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On the Hermitian positive definite solution of the nonlinear matrix equation $X + A^*X^{-1}A + B^*X^{-1}B = I$

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Abstract. In this paper, we study the matrix equation $X + A^*X^{-1}A + B^*X^{-1}B = I$, where A, B are square matrices, and obtain some conditions for the existence of the positive definite solution of this equation. Two iterative algorithms to find the positive definite solution are given. Some numerical results are reported to illustrate the effectiveness of the algorithms.

Keywords: nonlinear matrix equation, positive definite solution, iterative method.

Mathematical subject classification: 65F10, 65F30, 65H10, 15A24.

1 Introduction

In this paper, we consider the matrix equation

$$X + A^*X^{-1}A + B^*X^{-1}B = I, \tag{1.1}$$

where A, B are square matrices, I is the identity matrix and a Hermitian positive definite solution is required.

The solving of the matrix equation (1.1) is a problem of practical importance. In many physical applications, we must solve the system of linear equation [1]

$$Px = f \tag{1.2}$$

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where the positive definite matrix P arises from a finite difference approximation to an elliptic partial differential equation. As an example, let

$$P = \begin{pmatrix} I & 0 & A \\ 0 & I & B \\ A^* & B^* & I \end{pmatrix}. \tag{1.3}$$

We consider the matrix $P = \tilde{P} + D$, where

$$\tilde{P} = \begin{pmatrix} X & 0 & A \\ 0 & X & B \\ A^* & B^* & I \end{pmatrix}, \ D = \begin{pmatrix} I - X & 0 & 0 \\ 0 & I - X & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We may decompose the matrix $\tilde{P}$ as

$$\begin{pmatrix} X & 0 & A \\ 0 & X & B \\ A^* & B^* & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ A^*X^{-1} & B^*X^{-1} & I \end{pmatrix} \begin{pmatrix} X & 0 & A \\ 0 & X & B \\ 0 & 0 & X \end{pmatrix}. \tag{1.4}$$

For the decomposition (1.4) exists, the matrix X must be a solution of the equation $X + A^*X^{-1}A + B^*X^{-1}B = I$. The solving of the system $\tilde{P}y = f$ is transformed to the solving of two linear systems that have lower triangular block coefficient matrix and upper triangular block coefficient matrix respectively. The Woodbury formula [2] can be applied to compute the solution of equation (1.2).

Recently, some authors [4-14] have studied the matrix equations

$$X + A^*X^{-1}A = Q, \tag{1.5}$$

and

$$X + A^*X^{-1}A = I. \tag{1.6}$$

In both cases, A, Q are square matrices, Q is a Hermitian positive definite matrix and the positive definite solutions are required. Several conditions for the existence of the positive definite solution were given in [3, 4, 5, 7], and some iterations were discussed to find the maximal positive definite solution in [8-14] for these two equations.

Obviously, the equation (1.1) generalizes the equation (1.6). Note that the equation (1.1) may have non-Hermitian or indefinite solutions. For example, if A=$(1/2)I_2$, $B = (\sqrt{3}/2)I_2$, then

$$X = \begin{pmatrix} 1 & -\alpha \\ \alpha^{-1} & 0 \end{pmatrix} \tag{1.7}$$

with any real number $\alpha \neq 0$, is always a solution. We will not consider this case in this paper. The matrix equation (1.1) arises in many application areas including control theory, ladder networks, dynamics programming, stochastic filtering and statistic (see references given in [3]).

Throughout this paper, we denote by $C^{n\times n}$ and $H^{n\times n}$ the set of $n \times n$ complex and $n \times n$ Hermitian matrices, respectively. For $A, B \in H^{n\times n}$, $A \geq 0(A > 0)$ means that A is positive semi-definite(positive definite). Moreover, $A \geq B(A > B)$ means that $A - B \geq 0(A - B > 0)$, and $X \in [A, B]$ means $A \leq X \leq B$. A^* and $r(A)$ denote the complex conjugate transpose and the spectral radius of A, respectively. $\lambda_{\max}(A^*A)$ and $\lambda_{\min}(A^*A)$ denote the maximal and the minimal eigenvalue of A^*A, respectively. Let $A \otimes B = (a_{ij}B)$, $\mathrm{vec}(A) = \left(a_1^T, a_2^T, \cdots, a_n^T\right)^T$, where $A = (a_{ij})$, $a_1, \cdots, a_n \in C^n$ are the the columns of A, $||A||_2 = \lambda_{\max}^{1/2}(A^*A)$, $||A||_F = (\mathrm{tr}(A^*A))^{1/2}$.

