(A) \subseteq \Lambda^u = \Lambda^{fu}.$$

Similarly $v^{-1}(B) \subseteq \Lambda^{kv}$. Therefore $u^{-1}(A) \cap v^{-1}(B) \subseteq \Lambda^{fu} \cap \Lambda^{kv}$, proving that the product does belong to $S(\Lambda)$. To prove associativity let (l, C, m) be a third element in $S(\Lambda)$ and we shall prove that

$$\big((f, A, g)(h, B, k)\big)(l, C, m) = (f, A, g)\big((h, B, k)(l, C, m)\big). \tag{14.16.1}$$

We leave it up to the reader to show that if either $g \perp h$, or $k \perp l$, then both sides reduce to zero. So we assume instead that $g \Cap h$, and $k \Cap l$, and write $\operatorname{lcm}(g, h) = gu = hv$, and $\operatorname{lcm}(k, l) = kx = ly$.

We now claim that if $v \perp x$, then both sides of (14.16.1) vanish. Assuming by contradiction that e.g. the left-hand side is nonzero then

$$0 \neq (f, A, g)(h, B, k) = (fu, u^{-1}(A) \cap v^{-1}(B), kv),$$

and moreover $kv \Cap l$. Write $\operatorname{lcm}(kv, l) = kvz = lw$. By (14.9) we deduce that $vz = xr$, and $w = yr$, for some $r \in \tilde{\Lambda}$. This contradicts the assumption that $v \perp x$, and a similar argument proves that the right-hand side vanishes as well, so (14.16.1) is proved under the hypothesis that $v \perp x$.

We are then left to treat the case in which $v \Cap x$. Write

$$\operatorname{lcm}(v, x) = vp = xq, \tag{14.16.2}$$

so we have built the diagram

$$\begin{tikzcd}[column sep=small]
 & & & \bullet \arrow[dl, "p"'] \arrow[dr, "q"] & & & \\
 & & \bullet \arrow[dl, "u"'] \arrow[dr, "v"] & & \bullet \arrow[dl, "x"'] \arrow[dr, "y"] & & \\
 & \overset{A}{\bullet} \arrow[dl, "f"'] \arrow[dr, "g"] & & \overset{B}{\bullet} \arrow[dl, "h"'] \arrow[dr, "k"] & & \overset{C}{\bullet} \arrow[dl, "l"'] \arrow[dr, "m"] & \\
\bullet & & \bullet & & \bullet & & \bullet
\end{tikzcd}$$

Even under all of the hypotheses assumed so far, it is still possible that either side of (14.16.1) vanish, given the role played by the sets A, B, and C. Obviously, if both sides vanish there is nothing to prove, so we shall assume without loss of generality that the left-hand side is nonzero. This entitles us to assume the following

(a) $u \in A \cup \{1\}$,

(b) $v \in B \cup \{1\}$.

Since $A \subseteq \Lambda^f$ and $B \subseteq \Lambda^k$ we have that fu and kv are indeed legal products, and in addition

(c) $kv \Cap l$.

As above let $\operatorname{lcm}(kv, l) = kvz = lw$ and, using (14.9), write $vz = xr$, and $w = yr$, for some $r \in \tilde{\Lambda}$. Using (14.16.2) we may further write $z = ps$, and $r = qs$, with $s \in \tilde{\Lambda}$.

We would very much like to be able to perform the multiplication "kvp", but even though kv and vp are known to be legal multiplications we cannot use the only clause of the associativity property which might fail if $v = 1$. Briefly assuming that $v = 1$, notice that

$$kvz = kz = k(ps),$$

which implies that kp is a legal multiplication, thus taking care of our concern. We next observe that

$$(kv)p = k(vp) = k(xq) = (kx)q = (ly)q = l(yq),$$

so both kv and l divide kvp, and hence

$$kvz = \operatorname{lcm}(kv, l) \mid kvp.$$

Since all elements are monic this implies that $z \mid p$, but we have seen above that $p \mid z$, and hence $p = z$ by (14.8). This gives $s = 1$, and hence $r = q$, and finally $w = yq$. Summarizing,

$$\operatorname{lcm}(kv, l) = kvp = lyq.$$

Recall that we are assuming the non-vanishing of the left-hand side of (14.16.1), which is given by

$$\begin{aligned}
&\big(fu, u^{-1}(A) \cap v^{-1}(B), kv\big)\,\big(l, C, m\big) \\
&\quad = \big(fup, p^{-1}\big(u^{-1}(A) \cap v^{-1}(B)\big) \cap (yq)^{-1}(C), myq\big) \qquad (14.16.3)\\
&\quad = \big(fup, p^{-1}\big(u^{-1}(A)\big) \cap p^{-1}\big(v^{-1}(B)\big) \cap q^{-1}\big(y^{-1}(C)\big), myq\big).
\end{aligned}$$

Using (14.14) we have that $p \in \Lambda^u \cup \{1\}$, so up is a legal multiplication. Moreover, by (14.13) we have that $up \in A \cup \{1\}$. Since $A \subseteq \Lambda^g$ we are allowed to set

$$t = gup.$$

Speaking of the right-hand side of (14.16.1), we have

$$(f, A, g)\big((h, B, k)(l, C, m)\big) = (f, A, g)\big(hx, x^{-1}(B) \cap y^{-1}(C), my\big), \quad (14.16.4)$$

and we claim that t, defined just above, is the least common multiple of g and hx. It is clear that $g \mid t$ and

$$t = gup = (hv)p = h(vp) = h(xq) = (hx)q, \qquad (14.16.5)$$

so $hx \mid t$, as well. Let s be a common multiple of g and hx, and write $s = ga = hxb$, for suitable $a, b \in \tilde{\Lambda}$. Using (14.9) there is $c \in \tilde{\Lambda}$ such that $a = uc$, and $xb = vc$. Observing that

$$h(xb) = ga = g(uc) = (gu)c = (hv)c = h(vc),$$

and that h is monic, we have $vc = xb$, so we may write $c = pd$, and $b = qd$, for some $d \in \tilde{\Lambda}$. Thus

$$s = hxb = (hx)(qd) = ((hx)q)d \overset{(14.16.5)}{=} td,$$

proving that $t \mid s$. This shows that $t = \operatorname{lcm}(g, hx)$, and by (14.16.5) we have that (14.16.4) equals

$$\begin{aligned}&\big(fup, (up)^{-1}(A) \cap q^{-1}\big(x^{-1}(B) \cap y^{-1}(C)\big), myq\big)\\ &\quad = \big(fup, p^{-1}\big(u^{-1}(A)\big) \cap q^{-1}\big(x^{-1}(B)\big) \cap q^{-1}\big(y^{-1}(C)\big), myq\big)\end{aligned}$$

which coincides with (14.16.3) because $pv = qx$. This concludes the proof of associativity, and it remains to prove that $S(\Lambda)$ is an inverse semigroup with the indicated adjoint operation. The reader will find no difficulty in proving that

$$(f, A, g)(g, A, f)(f, A, g) = (f, A, g),$$

so what is really at stake is the uniqueness of the adjoint. So, suppose that we are given s and t in $S(\Lambda)$ such that

$$sts = s, \quad \text{and} \quad tst = t.$$

If either s of t vanishes it is immediate that $t = s^*$, so we will suppose that $s, t \neq 0$. This also implies that all products involved are nonzero. Write $s = (f, A, g)$ and $t = (h, B, k)$ and, observing that $g \Cap h$ and $k \Cap f$, write

$$\operatorname{lcm}(g, h) = gu = hv, \quad \text{and} \quad \operatorname{lcm}(k, f) = kx = fy.$$

We then have

$$s = sts = (f, A, g)(h, B, k)(f, A, g) = \big(fu, u^{-1}(A) \cap v^{-1}(B), kv\big)\ (f, A, g) = \dots$$

Further writing $\mathrm{lcm}(kv, f) = kvz = fw$, the above equals

$$\dots = \Big(fuz, z^{-1}\big(u^{-1}(A) \cap v^{-1}(B)\big) \cap w^{-1}(A), gw\Big). \tag{14.16.6}$$

Since this coincides with s we have that $fuz = f$, and $gw = g$, and hence $u = z = w = 1$, because all elements are monic. Therefore $h \mid g$ and $k \mid f$. Applying the same reasoning to the equation $tst = t$ we deduce that $g \mid h$ and $f \mid k$, so $f = k$, and $g = h$, by (14.8). This also implies that $v = 1$, and turning to the middle coordinate of (14.16.6) we conclude that $A \cap B = A$, so $A \subseteq B$. By symmetry we also have that $B \subseteq A$, so in fact $A = B$, and this finally gives $t = s^*$. □

It is not hard to see that the idempotent semilattice $E(S(\Lambda))$ of $S(\Lambda)$ is formed by the elements $(f, A, g) \in S(\Lambda)$, for which $f = g$. Given the importance of the order relation in $E(S(\Lambda))$ we shall now describe it in explicit terms:

Proposition 14.17. *Let (f, A, f) and (g, B, g) be idempotents in $E(S(\Lambda))$, with $A \neq \varnothing$. Then*

(i) *$(f, A, f) \leqslant (g, B, g)$, if and only if $g \mid f$ and, writing $f = gh$, for $h \in \tilde{\Lambda}$, one has that $A \subseteq h^{-1}(B)$,*

(ii) *if $f = g$, then $(f, A, f) \leqslant (f, B, f)$, if and only if $A \subseteq B$,*

(iii) *if $g \in \Lambda$, and $B = \Lambda^g$, then $(f, A, f) \leqslant (g, \Lambda^g, g)$, if and only if $g \mid f$.*

(iv) *if $g = 1$, and $f \in \Lambda$, then $(f, A, f) \leqslant (1, B, 1)$, if and only if $f \in B$.*

Proof. Beginning with (i), supposing that $f = gh$, and that $A \subseteq h^{-1}(B)$, we have that $\mathrm{lcm}(f, g) = f = f1 = gh$, so

$$(f, A, f)(g, B, g) = (f, A \cap h^{-1}(B), gh) = (f, A, f),$$

so $(f, A, f) \leqslant (g, B, g)$, as desired. Conversely, assuming that $(f, A, f) \leqslant (g, B, g)$ we have that

$$(f, A, f)(g, B, g) = (f, A, f) \neq 0,$$

because of the assumption that $A \neq \varnothing$. This implies that $f \Cap g$, so we write $\operatorname{lcm}(f, g) = fk = gh$, with $k, h \in \tilde{\Lambda}$, and then

$$0 \neq (f, A, f) = (f, A, f)(g, B, g) = (fk, k^{-1}(A) \cap h^{-1}(B), gh).$$

Notice that since the elements we are comparing above are nonzero, there is no identification involved, meaning that equality only holds when the correspondent components agree. This implies in particular that $f = fk = gh$, and hence $k = 1$, by (14.5). This also proves that $g \mid f$. Another conclusion to be drawn from the equation displayed above is that

$$A = k^{-1}(A) \cap h^{-1}(B) = A \cap h^{-1}(B),$$

which implies that $A \subseteq h^{-1}(B)$. The other points follow easily from (i). □

Referring to (14.17), observe that if $A = \varnothing$, then $(f, A, f) = 0$, and hence $(f, A, f) \leqslant (g, B, g)$, regardless of any other relationship between f, g, and B.

Proposition 14.18. *Let (f, A, f) and (g, B, g) be idempotent elements in $E(S(\Lambda))$. Then $(f, A, f) \Cap (g, B, g)$ if and only if there are $u, v \in \tilde{\Lambda}$ such that $fu = gv$, and $u^{-1}(A) \cap v^{-1}(B)$ is nonempty.*

Proof. Supposing that $(f, A, f) \Cap (g, B, g)$ we have that

$$0 \neq (f, A, f)(g, B, g) = \big(fu, u^{-1}(A) \cap v^{-1}(B), gv\big),$$

where $\operatorname{lcm}(f, g) = fu = gv$. Obviously $u^{-1}(A) \cap v^{-1}(B)$ is nonempty.

Conversely, suppose that u and v exist as in the statement. Since $u^{-1}(A)$ is nonempty we have by (14.13) that

$$u \in A \cup \{1\} \subseteq \Lambda^f \cup \{1\},$$

so fu is meaningful, and so is gv. Moreover it is clear that $u^{-1}(A) \subseteq \Lambda^{fu}$, and $v^{-1}(B) \subseteq \Lambda^{gv}$, so we have that

$$u^{-1}(A) \cap v^{-1}(B) \subseteq \Lambda^{fu} \cap \Lambda^{gv},$$

proving that $\big(fu, u^{-1}(A) \cap v^{-1}(B), gv\big)$, is an element of $E(S(\Lambda))$, which is clearly nonzero. Using (14.17.i) we see that this element is smaller than both (f, A, f) and (g, B, g), and hence

$$0 \neq \big(fu, u^{-1}(A) \cap v^{-1}(B), gv\big) \leqslant (f, A, f)(g, B, g),$$

proving that $(f, A, f) \Cap (g, B, g)$. □

The idempotents $(1, B, 1)$ have an interesting property which is described in our next result:

Proposition 14.19. *Let (f, A, f) be an idempotent such that $f \neq 1$, and let $B \in \mathcal{Q}$.*

(i) If $f \in B$, then $(f, A, f) \leqslant (1, B, 1)$.

(ii) If $f \notin B$, then $(f, A, f) \perp (1, B, 1)$.

Proof. We have

$$(1, B, 1)(f, A, f) = (f, f^{-1}(B) \cap A, f) = \ldots$$

Assuming that $f \in B$ we have by (14.13), that $f^{-1}(B) = \Lambda^f$. In addition it is implicit that $A \subseteq \Lambda^f$, so the above equals

$$\ldots = (f, \Lambda^f \cap A, f) = (f, A, f),$$

proving (i). On the other hand, if f is not in B, we have that $f^{-1}(B) = \varnothing$, and hence $(1, B, 1)(f, A, f) = 0$. □

In view of the relevance of E^*-unitary inverse semigroups in the characterization of the Hausdorff property for the groupoid of germs given in (6.2) and (6.4), it is interesting to find sufficient conditions for $S(\Lambda)$ to be E^*-unitary. By analogy with (14.5) we will say that an element $f \in \Lambda$ is *epic* if for every $g, h \in \tilde{\Lambda}$ we have

$$gf = hf \Rightarrow g = h.$$

Proposition 14.20. *Let Λ be a semigroupoid satisfying* (14.7)*, and such that all of its elements are epic. Then $S(\Lambda)$ is E^*-unitary.*

Proof. Suppose that an element (f, A, g) in $S(\Lambda)$ dominates a nonzero idempotent (h, B, h). Then

$$0 \neq (h, B, h) = (f, A, g)(h, B, h),$$

which implies that $g \Cap h$, so we may write $\operatorname{lcm}(g, h) = gu = hv$, for suitable elements $u, v \in \tilde{\Lambda}$, and

$$(h, B, h) = (fu, u^{-1}(A) \cap v^{-1}(B), hv).$$

Since this is nonzero we conclude that $h = fu = hv$. It follows that

$$fu = hv = gu,$$

and since u is epic, we conclude that $f = g$, thus proving that (f, A, g) is an idempotent. □

15 Representations of semigroupoids

As before we fix a semigroupoid Λ satisfying (14.7). We shall begin this section by introducing several important notions inspired in [10], most of which are homonyms of similar notions introduced earlier in this work in the context of inverse semigroups, such as *representations*, *covers*, and *tight representations*. Once the appropriate context is clear we believe the double meanings will cause no confusion.

Definition 15.1. A representation of the semigroupoid Λ in an inverse semigroup with zero S is a map $\pi : \Lambda \to S$, such that for every $f, g \in \Lambda$, one has that:

(i) $\pi_f \pi_g = \begin{cases} \pi_{fg}, & \text{if } (f, g) \in \Lambda^{(2)}, \\ 0, & \text{otherwise.} \end{cases}$

Moreover the initial and final projections

$$q_f^\pi = \pi_f^* \pi_f, \quad \text{and} \quad p_g^\pi = \pi_g \pi_g^*, \tag{15.1.1}$$

respectively, are required to satisfy

(ii) $p_f^\pi p_g^\pi = 0$, if $f \perp g$,

(iii) $q_f^\pi p_g^\pi = p_g^\pi$, if $(f, g) \in \Lambda^{(2)}$.

