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Surfaces with constant curvature in $S^2 \times R$ and $H^2 \times R$. Height estimates and representation

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Abstract. We obtain optimal height estimates for surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ with constant Gaussian curvature $K(I)$ and positive extrinsic curvature, characterizing the extreme cases as the revolution ones. Moreover, we get a representation for surfaces with constant Gaussian curvature in such ambient spaces, paying special attention to the cases of $K(I) = 1$ in $\mathbb{S}^2 \times \mathbb{R}$ and $K(I) = -1$ in $\mathbb{H}^2 \times \mathbb{R}$.

Keywords: surfaces with constant curvature, height estimates, representation formula.

Mathematical subject classification: 53C42.

1 Introduction

In this work we deal with surfaces of constant Gaussian curvature in the ambient spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$. This kind of surfaces was already studied by the authors in [AEG], where the following question posed by Alencar, Do Carmo and Tribuzy in [ACT] was studied:

> What are the closed surfaces of constant Gaussian curvature in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$?

This question became natural after the classification of the surfaces with constant mean curvature and genus zero due to Abresch and Rosenberg in [AR]. In particular, it was shown in [AEG] that the only complete surfaces of constant Gaussian curvature $K(I) > 0$ in $\mathbb{H}^2 \times \mathbb{R}$ ($K(I) > 1$ in $\mathbb{S}^2 \times \mathbb{R}$) are rotational surfaces. In addition, the authors proved the non existence of complete surfaces with constant Gaussian curvature $K(I) < -1$ in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$.

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Here we focus on two aspects of the surfaces of constant Gaussian curvature. On the one hand, in Section 3 we obtain optimal height estimates for these surfaces when they have positive extrinsic curvature, characterizing the rotational complete examples as the only surfaces for which the bounds are attained. In general, height estimates for surfaces with any constant curvature (mean, Gaussian, Weingarten surfaces, etc.) are a powerful tool to get geometrical information about such surfaces (see, for instance, [AEG2], [H], [HLR], [KKMS], [KKS] and [RS]).

On the other hand, we provide a representation for the surfaces of constant Gaussian curvature $K(I)$ in terms of the height function, h, and the last coordinate of its unit normal, ν (see Sections 4 and 5). We pay special attention to the particular cases when $K(I)$ agrees with the sectional curvature of the base of the ambient space, that is, $K(I) = 1$ in $\mathbb{S}^2 \times \mathbb{R}$ (resp. $K(I) = -1$ in $\mathbb{H}^2 \times \mathbb{R}$), because then the integrability equations reduce to an elliptic (resp. hyperbolic) sinh-Gordon equation. We also provide complete examples of surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with $K(I) = -1$ by means of its representation.

2 Preliminaries

Let us consider the homogeneous space $\mathbb{S}^2 \times \mathbb{R}$ as the hypersurface of the usual Euclidean 4-space $\mathbb{R}^4$ given by

$$\mathbb{S}^2 \times \mathbb{R} = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

Let us also denote $\mathbb{H}^2 \times \mathbb{R}$ as the Riemannian submanifold of the Lorentzian 4-space $\mathbb{L}^4$, with induced metric $-x_1^2 + x_2^2 + x_3^2 + x_4^2$, given by

$$\mathbb{H}^2 \times \mathbb{R} = \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ -x_1^2 + x_2^2 + x_3^2 = -1, \ x_1 > 0\right\}.$$

As long as we do not need to distinguish between them explicitly, we will embrace both $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ under the usual notation $\mathbb{M}^2(\varepsilon) \times \mathbb{R}$, with $\varepsilon = 1, -1$, where $\mathbb{M}^2(1) = \mathbb{S}^2$ and $\mathbb{M}^2(-1) = \mathbb{H}^2$.