This paper is organized as follows, in section 2, we derive some necessary conditions, some sufficient conditions and a necessary and sufficient condition for the existence of the positive definite solution of equation (1.1), respectively. Section 3 contains two iterative algorithms for obtaining the positive definite solution of equation (1.1). For the sake of illustrating the effectiveness of our algorithms, several numerical examples are presented in section 4. We draw conclusions in section 5.

2 Conditions for existence of the positive definite solution

Theorem 2.1.

(1) *If equation* (1.1) *has a positive definite solution X, then $X \leq I$.*

Proof. First assume that equation (1.1) has a solution $X > 0$, then $X^{-1} > 0$. This implies that $A^*X^{-1}A + B^*X^{-1}B \geq 0$, which gives

$$X = I - A^*X^{-1}A - B^*X^{-1}B \leq I. \qquad \square$$

Theorem 2.2. *If equation* (1.1) *has a positive definite solution X, then*

(1) $A^*A + B^*B < I$,

(2) $X > AA^*$ *and* $X > BB^*$.

Proof. Let X be a positive definite solution of equation (1.1), by Theorem 2.1, $X \leq I$. Hence, we obtain

$$A^*A + B^*B \leq A^*X^{-1}A + B^*X^{-1}B = I - X < I.$$

Rewriting (1.1) yields that

$$X + A^*X^{-1}A = I - B^*X^{-1}B. \tag{2.1}$$

Since X is positive definite, $X + A^*X^{-1}A$ is invertible. Applying Schur's lemma to (2.1) yields that

$$\begin{aligned} (X + A^*X^{-1}A)^{-1} &= (I - B^*X^{-1}B)^{-1} \\ &= I - B^*(BB^* - X)^{-1}B. \end{aligned} \tag{2.2}$$

Hence, $X - BB^*$ is invertible. Now consider $(X - BB^*)^{-1}$. Applying Schur's lemma once again yields that

$$\begin{aligned} (X - BB^*)^{-1} &= X^{-1} - X^{-1}B(B^*X^{-1}B - I)^{-1}B^*X^{-1} \\ &= X^{-1} + X^{-1}B(X + A^*X^{-1}A)^{-1}B^*X^{-1}, \end{aligned} \tag{2.3}$$

which is clearly positive definite. Hence $X > BB^*$. Analogously, we can also prove $X > AA^*$. □

Remark. Theorem 2.2 generalizes the Theorem 3.1 and Theorem 3.2 (i) in [6].

Lemma 2.1. *Let P, Q, R and $S \in C^{n\times n}$, then*

$$r(P^*R - R^*P + Q^*S - S^*Q) \leq r(P^*P + Q^*Q + R^*R + S^*S). \tag{2.4}$$

Proof. First we introduce the following notations,

$$U = (P^*, R^*, Q^*, S^*), \qquad V = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \end{pmatrix}.$$

where I is the $n \times n$ identity matrix. By elementary calculation we have

$$r(P^*R - R^*P + Q^*S - S^*Q) = r(UVU^*). \tag{2.5}$$

Since $r(AB) = r(BA)$ for $\forall A, B \in C^{n\times n}$, we get

$$r(UVU^*) = r(VU^*U).$$

Since $r(A) \leq ||A||_2$, we obtain

$$r(VU^*U) \leq ||VU^*U||_2 \leq ||V||_2\, ||U^*U||_2.$$

Since U^*U is a normal matrix, we have $||U^*U||_2 = r(U^*U)$ and

$$\begin{aligned} r(P^*R - R^*P + Q^*S - S^*Q) &= r(VU^*U) \\ &\leq ||V||_2\, ||U^*U||_2 \\ &= 1 * r(UU^*) \\ &= r(P^*P + Q^*Q + R^*R + S^*S). \end{aligned}$$

□

Lemma 2.2. *If $A \in H^{n\times n}$ satisfies $-I \leq A \leq I$, then $r(A) \leq 1$.*

Proof. Since $A \in H^{n\times n}$, there exists a unitary matrix $U \in C^{n\times n}$ such that $U^*AU = \Sigma = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $\lambda_i \in R$ is a eigenvalue of A, $i = 1, 2, \cdots, n$. Since $I - A \geq 0$, $U^*IU - U^*AU \geq 0$, that is, $I - \Sigma \geq 0$, thus

$$1 - \lambda_i \geq 0,\ i = 1, 2, \ldots, n. \tag{2.6}$$

Analogously, we can prove

$$1 + \lambda_i \geq 0,\ i = 1, 2, \ldots, n. \tag{2.7}$$

It follows from (2.6)–(2.7) that $-1 \leq \lambda_i \leq 1$, $i = 1, 2, \ldots, n$, thus $r(A) \leq 1$. □

Theorem 2.3. *If equation* (1.1) *has a positive definite solution, then the matrix A, B satisfy the following inequalities:*

(1) $r(A + A^*) \leq 1$, $r(B + B^*) \leq 1$,

(2) $r(A - A^*) \leq 1$, $r(B - B^*) \leq 1$.