In case S is an inverse semigroup formed by partial isometries on a Hilbert space H, and containing the zero operator, we will say that π is a *representation of* Λ *on* H.

We insist that, since the symbol "$\perp$" is used in (ii) in the context of semigroupoids, its meaning is to be taken from (14.4), and not from (11.1).

One might wonder what happens to the element appearing in the left-hand side of the equation in (15.1.iii), in case (f, g) is not in $\Lambda^{(2)}$. The answer is provided by (15.1.i), since in this case

$$q_f^\pi p_g^\pi = \pi_f^*(\pi_f \pi_g)\pi_g^* = 0. \tag{15.2}$$

Should the context leave no room for confusion we will abbreviate the notations q_f^π and p_g^π, to q_f and p_g, respectively.

Definition 15.3. Let Γ be any subset of the semigroupoid Λ. A subset $H \subseteq \Gamma$ will be called a *cover* for Γ if for every $f \in \Gamma$ there exists $h \in H$ such that

$h \Cap f$. If moreover the elements of H are mutually disjoint then H will be called a *partition* of Γ.

Definition 15.4. Let S be an inverse semigroup of partial isometries on a Hilbert space H, containing the identically zero operator. A representation π of Λ on S is said to be *tight* if for every finite subsets $F, G \subseteq \Lambda$, and for every finite covering H of

$$\Lambda^{F,G} := \bigcap_{f\in F} \Lambda^f \cap \bigcap_{g\in G} \Lambda \setminus \Lambda^g,$$

one has that $\displaystyle\bigvee_{h\in H} p_h = q_{F,G}$, where

$$q_{F,G} := \prod_{f\in F} q_f \prod_{g\in G} (1 - q_g).$$

Observe that the Definition given in [10, 4.5] requires that the above holds for every finite subsets F and G of $\tilde{\Lambda}$, as opposed to Λ. However notice that since $\Lambda^1 = \Lambda$, and the convention adopted there says that $q_1 = 1$, one has that $\Lambda^{F,G} = \varnothing$, whenever $1 \in G$, and the above condition holds vacuously. If, on the other hand, 1 is in F, then $\Lambda^{F,G} = \Lambda^{F',G}$, where $F' = F \setminus \{1\}$, at the same time that

$$\prod_{f\in F} q_f = \prod_{f\in F'} q_f.$$

Therefore we see that the above definition is equivalent to [10, 4.5], regardless of our use of Λ in place of $\tilde{\Lambda}$.

In the above definition the recipient inverse semigroup S needs to be embedded in $B(H)$ or otherwise neither the supremum $\bigvee_{h\in H} p_h$, nor the term $1-q_g$, would make sense. This situation may however be generalized by assuming that $E(S)$ admits the structure of a Boolean algebra which is compatible with the order of $E(S)$, in which case one might say that S is a *Boolean inverse semigroup*. This and related results may be found in [11].

Tight representations have the following good behavior with respect to least common multiples:

Proposition 15.5. *Suppose that for every $f \in \Lambda$, and every $h \in \Lambda^f$, there exists a finite partition H of Λ^f, such that $h \in H$. If $f, g \in \Lambda$ are such that $f \Cap g$, and π is a tight representation of Λ one has that*

$$p_f p_g = p_m,$$

where $m = \mathrm{lcm}(f, g)$.

Proof. In case $f \mid g$, write $g = fh$, with $h \in \tilde{\Lambda}$, and notice that

$$p_f p_g = \pi_f \pi_f^* \pi_f \pi_h \pi_h^* \pi_f^* = \pi_f \pi_h \pi_h^* \pi_f^* = \pi_g \pi_g^* = p_g = p_m,$$

since $m = g$. The case $g \mid f$ may be treated similarly so we next assume that there are u and v in Λ (as opposed to $\tilde{\Lambda}$), such that $fu = gv = m$. By hypothesis let H and K be finite partitions of Λ^f and Λ^g, respectively, such that $u \in H$ and $v \in K$. Given that our representation is tight we have

$$q_f = \bigvee_{h\in H} p_h = \sum_{h\in H} p_h,$$

where the last equality follows from the fact that the p_h are pairwise orthogonal projections by (15.1.ii). Therefore

$$p_f = \pi_f \pi_f^* \pi_f \pi_f^* = \pi_f q_f \pi_f^* = \sum_{h\in H} \pi_f p_h \pi_f^* = \sum_{h\in H} p_{fh},$$

and similarly $p_g = \sum\limits_{k\in K} p_{gk}$. So

$$p_f p_g = \sum_{h\in H} \sum_{k\in K} p_{fh} p_{gk}.$$

Among the pairs $(h,k) \in H \times K$ one clearly has the pair (u,v) for which $p_{fu} p_{gv} = p_m$, and the proof will be complete once we show that $p_{fh} p_{gk} = 0$, for all other pairs (h,k). Thus assume that $(h,k) \in H \times K$ is such that either $h \neq u$ or $k \neq v$. We will in fact prove that $fh \perp gk$, and hence the conclusion will follow from (15.1.ii). Arguing by contradiction suppose that $fhx = gky$, where $x, y \in \tilde{\Lambda}$. By (14.9) we have that $hx = ur$, and $ky = vr$, for some $r \in \tilde{\Lambda}$, but this says that $h \Cap u$ and $k \Cap v$, a contradiction. □

This result motivates the following:

Definition 15.6. A representation π of Λ in an inverse semigroup S is said to *respect least common multiples* if for every intersecting pair of elements f and g in Λ, one has that

$$p_f p_g = p_{\mathrm{lcm}(f,g)}.$$

The following is an important property of these representations. It is related to equation [16, 3.1] in the context of finitely aligned higher rank graphs.

Proposition 15.7. *Suppose that π is a map from Λ into a *-semigroup*[15] *S, satisfying* (15.1.i–iii) *and the equation displayed in* (15.6). *Suppose moreover that*

$$\pi_f \pi_f^* \pi_f = \pi_f, \quad \forall f \in \Lambda.$$

(This is clearly the case if S is an inverse semigroup and π is a representation of Λ in S respecting least common multiples). Given a pair of intersecting elements $f, g \in \Lambda$, with $f \neq g$, write $\operatorname{lcm}(f, g) = m = fh = gk$*, with $h, k \in \tilde{\Lambda}$. Then*

$$\pi_f^* \pi_g = \pi_h \pi_k^*.$$

Proof. It is conceivable that h or k be equal to 1, in which case the right hand side of the equation above needs clarification. First observe that since we are assuming that $f \neq g$, the situation in which both h and k are equal to 1 will never arise. If $h = 1 \neq k$, then $\pi_h \pi_k^*$ is supposed to mean π_k^*, and vice versa. The best way to deal with this problem is to think that $\pi_1 = 1$, where the last occurrence of 1 is a *multiplier* of S, meaning an element which may not belong to S, but which is allowed to multiply elements of S in such a way that

$$1s = s1 = s, \quad \forall s \in S.$$

Also we set $1^* = 1$.

Observe that whenever $(f, g) \in \Lambda^{(2)}$, we have by (15.1.iii) that

$$\pi_f^* \pi_f \pi_h = \pi_f^* \pi_f \pi_h \pi_h^* \pi_h = q_f p_h \pi_h = p_h \pi_h = \pi_h. \tag{15.7.1}$$

Let us now prove the statement under the special assumption that $f \mid g$. In this case we have $m = g$, and $k = 1 \neq h$. Therefore

$$\pi_f^* \pi_g = \pi_f^* \pi_f \pi_h \overset{(15.7.1)}{=} \pi_h = \pi_h \pi_k^*.$$

Assuming instead that $g \mid f$ one may give a similar proof, or just use adjoints, so we next suppose that f and g do not divide each other. This implies that $h, k \neq 1$, so $h \in \Lambda^f$ and $k \in \Lambda^g$. We then have

$$\begin{aligned} \pi_f^* \pi_g &= \pi_f^* \pi_f \pi_f^* \pi_g \pi_g^* \pi_g = \pi_f^* p_f p_g \pi_g = \pi_f^* p_m \pi_g = \pi_f^* \pi_m \pi_m^* \pi_g \\ &= \pi_f^* \pi_f \pi_h \pi_k^* \pi_g^* \pi_g = \pi_f^* \pi_f \pi_h (\pi_g^* \pi_g \pi_k)^* \overset{(15.7.1)}{=} \pi_h \pi_k^*. \end{aligned}$$

□

[15] A *-semigroup is a semigroup equipped with an involution $s \mapsto s^*$, which satisfies $(st)^* = t^* s^*$. For reasons which will soon become apparent we do not suppose that S is an inverse semigroup here.

We will often deal with representations of Λ in inverse semigroups of partial isometries on a Hilbert space and in the lcm-preserving case it is possible to omit any reference to that semigroup:

Proposition 15.8. *Let H be a Hilbert space and let $\pi : \Lambda \to B(H)$ be a map satisfying* (15.1.i–iii). *Suppose moreover that, for every $f, g \in \Lambda$, one has that*

(i) *$\pi(f)$ is a partial isometry,*

(ii) *q_f and q_g commute,*

(iii) *if $f \Cap g$, then $p_f p_g = p_{\mathrm{lcm}(f,g)}$.*

Then the smallest multiplicative subsemigroup of $B(H)$ which is closed under adjoints and contains the range of π is an inverse semigroup and moreover π is a representation of Λ in it.

Proof. For each finite nonempty subset $F \subseteq \Lambda$, let

$$q_F = \prod_{f \in F} q_f.$$

The order in which the above elements are multiplied is irrelevant in view of (ii). Extending π to $\tilde{\Lambda}$ by setting $\pi_1 = 1$, let

$$\mathscr{S} = \left\{\pi_f q_F \pi_g^* : f, g \in \tilde{\Lambda},\ F \subseteq \Lambda \text{ is finite and nonempty}\right\} \cup \{0\}.$$

Notice that for every $(f, g) \in \tilde{\Lambda} \times \tilde{\Lambda} \setminus \{(1, 1)\}$, we have that

$$\pi_f \pi_g^* = \pi_f \pi_f^* \pi_f \pi_g^* \pi_g \pi_g^* = \pi_f q_{\{f,g\}} \pi_g^* \in \mathscr{S},$$

so in particular $\mathscr{S}$ contains the range of π. We next then claim that $\mathscr{S}$ is a multiplicative subsemigroup of $B(H)$. To prove it let us be given $f, g, h, k \in \tilde{\Lambda}$, and finite nonempty subsets $F, G \subseteq \Lambda$. We will prove that

$$\pi_f q_F \pi_g^* \, \pi_h q_G \pi_k^* \tag{15.8.1}$$

either vanishes or equals $\pi_u q_H \pi_v^*$, for suitable $u, v \in \tilde{\Lambda}$, and $H \subseteq \Lambda$. We divide the proof in several cases, according to the values of g and h:

Case 1: $g = h = 1$. Take $u = f$, $H = F \cup G$, and $v = k$.

Case 2: $g = 1,\ h \neq 1$. Notice that $q_F \pi_h = q_F p_h \pi_h$, while

$$q_F p_h = \begin{cases} p_h, & \text{if } (f, h) \in \Lambda^{(2)},\ \forall f \in F, \\ 0, & \text{otherwise,} \end{cases}$$

by (15.1.iii) and (15.2). Thus $q_F \pi_h$ either vanishes or agrees with π_h. Therefore (15.8.1) either vanishes or equals

$$\pi_f q_F \pi_h q_G \pi_k^* = \pi_f \pi_h q_G \pi_k^* = \pi_{fh} q_G \pi_k^*,$$

where we have also assumed that $(f, h) \in \Lambda^{(2)}$, or else (15.8.1) again vanishes.

Case 3: $g \neq 1,\ h = 1$. Follows from case 2, and taking adjoints.

Case 4: $g, h \in \Lambda$, and $g \perp h$. Then $\pi_g^* \pi_h = \pi_g^* p_g p_h \pi_h = 0$, by (15.1.ii), and hence (15.8.1) vanishes.

Case 5: $g = h \neq 1$. Take $u = f$, $H = F \cup \{g\} \cup G$, and $v = k$.

Case 6: $g, h \in \Lambda$, $g \neq h$, and $g \Cap h$. Applying (15.7) to the multiplicative *-semigroup of all bounded operators on H, we have that $\pi_g^* \pi_h = \pi_u \pi_v^*$, with $u, v \in \tilde{\Lambda}$. Then (15.8.1) equals

$$\pi_f q_F \pi_u \pi_v^* q_G \pi_k^*,$$

and the result follows as in case 2. It is now clear that $\mathscr{S}$ is the *-subsemigroup of $B(H)$ generated by the range of π. We will now prove that $\mathscr{S}$ consists of partial isometries. For this let $u \in \mathscr{S}$ be a generic element and write $u = \pi_f q_F \pi_g^*$. Observing that

$$u = \pi_f q_f q_F q_g \pi_g^* = \pi_f q_{\{f\} \cup F \cup \{g\}} \pi_g^*,$$

we may assume that $f, g \in F$. We then have

$$uu^*u = \pi_f q_F \pi_g^*\ \pi_g q_F \pi_f^*\ \pi_f q_F \pi_g^* = \pi_f q_F q_g q_F q_f q_F \pi_g^* = \pi_f q_F \pi_g^* = u,$$

so u is a partial isometry as claimed. It is well known that any subsemigroup of $B(H)$ consisting of partial isometries, and which is closed under adjoints, is an inverse semigroup. It is obvious that $\mathscr{S}$ is closed under adjoints, so it is an inverse semigroup. Obviously π is then a representation of Λ in $\mathscr{S}$. ◻

Our next long term goal is to show a close relationship between tight representations of Λ and tight representations of $S(\Lambda)$ (which in turn are related to representations of the groupoid of germs, by (13.3)). An important ingredient in this relationship is a representation of Λ in $S(\Lambda)$ to be introduced next.

Proposition 15.9. *The map $\tau : \Lambda \to S(\Lambda)$ defined by $\tau_f = (f, \Lambda^f, 1)$, for all $f \in \Lambda$, is a representation of Λ in $S(\Lambda)$, which respects least common multiples and moreover satisfies*

$$q_f^\tau = \tau_f^* \tau_f = (1, \Lambda^f, 1), \quad \textit{and} \quad p_f^\tau = \tau_f \tau_f^* = (f, \Lambda^f, f),$$

for all $f \in \Lambda$.

Proof. For $f \in \Lambda$ one has that

$$\tau_f^* \tau_f = (1, \Lambda^f, f)(f, \Lambda^f, 1) = \big(1, 1^{-1}(\Lambda^f) \cap 1^{-1}(\Lambda^f), 1\big) = (1, \Lambda^f, 1),$$

and similarly one proves that $\tau_f \tau_f^* = (f, \Lambda^f, f)$. If we are also given $g \in \Lambda$, then

$$\tau_f \tau_g = (f, \Lambda^f, 1)(g, \Lambda^g, 1) = \big(fg, g^{-1}(\Lambda^f) \cap \Lambda^g, 1\big).$$

If $(f, g) \in \Lambda^{(2)}$ then $g \in \Lambda^f$ and hence $g^{-1}(\Lambda^f) = \Lambda^g$, by (14.13), so

$$\tau_f \tau_g = \big(fg, \Lambda^g, 1\big) = \big(fg, \Lambda^{fg}, 1\big) = \tau_{fg}.$$

If $(f, g) \notin \Lambda^{(2)}$ then $g \notin \Lambda^f$ and using (14.13) again we have that $g^{-1}(\Lambda^f) = \varnothing$, so

$$\tau_f \tau_g = (fg, \varnothing, 1) = 0,$$

regardless of the fact that fg is meaningless. With respect to (15.1.ii) assume that $f \perp g$. Then

$$\tau_f^* \tau_g = (1, \Lambda^f, f)(g, \Lambda^g, 1) = 0,$$

by definition, from which one sees that $p_f p_g = 0$. If $g \in \Lambda^f$ then

$$\tau_f^* \tau_f \tau_g = (1, \Lambda^f, 1)(g, \Lambda^g, 1) = (g, g^{-1}(\Lambda^f) \cap \Lambda^g, 1) = (g, \Lambda^g, 1) = \tau_g,$$

and hence (15.1.iii) follows. To conclude we must show that τ respects least common multiples, so let $f, g \in \Lambda$ be intersecting elements. Write $\mathrm{lcm}(f, g) = m = fu = gv$, for $u, v \in \tilde{\Lambda}$, and notice that

$$u^{-1}(\Lambda^f) = u^{-1}(f^{-1}(\Lambda)) \overset{(14.14.\mathrm{ii})}{=} (fu)^{-1}(\Lambda) = \Lambda^{fu} = \Lambda^m,$$

and similarly $v^{-1}(\Lambda^g) = \Lambda^m$. So

$$\begin{aligned} p^\tau_f p^\tau_g &= (f, \Lambda^f, f)(g, \Lambda^g, g) = (fu, u^{-1}(\Lambda^f) \cap v^{-1}(\Lambda^g), gv) \\ &= (m, \Lambda^m, m) = p^\tau_m. \end{aligned} \qquad \square$$

It is interesting to notice that if f is a spring, that is, if $\Lambda^f = \varnothing$, then $\tau_f = 0$. This is partly the reason why springs are cumbersome elements to deal with.