Let η be the unit normal of $\mathbb{M}^2(\varepsilon) \times \mathbb{R}$ in the ambient space $\mathbb{R}^4$ or $\mathbb{L}^4$ given by

$$\eta(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, 0).$$

Let us consider S as an orientable surface and $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ an immersion with unit normal vector field N and associated shape operator $\mathcal{A}$. We will denote by h the height function on S, that is, the fourth coordinate immersion of ψ, and by ν the fourth coordinate of the unit normal N.

Unlike the case of immersions in a space form, the immersion ψ is not totally determined by the Gauss and Codazzi equations. More precisely, the compatibility equations for such an immersion can be written as (see [Da])

$$\text{Gauss} \quad K(I) = K + \varepsilon \nu^2 \tag{2.1}$$

$$\text{Codazzi} \quad \nabla_X \mathcal{A} Y - \nabla_Y \mathcal{A} X - \mathcal{A}[X, Y] = \varepsilon \nu (\langle Y, T \rangle X - \langle X, T \rangle Y) \tag{2.2}$$

$$\nabla_X T = \nu \mathcal{A} X \tag{2.3}$$

$$d\nu(X) = -\langle \mathcal{A} X, T \rangle \tag{2.4}$$

$$||T||^2 + \nu^2 = 1 \tag{2.5}$$

for all differentiable vector fields X, Y on S. Here, ∇ is the induced Levi-Civita connection on S, T is the gradient of the height function h, $||T||$ its modulus for the induced metric, $K(I)$ is the Gaussian curvature and K the extrinsic curvature of the immersion.

Throughout this paper we will deal with surfaces in $\mathbb{M}^2(\varepsilon) \times \mathbb{R}$ with constant Gaussian curvature. We will refer to such surfaces as $K(I)-$*surfaces*.

3 Height estimates for $K(I)$-surfaces

Let S be a surface and $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ an immersion with positive extrinsic curvature, $K > 0$. Then its second fundamental form, II, is definite. We will choose N such that II is positive definite, that is, N is the inner normal.

Let z be a conformal complex parameter for the Riemannian metric II. Then, we can write

$$\begin{aligned} I &= \langle d\psi, d\psi \rangle = E dz^2 + 2F\,|dz|^2 + \bar{E} d\bar{z}^2, \\ II &= \langle d\psi, -dN \rangle = 2\rho\,|dz|^2 . \end{aligned}$$

In order to make some computations on S with respect to the second fundamental form, we will rewrite the compatibility equations (2.1)-(2.5) in terms of the parameter z.

First, observe that the extrinsic curvature is given by

$$K = -\frac{\rho^2}{D}, \tag{3.1}$$

where $D = |E|^2 - F^2 < 0$.

We can put

$$T = \frac{1}{D}\big(\alpha \partial_z + \bar{\alpha} \partial_{\bar{z}}\big) \tag{3.2}$$

where

$$\alpha = \bar{E}h_z - Fh_{\bar{z}} \tag{3.3}$$

and, for instance, ∂_z denotes the field $\frac{\partial}{\partial z}$.

Now, taking $X = \partial_{\bar{z}}$ and $Y = \partial_z$ in (2.2), we have

$$\nabla_{\partial_{\bar{z}}} \mathcal{A}\partial_z - \nabla_{\partial_z} \mathcal{A}\partial_{\bar{z}} = \varepsilon\nu\big(h_z\partial_{\bar{z}} - h_{\bar{z}}\partial_z\big).$$

Thus, in order to obtain the information given by the previous equation (or equivalently (2.2)) we consider, for instance, the inner product with $\partial_{\bar{z}}$

$$\partial_{\bar{z}}\langle \mathcal{A}\partial_z, \partial_{\bar{z}}\rangle - \langle \mathcal{A}\partial_z, \nabla_{\partial_{\bar{z}}}\partial_{\bar{z}}\rangle + \langle \mathcal{A}\partial_{\bar{z}}, \nabla_{\partial_{\bar{z}}}\partial_z\rangle = \varepsilon\nu(h_z\bar{E} - h_{\bar{z}}F) = \varepsilon\nu\alpha.$$