Proof. Let X be the positive definite solution of equation (1.1). First, we introduce the following notations:

$$\begin{cases} P := X^{1/2} - X^{-1/2}A + X^{-1/2}B, \\ Q := X^{1/2} - X^{-1/2}A - X^{-1/2}B, \\ R := X^{1/2} + X^{-1/2}A + X^{-1/2}B, \\ S := X^{1/2} + X^{-1/2}A - X^{-1/2}B. \end{cases}$$

Hence, we get the following equalities:

$$\begin{cases} P^*P + Q^*Q = 2(I - A - A^*), \; R^*R + S^*S = 2(I + A + A^*), \\ Q^*Q + S^*S = 2(I - B - B^*), \; P^*P + R^*R = 2(I - B - B^*), \end{cases} \tag{2.8}$$

$$\begin{cases} P^*R - R^*P + Q^*S - S^*Q = 4A - AA^*, \\ Q^*R - R^*Q + S^*P - P^*S = 4B - 4B^*. \end{cases} \tag{2.9}$$

Since $P^*P + Q^*Q$, $R^*R + S^*S$, $P^*P + R^*R$ and $Q^*Q + S^*S$ are positive semi-definite, from (2.8) we know that $-I \leq A + A^* \leq I$, $-I \leq B + B^* \leq I$. By Lemma 2.2, the assertion (1) is proved.

Lemma 2.1 and (2.7) yield that

$$\begin{aligned} r(A - A^*) &= 1/4 * r(P^*R - R^*P + Q^*S - S^*Q) \\ &\leq 1/4 * r(P^*P + Q^*Q + R^*R + S^*S) \\ &= 1/4 * r(4I) = 1, \end{aligned} \tag{2.10}$$

$$\begin{aligned} r(B - B^*) &= 1/4 * r(Q^*R - R^*Q + S^*P - P^*S) \\ &\leq 1/4 * r(P^*P + Q^*Q + R^*R + S^*S) \\ &= 1/4 * r(4I) = 1. \end{aligned} \tag{2.11}$$

These proves assertion (2). □

Remark. Theorem 2.3 generalizes the Theorem 7 in [5].

Theorem 2.4. *Equation* (1.1) *has a positive definite solution X if and only if A, B admit the following factorization:*

$$A = W^*Z_1, \quad B = W^*Z_2, \tag{2.12}$$

where W is a nonsingular matrix and the columns of $\begin{pmatrix} W \\ Z_1 \\ Z_2 \end{pmatrix}$ *are orthonormal. In this case $X = W^*W$ is a solution of equation* (1.1).

Proof. If equation (1.1) has a positive definite solution X, then $X = W^*W$ for some nonsingular matrix W. Rewrite equation (1.1) as

$$\begin{gathered} W^*W + A^*(W^*W)^{-1}A + B^*(W^*W)^{-1}B = I \\ W^*W + (W^{-*}A)^*(W^{-*}A) + (W^{-*}B)^*(W^{-*}B) = I \end{gathered}$$

or equivalently

$$\begin{pmatrix} W \\ W^{-*}A \\ W^{-*}B \end{pmatrix}^* \begin{pmatrix} W \\ W^{-*}A \\ W^{-*}B \end{pmatrix} = I. \tag{2.13}$$

Let $W^{-*}A = Z_1$, $W^{-*}B = Z_2$, then $A = W^*Z_1$, $B = W^*Z_2$ and (2.13) means that the columns $\begin{pmatrix} W \\ Z_1 \\ Z_2 \end{pmatrix}$ are orthonormal. Conversely, suppose that A, B has the decomposition (2.12). Set $X = W^*W$, then

$$\begin{aligned} X + A^*X^{-1}A + B^*X^{-1}B &= W^*W + (W^*Z_1)^*(W^*W)^{-1}(W^*Z_1) \\ &\quad + (W^*Z_2)^*(W^*W)^{-1}(W^*Z_2) \\ &= W^*W + Z_1^*Z_1 + Z_2^*Z_2 = I, \end{aligned}$$