Given $A, B \in \mathcal{Q}$ it is immediate that

$$(1, A, 1)(1, B, 1) = (1, A \cap B, 1).$$

So, given any $A \in \mathcal{Q}$, say $A = \bigcap\limits_{h \in H} \Lambda^h$, where H is a nonempty finite subset of Λ, we have that

$$(1, A, 1) = \Big(1, \bigcap_{h \in H} \Lambda^h, 1\Big) = \prod_{h \in H} (1, \Lambda^h, 1) = \prod_{h \in H} \tau^*_h \tau_h.$$

In addition, if $f, g \in \Lambda$ are such that $\Lambda^f \cap \Lambda^g \supseteq A$, we have

$$(f, \Lambda^f, 1)(1, A, 1)(1, \Lambda^g, g) = (f, \Lambda^f \cap A \cap \Lambda^g, g) = (f, A, g),$$

so we have proved that:

Proposition 15.10. *Let $(f, A, g) \in S(\Lambda)$, and write $A = \bigcap_{h \in H} \Lambda^h$, for some nonempty finite subset $H \subseteq \Lambda$. Then*

$$(f, A, g) = \tau_f \Big(\prod_{h \in H} \tau^*_h \tau_h\Big) \tau^*_g.$$

We therefore see that the range of τ generates $S(\Lambda)$ as an inverse semigroup. Our next result uses τ to express the first relationship between representations of Λ and representations of $S(\Lambda)$.

Proposition 15.11. *If σ is a representation of $S(\Lambda)$ on a Hilbert space H (in the sense of 10.4) such that $\sigma(0) = 0$, then the composition $\pi = \sigma \circ \tau$ is a representation of Λ (in the sense of 15.1), which respects least common multiples. If moreover Λ has no springs and σ is tight (in the sense of 13.1), then π is tight (in the sense of 15.4).*

Proof. That π is a representation preserving least common multiples follows immediately from (15.9), and the fact that $\sigma(0) = 0$. Next, assuming that σ is tight, let us prove that the same applies to π. So let $F, G \subseteq \Lambda$ be finite sets and let H be a cover for $\Lambda^{F,G}$ in the sense of (15.3). We must prove that

$$\bigvee_{h\in H} \pi_h\pi_h^* = \prod_{f\in F} \pi_f^*\pi_f \prod_{g\in G}(1-\pi_g^*\pi_g). \tag{15.11.1}$$

Letting

$$\begin{aligned} X &= \big\{(1, \Lambda^f, 1) : f \in F\big\}, \\ Y &= \big\{(1, \Lambda^g, 1) : g \in G\big\}, \\ Z &= \big\{(h, \Lambda^h, h) : h \in H\big\}, \end{aligned}$$

we claim that Z is a cover of $E(S(\Lambda))^{X,Y}$, in the sense of (11.5). In order to prove our claim let (k, C, k) be a nonzero idempotent in $E(S(\Lambda))^{X,Y}$. Therefore C is nonempty, so we pick some $c \in C$. Given that $C \subseteq \Lambda^k$, we may speak of kc, and it is easy to see, based on the fact that $\Lambda^{kc} = \Lambda^c$, and (14.17.i), that

$$(kc, \Lambda^{kc}, kc) \leqslant (k, C, k).$$

Since (k, C, k) is in $E(S(\Lambda))^{X,Y}$, the same applies to (kc, Λ^{kc}, kc), and hence for every $f \in F$, and $g \in G$, we have

$$(kc, \Lambda^{kc}, kc) \leqslant (1, \Lambda^f, 1), \quad \text{and} \quad (kc, \Lambda^{kc}, kc) \perp (1, \Lambda^g, 1).$$

Noticing that (kc, Λ^{kc}, kc) is nonzero because Λ has no springs, and hence it cannot be simultaneously orthogonal and smaller than any other element, we have by (14.19) that $kc \in \Lambda^f$ and $kc \notin \Lambda^g$. This says that

$$kc \in \bigcap_{f\in F} \Lambda^f \cap \bigcap_{g\in G} \Lambda \setminus \Lambda^g = \Lambda^{F,G},$$

so there exists some $h \in H$ such that $kc \Cap h$, and hence we may write $kcx = hy$, for some $x, y \in \tilde{\Lambda}$. Using (14.17.iii) one has that $(kcx, \Lambda^{kcx}, kcx)$ is simultaneously smaller than (h, Λ^h, h) and (kc, Λ^{kc}, kc). This implies that

$$0 \neq (kcx, \Lambda^{kcx}, kcx) \leqslant (h, \Lambda^h, h)(kc, \Lambda^{kc}, kc) \leqslant (h, \Lambda^h, h)(k, C, k),$$

proving that $(h, \Lambda^h, h) \Cap (k, C, k)$. This shows that Z is indeed a cover for $E(S(\Lambda))^{X,Y}$, and because σ is assumed to be a tight representation of $S(\Lambda)$ we have

$$\bigvee_{z\in Z} \sigma_z = \prod_{x\in X} \sigma_x \prod_{y\in Y}(1-\sigma_y). \tag{15.11.2}$$

For every $z = (h, \Lambda^h, h) \in Z$, with $h \in H$, notice that

$$\sigma_z = \sigma(h, \Lambda^h, h) = \sigma(\tau_h\tau_h^*) = \pi_h\pi_h^*,$$

and similarly

$$\sigma_x = \pi_f^*\pi_f, \quad \text{and} \quad \sigma_y = \pi_g^*\pi_g,$$

for every $x = (1, \Lambda^f, 1) \in X$, and $y = (1, \Lambda^g, 1) \in Y$, so we see that (15.11.1) follows from (15.11.2). This proves that π is tight. □

16 The Boolean algebra of Domains

In our pursuit of a bijective correspondence between representations of the semigroupoid Λ and of its associated inverse semigroup $S(\Lambda)$ we would like to show that any representation π of Λ may be *extended* to $S(\Lambda)$, meaning that there exists a representation σ of $S(\Lambda)$, such that $\pi = \sigma \circ \tau$, thus obtaining a converse to (15.11). In order to understand the difficulties in doing so let us temporarily suppose that σ has been found. If F is a nonempty finite subset of Λ, then

$$\begin{aligned}\prod_{f\in F} \pi_f^*\pi_f &= \prod_{f\in F} \sigma(\tau_f^*\tau_f) = \sigma\Big(\prod_{f\in F} \tau_f^*\tau_f\Big) = \sigma\Big(\prod_{f\in F} (1, \Lambda^f, 1)\Big) \\ &= \sigma\Big(1, \bigcap_{f\in F} \Lambda^f, 1\Big) = \sigma\Big(1, Q^F, 1\Big),\end{aligned}$$

where our use of Q^F is the same as in (14.10). Implicit in the above calculation is the fact that $\prod_{f\in F} \pi_f^*\pi_f$ depends only on Q^F, and not on F. While for a general representation π this may fail, we will prove that this does hold provided π is tight. Under that hypothesis we will not only prove that σ exists, but also that it is tight. Our correspondence will therefore involve tight representations only.

A large part of the effort in accomplishing our goal will be spent on studying the behavior of tight representations of Λ with respect to a certain Boolean algebra of subsets of Λ.

Throughout this section we therefore fix a semigroupoid Λ satisfying (14.7) and a tight representation π of Λ on a Hilbert space H.

Definition 16.1. A subset $X \subseteq \Lambda$ will be called a *domain*, provided it belongs to the Boolean subalgebra $\mathscr{D}$ of $\mathscr{P}(\Lambda)$ generated by $\{\Lambda^f : f \in \Lambda\}$.

If F and G are finite subsets of Λ, then the set

$$\Lambda^{F,G} = \bigcap_{f\in F} \Lambda^f \cap \bigcap_{g\in G} \Lambda \setminus \Lambda^g,$$

already employed in (15.4), is clearly a domain. Moreover it is easy to see that any member of $\mathscr{D}$ may be written as the union of a finite collection of sets each of which has the above form.

If one is to decide whether or not some h in Λ belongs to a given domain $D \in \mathscr{D}$, that task will consist of a perhaps logically complicated check depending on whether or not $h \in \Lambda^f$, for several elements f in Λ. It is therefore easy to see that for every $k \in \Lambda^h$ one has

$$h \in D \iff hk \in D. \tag{16.2}$$

In this section we will not be dealing with any representation of Λ other than π, so we will drop the superscripts in the q_f^π and p_g^π of (15.1.1).

We wish to define a map $Q : \mathscr{D} \to B(H)$ such that $Q(\Lambda^f) = q_f$, and which is a Boolean algebra homomorphism in the sense that

- $Q(\varnothing) = 0$,
- $Q(C \cap D) = Q(C)Q(D)$, and
- $Q(\tilde{C}) = 1 - Q(C)$,

for every $C, D \in \mathscr{D}$, where $\tilde{C}$ denotes the complement of C in Λ. Clearly we will have as a consequence that

$$\begin{aligned} Q(C \cup D) &= 1 - Q(\tilde{C} \cap \tilde{D}) = 1 - \big(1 - Q(C)\big)\big(1 - Q(D)\big) \\ &= Q(C) + Q(D) - Q(C)Q(D), \end{aligned}$$

which is precisely the join, or supremum $Q(C) \vee Q(D)$, of the commuting projections $Q(C)$ and $Q(D)$ in $B(H)$. If we are to succeed in obtaining Q then for every finite subsets $F, G \subseteq \Lambda$ we must have

$$Q(\Lambda^{F,G}) = \prod_{f \in F} Q(\Lambda^f) \prod_{g \in G} \big(1 - Q(\Lambda^g)\big) = \prod_{f \in F} q_f \prod_{g \in G} (1 - q_g) = q_{F,G},$$

where $q_{F,G}$ was already employed in (15.4).

In the next result we will take a first step in the direction of the goal stated above by showing that $q_{F,G}$ does indeed depend only on the set $\Lambda^{F,G}$, and not on F and G.

Proposition 16.3. *Let F, G, H, and K be finite subsets of Λ.*

(i) *If $\Lambda^{F,G} \subseteq \Lambda^{H,K}$, then $q_{F,G} \leqslant q_{H,K}$.*

(ii) *If $\Lambda^{F,G} = \Lambda^{H,K}$, then $q_{F,G} = q_{H,K}$.*

Proof. Assume that $\Lambda^{F,G} \subseteq \Lambda^{H,K}$. For each fixed $h \in H$ notice that $\Lambda^{F,G} \subseteq \Lambda^{H,K} \subseteq \Lambda^h$, and hence

$$\varnothing = \Lambda^{F,G} \cap (\Lambda \setminus \Lambda^h) = \Lambda^{F,G\cup\{h\}}.$$

Since π is tight we deduce that

$$0 = q_{F,G\cup\{h\}} = q_{F,G}(1 - q_h),$$

so that $q_{F,G} \leqslant q_h$. On the other hand, for every $k \in K$, we have that $\Lambda^{F,G} \subseteq \Lambda \setminus \Lambda^k$, and hence

$$\varnothing = \Lambda^{F,G} \cap \Lambda^k = \Lambda^{F\cup\{k\},G}.$$

Since π is tight we deduce that $0 = q_{F\cup\{k\},G} = q_{F,G}\, q_k$, so that $q_{F,G} \leqslant 1 - q_k$. Therefore

$$q_{F,G} \leqslant \prod_{h\in H} q_h \prod_{k\in K} (1 - q_k) = q_{H,K},$$

proving (i), and hence also (ii). □

If a domain D has the form $\Lambda^{F,G}$, we may then define $Q(D) = q_{F,G}$, without worrying about other possible descriptions of D in the form $\Lambda^{H,K}$. In the next result we shall consider the possibility that some domains may be described in several ways as unions of sets of the form $\Lambda^{F,G}$.

Proposition 16.4. *Let $\{F_i\}_{i=0}^n$ and $\{G_i\}_{i=0}^n$ be two collections of finite subsets of Λ, such that*

$$\Lambda^{F_0,G_0} = \Lambda^{F_1,G_1} \cup \Lambda^{F_2,G_2} \cup \ldots \cup \Lambda^{F_n,G_n},$$

Then $q_{F_0,G_0} = \bigvee_{i=1}^n q_{F_i,G_i}$.

Proof. Let $H = F_0 \cup G_0 \cup F_1 \cup G_1 \cup \ldots \cup F_n \cup G_n$. For each subset $X \subseteq H$ we let

$$E^X = \Lambda^{X,H\setminus X}.$$

It is then easy to see that the E_X are pairwise disjoint and that $\bigcup_{X\in\mathscr{P}(H)} E^X = \Lambda$. Likewise, letting

$$e_X = q_{X,H\setminus X},$$

it is easy to see that the e_X are pairwise orthogonal projections such that $\sum_{X\in\mathscr{P}(H)} e_X = 1$. In order to prove the statement it is therefore enough to show that for every $X \in \mathscr{P}(H)$ one has that

$$q_{F_0,G_0} e_X = \bigvee_{i=1}^n q_{F_i,G_i} e_X. \tag{16.4.1}$$

Since $e_X = 0$ whenever $E^X = \varnothing$, by (16.3), we need only consider those X for which E^X is nonempty. Let us thus fix $X \subseteq H$, with $E^X \neq \varnothing$. For each $i = 0, \ldots, n$, observe that if $F_i \subseteq X$, and $G_i \subseteq H \setminus X$, then $E^X \subseteq \Lambda^{F_i, G_i}$ and $e_X \leqslant q_{F_i,G_i}$. On the other hand if either $F_i \nsubseteq X$, or $G_i \nsubseteq H \setminus X$, then necessarily $E^X \cap \Lambda^{F_i,G_i} = \varnothing$, and $e_X \perp q_{F_i,G_i}$.

Case 1: Assume that there exists some $i \geqslant 1$ such that $F_i \subseteq X$, and $G_i \subseteq H \setminus X$. Then the right-hand side of (16.4.1) equals e_X. Moreover $E^X \subseteq \Lambda^{F_i,G_i} \subseteq \Lambda^{F_0,G_0}$, from where we deduce that $F_0 \subseteq X$, and $G_0 \subseteq H \setminus X$, and hence $e_X \leqslant q_{F_0,G_0}$, so the left-hand side of (16.4.1) also equals e_X.