Therefore, (2.2) is equivalent to

$$\frac{\rho_{\bar{z}}}{\rho} + \left(\Gamma^1_{12} - \Gamma^2_{22}\right) = \varepsilon\alpha\frac{\nu}{\rho},$$

where Γ^k_{ij}, $i, j, k = 1, 2$, stand for the Christoffel symbols associated to z. Thus, for instance, the tangent part of $\nabla_{\partial_z}\partial_z$ is given by $\Gamma^1_{11}\partial_z + \Gamma^2_{11}\partial_{\bar{z}}$ and the tangent part of $\nabla_{\partial_z}\partial_{\bar{z}}$ is $\Gamma^1_{12}\partial_z + \Gamma^2_{12}\partial_{\bar{z}}$.

On the other hand, taking $X = \partial_z$, (2.3) becomes

$$\nabla_{\partial_z} T = \nu\mathcal{A}\partial_z. \tag{3.4}$$

Considering the inner product with $\partial_{\bar{z}}$

$$\begin{aligned}\langle \nabla_{\partial_z} T, \partial_{\bar{z}}\rangle &= \partial_z\langle T, \partial_{\bar{z}}\rangle - \langle T, \nabla_{\partial_z}\partial_{\bar{z}}\rangle = h_{z\bar{z}} - \Gamma^1_{12}h_z - \Gamma^2_{12}h_{\bar{z}}\\ \nu\langle \mathcal{A}\partial_{\bar{z}}, \partial_z\rangle &= \nu\rho\end{aligned}$$

and so

$$h_{z\bar{z}} = \Gamma^1_{12}h_z + \Gamma^2_{12}h_{\bar{z}} + \nu\rho.$$

And considering the inner product of (3.4) with ∂_z one gets

$$h_{zz} = \Gamma^1_{11}h_z + \Gamma^2_{11}h_{\bar{z}}.$$

The two previous equations are equivalent to (2.3).

Now, writing (2.4) for $X = \partial_{\bar{z}}$, and using (3.1) and (3.2), we have

$$\nu_{\bar{z}} = -\langle \mathcal{A}\partial_{\bar{z}}, T\rangle = -\frac{\alpha}{D}\langle \mathcal{A}\partial_{\bar{z}}, \partial_z\rangle = -\frac{\alpha}{D}\rho = \frac{\alpha K}{\rho},$$

expression equivalent to (2.4).

Finally, from (3.2) and (3.3) we have

$$\langle T, T\rangle = \frac{1}{D}(\alpha h_z + \overline{\alpha}h_{\bar{z}}).$$

Bearing in mind the above comments, we get the following:

Lemma 3.1. *The compatibility equations for an immersion ψ with positive extrinsic curvature can be written in a conformal parameter for the second fundamental form as*

$$\text{Gauss} \qquad K(I) = K + \varepsilon\nu^2 \tag{3.5}$$

$$\text{Codazzi} \qquad \frac{\rho_{\bar z}}{\rho} + (\Gamma_{12}^1 - \Gamma_{22}^2) = \varepsilon\alpha\frac{\nu}{\rho} \tag{3.6}$$

$$h_{zz} = \Gamma_{11}^1 h_z + \Gamma_{11}^2 h_{\bar z} \tag{3.7}$$

$$h_{z\bar z} = \Gamma_{12}^1 h_z + \Gamma_{12}^2 h_{\bar z} + \nu\rho \tag{3.8}$$

$$\nu_{\bar z} = \frac{\alpha K}{\rho} \tag{3.9}$$

$$\frac{1}{D}(\alpha h_z + \overline{\alpha} h_{\bar z}) + \nu^2 = 1. \tag{3.10}$$

We will focus our attention on the surfaces with constant Gaussian curvature. Thus, from now on we will suppose that ψ has constant Gaussian curvature $K(I)$. In order to obtain some estimates for these surfaces we compute the Laplacian of h and ν with respect to the second fundamental form.