that is $X = W^*W$ being a positive definite solution of equation (1.1). □

Consider the following two equations,

$$x^2 - x + \lambda_{\max}(A^*A) + \lambda_{\max}(B^*B) = 0, \tag{2.14}$$

$$x^2 - x + \lambda_{\min}(A^*A) + \lambda_{\min}(B^*B) = 0. \tag{2.15}$$

If

$$\lambda_{\max}(A^*A) + \lambda_{\max}(B^*B) < \frac{1}{4}, \tag{2.16}$$

then equation (2.14) has two positive real roots $\alpha_2 < \beta_1$, equation (2.15) has two positive real roots $\alpha_1 < \beta_2$. Easily prove that

$$0 < \alpha_1 \leq \alpha_2 < \frac{1}{2} < \beta_1 \leq \beta_2. \tag{2.17}$$

We define some matrix sets as follows,

$$\begin{aligned} \varphi_1 &= \{X^* = X \mid 0 < X < \alpha_1 I\}, \\ \varphi_2 &= \{X^* = X \mid \alpha_1 I \leq X \leq \alpha_2 I\}, \\ \varphi_3 &= \{X^* = X \mid \alpha_2 I < X < \beta_1 I\}, \\ \varphi_4 &= \{X^* = X \mid \beta_1 I \leq X \leq \beta_2 I\}, \\ \varphi_5 &= \{X^* = X \mid \beta_2 I < X\}. \end{aligned}$$

Theorem 2.5. *Suppose that A, B satisfy* (2.16)*, then equation* (1.1) *has*

(1) *positive definite solutions in* φ_4*,*

(2) *no positive definite solution in* $\varphi_1, \varphi_3, \varphi_5$*.*

Proof. Consider the mapping $\phi_1 : \phi_1(X) = I - A^*X^{-1}A - B^*X^{-1}B$, which is continuous in φ_4. Obviously, φ_4 is a bounded closed convex set. If $X \in \varphi_4$, then

$$\begin{aligned}
\lambda_{\min}(\phi_1(X)) &= \lambda_{\min}(I - A^*X^{-1}A - B^*X^{-1}B)\\
&\geq \lambda_{\min}(I - (A^*A + B^*B)/\beta_1)\\
&\geq 1 - (\lambda_{\max}(A^*A) + \lambda_{\max}(B^*B))/\beta_1\\
&= \beta_1,\\
\lambda_{\max}(\phi_1(X)) &= \lambda_{\max}(I - A^*X^{-1}A - B^*X^{-1}B)\\
&\leq \lambda_{\max}(I - (A^*A + B^*B)/\beta_2)\\
&\leq 1 - (\lambda_{\min}(A^*A) + \lambda_{\min}(B^*B))/\beta_2\\
&= \beta_2.
\end{aligned}$$

Hence ϕ_1 maps φ_4 into itself. By Brouwer fixed point theorem, ϕ_1 has fixed point in φ_4. Thus equation (1.1) has positive definite solution in φ_4.

Assume X is the positive definite solution of (1.1), then

$$\begin{aligned}
\lambda_{\min}(X) &= \lambda_{\min}(I - A^*X^{-1}A - B^*X^{-1}B)\\
&\geq 1 - \lambda_{\max}(A^*X^{-1}A) - \lambda_{\max}(B^*X^{-1}B)\\
&\geq 1 - \lambda_{\max}(A^*A)/\lambda_{\min}(X) - \lambda_{\max}(B^*B)/\lambda_{\min}(X),
\end{aligned}$$

that is, $\lambda^2_{\min}(X) - \lambda_{\min}(X) + \lambda_{\max}(A^*A) + \lambda_{\max}(B^*B) \geq 0$. So, $\lambda_{\min}(X) \leq \alpha_2$ or $\lambda_{\min}(X) \geq \beta_1$, and equation (1.1) has no positive definite solution in φ_3.

$$\begin{aligned}
\lambda_{\max}(X) &= \lambda_{\max}(I - A^*X^{-1}A - B^*X^{-1}B)\\
&\leq 1 - \lambda_{\min}(A^*X^{-1}A) - \lambda_{\min}(B^*X^{-1}B)\\
&\leq 1 - \lambda_{\min}(A^*A)/\lambda_{\max}(X) - \lambda_{\min}(B^*B)/\lambda_{\max}(X),
\end{aligned}$$

that is, $\lambda^2_{\max}(X) - \lambda_{\max}(X) + \lambda_{\min}(A^*A) + \lambda_{\min}(B^*B) \leq 0$. So, $\alpha_1 \leq \lambda_{\max}(X) \leq \beta_2$, and equation (1.1) has no positive definite solution in $\varphi_1,\ \varphi_5$. □

Theorem 2.6. *If A and B satisfy the following inequality*

$$||A^T \otimes A^* + B^T \otimes B^*||_2 < \alpha_1^2, \tag{2.18}$$

where α_1 *is the smaller positive root of* (2.15)*, then equation* (1.1) *has a unique positive definite solution in* φ_2*.*