Case 2: Assume that there is no $i \geqslant 1$ such that $F_i \subseteq X$, and $G_i \subseteq H \setminus X$. Then $e_X \perp q_{F_i,G_i}$, for all $i \geqslant 1$, and hence the right-hand side of (16.4.1) vanishes. Moreover E^X is disjoint from each Λ^{F_i,G_i}, with $i \geqslant 1$, and hence it is also disjoint from Λ^{F_0,G_0}. Thus, it cannot be that $F_0 \subseteq X$, and $G_0 \subseteq H \setminus X$, and hence $e_X \perp q_{F_0,G_0}$, proving that the left-hand side of (16.4.1) also vanishes. ◻

The next result will finally allow us to define the map we are seeking:

Proposition 16.5. *For every $D \in \mathscr{D}$, write $D = \bigcup_{j=1}^n \Lambda^{F_j,G_j}$, where the F_j and G_j are finite subsets of Λ, and define $Q(D) = \bigvee_{j=1}^n q_{F_j,G_j}$. Then $Q : \mathscr{D} \to B(H)$ is a well defined map which moreover satisfies*

(i) $Q(\Lambda^f) = q_f$,

(ii) $Q(\varnothing) = 0$,

(iii) $Q(C \cap D) = Q(C)Q(D)$,

(iv) $Q(C \cup D) = Q(C) \vee Q(D)$,

(v) $Q(\tilde{D}) = 1 - Q(D)$,

for every $f \in \Lambda$, and every $C, D \in \mathscr{D}$.

Proof. To show well definedness suppose that D is a domain which may be written in two ways as

$$D = \bigcup_{j=1}^{n_1} \Lambda^{F_j^1,G_j^1} = \bigcup_{j=1}^{n_2} \Lambda^{F_j^2,G_j^2},$$

where the F^i_j and G^i_j are finite subsets of Λ. Fix $k \leqslant n_1$ and notice that

$$\Lambda^{F^1_k,G^1_k} = \Lambda^{F^1_k,G^1_k} \cap D = \bigcup_{j=1}^{n_2} \Lambda^{F^1_k,G^1_k} \cap \Lambda^{F^2_j,G^2_j} = \bigcup_{j=1}^{n_2} \Lambda^{F^1_k \cup F^2_j, G^1_k \cup G^2_j}.$$

By (16.4) we conclude that

$$q_{F^1_k,G^1_k} = \bigvee_{j=1}^{n_2} q_{F^1_k \cup F^2_j, G^1_k \cup G^2_j} = \bigvee_{j=1}^{n_2} q_{F^1_k,G^1_k} q_{F^2_j,G^2_j} = q_{F^1_k,G^1_k}\Big(\bigvee_{j=1}^{n_2} q_{F^2_j,G^2_j}\Big),$$

showing that

$$q_{F^1_k,G^1_k} \leqslant \bigvee_{j=1}^{n_2} q_{F^2_j,G^2_j}.$$

Since k is arbitrary we deduce that

$$\bigvee_{k=1}^{n_1} q_{F^1_k,G^1_k} \leqslant \bigvee_{j=1}^{n_2} q_{F^2_j,G^2_j},$$

and by symmetry we obtain the reverse inequality, hence proving that the two possibly different descriptions of D lead to the same proposed value of $Q(D)$.

We leave it for the reader to prove (i–iii) and we will verify (v) next. Supposing initially that D has the form $D = \Lambda^{F,G}$, we have

$$\tilde{D} = \bigcup_{f\in F} \Lambda \setminus \Lambda^f \cup \bigcup_{g\in G} \Lambda^g,$$

hence

$$Q(\tilde{D}) = \bigvee_{f\in F}(1-q_f) \vee \bigvee_{g\in G} q_g = 1 - \prod_{f\in F} q_f \prod_{g\in G}(1-q_g) = 1 - q_{F,G} = 1 - Q(D).$$

In the general case write $D = \bigcup_{j=1}^n D_j$, where each D_j is of the above form, then

$$\begin{aligned} Q(\tilde{D}) &= Q\Big(\bigcap_{j=1}^n \tilde{D}_j\Big) = \prod_{j=1}^n Q(\tilde{D}_j) = \prod_{j=1}^n \big(1 - Q(D_j)\big) \\ &= 1 - \bigvee_{j=1}^n Q(D_j) = 1 - Q(D). \end{aligned}$$

As already seen, (iv) follows from (iii) and (v). □

For the record we notice the following:

Corollary 16.6. *If* $H \subseteq \Lambda$ *is a finite nonempty subset and* $A = \bigcap_{h \in H} \Lambda^h$ *then* $Q(A) = \prod_{h \in H} \pi_h^* \pi_h$.

Recall that the condition for a representation of Λ to be tight is that

$$\bigvee_{h \in H} p_h = \prod_{f \in F} q_f \prod_{g \in G} (1 - q_g),$$

whenever $F, G \subseteq \Lambda$ are finite sets and H is a finite cover for $\Lambda^{F,G}$. If we denote by D the domain $D = \Lambda^{F,G}$, then the above condition may be expressed as $\bigvee_{h \in H} p_h = Q(D)$. One may therefore ask if the same is true for every domain. The next result proves that this is in fact true.

Proposition 16.6. *Let* D *be a domain and let* H *be a finite cover for* D *in the sense of* (15.3)*, then*

$$\bigvee_{h \in H} p_h = Q(D).$$

Proof. Write $D = \bigcup_{j=1}^n \Lambda^{F_j, G_j}$, where the F_j and G_j are finite subsets of Λ. By assumption we have that $H \subseteq D$, so if we put

$$H_j = H \cap \Lambda^{F_j, G_j},$$

we will have that $H = \bigcup_{j=1}^n H_j$. We claim that H_j is a cover for Λ^{F_j, G_j} for each $j \leqslant n$. In fact, given any $k \in \Lambda^{F_j, G_j}$, we in particular have that $k \in D$. By hypothesis there exists some $h \in H$ such that $h \Cap k$, and we may therefore choose $u, v \in \tilde{\Lambda}$ such that $hu = kv$. By (16.2) we have that $h \in \Lambda^{F_j, G_j}$, so $h \in H_j$, and the claim is proved. Since we are assuming π to be tight it follows that

$$\bigvee_{h \in H_j} p_h = q_{F_j, G_j},$$

and therefore that

$$\bigvee_{h \in H} p_h = \bigvee_{j=1}^n \bigvee_{h \in H_j} p_h = \bigvee_{j=1}^n q_{F_j, G_j} = Q(D). \qquad \square$$

17 Extending representations

The sole aim of this section is to prove the following:

Theorem 17.1. *Let Λ be a semigroupoid without springs in which every element is monic, and such that every intersecting pair of elements admits a least common multiple. Given a tight representation π of Λ on a Hilbert space H, which respects least common multiples, there exists a unique representation σ of the inverse semigroup $S(\Lambda)$ such that $\sigma = \pi \circ \tau$. Moreover σ is tight.*

Observing that the representation of Λ given by (15.11) necessarily respects least common multiples, the above result cannot survive without assuming that π also has this property.

Given (f, A, g) in $S(\Lambda)$ write $A = \bigcap_{h \in H} \Lambda^h$, where $H \subseteq \Lambda$ is a finite nonempty subset. Recall from (15.10) that

$$(f, A, g) = \tau_f \Big(\prod_{h \in H} \tau_h^* \tau_h \Big) \tau_g^*,$$

so if we want a representation σ of $S(\Lambda)$ such that $\sigma \circ \tau = \pi$, we have no choice but to define $\sigma(f, A, g) = \pi_f \Big(\prod_{h \in H} \pi_h^* \pi_h \Big) \pi_g^*$. This immediately gives uniqueness and, in view of (16.6), it also suggests that we define

$$\sigma(f, A, g) = \pi_f Q(A) \pi_g^*,$$

where Q is given by (16.5). For $f \in \Lambda$ we then have that

$$\sigma(\tau_f) = \sigma(f, \Lambda^f, 1) = \pi_f Q(\Lambda^f) = \pi_f \pi_f^* \pi_f = \pi_f,$$

so $\pi = \sigma \circ \tau$, as required. It is also clear that σ preserves the star operation.

We next claim that if $A \in \mathcal{Q}$ and $f \in \Lambda$ one has that

$$Q(A)\pi_f = \begin{cases} \pi_f, & \text{if } f \in A, \\ 0, & \text{otherwise.} \end{cases} \tag{17.2}$$

To prove it write $A = \bigcap_{h \in H} \Lambda^h$, for some finite nonempty subset H of Λ. Assuming initially that $f \in A$, we then have that $f \in \Lambda^h$, for all $h \in H$, and hence $\pi_h^* \pi_h \pi_f = \pi_f$, by (15.1.iii). So

$$Q(A)\pi_f = \Big(\prod_{h \in H} \pi_h^* \pi_h \Big) \pi_f = \pi_f.$$

On the other hand, if $f \notin A$, then $f \notin \Lambda^h$ for some $h \in H$, and hence $\pi_h^* \pi_h \pi_f = 0$, by (15.1.i), so our claim is proved. Using (14.13), we may express (17.2) alternatively as

$$Q(A)\pi_f = \pi_f Q\big(f^{-1}(A)\big). \tag{17.3}$$

The advantage of this over (17.2) is that it holds inclusively for $f = 1$, while the former does not.

We are now prepared to prove that σ is multiplicative. Given (f, A, g) and (h, B, k) in $S(\Lambda)$ we then have to show that

$$\sigma(f, A, g)\sigma(h, B, k) = \sigma\big((f, A, g)(h, B, k)\big). \tag{17.4}$$

Observing that the left-hand side equals

$$\pi_f Q(A)\pi_g^* \pi_h Q(B)\pi_k^*,$$

we see that it vanishes whenever $g \perp h$, because

$$\pi_g^*\pi_h = \pi_g^*\pi_g\pi_g^*\pi_h\pi_h^*\pi_h = \pi_g^* p_g p_h \pi_h = 0,$$

by (15.1.ii). Still under the assumption that $g \perp h$, we have that $(f, A, g)(h, B, k) = 0$, by definition, and since $\sigma(0) = 0$, the right-hand side of (17.4) also vanishes. Thus (17.4) is true provided $g \perp h$, and we may then suppose that $g \Cap h$, writing $\operatorname{lcm}(g, h) = m = gu = hv$, with $u, v \in \tilde{\Lambda}$.

Assuming, as we are, that π respects least common multiples, we wish to apply (15.7) to describe $\pi_g^*\pi_h$, but for this we also need to assume we are in the special case in which $g \neq h$. Under this premise we have that $\pi_g^*\pi_h = \pi_u\pi_v^*$, and hence the left-hand side of (17.4) equals

$$\begin{aligned} \pi_f Q(A)\pi_u\pi_v^* Q(B)\pi_k^* &\overset{(17.3)}{=} \pi_f\pi_u Q\big(u^{-1}(A)\big)Q\big(v^{-1}(B)\big)\pi_v^*\pi_k^* \\ &= \pi_f\pi_u Q\big(u^{-1}(A) \cap v^{-1}(B)\big)\pi_v^*\pi_k^*. \end{aligned} \tag{17.5}$$

If $u^{-1}(A) \cap v^{-1}(B) = \varnothing$, then the above is zero, but so is the right-hand side of (17.4), and hence equality is established. In the event that $u^{-1}(A) \cap v^{-1}(B)$ is nonempty we have by (14.13) that

$$u \in A \cup \{1\} \subseteq \Lambda^f \cup \{1\}, \quad \text{and} \quad v \in B \cup \{1\} \subseteq \Lambda^k \cup \{1\}.$$

Thus $\pi_f\pi_u = \pi_{fu}$ and $\pi_k\pi_v = \pi_{kv}$, so (17.5) equals

$$\begin{aligned} \pi_{fu} Q\big(u^{-1}(A) \cap v^{-1}(B)\big)\pi_{kv}^* &= \sigma\big(fu, u^{-1}(A) \cap v^{-1}(B), kv\big) \\ &= \sigma\big((f, A, g)(h, B, k)\big), \end{aligned}$$

proving (17.4) under the assumption that $g \neq h$, and the only case left to be discussed is that in which $g = h$. Under this assumption observe that, since $A \subseteq \Lambda^g$, we have by (16.5) that

$$Q(A)\pi_g^*\pi_g = Q(A)Q(\Lambda^g) = Q(A \cap \Lambda^g) = Q(A),$$

and hence the left-hand side of (17.4) equals

$$\begin{aligned}\pi_f Q(A)\pi_g^*\pi_g Q(B)\pi_k^* &= \pi_f Q(A)Q(B)\pi_k^* = \pi_f Q(A\cap B)\pi_k^* \\ &= \sigma(f, A\cap B, k) = \sigma\big((f, A, g)(g, B, k)\big),\end{aligned}$$

proving that σ is indeed a representation of $S(\Lambda)$. Summarizing our findings so far we have:

Lemma 17.6. *Under the hypotheses of* (17.1) *the map* $\sigma : S(\Lambda) \to B(H)$*, defined by*

$$\sigma(f, A, g) = \pi_f Q(A)\pi_g^*, \quad \forall (f, A, g) \in S(\Lambda),$$

is a representation of $S(\Lambda)$ *satisfying* $\sigma \circ \tau = \pi$.

The remaining of this section will be dedicated to proving the last sentence of (17.1), namely that σ is tight. The characterization of tightness given in (11.8) will prove itself useful, but to employ it we must first check either (i) or (ii) of (11.7). We therefore suppose that (11.7.ii) fails, meaning that $E(S(\Lambda))$ admits a finite cover, say Z. Let us classify the elements (f, A, f) of Z according to whether $f = 1$ or not by setting

$$Z' = \big\{(f, A, f) \in Z : f \in \Lambda\big\}, \quad \text{and} \quad Z'' = \big\{(f, A, f) \in Z : f = 1\big\}.$$

For each $(1, A, 1)$ in Z'' write $A = \bigcap\limits_{g\in G} \Lambda^g$, where $G \subseteq \Lambda$ is finite and nonempty, and choose at random some $g_A \in G$. Once this is done we have that $A \subseteq \Lambda^{g_A}$ and hence

$$(1, A, 1) \leqslant (1, \Lambda^{g_A}, 1) = \tau_{g_A}^*\tau_{g_A}.$$

With respect to the elements (f, A, f) in Z' notice that $A \subseteq \Lambda^f$ and hence

$$(f, A, f) \leqslant (f, \Lambda^f, f) = \tau_f\tau_f^*.$$

Substituting each element of Z appearing in the left-hand side of the two inequalities displayed above by the respective right-hand side we therefore obtain a set of the form

$$W = \big\{\tau_g^*\tau_g : g \in G\big\} \cup \big\{\tau_f\tau_f^* : f \in F\big\},$$

which is clearly also a cover for $E(S(\Lambda))$. We next claim that F is a cover for

$$\Lambda^{\varnothing,G} = \bigcap_{g\in G} \Lambda \setminus \Lambda^g,$$

in the sense of (15.3). To prove this let $h \in \Lambda^{\varnothing,G}$, and notice that $\tau_h\tau_h^*$ must necessarily intersect some element of W. If that element is of the form $\tau_g^*\tau_g$, for some $g \in G$, then

$$0 \neq \tau_h\tau_h^*\,\tau_g^*\tau_g = (h, \Lambda^h, h)(1, \Lambda^g, 1) = (h, \Lambda^h \cup h^{-1}(\Lambda^g), h),$$

so $h^{-1}(\Lambda^g)$ is nonempty, and hence $h \in \Lambda^g$ by (14.13.iii), but this contradicts the fact that $h \in \Lambda^{\varnothing,G}$. The conclusion is that the element of W which intersects $\tau_h\tau_h^*$ must be some $\tau_f\tau_f^*$, with $f \in F$. In this case

$$0 \neq \tau_h\tau_h^*\,\tau_f\tau_f^* = (h, \Lambda^h, h)(f, \Lambda^f, f)$$

which implies that $h \Cap f$, concluding the proof of our claim. Since we are assuming that π is tight we have that

$$\bigvee_{f\in F}\pi_f\pi_f^* = \prod_{g\in G}(1-\pi_g^*\pi_g) = 1 - \bigvee_{g\in G}\pi_g^*\pi_g,$$

and hence that

$$\Big(\bigvee_{f\in F}\pi_f\pi_f^*\Big) \vee \Big(\bigvee_{g\in G}\pi_g^*\pi_g\Big) = 1.$$

This implies that

$$\begin{aligned}\bigvee_{w\in W}\sigma(w) &= \Big(\bigvee_{f\in F}\sigma(\tau_f\tau_f^*)\Big) \vee \Big(\bigvee_{g\in G}\sigma(\tau_g^*\tau_g)\Big)\\ &= \Big(\bigvee_{f\in F}\pi_f\pi_f^*\Big) \vee \Big(\bigvee_{g\in G}\pi_g^*\pi_g\Big) = 1.\end{aligned}$$

We have therefore proven:

Lemma 17.7. *Either $E(S(\Lambda))$ does not admit any finite cover or there exists a finite cover W such that $\bigvee_{w\in W}\sigma(w) = 1$.*

As already mentioned this result enables us to use (11.8) to attempt a proof that σ is tight. Therefore, given $(f, A, f) \in E(S(\Lambda))$ and a finite cover Z for (f, A, f) we need to prove that

$$\bigvee_{z\in Z}\sigma(z) \geqslant \sigma(f, A, f). \tag{17.8}$$

We will argue in two different ways according to whether $f = 1$ or not, so let us begin by assuming that $f = 1$. As before write $Z = Z' \cup Z''$, where

$$Z' = \big\{(h, C, h) \in Z : h \in \Lambda\big\} = \big\{(h_i, C_i, h_i)\big\}_{i=1}^n,$$

and

$$Z'' = \big\{(h, C, h) \in Z : h = 1\big\} = \big\{(1, D_i, 1)\big\}_{i=1}^{m}.$$

Our proof will be by induction on $|Z'| = n$, so let us first treat the case in which $n = 0$.