Proposition 3.1. *Let $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ be an immersion with constant Gaussian curvature $K(I)$ and positive extrinsic curvature. Let us consider a conformal parameter z for its second fundamental form such that*

$$I = \langle d\psi, d\psi\rangle = E dz^2 + 2F\,|dz|^2 + \bar E d\bar z^2,$$
$$II = \langle d\psi, -dN\rangle = 2\rho\,|dz|^2, \quad \rho > 0.$$

Then one has

$$h_{z\bar z} = (K(I) - \varepsilon)\frac{\nu\rho}{K}, \tag{3.11}$$

$$\nu_{z\bar z} = \varepsilon\nu\frac{|\alpha|^2}{D} - \nu F K. \tag{3.12}$$

Proof. A straightforward computation (see [Mi, Lemma 8]) gives

$$\Gamma_{12}^1 + \Gamma_{22}^2 = \frac{D_{\bar z}}{2D}. \tag{3.13}$$

Then (3.6) and (3.13) allow us to obtain

$$\frac{D_{\bar z}}{2D} - \frac{\rho_{\bar z}}{\rho} = 2\Gamma_{12}^1 - \varepsilon\alpha\frac{\nu}{\rho}.$$

Now, differentiating (3.1) with respect to $\bar{z}$ it follows

$$\frac{D_{\bar{z}}}{2D} - \frac{\rho_{\bar{z}}}{\rho} = -\frac{K_{\bar{z}}}{2K} \tag{3.14}$$

and so we get

$$\Gamma_{12}^{1} = -\frac{K_{\bar{z}}}{4K} + \varepsilon\alpha\frac{\nu}{2\rho}.$$

On the other hand, since the Gaussian curvature of ψ is constant, differentiating the Gauss equation (3.5) we have, using (3.9), that

$$K_{\bar{z}} = -2\varepsilon\nu\frac{\alpha K}{\rho}. \tag{3.15}$$

Therefore,

$$\Gamma_{12}^{1} = \varepsilon\frac{\nu\alpha}{\rho}. \tag{3.16}$$

From (3.8) and bearing in mind that $\overline{\Gamma_{12}^{1}} = \Gamma_{12}^{2}$ one has

$$h_{z\bar{z}} = \varepsilon\frac{\nu\alpha}{\rho}h_z + \varepsilon\frac{\nu\overline{\alpha}}{\rho}h_{\bar{z}} + \nu\rho = \nu\rho\left(1 + \varepsilon\frac{\alpha h_z + \overline{\alpha}h_{\bar{z}}}{\rho^2}\right)$$

and using (3.1) and (3.10)

$$h_{z\bar{z}} = \nu\rho\left(1 + \varepsilon\frac{D(1-\nu^2)}{\rho^2}\right) = \frac{\nu\rho}{K}\left(K - \varepsilon(1-\nu^2)\right).$$

Thus (3.11) follows from the Gauss equation (3.5).

Now, if we differentiate (3.9) and use (3.1), (3.14) and (3.15) then one has

$$\nu_{z\bar{z}} = \frac{K}{\rho}\alpha_z + K_z\frac{\alpha}{\rho} - K\alpha\frac{\rho_z}{\rho^2} = \frac{K}{\rho}\alpha_z + \varepsilon\nu\frac{|\alpha|^2}{D} - \frac{\alpha K}{\rho}\frac{D_z}{2D}. \tag{3.17}$$