Proof. $\forall X, Y \in \varphi_2$, we get $X^{-1} \leq \frac{I}{\alpha_1}$, $Y^{-1} \leq \frac{I}{\alpha_1}$ and

$$\begin{aligned} ||Y^{-1} - X^{-1}||_F &= ||X^{-1}(X - Y)Y^{-1}||_F \\ &= ||(Y^{-1} \otimes X^{-1}) \text{ vec}(X - Y)||_2 \\ &\leq \frac{1}{\alpha_1^2}||X - Y||_F. \end{aligned}$$

Hence,

$$\begin{aligned} ||\phi_1(X) - \phi_1(Y)||_F &= ||A^*(Y^{-1} - X^{-1})A + B^*(Y^{-1} - X^{-1})B||_F \\ &= ||(A^T \otimes A^* + B^T \otimes B^*)\text{vec}(Y^{-1} - X^{-1})||_2 \\ &\leq ||(A^T \otimes A^* + B^T \otimes B^*)||_2 \, ||(Y^{-1} - X^{-1})||_F \\ &\leq ||(A^T \otimes A^* + B^T \otimes B^*)||_2 \, ||X - Y||_F/\alpha_1^2 \\ &< ||X - Y||_F. \end{aligned}$$

where the mapping ϕ_1 is the same as in Theorem 2.5. Therefore the mapping ϕ_1 is a contraction mapping. By the contraction mapping principle, the mapping ϕ_1 has a unique fixed point in φ_2. □

Theorem 2.7. *Assume A, B satisfy* (2.16)*, then equation* (1.1) *has a unique positive definite solution in φ_4.*

Proof. By Theorem 2.4, the set of solution, called S, is not empty in φ_4. Suppose $X, Y \in S$ and $X \neq Y$, then

$$X - Y = A^*(Y^{-1} - X^{-1})A + B^*(Y^{-1} - X^{-1})B. \tag{2.19}$$

Since

$$\begin{aligned} ||Y^{-1} - X^{-1}||_F &= ||X^{-1}(X - Y)Y^{-1}||_F \\ &= ||(Y^{-1} \otimes X^{-1}) \text{ vec}(X - Y)||_2 \\ &\leq \frac{1}{\beta_1^2}||X - Y||_F, \end{aligned} \tag{2.20}$$

$$\begin{aligned} ||X - Y||_F &\leq (||A||_2^2 + ||B||_2^2)||Y^{-1} - X^{-1}||_F \\ &\leq ((||A||_2^2 + ||B||_2^2)/\beta_1^2)||X - Y||_F \\ &< ||X - Y||_F, \end{aligned}$$

which is a contradiction. Thus equation (1.1) has a unique positive solution in φ_4. □

3 Iterative methods

In this section, we consider two iterative algorithms for finding the positive definite solution of equation (1.1).

The following lemma considers the linear matrix equation of the type

$$X - A_1^* X A_1 - A_2^* X A_2 = Q, \tag{3.1}$$

where Q is a positive semi-definite matrix, $A_1,\ A_2$ are square matrices and X is unknown matrix.

Lemma 3.1 [15]. *If there exists a positive definite matrix $\tilde{Q}$ satisfying $\tilde{Q} - A_1^* \tilde{Q} A_1 - A_2^* \tilde{Q} A_2 > 0$, then equation* (3.1) *has a unique solution which is positive semi-definite.*

Algorithm 3.1. *(Basic fixed point iteration)*

$$\begin{cases} X_0 = \delta I, \delta \in \left[\frac{1}{2},\ 1\right], \\ X_{n+1} = I - A^* X_n^{-1} A - B^* X_n^{-1} B, \end{cases}$$

Theorem 3.1. *Assume that equation* (1.1) *has a positive definite solution, then the Algorithm* 3.1 *with*

$$\delta(1-\delta) \leq \lambda_{\min}(A^*A)+\lambda_{\min}(B^*B), \quad and \quad \delta^2 > \lambda_{\max}(A^*A)+\lambda_{\max}(B^*B) \tag{3.2}$$

defines a monotonically decreasing matrix sequence $\{X_n\}$ which converges to the positive definite solution X of equation (1.1).