Thus Z'' is a cover for $(1, A, 1)$, and we claim that $A = \bigcup_{i=1}^{m} D_i$. Since each $(1, D_i, 1) \leqslant (1, A, 1)$ we have that $D_i \subseteq A$. On the other hand, given $f \in A$, we have that $(f, \Lambda^f, f) \leqslant (1, A, 1)$. Assuming that Λ has no springs we see that (f, Λ^f, f) is nonzero, so there is some $(1, D_i, 1) \in Z''$, such that $(f, \Lambda^f, f) \Cap (1, D_i, 1)$, which means that $f \in D_i$, by (14.19). This proves our claim, therefore

$$\begin{aligned}\bigvee_{z\in Z} \sigma(z) &= \bigvee_{i=1}^{m} \sigma(1, D_i, 1) = \bigvee_{i=1}^{m} Q(D_i) \overset{(16.5.\mathrm{iv})}{=} Q\Big(\bigcup_{i=1}^{m} D_i\Big)\\ &= Q(A) = \sigma(1, A, 1),\end{aligned}$$

thus proving (17.8) for $f = 1$, and $n = 0$.

Still assuming that $f = 1$, but now that $n \geqslant 1$, pick any $j \leqslant n$, and let $(h, C, h) = (h_j, C_j, h_j)$. Since $(h, C, h) \leqslant (1, A, 1)$ we have by (14.17.iv) that $h \in A$. Incidentally notice that (14.17.iv) requires that C be nonempty, which we may assume, since otherwise (h, C, h) may be deleted from the covering Z without altering the left-hand side of (17.8). Given that $h \in A$, we have that

$$(h, \Lambda^h, h) \leqslant (1, A, 1).$$

We next claim that

$$Z_h := \tau_h^* \, Z \, \tau_h$$

is cover for $\tau_h^*\tau_h = (1, \Lambda^h, 1)$. To prove the claim let $0 \neq \gamma \leqslant \tau_h^*\tau_h$ and observe that

$$\tau_h \gamma \tau_h^* \leqslant \tau_h \tau_h^* \tau_h \tau_h^* = \tau_h \tau_h^* = (h, \Lambda^h, h) \leqslant (1, A, 1).$$

Since Z is a cover for $(1, A, 1)$, and $\tau_h \gamma \tau_h^*$ is nonzero (or else $\gamma = \tau_h^* \tau_h \gamma \tau_h^* \tau_h = 0$), there exists $z \in Z$ such that $\tau_h \gamma \tau_h^* \Cap z$, so

$$0 \neq \tau_h \gamma \tau_h^* z = (\tau_h \tau_h^*) \tau_h \gamma \tau_h^* z = \tau_h \gamma \tau_h^* z (\tau_h \tau_h^*),$$

which implies that $\gamma \tau_h^* z \tau_h \neq 0$, and hence that $\gamma \Cap \tau_h^* z \tau_h$, proving the claim.

Let us now decompose Z_h as the union $Z_h' \cup Z_h''$, where

$$Z_h' = \big\{(g, B, g) \in Z_h : g \in \Lambda\big\}, \quad \text{and} \qquad Z_h'' = \big\{(g, B, g) \in Z_h : g = 1\big\},$$

in the same way we did with Z, because we are interested in the number of elements of Z'_h, given that our proof is by induction on this parameter. Notice that for every $i \leqslant m$

$$\begin{aligned}\tau_h^*(1, D_i, 1)\tau_h &= (1, \Lambda^h, h)(1, D_i, 1)(h, \Lambda^h, 1) \\ &= \big(1, \Lambda^h \cap h^{-1}(D_i), h\big)(h, \Lambda^h, 1) \\ &= \big(1, \Lambda^h \cap h^{-1}(D_i) \cap \Lambda^h, 1\big) \in Z''_h,\end{aligned}$$

which means that

$$\tau_h^* Z'' \tau_h \subseteq Z''_h.$$

If the reader is expecting a similar inclusion with single primes replacing double primes, he or she will be surprised to find that there is an element of Z' which migrates to Z''_h when *conjugated* by τ_h, namely

$$\begin{aligned}\tau_h^*(h, C, h)\tau_h &= (1, \Lambda^h, h)(h, C, h)(h, \Lambda^h, 1) \\ &= (1, \Lambda^h \cap C \cap \Lambda^h, 1) = (1, C, 1) \in Z''_h.\end{aligned}$$

It follows that Z'_h has at most $n-1$ elements and hence the induction hypothesis applies to give

$$\bigvee_{z \in Z} \sigma(\tau_h^* z \tau_h) \geqslant \sigma(\tau_h^* \tau_h),$$

which translates into

$$\bigvee_{z \in Z} \pi_h^* \sigma(z) \pi_h \geqslant \pi_h^* \pi_h.$$

If this is left-multiplied by π_h, and right-multiplied by π_h^*, we get

$$\bigvee_{z \in Z} \pi_h \pi_h^* \sigma(z) \pi_h \pi_h^* \geqslant \pi_h \pi_h^* \pi_h \pi_h^* = \pi_h \pi_h^*,$$

which means that

$$\pi_h \pi_h^* \leqslant \bigvee_{z \in Z} \sigma(z). \tag{17.9}$$

Leaving this aside for a moment consider the domain $D = A \setminus \bigcup_{i=1}^m D_i$, and let K be the set of all h_i's belonging to D. So far we have been discussing several covers in the sense of semilattices (11.5), but now we claim that K is a cover for D, in the sense of semigroupoids (15.3). To see this let $g \in D$, and notice that since $g \in A$, we have that $(g, \Lambda^g, g) \leqslant (1, A, 1)$. It follows that there is some $z \in Z$ such that $z \Cap (g, \Lambda^g, g)$, but notice that such a z may not be in Z'', since

$$(g, \Lambda^g, g) \perp (1, D_i, 1),$$

by (14.19), because $g \notin D_i$. Therefore $(h_i, C_i, h_i)(g, \Lambda^g, g) \neq 0$, for some $i \leqslant n$. In particular this implies that $h_i u = gv$, for some $u, v \in \tilde{\Lambda}$. Since $g \in D$, we have by (16.2) that $h_i \in D$, so in fact $h_i \in K$. This proves our claim that K is a cover for D, so

$$Q(D) \overset{(16.7)}{=} \bigvee_{h \in K} \pi_h \pi_h^* \overset{(17.9)}{\leqslant} \bigvee_{z \in Z} \sigma(z). \tag{17.10}$$

Since $A \subseteq D \cup \bigcup_{i=1}^m D_i$, we have

$$\begin{aligned} \sigma(1, A, 1) &= Q(A) \leqslant Q(D) \vee \bigvee_{i=1}^m Q(D_i) \\ &\overset{(17.10)}{\leqslant} \bigvee_{z \in Z} \sigma(z) \vee \bigvee_{i=1}^m \sigma(1, D_i, 1) = \bigvee_{z \in Z} \sigma(z), \end{aligned}$$

proving (17.8) for $f = 1$, and arbitrary n. Summarizing:

Lemma 17.11. *If $A \in \mathcal{Q}$ and Z is a cover for $(1, A, 1)$ then*

$$\bigvee_{z \in Z} \sigma(z) \geqslant \sigma(1, A, 1).$$

Let us now face (17.8) in the most general situation, so we assume that (f, A, f) is an arbitrary element of $S(\Lambda)$ and that $Z = \{(h_i, C_i, h_i)\}_{i=1}^n$ is a cover for (f, A, f).

Since $(h_i, C_i, h_i) \leqslant (f, A, f)$, for every i, we have by (14.17) that $f \mid h_i$, so we may write $h_i = f g_i$, with $g_i \in \tilde{\Lambda}$, and in addition we have that $C_i \subseteq g_i^{-1}(A)$. Observe that

$$C_i \subseteq \Lambda^{h_i} = \Lambda^{f g_i} \subseteq \Lambda^{g_i},$$

so $(g_i, C_i, g_i) \in S(\Lambda)$. Notice that

$$(g_i, C_i, g_i)(1, A, 1) = (g_i, C_i \cap g_i^{-1}(A), g_i) = (g_i, C_i, g_i),$$

so $(g_i, C_i, g_i) \leqslant (1, A, 1)$. We then claim that $\{(g_i, C_i, g_i)\}_{1=1}^n$ is a cover for $(1, A, 1)$. In order to prove it let (k, B, k) be a nonzero element with $(k, B, k) \leqslant (1, A, 1)$, so $B \subseteq k^{-1}(A)$. Given that B is nonempty, the same is true for $k^{-1}(A)$, so (14.13) applies and gives $k \in A \cup \{1\} \subseteq \Lambda^f \cup \{1\}$, so fk is a well defined element of $\tilde{\Lambda}$. Since $A \subseteq \Lambda^f = f^{-1}(\Lambda)$, we have

$$B \subseteq k^{-1}(A) \subseteq k^{-1}(f^{-1}(\Lambda)) \overset{(14.11.\text{ii})}{=} (fk)^{-1}(\Lambda) = \Lambda^{fk},$$

hence $(fk, B, fk) \in S(\Lambda)$, and clearly $(fk, B, fk) \leqslant (f, A, f)$. Therefore there exists $i \leqslant n$ such that $(fk, B, fk) \Cap (h_i, C_i, h_i)$, which may be interpreted via (14.18), by saying that there are $x, y \in \tilde{\Lambda}$, such that

$$fkx = h_i y = fg_i y, \quad \text{and} \quad x^{-1}(B) \cap y^{-1}(C_i) \neq \varnothing.$$

Since f is monic we conclude that $kx = g_i y$ and, again by (14.18), that $(k, B, k) \Cap (g_i, C_i, g_i)$, proving our claim. Therefore

$$\bigvee_{i=1}^{n} \pi_{g_i} Q(C_i)\pi_{g_i}^* = \bigvee_{i=1}^{n} \sigma(g_i, C_i, g_i) \overset{(17.11)}{\geqslant} \sigma(1, A, 1) = Q(A),$$

which leads to

$$\begin{aligned}\sigma(f, A, f) &= \pi_f Q(A)\pi_f^* \leqslant \pi_f\Big(\bigvee_{i=1}^{n} \pi_{g_i} Q(C_i)\pi_{g_i}^*\Big)\pi_f^* \\ &= \bigvee_{i=1}^{n} \pi_f \pi_{g_i} Q(C_i)\pi_{g_i}^*\pi_f^* = \bigvee_{i=1}^{n} \pi_{h_i} Q(C_i)\pi_{h_i}^* \\ &= \bigvee_{i=1}^{n} \sigma(h_i, C_i, h_i) = \bigvee_{z\in Z} \sigma(z),\end{aligned}$$

and we are done!

18 The C*-algebra of a semigroupoid

In this section we fix a countable semigroupoid Λ without springs, in which every element is monic, and such that every intersecting pair of elements admits a least common multiple. Our goal will be to study the universal C*-algebra for representations of Λ.

To single out the special kind of representations of Λ which we will focus on we give the following:

Definition 18.1. A representation π of Λ on a Hilbert space H will be called *normal*, provided it is tight and respects least common multiples.

Recall from [10] that the C*-algebra of Λ, denoted $\mathcal{O}_\Lambda$, is the C*-algebra generated by a universal tight representation of Λ. By definition we therefore see that *-representations of $\mathcal{O}_\Lambda$ correspond bijectively to tight representations of Λ. If Λ satisfies the hypothesis of (15.5), then it is automatic that the tight representations we are talking about respect least common multiples, and hence are normal representations.

It is not clear to me whether or not one really needs the hypothesis of (15.5) to obtain that conclusion but, given the dependence of our previous results on least common multiples, we simply cannot live without it. So much so that we are willing to impose it from the outside:

Definition 18.2. We will denote by $\mathcal{O}_\Lambda^{\text{lcm}}$ the C*-algebra generated by the range of a universal normal representation π^u of Λ (such as the direct sum of all normal representations of Λ on subspaces of Hilbert's space l_2).

If Λ satisfies the hypothesis of (15.5) it is then obvious that $\mathcal{O}_\Lambda^{\text{lcm}} = \mathcal{O}_\Lambda$, but in general all we can say is that $\mathcal{O}_\Lambda^{\text{lcm}}$ is a quotient of $\mathcal{O}_\Lambda$.

Observe that π^u is not necessarily injective by (11.11). The reader is referred to [11] for a thorough treatment of the injectivity question.

Restricting our attention to $\mathcal{O}_\Lambda^{\text{lcm}}$ we therefore see that its *-representations correspond bijectively to normal representations of Λ, and in view of (15.11) and (17.1), they also correspond bijectively to tight representations of $S(\Lambda)$. Furthermore these correspond to representations of the C*-algebra of the groupoid described in (13.3).

Definition 18.3. We will denote by $\mathcal{G}_\Lambda$ the the groupoid of germs associated to the restriction of the action θ of (10.3.iv) to the tight part of the spectrum of $E(S(\Lambda))$.

We thus arrive at one of the main results of this work:

Theorem 18.4. *Let Λ be a countable semigroupoid with no springs, in which every element is monic, and such that every intersecting pair of elements admits a least common multiple. Then $\mathcal{O}_\Lambda^{\text{lcm}}$ is naturally isomorphic to $C^*(\mathcal{G}_\Lambda)$.*

Proof. Let $X = \widehat{E(S(\Lambda))}_{tight}$, as defined in (12.8). By (10.13) we have that the map

$$\sigma^u : s \in S(\Lambda) \mapsto i(1_{ss^*}^X \delta_s) \in C^*(\mathcal{G}_\Lambda),$$

is a representation of $S(\Lambda)$ in $C^*(\mathcal{G}_\Lambda)$ (assumed to be an algebra of operators via any faithful non-degenerated representation), which is supported in $\widehat{E(S(\Lambda))}_{tight}$, and hence is tight by (13.2). The superscript "u" in σ^u is justified by its universal property (10.14).

Employing (15.11) we deduce that the composition

$$\pi : \Lambda \xrightarrow{\tau} S(\Lambda) \xrightarrow{\sigma^u} C^*(\mathcal{G}_\Lambda)$$

is a tight representation of Λ which respects least common multiples, i.e., π is a normal representation. Invoking the universal property of $\mathcal{O}_\Lambda^{\text{lcm}}$ there exists a *-homomorphism

$$\phi : \mathcal{O}_\Lambda^{\text{lcm}} \to C^*(\mathcal{G}_\Lambda),$$

such that the diagram

$$\begin{tikzcd} & S(\Lambda) \arrow[dr, "\sigma^u"] & \\ \Lambda \arrow[ur, "\tau"] \arrow[rr, "\pi"] \arrow[dr, "\pi^u"'] & & C^*(\mathcal{G}_\Lambda) \\ & \mathcal{O}_\Lambda^{\text{lcm}} \arrow[ur, "\phi"'] & \end{tikzcd}$$

commutes, where π^u was defined in (18.2). To define an inverse for ϕ recall that π^u is a normal representation of Λ, and hence by (17.1) there exists a tight representation σ of $S(\Lambda)$ such that $\pi^u = \sigma \circ \tau$. The space of σ is evidently the same as the space H^u of π^u. Since $\tau(\Lambda)$ generates $S(\Lambda)$, by (15.10), we deduce that the range of σ is contained in the inverse semigroup of partial isometries on H^u generated by the range of π^u, which is obviously contained in $\mathcal{O}_\Lambda^{\text{lcm}}$. We may therefore regard σ as a map from $S(\Lambda)$ to $\mathcal{O}_\Lambda^{\text{lcm}}$.