From (3.3), (3.7) and (3.8) one gets

$$\begin{aligned}
\alpha_z &= 2\langle\psi_{z\bar{z}}, \psi_{\bar{z}}\rangle h_z + \bar{E}h_{zz} - \langle\psi_{zz}, \psi_{\bar{z}}\rangle h_{\bar{z}} - \langle\psi_z, \psi_{z\bar{z}}\rangle h_{\bar{z}} - Fh_{z\bar{z}} \\
&= 2(\Gamma_{12}^{1}F + \Gamma_{12}^{2}\bar{E})h_z + \bar{E}(\Gamma_{11}^{1}h_z + \Gamma_{11}^{2}h_{\bar{z}}) - (\Gamma_{11}^{1}F + \Gamma_{11}^{2}\bar{E})h_{\bar{z}} \\
&\quad -(\Gamma_{12}^{1}E + \Gamma_{12}^{2}F)h_{\bar{z}} - F(\Gamma_{12}^{1}h_z + \Gamma_{12}^{2}h_{\bar{z}} + \rho\nu) \\
&= \alpha(\Gamma_{12}^{2} + \Gamma_{11}^{1}) + \alpha\Gamma_{12}^{2} - \bar{\alpha}\Gamma_{12}^{1} - \nu\rho F.
\end{aligned}$$

We observe that from (3.16) one has $\alpha\Gamma_{12}^{2} - \bar{\alpha}\Gamma_{12}^{1} = 0$ and by conjugating (3.13) one gets $\Gamma_{12}^{2} + \Gamma_{11}^{1} = D_z/(2D)$. Thus,

$$\alpha_z = \alpha\frac{D_z}{2D} - \nu\rho F.$$

Hence, (3.12) follows from (3.17). □

Let us consider the quadratic form

$$Q\,dz^2 = \big((K(I)-\varepsilon)\langle\psi_z,\psi_z\rangle + \varepsilon h_z^2\big)\,dz^2, \tag{3.18}$$

which is well defined for surfaces of constant Gaussian curvature and positive extrinsic curvature [AEG].

From (3.1), (3.3), (3.11) and (3.16) we have

$$\begin{aligned}
Q_{\bar z} &= 2(K(I)-\varepsilon)\langle\psi_{z\bar z},\psi_z\rangle + 2\varepsilon h_z h_{z\bar z}\\
&= 2(K(I)-\varepsilon)\left(\Gamma^1_{12}E + \Gamma^2_{12}F\right) + 2\varepsilon h_z (K(I)-\varepsilon)\frac{\nu\rho}{K}\\
&= 2\varepsilon(K(I)-\varepsilon)\left(\frac{\nu\alpha}{\rho}E + \frac{\nu\bar\alpha}{\rho}F - \frac{\nu D}{\rho}h_z\right)\\
&= 2\varepsilon(K(I)-\varepsilon)\frac{\nu}{\rho}\left(\alpha E + \bar\alpha F - D h_z\right) \;=\; 0.
\end{aligned}$$

That is, $Q\,dz^2$ is a holomorphic quadratic form. This result was proven in [AEG] in order to classify the complete surfaces with constant Gaussian curvature. We also have as a consequence

Corollary 3.1. *Let Σ be a surface satisfying*

1. *Σ is a flat surface in $\mathbb{H}^2\times\mathbb{R}$, which is locally a graph on $\mathbb{H}^2$. Then the projection $\eta : \mathbb{H}^2\times\mathbb{R} \longrightarrow \mathbb{H}^2\times\{0\} \equiv \mathbb{H}^2$ is a harmonic map for the second fundamental form.*
2. *Σ is a surface of constant Gaussian curvature one in $\mathbb{S}^2\times\mathbb{R}$ with $\|T\|\neq 0$ everywhere. Then the height function is harmonic for the second fundamental form.*

Proof. We remark that, from (2.1) and (2.5), a flat surface in $\mathbb{H}^2\times\mathbb{R}$ has positive extrinsic curvature if and only if $\|T\|\neq 1$ everywhere. Or equivalently, it is locally a graph on $\mathbb{H}^2$.

In addition, for the flat surface Σ

$$Q\,dz^2 = (\langle\psi_z,\psi_z\rangle - h_z^2)\,dz^2 = \langle\eta_z,\eta_z\rangle\,dz^2$$

is holomorphic. That is, η is a harmonic map.

The proof is similar in the second case. □

These results should be understood as an analogue to the fact that η and h are harmonic maps for the induced metric of a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. More generically, the height function is a harmonic map for the induced metric of a minimal surface in every product space $N^2 \times \mathbb{R}$.