Proof. Let X_l be a positive definite solution of equation (1.1). We first show by induction that $X_k \geq X_l$ for any k. The formulas (3.2) implies that $A^*A + B^*B \geq \delta(1-\delta)$ and

$$\begin{aligned} X_0 + A^* X_0^{-1} A + B^* X_0^{-1} B &= \delta I + \frac{1}{\delta}(A^*A + B^*B) \\ &\geq \delta I + (1-\delta) I \\ &= X_l + A^* X_l^{-1} A + B^* X_l^{-1} B. \end{aligned}$$

Hence

$$X_0 - X_l - A^* X_l^{-1}(X_0 - X_l) X_0^{-1} A - B^* X_l^{-1}(X_0 - X_l) X_0^{-1} B \geq 0$$

and

$$\begin{aligned} &X_0 - X_l - [A^*X_0^{-1}(X_0 - X_l)X_0^{-1}A + A^*X_0^{-1}(X_0 - X_l)X_l^{-1}(X_0 - X_l)X_0^{-1}A] \\ &\quad -[B^*X_0^{-1}(X_0 - X_l)X_0^{-1}B + B^*X_0^{-1}(X_0 - X_l)X_l^{-1}(X_0 - X_l)X_0^{-1}B] \geq 0. \end{aligned} \tag{3.3}$$

Since $X_l > 0$, we have that

$$A^*X_0^{-1}(X_0 - X_l)X_l^{-1}(X_0 - X_l)X_0^{-1}A \geq 0, \quad \text{and}$$
$$B^*X_0^{-1}(X_0 - X_l)X_l^{-1}(X_0 - X_l)X_0^{-1}B \geq 0.$$

From (3.3), there exists a matrix $C \geq 0$ such that

$$X_0 - X_l - A^*X_0^{-1}(X_0 - X_l)X_0^{-1}A - B^*X_0^{-1}(X_0 - X_l)X_0^{-1}B = C. \tag{3.4}$$

Let $Y = X_0 - X_l$. Equation (3.4) is equivalent to

$$Y - A^*X_0^{-1}YX_0^{-1}A - B^*X_0^{-1}YX_0^{-1}B = C. \tag{3.5}$$

We take $\tilde{Y} = X_0^2$. We get $A^*A + B^*B < \delta^2 I$ from (3.2). Therefore,

$$\tilde{Y} - A^*X_0^{-1}\tilde{Y}X_0^{-1}A - B^*X_0^{-1}\tilde{Y}X_0^{-1}B = \delta^2 I - A^*A - B^*B > 0.$$

By Lemma 3.1, we get that equation (3.5) has a unique positive semi-definite solution Y. Hence $Y = X_0 - X_l \geq 0$, that is $X_0 \geq X_l$. Now assume that $X_k \geq X_l$ holds for $k = n$. Then

$$X_{k+1} - X_l = A^*\big(X_l^{-1} - X_k^{-1}\big)A + B^*\big(X_l^{-1} - X_k^{-1}\big)B, \tag{3.6}$$

since $X_k \geq X_l > 0$, it is obvious that $X_{k+1} \geq X_l$.

Next we show that $\{X_k\}$ is a monotonically decreasing sequence. First, we consider that

$$\begin{aligned} X_0 - X_1 &= X_0 - (I - A^*X_0^{-1}A - B^*X_0^{-1}B) \\ &= (\delta - 1)I + \frac{1}{\delta}(A^*A + B^*B) \\ &\geq (\delta - 1)I + \frac{1}{\delta}(\lambda_{\min}(A^*A) + \lambda_{\min}(B^*B))I \\ &\geq 0, \end{aligned}$$

So the statement holds for $k = 0$, Next, assume that $X_k - X_{k+1} \geq 0$ for $k = n$. Using the induction argument and the fact that $X_k > 0$ for any k, we have

$$X_{n+1} - X_{n+2} = A^*\big(X_{n+1}^{-1} - X_n^{-1}\big)A + B^*\big(X_{n+1}^{-1} - X_n^{-1}\big)B \geq 0. \tag{3.7}$$

So $\{X_n\}$ is a monotonically decreasing sequence. Combination of both results yields that $\{X_n\}$ converges to a matrix X which satisfies $X = I - A^*X^{-1}A - B^*X^{-1}B$ and $X \geq X_l$. □

Algorithm 3.2. *(Inversion free variant of basic fixed point iteration)*

$$\begin{cases} X_0 = I,\, Y_0 = I, \\ X_{n+1} = I - A^* Y_n A - B^* Y_n B, \\ Y_{n+1} = Y_n(2I - X_n Y_n). \end{cases}$$

Lemma 3.2 [8]. *Let C and P be Hermitian matrices of the same order and let $P > 0$, then*

$$CPC + P^{-1} \geq 2C. \tag{3.8}$$

Theorem 3.2. *Assume that equation* (1.1) *has a positive definite solution, then the Algorithm* 3.2 *defines a monotonically decreasing matrix sequence $\{X_n\}$ converging to the positive definite matrix X which is a solution of equation* (1.1).