Being tight, σ is supported in $\widehat{E(S(\Lambda))}_{tight}$ by (13.2) and hence we may use (10.14) to conclude that there exists a *-representation ρ of $C^*(\mathcal{G}_\Lambda)$ on H^u, such that $\rho \circ \sigma^u = \sigma$. The range of σ^u may be shown to generate $C^*(\mathcal{G}_\Lambda)$ as a C*-algebra, and hence we conclude as above that ρ may be regarded as a map from $C^*(\mathcal{G}_\Lambda)$ to $\mathcal{O}_\Lambda^{\text{lcm}}$.

$$\begin{tikzcd} & S(\Lambda) \arrow[dr, "\sigma^u"] \arrow[dd, "\sigma"] & \\ \Lambda \arrow[ur, "\tau"] \arrow[dr, "\pi^u"'] & & C^*(\mathcal{G}_\Lambda) \arrow[dl, "\rho"] \\ & \mathcal{O}_\Lambda^{\text{lcm}} & \end{tikzcd}$$

We therefore have

$$\rho \circ \phi \circ \pi^u = \rho \circ \pi = \rho \circ \sigma^u \circ \tau = \sigma \circ \tau = \pi^u,$$

so $\rho\circ\phi$ coincides with the identity on the range of π^u, which is known to generate $\mathcal{O}_\Lambda^{\rm lcm}$. This proves that $\rho \circ \phi$ is the identity map. On the other hand

$$\phi \circ \rho \circ \pi = \phi \circ \pi^u = \pi,$$

so $\phi \circ \rho$ coincides with the identity on the range of π, which again generates $C^*(\mathcal{G}_\Lambda)$. Therefore $\phi \circ \rho$ is the identity map, proving that ϕ and ρ are each other's inverse, and hence isomorphisms. □

It is interesting to notice that since $\tau(\Lambda)$ generates $S(\Lambda)$, the groupoid $\mathcal{G}_\Lambda$, which consists of germs for the action of $S(\Lambda)$ on $\widehat{E(S(\Lambda))}_{tight}$, is also in a sense generated by the action of Λ, via τ.

A concrete understanding of this groupoid clearly depends on the ability to describe $\widehat{E(S(\Lambda))}_{tight}$ in clear terms. That is the purpose of our next section.

19 Categorical semigroupoids

In this section we will give a concrete description for the space of tight characters on the idempotent semilattice of $S(\Lambda)$, where Λ is a semigroupoid. To reduce the technical difficulties to a minimum we will assume that Λ possesses a crucial property well known to hold on categories.

Definition 19.1. A semigroupoid Λ is said to be *categorical*, if for every $f, g \in \Lambda$ one has that Λ^f and Λ^g are either equal or disjoint.

With this notion we wish to capture the essential characteristic of categories which is relevant to our work. Obviously any small category is a categorical semigroupoid.

In order to apply our results to a categorical semigroupoid Λ we must assume that it satisfies our crucial working hypotheses, namely the conditions listed in (14.7), often adding the absence of springs. With respect to the requirement that every element is monic we should stress that, although the term we use is inspired in the Theory of Categories, our use of it is strictly different. In particular, requiring an element f to be monic impedes the existence of a right unit to f, namely an element u such that $fu = f$. According to Definition (14.5), the only element u which is allowed to satisfy such an equation is the added unit 1, as in $\tilde{\Lambda} = \Lambda \,\dot{\cup}\, \{1\}$.

If one is to apply our theory to a classical small category, one should therefore first remove all of its identities, and then hope that the products of the remaining elements never come out to being an identity. See below for a discussion of this issue in the context of higher rank graphs.

Proposition 19.2. *Let $\mathscr{C}$ be a small category such that no morphism is right-invertible, except for the identities. Then the set Λ of all non-identity morphisms admits the structure of a categorical semigroupoid. In addition:*

(i) *If every morphism in $\mathscr{C}$ is a monomorphism (in the usual sense of the word), then every element of Λ is monic (in the sense of Definition* (14.5)*).*

(ii) *If for every object v in $\mathscr{C}$ there exists a morphism $f \neq id_v$, such that $\mathbf{r}(f) = v$, then Λ has no springs.*

(iii) *Suppose that whenever $fu = gv$ in $\mathscr{C}$, there exists a pull-back for the pair (f, g). Then Λ admits least common multiples.*

Proof. Given $f, g \in \Lambda$ suppose that fg is an identity morphism, necessarily the identity on $v := \mathbf{r}(f)$. Then f is right-invertible and hence by hypothesis, $f = id_v \notin \Lambda$, a contradiction. So whenever $f, g \in \Lambda$, and fg is defined in $\mathscr{C}$, one has that $fg \in \Lambda$. We may then put

$$\Lambda^{(2)} = \{(f, g) \in \Lambda \times \Lambda : \mathbf{d}(f) = \mathbf{r}(g)\},$$

and it is clear that Λ is a categorical semigroupoid with composition as multiplication.

Under hypothesis (i) suppose that $fg = fh$, for $f \in \Lambda$, and $g, h \in \tilde{\Lambda}$. If $g, h \in \Lambda$, we have that $g = h$ because f is a monomorphism, by hypothesis. If $g \in \Lambda$ and $h = 1$, then

$$fg = f = f\, id_{\mathbf{d}(f)}.$$

Using again that f is monic we deduce that $g = id_{\mathbf{d}(f)} \notin \Lambda$, a contradiction. This shows that every element is monic in the sense of (14.5).

Point (ii) is elementary. With respect to (iii) let $f, g \in \Lambda$ be such that $f \Cap g$. If $g \mid f$ then it is obvious that $f = \operatorname{lcm}(f, g)$, and similarly $g = \operatorname{lcm}(f, g)$, if $f \mid g$. Otherwise, assuming that neither $g \mid f$, nor $f \mid g$, there are u and v in Λ (as opposed to $\tilde{\Lambda}$) such that $fu = gv$. So there let (p, q) be a pull back for (f, g), which in particular entails $fp = gq$. Since f and g do not divide each other we have that neither p nor q are identity morphisms. Setting $m = fp$, notice that m is not an identity either because f is not right-invertible, and so $m \in \Lambda$. It is then clear that m is a common multiple of f and g (relative to Λ). If $n \in \Lambda$ is another common multiple of f and g, then $n = fx = gy$, for some $x, y \in \tilde{\Lambda}$. But, since f and g do not divide each other we see that $x, y \in \Lambda$, and hence the equation $fx = gy$ makes sense in $\mathscr{C}$. By definition of pull-backs, there is a morphism r such that $x = pr$, and $y = qr$. Therefore $n = fx = fpr = mr$, and hence $m \mid n$ in Λ, regardless of whether or not $r \in \Lambda$. □

From now on we assume that Λ is a fixed categorical semigroupoid satisfying (14.7), and having no springs.

The greatest simplification brought about by restricting one's attention to such semigroupoids is in the structure of the semilattice $\mathscr{Q}$ of elementary domains defined in (14.10), which is easily seen to be just

$$\mathscr{Q} = \{\Lambda^f : f \in \Lambda\} \cup \{\varnothing\}.$$

If f lies in Λ^{g_1} and Λ^{g_2}, for two elements $g_1, g_2 \in \Lambda$, then evidently $\Lambda^{g_1} \cap \Lambda^{g_2}$ is nonempty, and hence by hypothesis $\Lambda^{g_1} = \Lambda^{g_2}$. We may then define the *range* of f, denoted

$$\mathbf{r}(f),$$

to be the only element $A \in \mathscr{Q}$ for which $f \in A$. It is possible that some $f \in \Lambda$ is not in any Λ^g, in which case $\mathbf{r}(f)$ will not be defined. We then conclude that

$$(f, g) \in \Lambda^{(2)} \iff \Lambda^f = \mathbf{r}(g), \quad \forall\, f, g \in \Lambda,$$

where we consider the expression in the right-hand side to be false if $\mathbf{r}(g)$ is not defined. The reader is invited to compare this with the criteria for two morphisms in a category to be composable.

The above simple form of $\mathscr{Q}$ leads to a simplified $S(\Lambda)$, which then consists of the disjoint union of the following sets:

$$\big\{(f, A, g) : f, g \in \Lambda,\ \Lambda^f = \Lambda^g = A\big\}, \quad \big\{(f, \Lambda^f, 1) : f \in \Lambda\big\},$$
$$\big\{(1, \Lambda^g, g) : g \in \Lambda\big\}, \quad \big\{(1, \Lambda^f, 1) : f \in \Lambda\big\}, \quad \text{and} \quad \{0\}.$$

The all important semilattice $E\big(S(\Lambda)\big)$ is then simply the disjoint union of the sets[16]

$$E(S(\Lambda)) = \big\{(f, \Lambda^f, f) : f \in \Lambda\big\} \cup \big\{(1, A, 1) : A \in \mathscr{Q}\big\}.$$

[16] In case Λ is obtained from a category $\mathscr{C}$, as in (19.2), then $\mathscr{Q}$ is in one-to-one correspondence with the objects in $\mathscr{C}$, or at least those which are the co-domain of a non-identity morphism. It is therefore curious that, after the identities have been put to sleep in (19.2), they were suddenly awakened by this expression for $E(S(\Lambda))$.

Notations 19.3. From now on we shall adopt the following shorthand notations:

(i) $E = E\big(S(\Lambda)\big)$,

(ii) $E_p = \big\{(f, \Lambda^f, f) : f \in \Lambda\big\}$,

(iii) $E_q = \big\{(1, A, 1) : A \in \mathcal{Q}\}$,

(iv) $p_f = p_f^\tau = (f, \Lambda^f, f)$, for all $f \in \Lambda$,

(v) $q_A = (1, A, 1)$, for all $A \in \mathcal{Q}$.

This in turn evokes the notations $\hat{E}$ from (10.1), $\hat{E}_0$ from (12.4), in addition to $\hat{E}_\infty$ and $\hat{E}_{tight}$ from (12.8). It is our purpose here to describe the most important of these, namely $\hat{E}_{tight}$. Recall from (12.9) that $\hat{E}_{tight}$ is the closure of $\hat{E}_\infty$ in $\hat{E}_0$. Being left out of this equation, $\hat{E}$ will not matter much to us.

The following is a compilation of properties relating to the order relation on E, some of which we have already encountered in (14.17) and (14.19), and which completely describes the structure of E, as a semilattice.

Proposition 19.4. *If $f, g \in \Lambda$, and $A, B \in \mathcal{Q}$. Then*

(i) $p_f p_g = p_{\text{lcm}(f,g)}$, *if* $f \Cap g$,

(ii) $p_f \leqslant p_g$, *if and only if* $g \mid f$,

(iii) $p_f \perp p_g$, *if* $f \perp g$,

(iv) $p_f \leqslant q_A$, *if* $f \in A$,

(v) $p_f \perp q_A$, *if* $f \notin A$,

(vi) $q_A \perp q_B$, *if* $A \neq B$.

Definition 19.5. Let ξ be a filter in E. We will say that ξ is of

(i) p-type, if $\xi \subseteq E_p$,

(ii) q-type, if $\xi \subseteq E_q$,

(iii) pq-type, if $\xi \cap E_p$, and $\xi \cap E_q$ are nonempty.

If ξ is a filter of q-type then all of its elements are of the form q_A, for some nonempty $A \in \mathcal{Q}$. But since any two of these are disjoint by (19.4.vi), only one such element is allowed. We thus see that $\xi = \{q_A\}$, for a single nonempty

$A \in \mathcal{Q}$. On the other hand, given any $A \in \mathcal{Q}$, with $A \neq \varnothing$, it is easy to see that the singleton

$$\xi_A = \{q_A\} \tag{19.6}$$

is a filter of q-type.

The next concept is borrowed from [12, 5.5].

Definition 19.7. Let ξ be a filter in E. We will say that the *stem* of ξ is the set

$$\omega_\xi = \{f \in \Lambda : p_f \in \xi\}.$$

It is clear that a filter is of q-type if and only if its stem is empty. The following elementary result describes all filters according to their type:

Proposition 19.8. *Let ξ be a filter in E.*

(i) *If ξ is of q-type, then $\xi = \xi_A := \{q_A\}$, for some $A \in \mathcal{Q}$, with $A \neq \varnothing$.*

(ii) *If ξ is of p-type, then $\xi = \{p_f : f \in \omega_\xi\}$, and moreover $\mathbf{r}(f)$ is not defined for any $f \in \omega_\xi$.*

(iii) *If ξ is of pq-type, then there is some $A \in \mathcal{Q}$, such that $\omega_\xi \subseteq A$. In addition $\xi = \{p_f : f \in \omega_\xi\} \cup \{q_A\}$.*

Proof. Point (i) was already discussed above. Under the hypothesis of (ii), suppose that f is an element of ω_ξ such that $f \in A$, for some $A \in \mathcal{Q}$. Then $p_f \leqslant q_A$, by (19.4.iv) and hence $q_A \in \xi$, by (12.1.ii). This contradicts the fact that ξ is of p-type, and hence f does not belong to any A, which means that $\mathbf{r}(f)$ is not defined. The first sentence of (ii) is obvious.

If ξ is of pq-type, then by assumption ξ contains some q_A, for $A \in \mathcal{Q}$. As already argued, only one such element is allowed and hence $\xi \cap E_q$ must be a singleton $\{q_A\}$. The other elements of ξ must be of the form p_f, for $f \in \Lambda$, and hence $\xi = \{p_f : f \in \omega_\xi\} \cup \{q_A\}$. Given any $f \in \omega_\xi$, we have that both p_f and q_A lie in ξ, and hence $p_f \Cap q_A$, by (12.1.iii). It then follows from (19.4.iv) that $f \in A$. □

Proposition 19.9. *Given a filter ξ on E one has that:*

(i) *if $f \in \omega_\xi$ and $g \in \Lambda$ is such that $g \mid f$, then $g \in \omega_\xi$,*

(ii) *for every $f, g \in \omega_\xi$, one has that $f \Cap g$, and moreover $\mathrm{lcm}(f, g) \in \omega_\xi$.*

Proof. If $g \mid f \in \omega_\xi$, we have that $\xi \ni p_f \leqslant p_g$, so $p_g \in \xi$, and hence $g \in \omega_\xi$. In order to prove (ii) let us be given $f, g \in \omega_\xi$ and suppose by contradiction that $f \perp g$. Then

$$0 = p_f p_g \in \xi,$$

which is impossible. This proves that $f \Cap g$. Moreover,

$$\xi \ni p_f p_g = p_{\mathrm{lcm}(f,g)},$$

so $\mathrm{lcm}(f, g) \in \omega_\xi$. □

Based on the findings of the above result we introduce the following generalization of the notion of paths in a graph:

Definition 19.10. A *path* in a semigroupoid Λ is a subset $\omega \in \Lambda$ such that,

(i) if $f \in \omega$, and $g \in \Lambda$ is such that $g \mid f$, then $g \in \omega$,

(ii) for every $f, g \in \omega$, one has that $f \Cap g$, and moreover $\mathrm{lcm}(f, g) \in \omega$.

An *ultra-path* is a path which is not properly contained in any other path.

It is therefore obvious that ω_ξ is a path for every filter ξ, possibly the empty path if ξ is of q-type.

Given a nonempty path ω, suppose that $f, g \in \omega$ and that $f \in A$, for some $A \in \mathcal{Q}$. By (19.10.ii) we may write $fu = gv$, for suitable $u, v \in \tilde{\Lambda}$, and hence $g \in A$, by (14.2). This means that, if $\mathbf{r}(f)$ is defined for some $f \in \omega$, then $\mathbf{r}(g) = \mathbf{r}(f)$ for every $g \in \omega$. In this case we say that A is the range of ω, in symbols

$$\mathbf{r}(\omega) = A.$$

Otherwise $\mathbf{r}(\omega)$ is not defined.