In our height estimates we will see that the extreme cases are the complete surfaces of constant Gaussian curvature in $\mathbb{M}^2(\varepsilon) \times \mathbb{R}$ (see [AEG]). Thus, we will briefly describe them.

Example 3.1 (Complete $K(I)$-surfaces in $\mathbb{H}^2 \times \mathbb{R}$). Let $K(I)$ be a positive constant and

$$\psi(u, v) = \big(\cosh k(v), \sinh k(v) \cos u, \sinh k(v) \sin u, h(v)\big).$$

the revolution surface in $\mathbb{H}^2 \times \mathbb{R}$ given by

$$k(v) = \operatorname{arcsinh}\left(\frac{1}{\sqrt{K(I)}} \sin(\sqrt{K(I)}\, v)\right) \quad \text{and}$$

$$h(v) = -\sqrt{\frac{1 + K(I)}{K(I)}} \arctan\left(\frac{\cos(\sqrt{K(I)}\, v)}{\sqrt{K(I) + \sin^2(\sqrt{K(I)}\, v)}}\right)$$

with $v \in [0, \pi/\sqrt{K(I)}]$.

Then, ψ is, up to isometries, the parametrization of the only complete surface with constant Gaussian curvature $K(I)$ in $\mathbb{H}^2 \times \mathbb{R}$.

Example 3.2 (Complete $K(I)$-surfaces in $\mathbb{S}^2 \times \mathbb{R}$). Let us consider a constant $K(I) > 1$ and

$$\psi(u, v) = \big(\sin k(v), \cos k(v) \cos u, \cos k(v) \sin u, h(v)\big)$$

the revolution surface in $\mathbb{S}^2 \times \mathbb{R}$ given by

$$k(v) = \arccos\left(\frac{1}{\sqrt{K(I)}} \sin(\sqrt{K(I)}\, v)\right) \quad \text{and}$$

$$h(v) = -\sqrt{\frac{K(I) - 1}{K(I)}} \log\left(\frac{\cos(\sqrt{K(I)}\, v) + \sqrt{K(I) - \sin^2(\sqrt{K(I)}\, v)}}{1 + \sqrt{K(I)}}\right)$$

with $v \in \big[0, \pi/\sqrt{K(I)}\big]$.

Then, ψ is, up to isometries, the parametrization of the only complete surface with constant Gaussian curvature $K(I)$ in $\mathbb{S}^2 \times \mathbb{R}$.

We will prove that our height estimates are reached when the quadratic form $Q\,dz^2$ vanishes identically. Thus, in order to characterize these examples we have the next Theorem. We observe that this result is local and a global version of it was given in [AEG].

Theorem 3.1. *Let $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ be an immersion of constant Gaussian curvature $K(I) > 0$ if $\varepsilon = -1$ (resp. $K(I) > 1$ if $\varepsilon = 1$). Let us assume that $Q\,dz^2$ vanishes identically on S. Then ψ is a piece of a complete $K(I)$-surface.*

Proof. For exposition clarity we will divide this proof into two parts. First, we will show that the immersion must be helicoidal, that is, ψ is invariant under a continuous group of isometries of the ambient space. Then, in the second part, we will demonstrate that the orbits of this helicoidal movement are circles on the slices $\mathbb{M}^2(\varepsilon) \times \{h_0\}$. Thus, we will conclude that ψ is a revolution surface.

Following [AEG] we define the metric

$$A = I + \frac{\varepsilon}{K(I) - \varepsilon}\, dh^2.$$

From (2.1) and (2.5), K is positive and so II is definite. In addition, since $Q\,dz^2 \equiv 0$, it follows that the $(2,0)$-part of A with respect to II vanishes. Therefore, A and II are conformal. In particular, there exists a positive function λ on S such that $II = \lambda A$. And, from [AEG, Lemma 1], $\lambda^2 = K(I) - \varepsilon$.