Proof. Let X_l be a positive definite solution of equation (1.1). We show by induction that $\{X_n\}$ is a monotonically decreasing sequence bounded from below. We first prove by induction that $X_n \geq X_l$, $Y_n \geq Y_{n-1}$. Since X_l is a positive definite solution, $X_l = I - A^* X_l^{-1} A - B^* X_l^{-1} B$.

By Algorithm 3.2, it is easy to compute that $X_0 = I \geq X_l$, $X_1 = I - A^* Y_0 A - B^* Y_0 B = I - A^* A - B^* B$, $Y_0 = I$, $Y_1 = Y_0(2I - X_0 Y_0) = I$, $X_2 = I - A^* Y_1 A - B^* Y_1 B = I - A^* A - B^* B$, $Y_2 = Y_1(2I - X_1 Y_1) = I + A^* A + B^* B$. Hence, $X_1 = X_2 = I - A^* A - B^* B \geq I - A^* X_l^{-1} A - B^* X_l^{-1} B = X_l$, and $Y_0 = Y_1 \leq I + A^* A + B^* B = Y_2$. Now assume that

$$X_k \geq X_l, \quad Y_k \geq Y_{k-1}, \tag{3.9}$$

for all $k \leq n$,$n \geq 2$. Using Lemma 3.2 we have

$$X_n^{-1} \geq 2Y_{n-1} - Y_{n-1} X_n Y_{n-1}. \tag{3.10}$$

It is obvious that

$$X_{n-1} - X_n = A^*(Y_{n-1} - Y_{n-2})A + B^*(Y_{n-1} - Y_{n-2})B. \tag{3.11}$$

By the assumption (3.9), we have $Y_{n-1} \geq Y_{n-2}$, so $X_{n-1} \geq X_n$, therefore

$$2Y_{n-1} - Y_{n-1} X_n Y_{n-1} \geq 2Y_{n-1} - Y_{n-1} X_{n-1} Y_{n-1} = Y_n. \tag{3.12}$$

We have $X_n^{-1} \geq Y_n$ or $Y_n^{-1} \geq X_n$ from (3.10)–(3.12). Thus

$$\begin{aligned} X_{n+1} &= I - A^* Y_n A - B^* Y_n B \\ &\geq I - A^* X_n^{-1} A - B^* X_n^{-1} B \\ &\geq I - A^* X_l^{-1} A - B^* X_l^{-1} B = X_l. \end{aligned}$$

Rewrite the second formula of Algorithm 3.2 as

$$Y_{n+1} - Y_n = Y_n(Y_n^{-1} - X_n)Y_n. \tag{3.13}$$

hence $Y_{n+1} \geq Y_n$, this completes the induction. From above process, we know easily that $\{X_n\}$ is monotone decreasing sequence and bounded from below X_l, $\{Y_n\}$ is monotone increasing sequence and bounded from above X_l^{-1}. Thus $\lim_{n\to\infty} X_n = X$ and $\lim_{n\to\infty} Y_n = Y$ exist. Taking limits in Algorithm 3.2 gives $Y = X^{-1}$ and $X = I - A^*X^{-1}A - B^*X^{-1}B$. From $X_n \geq X_l$, we have $X \geq X_l$, that is $\lim_{n\to\infty} X_n = X \geq X_l$. □

4 Numerical results

In this section, we report some numerical examples to compute the positive definite solution of equation (1.1) by Algorithm 3.1 and Algorithm 3.2.

Example 4.1. Consider the equation (1.1) with

$$A = \begin{pmatrix} 0.010 & -0.150 & -0.259 \\ 0.015 & 0.212 & -0.064 \\ 0.025 & -0.069 & 0.138 \end{pmatrix}, \quad B = \begin{pmatrix} 0.160 & -0.025 & 0.020 \\ -0.025 & -0.288 & -0.060 \\ 0.004 & -0.016 & -0.120 \end{pmatrix}.$$