Notice that if we are given some $f \in \Lambda$ then

$$\omega^f := \{g \in \Lambda : g \mid f\}$$

is clearly a path. By Zorn's Lemma there exists an ultra-path containing ω^f, and hence any element of Λ belongs to some ultra-path. Another consequence of this is that even if the definition allows for paths to be empty, the empty path is never an ultra-path (unless $\Lambda = \varnothing$).

A filter of the form ξ_A, as defined in (19.8.i), is never an ultra-filter because if $f \in A$, then the set of all elements in E which are bigger than or equal to p_f forms a filter properly containing ξ_A. For that reason the filters of q-type are left out of the following characterization of ultra-filters.

Proposition 19.11. *The correspondence* $\xi \mapsto \omega_\xi$ *is a bijection from the set of all filters in* E*, bar the* q*-types, and the set of all nonempty paths. This also gives a one-to-one correspondence from the set of all ultra-filters to the set of all ultra-paths.*

Proof. Given a nonempty path ω consider the subset ξ_ω of E defined by

$$\xi_\omega = \begin{cases} \{p_f : f \in \omega\} \cup \{q_{\mathbf{r}(\omega)}\}, & \text{if } \mathbf{r}(\omega) \text{ is defined,} \\ \{p_f : f \in \omega\}, & \text{otherwise.} \end{cases}$$

It is then easy to see that ξ_ω is a filter and that the resulting map $\omega \to \xi_\omega$ gives the inverse of the correspondence in the statement. Since the two correspondences referred to preserve inclusion, it is clear that ultra-filters correspond to ultra-paths. ◻

From now on we will use (12.5) and (12.6) to identify characters with filters, without further warnings. Therefore $\hat{E}_0$ will be seen as the set of all filters in E. This said, $\hat{E}_\infty$ corresponds to ultra-filters, and the filters corresponding to the elements of $\hat{E}_{tight}$ will be referred to as *tight-filters*.

Were we only interested in $\hat{E}_\infty$, it would be sensible to use the above result to replace $\hat{E}_\infty$ by the set of all ultra-paths. However our primary interest is in tight filters, and unfortunately paths fail to capture the topological complexity of filters.

Proposition 19.12. *Let* ξ *be a filter in* E*.*

(a) *Suppose that* ξ *is of* q*-type, and write* $\xi = \xi_A$*, as in* (19.6)*. Then* ξ *is tight if and only if* A *admits no finite cover (in the sense of* (15.3)*),*

(b) *Suppose that* ξ *is of* p*-type. Then* ξ *is tight if and only if for every* $f \in \omega_\xi$*, and every finite cover* H *for* Λ^f *(again in the sense of* (15.3)*), there is some* $h \in H$ *such that* $fh \in \omega_\xi$*.*

(c) *Suppose that* ξ *is of* pq*-type. Then* ξ *is tight if and only if the condition in* (b) *is satisfied and moreover for every finite cover (ditto)* H *of* $\mathbf{r}(\omega_\xi)$*, one has that* $h \in \omega_\xi$*, for some* $h \in H$*.*

Proof. Before we begin it is convenient to notice the following auxiliary result: if $A \in \mathscr{Q}$ is nonempty then the covers of q_A (in the sense of (11.5)) which do not contain q_A itself, correspond to the covers of A (in the sense of (15.3)) in the following way: given a cover H of A, the set $\{p_h : h \in A\}$ is a cover for

q_A. On the other hand, given a cover Z of q_A which does not contain q_A, the set $\{h \in \Lambda : p_h \in Z\}$ is a cover for A.

To prove it let H be a cover for A. Then every nonzero element $z \in E$, which is smaller than q_A is either q_A itself, in which case z intercepts every p_h, or $z = p_g$, for some $g \in A$ by (19.4). In the latter case $g \Cap h$, for some $h \in H$, and hence

$$zp_h = p_g p_h = p_{\mathrm{lcm}(g,h)} \neq 0,$$

so $z \Cap p_h$. Conversely, assuming that Z is a cover for q_A not containing q_A, we have by (19.4) that Z must have the form

$$Z = \{p_h : h \in H\},$$

where H is a subset of A. To prove that H is a cover for A, let $f \in A$. Then $p_f \leqslant q_A$ by (19.4.iv), so $p_f p_h$ is nonzero for some $h \in H$, which means that $f \Cap h$.

Addressing (i) suppose that ξ_A is a tight filter. Arguing by contradiction let H be a finite cover for A, so that $\{p_h : h \in H\}$ is a cover for q_A. If ϕ is the tight character associated to ξ according to (12.6), then

$$1 = \phi(q_A) = \bigvee_{h \in H} \phi(p_h),$$

and hence $p_h \in \xi$, for some $h \in H$. This would seem to indicate that $h \in \omega_\xi$, which is a contradiction. Thus no cover for A may exist.

Conversely suppose that A admits no finite cover. Again denoting by ϕ the associated character, as in (12.6), notice that $\phi(q_A) = 1$, and hence condition (11.7.i) is satisfied so we may use (11.8) in order to prove that ϕ is tight. So let $x \in E$ and let Z be a finite cover for x. We must then prove that

$$\bigvee_{z \in Z} \phi(z) \geqslant \phi(x). \tag{19.12.1}$$

Observe that, except for $x = q_A$, one has that $\phi(x) = 0$, in which case the above inequality holds trivially. We may then restrict our attention to the case in which $x = q_A$. Excluding the trivial case in which q_A itself belongs to Z, we have that $Z = \{p_h : h \in H\}$, where H is a finite cover for A, but since A admits no finite cover by hypothesis, there is nothing to be proven.

Suppose now that ξ is a tight filter of p-type or pq-type, which implies that ω_ξ is nonempty. Let $f \in \omega_\xi$ and let us be given a finite cover H for Λ^f. This time we claim that $\{p_{fh} : h \in H\}$ is a cover for p_f. In fact, any element of E

which is smaller than p_f is necessarily of the form p_g, for some $g \in \Lambda$ which is a multiple of f, by (19.4). So write $g = fk$, with $k \in \tilde{\Lambda}$. If $k = 1$ then obviously $p_g \Cap p_{fh}$, for all $h \in H$. Otherwise $k \in \Lambda^f$, and hence $h \Cap k$, for some $h \in H$. This implies that $fh \Cap fk$, or equivalently that $fh \Cap g$, whence

$$p_{fh} p_g = p_{\mathrm{lcm}(fh,g)} \neq 0,$$

proving the claim. Because the character ϕ associated to ξ is tight we deduce that

$$1 = \phi(p_f) = \bigvee_{h \in H} \phi(p_{fh}),$$

so there exists some $h \in H$, such that $p_{fh} \in \xi$, and hence $fh \in \omega_\xi$. In the special case in which ξ is a tight filter of pq-type we must still address the last assertion in (c). Let $A = \mathbf{r}(\omega_\xi)$, and picking any $f \in \omega_\xi$ we have that $f \in A$, so

$$\xi \ni p_f \leqslant q_A,$$

whence $q_A \in \xi$, which is to say that the associated character ϕ satisfies $\phi(q_A) = 1$. Let H be a finite cover for A. By the auxiliary result proved above one has that $\{p_h : h \in H\}$ is a cover for q_A and hence

$$1 = \phi(q_A) = \bigvee_{h \in H} \phi(p_h),$$

from where we deduce that $\phi(p_h) = 1$, for some $h \in H$, meaning that $h \in \omega_\xi$.

Let us now address the converse implications in (b) and (c) simultaneously. So let ξ be a filter with nonempty stem satisfying the condition in (b). In case ξ is of pq-type, we assume in addition that it also satisfies the condition in (c).

Since the associated character ϕ is nonzero, there must exist some $x \in E$, such that $\phi(x) = 1$, and hence we may again use (11.8) in order to prove that ϕ is tight. So let $x \in E$, and let Z be a finite cover for x. We must prove (19.12.1). Excluding the trivial case in which $\phi(x) = 0$, we suppose that $x \in \xi$.

The proof will be broken up in two cases, the first one corresponding to $x = p_f$, for some $f \in \omega_\xi$. Since Z is a cover for p_f we have by (19.4.ii) that,

$$Z = \{p_g : g \in G\},$$

where G is a finite subset of Λ consisting of multiples of f. We may therefore rewrite Z as

$$Z = \{p_{fh} : h \in H\},$$

where H is a finite subset of $\Lambda^f \cup \{1\}$. If p_f itself belongs to Z then obviously the right-hand side of (19.12.1) is 1, and the proof is finished. So assume that $H \subseteq \Lambda^f$. We then claim that H is a cover for Λ^f. To prove it let $k \in \Lambda^f$. Then $p_{fk} \leqslant p_f$, so that $p_{fk} \Cap p_{fh}$, for some $h \in H$, meaning that $fkx = fhy$, for suitable $x, y \in \tilde{\Lambda}$. Canceling out f we deduce that $kx = hy$, and hence that $k \Cap h$, proving our claim. The hypothesis therefore applies and we have that $fh \in \omega_\xi$, for some $h \in H$, which may be rephrased by saying that $\phi(p_{fh}) = 1$, proving that the left-hand side of (19.12.1) is 1.

Assume next that $x = q_A$, for some $A \in \mathcal{Q}$. As we are supposing that $\phi(x) = 1$, and hence that $q_A \in \xi$, this can only happen if ξ is of pq-type, in which case we moreover have that $A = \mathbf{r}(\omega_\xi)$. The fact that Z is a cover for q_A implies that either $q_A \in Z$, when (19.12.1) is readily proved, or $Z = \{p_h : h \in H\}$, where H is a cover for $A = \mathbf{r}(\omega_\xi)$. This may then be combined with our hypothesis to give $h \in \omega_\xi$, for some $h \in H$. Then $p_h \in \xi$ and hence $\phi(p_h) = 1$. Since p_h is in Z, we have that the left-hand side of (19.12.1) is 1, concluding the proof. □

20 Higher rank graphs

In this section we wish to apply our theory to higher rank graphs. The reader should consult the references listed in the introduction for more information on this subject.

From now on we assume that $k \geqslant 1$ is an integer and Λ is a k-graph, with *rank* map given by

$$\partial : \Lambda \to \mathbf{N}^k.$$

The well known *unique factorization property* states that for every morphism f in Λ, and for every $n, m \in \mathbf{N}^k$ such that $\partial(f) = n + m$, there exists a unique pair of morphisms (g, h) such that $f = gh$, $\partial(g) = n$, and $\partial(h) = m$.

As usual we will say that f is an *edge* if $\partial(f)$ is an element of the canonical basis $\{e_i\}_{i=1}^k$ of $\mathbf{N}^k$. For an edge f, one sometimes refer to $\partial(f)$ as the *color* of f. While one does not really have to attach "colors" to the e_i, it does make sense to say that two edges have, or do not have the same color.

The possibility of studying Λ with our tools naturally hinges on whether or not we may verify our working hypotheses, namely (14.7), and the absence of springs.

We will soon specialize to a situation in which we may apply all of the points in (19.2), hence obtaining our working hypotheses. We do so mainly to avoid technical complications, but we nevertheless believe that our methods, and Theorem (13.3) in special, may be applied to the inverse semigroup constructed in

[13] in the most general case, obtaining the same description of the C*-algebra of Λ as a groupoid C*-algebra.

Notice that the identities in Λ are precisely the morphisms with rank zero. Moreover if $f, g \in \Lambda$ are such that fg is an identity, then

$$0 = \partial(fg) = \partial(f) + \partial(g),$$

which implies that $\partial(f) = \partial(g) = 0$, so f and g are both identities. This says that no morphism other than the identities may be right-invertible, and hence we have by (19.2) that the set $\underline{\Lambda}$ of all non-identity morphisms is a categorical semigroupoid.

With respect to (19.2.i), if f, g, h are morphisms in Λ such that $fg = fh$, then $\partial(g) = \partial(h)$, and the uniqueness of the factorization implies that $g = h$. This says that every morphism in Λ is a monomorphism, so we may apply (19.2.i) to collect another of our working hypotheses.

For each vertex (object) v in Λ and each $n \in \mathbf{N}^k$ one usually denotes by Λ_n^v the set of all morphisms f in Λ with $\mathbf{r}(f) = v$ and $\partial(f) = n$.

Recall that Λ is said to be *row-finite* if Λ_n^v is finite for every v and n. If Λ_n^v is never empty then one says that Λ *has no sources*. Notice that in order for the associated semigroupoid $\underline{\Lambda}$ to have no springs one does not necessarily need to rule out all sources of Λ. It is clearly enough to suppose that

$$\underline{\Lambda}^v = \bigcup_{n\neq 0} \Lambda_n^v \neq \varnothing,$$

for every object v in Λ.

The last requirement we will impose on Λ is designed to allow for the use of (19.2.iii), and it is related to the question of finite alignment. Recall that Λ is said to be *finitely aligned*, if for every $f, g \in \Lambda$ one has that

$$\Lambda^{\min}(f, g) := \big\{(p, q) \in \Lambda \times \Lambda : fp = gq,\ \text{and } \partial(fp) = \partial(f) \vee \partial(g)\big\}$$

is finite.

Observe that for any pair (p, q) in $\Lambda^{\min}(f, g)$, one has that $m := fp$ is a common multiple of f and g. If (p', q') is another pair in $\Lambda^{\min}(f, g)$, then $m' = fp'$ is another common multiple but neither $m \mid m'$, nor $m' \mid m$, because $\partial(m) = \partial(m')$. Thus, unless $\Lambda^{\min}(f, g)$ has at most one element, $\underline{\Lambda}$ will not admit least common multiples.

Definition 20.1. We shall say that Λ is *singly aligned*, if $\Lambda^{\min}(f, g)$ has at most one element for every f and g in Λ.

We would like to reach the conclusion that Λ is singly aligned, and also that $\underline{\Lambda}$ admits least common multiples, starting with the following apparently weaker concept:

Definition 20.2. We shall say that Λ satisfies the *little pull-back property* if, given two commuting squares

$$\begin{tikzcd}[column sep=small] & \bullet \arrow[dr, "f_1"] & \\ \bullet \arrow[ur, "p_1"] \arrow[dr, "q_1"'] & & \bullet \\ & \bullet \arrow[ur, "g_1"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] & \bullet \arrow[dr, "f_2"] & \\ \bullet \arrow[ur, "p_2"] \arrow[dr, "q_2"'] & & \bullet \\ & \bullet \arrow[ur, "g_2"'] & \end{tikzcd}$$

such that

(i) all arrows involved are edges,

(ii) all northeast edges are of the same color,

(iii) all southeast edges are of the same color, but not the same as the northeast ones,

then

$$f_1 = f_2 \quad \text{and} \quad g_1 = g_2 \quad \Longrightarrow \quad p_1 = p_2 \quad \text{and} \quad q_1 = q_2.$$

It is obvious that a k-graph which does not satisfy the little pull-back property cannot be singly aligned.

Speaking of either one of the diagrams above, say the one on the left-hand side, one sometimes think of the two-dimensional figure formed by it as a geometrical representation of the element $f_1 p_1$ of Λ. The algebraic structure of Λ is based on the idea that this *square* is determined by the *sides* p_1 and f_1. In particular, there cannot be two different squares sharing these two sides. A similar observation clearly holds for the sides q_1 and g_1. The little pull-back property goes very much in this direction by stating that there cannot be two different squares sharing the sides f_1 and g_1. A similar property, which could be called the *little push-out property*, would say that two different squares cannot share the sides p_1 and q_1. That property may be shown to imply the existence of push-outs in Λ.

Proposition 20.3. *Suppose that Λ satisfies the little pull-back property, and let f_i, g_i, p_i and q_i be morphism (rather than edges) such that $f_i p_i = g_i q_i$, for $i = 1, 2$. Suppose also that $\partial(f_i) \wedge \partial(g_i) = 0$, and $\partial(p_i) \wedge \partial(q_i) = 0$, for $i = 1, 2$. Then the implication at the end of* (20.2) *holds true.*

Proof. First observe that

$$\partial(f_i) - \partial(g_i) = \partial(q_i) - \partial(p_i),$$

so one necessarily has

$$\partial(f_i) = \partial(q_i), \quad \text{and} \quad \partial(g_i) = \partial(p_i),$$

as a consequence of orthogonality.