Let (u, v) be local doubly orthogonal coordinates for the first and second fundamental form, that is,

$$\begin{aligned} I &= E du^2 + G dv^2 \\ II &= k_1 E du^2 + k_2 G dv^2 \end{aligned}$$

where k_1 and k_2 are the principal curvatures of the immersion ψ. Recall that these coordinates are available on a neighbourhood of every non umbilical point as well as on the interior set of umbilical points. Hence, we will consider these points and, using that this set is dense on S, we will deduce that the obtained properties can be extended to the whole surface by continuity.

Since

$$A = \left(E + \frac{\varepsilon}{K(I)-\varepsilon}\, h_u^2\right) du^2 + 2\,\frac{\varepsilon}{K(I)-\varepsilon}\, h_u\, h_v\, du\, dv + \left(G + \frac{\varepsilon}{K(I)-\varepsilon}\, h_v^2\right) dv^2$$

and $II = \sqrt{K(I)-\varepsilon}A$ we have

$$k_1 E = \sqrt{K(I)-\varepsilon}\left(E + \frac{\varepsilon}{K(I)-\varepsilon}\, h_u^2\right) \tag{3.19}$$

$$0 = h_u\, h_v \tag{3.20}$$

$$k_2 G = \sqrt{K(I)-\varepsilon}\left(G + \frac{\varepsilon}{K(I)-\varepsilon}\, h_v^2\right). \tag{3.21}$$

From (3.20) it is easy to see that the set given by the union of the interior of the set where $h_u = 0$ and the interior of the set where $h_v = 0$ is dense. Hence, we can assume that $h_u \equiv 0$ in our neighbourhood where (u, v) are available.

Then, from (3.19), the principal curvature $k_1 = \sqrt{K(I)-\varepsilon}$ is a positive constant. In addition, from (2.4)

$$\nu_u = d\nu(\partial_u) = -\langle k_1 \partial_u, T\rangle = -k_1\, h_u = 0.$$

Thus, from the Gauss equation (2.1), $(k_2)_u = 0$.

Now, (3.21) can be rewritten as

$$k_1(k_2 - k_1)G = \varepsilon h_v^2. \tag{3.22}$$

Since we are considering an umbilical free neighbourhood or totally umbilical neighbourhood, we observe that if $k_1 \equiv k_2$ then h_v also vanishes identically. Therefore, h must be constant, that is, the surface would lie on a slice and so $K(I) = \varepsilon$, which is a contradiction.

Hence, $k_1 \neq k_2$ in our neighbourhood and $G_u = 0$ from (3.22).

If we consider the Codazzi equation (2.2) for $X = \partial_u$ and $Y = \partial_v$, we get

$$\begin{aligned}(k_2 - k_1)\nabla_{\partial_u}\partial_v &= \nabla_{\partial_u} k_2 \partial_v - \nabla_{\partial_v} k_1 \partial_u \\ &= \varepsilon\nu\left(\langle\partial_v, T\rangle\partial_u - \langle\partial_u, T\rangle\partial_v\right) \\ &= \varepsilon\nu h_v \partial_u.\end{aligned}$$

Moreover, $\nabla_{\partial_u}\partial_v = \frac{E_v}{2E}\,\partial_u$, so $(\log E)_{uv} = (E_v/E)_u = 0$. That is, the function $E(u, v)$ can be written as $E(u, v) = E_1(u)\, E_2(v)$ for positive functions E_1 and E_2.

Finally, we take new parameters, (x, y) such that

$$dx = \sqrt{E_1(u)}\, du, \quad y = v.$$

Then, the first fundamental form, the second fundamental form, h and ν only depend on y, that is, the functions E, G, k_1, k_2, h and ν do not depend on (x, y) but only on y.