Algorithm 3.1 with $\delta = 1$ needs 12 iterations to obtain the solution

$$X = \begin{pmatrix} 0.9717897903 & -0.0049365696 & -0.0046035965 \\ -0.0049365696 & 0.8144332065 & -0.0388316596 \\ -0.0046035965 & -0.0388316596 & 0.8835618713 \end{pmatrix},$$

$$||X + A^*X^{-1}A + B^*X^{-1}B - I||_\infty = 2.72e - 010.$$

Algorithm 3.1 with $\delta = 0.85$ needs 10 iterations to get the solution

$$X = \begin{pmatrix} 0.9717897903 & -0.0049365696 & -0.004603596 \\ -0.0049365696 & 0.8144332064 & -0.0388316596 \\ -0.0046035965 & -0.0388316596 & 0.8835618713 \end{pmatrix},$$

$$||X + A^*X^{-1}A + B^*X^{-1}B - I||_\infty = 2.02e - 010.$$

Algorithm 3.2 needs 12 iterations to obtain the solution

$$X = \begin{pmatrix} 0.9717897903 & -0.0049365696 & -0.0046035965 \\ -0.0049365696 & 0.8144332068 & -0.0388316596 \\ -0.0046035965 & -0.0388316596 & 0.8835618713 \end{pmatrix},$$

$$||X + A^*X^{-1}A + B^*X^{-1}B - I||_\infty = 5.95e - 010.$$

Example 4.2. Consider the equation (1.1) with

$$A = 1/680 * \begin{pmatrix} 40 & 25 & 23 & 35 & 66 \\ 25 & 32 & 27 & 45 & 21 \\ 23 & 27 & 28 & 16 & 24 \\ 35 & 45 & 16 & 52 & 65 \\ 66 & 21 & 24 & 65 & 69 \end{pmatrix}, B = 1/400 * \begin{pmatrix} 11 & 21 & 23 & 25 & 32 \\ 21 & 31 & 60 & 42 & 33 \\ 23 & 60 & 34 & 18 & 26 \\ 25 & 42 & 18 & 44 & 30 \\ 32 & 33 & 26 & 30 & 50 \end{pmatrix}.$$

Algorithm 3.1 with $\delta = 1$ needs 40 iterations to obtain the solution

$$X = \begin{pmatrix} 0.9437370835 & -0.0642332338 & -0.0530308768 & -0.0690830561 & -0.0772109025 \\ -0.0642332338 & 0.9063186053 & -0.0738559550 & -0.0832893164 & -0.0906944974 \\ -0.0530308768 & -0.0738559550 & 0.9297460796 & -0.0716730460 & -0.0763116859 \\ -0.0690830561 & -0.0832893164 & -0.0716730460 & 0.9080246681 & -0.0969684430 \\ -0.0772109025 & -0.0906944974 & -0.0763116859 & -0.0969684430 & 0.8888791608 \end{pmatrix},$$

$$||X + A^*X^{-1}A + B^*X^{-1}B - I||_\infty = 1.04e - 009.$$

Algorithm 3.1 with $\delta = 0.61$ needs 25 iterations to get the solution

$$X = \begin{pmatrix} 0.9437370804 & -0.0642332375 & -0.0530308799 & -0.0690830601 & -0.0772109069 \\ -0.0642332375 & 0.9063186003 & -0.0738559593 & -0.0832893213 & -0.09069450285 \\ -0.0530308799 & -0.0738559593 & 0.9297460759 & -0.0716730502 & -0.0763116905 \\ -0.0690830601 & -0.0832893213 & -0.0716730502 & 0.9080246629 & -0.0969684486 \\ -0.0772109069 & -0.0906945028 & -0.0763116905 & -0.0969684486 & 0.8888791546 \end{pmatrix},$$

$$||X + A^*X^{-1}A + B^*X^{-1}B - I||_\infty = 8.32e - 009.$$

Algorithm 3.2 needs 40 iterations to obtain the solution

$$X = \begin{pmatrix} 0.9437370837 & -0.0642332336 & -0.0530308766 & -0.0690830559 & -0.0772109023 \\ -0.0642332336 & 0.9063186055 & -0.0738559548 & -0.0832893162 & -0.0906944972 \\ -0.0530308766 & -0.0738559548 & 0.9297460797 & -0.0716730458 & -0.0763116857 \\ -0.0690830559 & -0.0832893162 & -0.0716730458 & 0.9080246683 & -0.0969684427 \\ -0.0772109023 & -0.0906944972 & -0.0763116857 & -0.0969684427 & 0.8888791610 \end{pmatrix},$$

$$||X + A^*X^{-1}A + B^*X^{-1}B - I||_\infty = 1.45e - 009.$$

The above examples show that the different choices of δ occur different numerical results for Algorithm 3.1, and both Algorithm 3.1 and 3.2 are numerically reliable methods for computing the positive definite solution X.

5 Conclusion

In this paper we consider the nonlinear matrix equation

$$X + A^*X^{-1}A + B^*X^{-1}B = I.$$

Conditions for the existence of positive definite of this equation are derived. Two iterative algorithms for obtaining the positive definite solution of the equation are given. Moreover, several numerical results are reported to illustrate the effectiveness of the algorithms.

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