Letting $f = f_1 = f_2$, and $g = g_1 = g_2$, observe that it is enough to show that $p_1 = p_2$, since this would imply that

$$gq_1 = fp_1 = fp_2 = gq_2,$$

and the uniqueness of the factorization would give $q_1 = q_2$.

Suppose first that $\partial(f) = 0$. Then $\partial(q_1) = \partial(q_2) = 0$, so f, q_1, and q_2 are the identity morphisms on their respective domains. Therefore

$$p_1 = fp_1 = gq_1 = g = gq_2 = fp_2 = p_2.$$

A similar argument proves the result if $\partial(g) = 0$. We therefore suppose, from now on, that $\partial(f)$ and $\partial(g)$ are both nonzero.

We will now proceed by induction on $|\partial(f)| + |\partial(g)|$, observing that when $|\partial(f)| + |\partial(g)| \leqslant 1$, the conclusion follows from the above arguments.

If $|\partial(f)| + |\partial(g)| = 2$, since $|\partial(f)|, |\partial(g)| > 0$, we must have that $|\partial(f)| = |\partial(g)| = 1$, and hence f and g are edges. By hypothesis the p_i and q_i are also edges, so the conclusion follows from the little pull-back property.

We thus assume that $n \geqslant 2$, and $|\partial(f)| + |\partial(g)| = n + 1$. Hence either $|\partial(f)| \geqslant 2$, or $|\partial(g)| \geqslant 2$. Without loss of generality we assume that $|\partial(f)| \geqslant 2$. So by the factorization property there are morphisms f' and f'' such that $f = f'f''$, and $|\partial(f')|, |\partial(f'')| < |\partial(f)|$. Since, for $i = 1, 2$,

$$\partial(q_i) = \partial(f) = \partial(f') + \partial(f''),$$

we may write $q_i = q_i'q_i''$, with $\partial(q_i') = \partial(f')$, and $\partial(q_i'') = \partial(f'')$. We furthermore observe that

$$\partial(fp_i) = \partial(f') + \partial(f'') + \partial(p_i) = \partial(f') + \partial(p_i) + \partial(q_i''),$$

and hence we factorize $fp_i = \phi_i h_i \psi_i$, with

$$\partial(\phi_i) = \partial(f'), \quad \partial(h_i) = \partial(p_i), \quad \text{and} \quad \partial(\psi_i) = \partial(q_i'').$$

Notice that

$$f'f''p_i = fp_i = \phi_i h_i \psi_i.$$

By the uniqueness of the factorization we conclude that

$$f' = \phi_i, \quad \text{and} \quad f''p_i = h_i\psi_i. \tag{20.4}$$

On the other hand, notice that

$$f'h_i\psi_i = \phi_i h_i \psi_i = fp_i = gq_i = gq_i'q_i''.$$

Again by the uniqueness of the factorization we conclude that

$$f'h_i = gq_i', \quad \text{and} \quad \psi_i = q_i''.$$

Observe that $\partial(f') \wedge \partial(g) \leqslant \partial(f) \wedge \partial(g) = 0$, that $\partial(f') = \partial(q_i')$, and that $\partial(h_i) = \partial(g)$. By the induction hypothesis we have that $h_1 = h_2$, and $q_1' = q_2'$. Let us thus use the simplified notation $h = h_1 = h_2$, and $q' = q_1' = q_2'$. By (20.4) we then deduce that

$$f''p_i = h\psi_i.$$

Again we have $\partial(f'') \wedge \partial(h) \leqslant \partial(f) \wedge \partial(g) = 0$, $\partial(f'') = \partial(\psi_i)$, and $\partial(p_i) = \partial(h)$. By induction we conclude that $p_1 = p_2$, and $\psi_1 = \psi_2$, finishing the proof. □

While the result above deals with uniqueness, the next result will provide existence:

Lemma 20.5. *Let f_1, f_2, p_1, p_2 be morphisms such that $f_1p_1 = f_2p_2$. Then there are morphisms $r, \bar{p}_1, \bar{p}_2$, such that, for every $i = 1, 2$, one has*

(i) $f_1\bar{p}_1 = f_2\bar{p}_2$,

(ii) $p_i = \bar{p}_i r$,

(iii) $\partial(\bar{p}_1) \wedge \partial(\bar{p}_2) = 0$,

(iv) $\partial(f_i\bar{p}_i) = \partial(f_1) \vee \partial(f_2)$.

Proof. Since $\partial(f_1), \partial(f_2) \leqslant \partial(f_1p_1) = \partial(f_2p_2)$, we have that $\partial(f_1) \vee \partial(f_2) \leqslant \partial(f_ip_i)$, and hence there are morphisms s and r such that $sr = f_ip_i$, and $\partial(s) = \partial(f_1) \vee \partial(f_2)$. Notice that

$$\partial(s)+\partial(r) = \partial(f_ip_i) = \partial(f_i)+\partial(p_i) \leqslant \partial(f_1)\vee\partial(f_2)+\partial(p_i) = \partial(s)+\partial(p_i),$$

and hence $\partial(r) \leqslant \partial(p_i)$. By the factorization property we may factor $p_i = \bar{p}_i r_i$, with $\partial(r_i) = \partial(r)$. Notice that

$$f_i \bar{p}_i r_i = f_i p_i = sr.$$

By the uniqueness of the factorization we conclude that $f_i \bar{p}_i = s$, and $r_i = r$, hence proving (i) and (ii). In addition we have

$$\partial(f_i \bar{p}_i) = \partial(s) = \partial(f_1) \vee \partial(f_2),$$

taking care of (iv). In order to show (iii) suppose that $n \in \mathbf{N}^k$ is such that $n \leqslant \partial(\bar{p}_i)$, for all i, then

$$\partial(f_i) \leqslant \partial(f_i) + \partial(\bar{p}_i) - n = \partial(f_1) \vee \partial(f_2) - n,$$

whence $\partial(f_1) \vee \partial(f_2) \leqslant \partial(f_1) \vee \partial(f_2) - n$, so that $n = 0$. □

The following result can be proved by applying (20.5) to the opposite category Λ^{op}.

Lemma 20.6. *Let q_1, q_2, g_1, g_2 be morphisms such that $q_1 g_1 = q_2 g_2$. Then there are morphisms $s, \bar{q}_1, \bar{q}_2$, such that, for every $i = 1, 2$, one has*

(i) $\bar{q}_1 g_1 = \bar{q}_2 g_2$,

(ii) $q_i = s\bar{q}_i$,

(iii) $\partial(\bar{q}_1) \wedge \partial(\bar{q}_2) = 0$,

(iv) $\partial(\bar{q}_i g_i) = \partial(g_1) \vee \partial(g_2)$.

Proposition 20.7. *Assume that Λ satisfies the little pull-back property and let f_1, f_2, p_1, p_2, p'_1 and p'_2 be morphisms such that for all $i = 1, 2$,*

(i) $f_1 p_1 = f_2 p_2$, *and* $f_1 p'_1 = f_2 p'_2$,

(ii) $\partial(p_1) \wedge \partial(p_2) = 0$, *and* $\partial(p'_1) \wedge \partial(p'_2) = 0$.

Then $p_i = p'_i$, for $i = 1, 2$.

Proof. First observe that

$$\partial(p_1) - \partial(p_2) = \partial(f_2) - \partial(f_1) = \partial(p'_1) - \partial(p'_2),$$

so $\partial(p_i) = \partial(p_i')$, by (ii). Using (20.6) with $g_i = p_i$, and $q_i = f_i$, let $\bar{f}_i$ and s be such that $\bar{f}_1 p_1 = \bar{f}_2 p_2$, $f_i = s\bar{f}_i$, and $\partial(\bar{f}_1) \wedge \partial(\bar{f}_2) = 0$. Since

$$\partial(\bar{f}_1) - \partial(\bar{f}_2) = \partial(p_2) - \partial(p_1),$$

we have $\partial(\bar{f}_1) = \partial(p_2)$, and $\partial(\bar{f}_2) = \partial(p_1)$.

Replacing p_i by p_i' in our application of (20.6) just above we would get $\bar{f}_i'$ and s' such that $\bar{f}_1' p_1' = \bar{f}_2' p_2'$, $f_i = s'\bar{f}_i'$, $\partial(\bar{f}_1') \wedge \partial(\bar{f}_2') = 0$. As above we may also prove that $\partial(\bar{f}_1') = \partial(p_2')$, and $\partial(\bar{f}_2') = \partial(p_1')$. Therefore

$$\partial(\bar{f}_1) = \partial(p_2) = \partial(p_2') = \partial(\bar{f}_1'),$$

and

$$\partial(\bar{f}_2) = \partial(p_1) = \partial(p_1') = \partial(\bar{f}_2').$$

Furthermore

$$\partial(s) = \partial(f_i) - \partial(\bar{f}_i) = \partial(f_i) - \partial(\bar{f}_i') = \partial(s'),$$

and hence the identity $s\bar{f}_i = s'\bar{f}_i'$, together with the uniqueness of the factorization gives $s = s'$ and $\bar{f}_i = \bar{f}_i'$. The two identities

$$\bar{f}_1 p_1 = \bar{f}_2 p_2, \quad \text{and} \quad \bar{f}_1 p_1' = \bar{f}_2 p_2'$$

and (20.3) thus give the conclusion. □

So here is the result we were looking for:

Theorem 20.8. *A k-graph Λ satisfying the little pull-back property is singly aligned and the associated semigroupoid $\underline{\Lambda}$ admits least common multiples.*

Proof. Let f_1, f_2, p_1, p_2 be morphisms such that $f_1 p_1 = f_2 p_2$. Pick $r, \bar{p}_1, \bar{p}_2$ as in (20.5). We claim that $(\bar{p}_1, \bar{p}_2)$ is a pull-back for (f_1, f_2).

In order to prove this let q_1, q_2 be morphisms such that $f_1 q_1 = f_2 q_2$. Again pick $s, \bar{q}_1, \bar{q}_2$ as in (20.5), so that $f_1\bar{q}_1 = f_2\bar{q}_2$, $q_i = \bar{q}_i s$, and $\partial(\bar{q}_1) \wedge \partial(\bar{q}_2) = 0$. By (20.7) we deduce that $\bar{p}_i = \bar{q}_i$, and hence $q_i = \bar{p}_i s$, as desired. It is also clear that s is unique by the factorization property. If then immediately follows that Λ is singly aligned. That $\underline{\Lambda}$ admits least common multiples is then a consequence of (19.2.iii). □

The little pull-back property is the last restriction we need to impose on Λ in order to be able to apply all of the conclusions of (19.2).

In view of [13, 3.8.(3)] it is reasonable to restricts one's attention to representations of $\underline{\Lambda}$ which respects least common multiples. The following is the main result of this section.

Theorem 20.9. *Let Λ be a countable k-graph satisfying the little pull-back property and such that for every vertex v there is some morphism f, other than the identity on v, with $\mathbf{r}(f) = v$. Then, removing the identities from Λ we obtain a semigroupoid $\underline{\Lambda}$ which has no springs, contains only monic elements and in which every intersecting pair of elements admits a least common multiple. Moreover the the C*-algebra generated by the range of a universal tight representation of $\underline{\Lambda}$, respecting least common multiples, is naturally isomorphic to the C*-algebra of the groupoid $\mathcal{G}_{\underline{\Lambda}}$ of germs for the standard action of $S(\underline{\Lambda})$ on the tight part of the spectrum of its idempotent semilattice.*

Proof. Follows from (19.2) and (18.4). □

From now we fix a k-graph Λ satisfying the hypothesis of (20.9).

To conclude this section we will give a description of $\hat{E}_{tight}$, where E is the idempotent semilattice of $S(\underline{\Lambda})$. Given a path ω on $\underline{\Lambda}$, let $f, g \in \omega$ with $\partial(f) = \partial(g)$. Since $f \Cap g$, by (19.10.ii) we may write $fu = gv$. Extending ∂ to $\tilde{\underline{\Lambda}}$ by defining $\partial(1) = 0$, we then have that $\partial(u) = \partial(v)$, and then $f = g$ by the unique factorization property. This says that ω may contain at most one element f with $\partial(f) = n$, for each $n \in \mathbf{N}^k$.

Proposition 20.10. *Given a nonempty path ω on $\underline{\Lambda}$, let D be the image of ω under the rank function ∂, and for each $n \in D$, let $\mu(n)$ be the unique element f in ω with $\partial(f) = n$. Then*

(i) *$D \cup \{0\}$ is a hereditary subset of $\mathbf{N}^k$,*

(ii) *if $n, m \in D$, then $n \vee m \in D$,*

(iii) *if $n, m \in D$, and $n \leqslant m$, then $\mu(n) \mid \mu(m)$,*

(iv) *$\omega = \{\mu(n) : n \in D\}$.*

Proof. Let $n, m \in N^k$, with $m \in D$, and $0 \neq n \leqslant m$. Set $f = \mu(m)$, so that $\partial(f) = m$. Writing $m = n + (m - n)$, the unique factorization property implies that $f = gh$, with $\partial(g) = n$, and $\partial(h) = m - n$. Since $g \mid f$ we conclude that $g \in \omega$, and hence $n \in D$, proving (i). It is also clear that $g = \mu(n)$, so (iii) is also proved. To prove (ii) let $n, m \in D$, so $f := \mathrm{lcm}(\mu(n), \mu(m)) \in \omega$, and hence $\partial(f) \in D$. It may be proved that $\partial(f) = m \vee n$, but it suffices to notice that, since f is a common multiple of $\mu(n)$ and $\mu(m)$, one has that $n, m \leqslant \partial(f)$, and consequently $n \vee m \leqslant \partial(f)$. Thus (ii) follows from (i). The last point is trivial. □

The following is a converse to the above:

Proposition 20.11. *Let D be a subset of $\mathbf{N}^k$ not containing 0, but such that $D \cup \{0\}$ is a hereditary subset of $\mathbf{N}^k$. Assume that D is closed under "$\vee$" and let $\omega : D \to \underline{\Lambda}$ be any map such that for every $n, m \in D$,*

(i) $\partial(\mu(n)) = n$,

(ii) $\mu(n) \mid \mu(m)$, *if* $n \leqslant m$,

Then the set $\omega = \{\mu(n) : n \in D\}$ is a path in $\underline{\Lambda}$.

Proof. If $f \in \underline{\Lambda}$, and $f \mid \mu(m)$, for some $m \in D$, write $\mu(m) = fu$, for some $u \in \underline{\tilde{\Lambda}}$. This clearly implies that

$$n := \partial(f) \leqslant \partial(\mu(m)) = m,$$

so $n \in D$, and $\mu(m) = \mu(n)v$, for some $v \in \underline{\tilde{\Lambda}}$. By the unique factorization we have that $f = \mu(n) \in \omega$.

To prove (19.10.ii) suppose that $n, m \in D$. Then $\mu(n \vee m)$ is a common multiple of $\mu(n)$ and $\mu(m)$ and, recalling that under our assumptions $\underline{\Lambda}$ admits least common multiples, we have

$$\operatorname{lcm}(\mu(n), \mu(m)) \mid \mu(n \vee m).$$

However it is easy to see that $\partial\big(\operatorname{lcm}\big(\mu(n), \mu(m)\big)\big) \geqslant n \vee m$, so

$$\operatorname{lcm}(\mu(n), \mu(m)) = \mu(n \vee m) \in \omega. \qquad \square$$

Notice that for any set D as above one may define the supremum of D as an element

$$m \in (\mathbf{N} \cup \{\infty\})^k,$$

and hence $D = \Omega_{k,m} := \{n \in \mathbf{N}^k : n \leqslant m\}$, as defined in [13, 3.2].

It therefore follows that paths in $\underline{\Lambda}$ correspond to maps μ, as in (20.11), and hence also to the usual notion of paths in higher rank graphs [13, 5.1]. One may then use (19.12) to relate elements of $\hat{E}_{tight}$ to the boundary paths of [13].

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