Hence, the immersions $\psi(x, y)$ and $\varphi(x, y) = \psi(x + x_0, y)$, for a suitable x_0, have the same functions E, G, k_1, k_2, h and ν. So, $\psi(x, y)$ and $\psi(x + x_0, y)$ only differ from an isometry of the ambient space for each x_0 (see [Da]), that is, ψ is helicoidal and the orbits are given by $\beta(t) = \psi(x + t, y)$.

In the second part of the proof we will show that ψ is a rotation surface. First, we observe that $\beta(t)$ lies on a slice because the height function only depends on y.

In particular,

$$\beta(t) \subseteq \mathbb{M}^2(\varepsilon) \times \{y\} \equiv \mathbb{M}^2(\varepsilon)$$

is invariant under a continuous group of isometries of $\mathbb{M}^2(\varepsilon)$. Then, the curvature of β in $\mathbb{M}^2(\varepsilon)$ is constant.

Therefore, if $\varepsilon = 1$ then β lies on a circle of $\mathbb{S}^2$. Otherwise, if $\varepsilon = -1$, β lies on a circle of $\mathbb{H}^2$ if and only if its curvature is greater than one.

For the last case, an easy computation gives us

$$\nabla_{\partial_x}\partial_x = -\frac{1}{2}\,\frac{E_y}{G}\partial_y + k_1\, E\, N,$$

where ∇ denotes the Levi-Civita connection in $\mathbb{H}^2 \times \mathbb{R}$. In addition, β can be parametrized by the arc length as $\beta(s) = \psi(x + s/\sqrt{E(y)}, y)$. Then, the square of its curvature is given by

$$\begin{aligned}\langle \nabla_{\beta'(s)}\beta'(s), \nabla_{\beta'(s)}\beta'(s)\rangle &= \frac{1}{E(y)^2}\left(\frac{E_y(y)^2}{G(y)} + k_1^2 E(y)^2\right) \geq k_1^2 \\ &= K(I) + 1 > 1.\end{aligned}$$

Thus, β lies on a circle in any case and ψ must be a rotation surface. Finally, the proof finishes from the next Lemma. □

Lemma 3.2. *Let $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ be a rotation surface such that the principal curvatures k_1 associated with its parallels agree. If $k_1 > 1$ for $\varepsilon = -1$ or $k_1 > 0$ for $\varepsilon = 1$ then, up to isometries, it is a piece of the complete surfaces of constant Gaussian curvature described in Example* 3.1 *or Example* 3.2.

Proof. Let us parametrize

$$\psi(u,v) = (\cosh k(v), \sinh k(v)\cos u, \sinh k(v)\sin u, h(v)) \quad \text{if } \varepsilon = -1,$$
$$\psi(u,v) = (\sin k(v), \cos k(v)\cos u, \cos k(v)\sin u, h(v)) \quad \text{if } \varepsilon = 1,$$

with $k'(v)^2 + h'(v)^2 = 1$.

Then, a straightforward computation gives us that the principal curvatures associated with the parallels are

$$k_1 = h'(v)\coth(k(v)) \quad \text{if } \varepsilon = -1,$$
$$k_1 = h'(v)\tan(k(v)) \quad \text{if } \varepsilon = 1.$$

The solutions to these equations are given by

$$k(v) = k_0(\pm(v + c_0)), \qquad h(v) = \pm h_0(\pm(v + c_0)) + c_1,$$

where c_0, c_1 are two real constants and $k_0(v)$, $h_0(v)$ are the ones of Example 3.1 and Example 3.2.

Therefore, ψ is, up to reparametrizations and vertical translations, a piece of the complete surfaces of constant Gaussian curvature. □

Now, we are ready to obtain our height estimate.

Theorem 3.2. *Let $\Sigma \subseteq \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ be a compact graph on a set $\Omega \subseteq \mathbb{M}^2(\varepsilon)$, with positive constant Gaussian curvature $K(I) > \varepsilon$ and whose boundary is contained on the slice $\mathbb{M}^2(\varepsilon) \times \{0\}$. Then the maximum height that Σ can attain on $\mathbb{M}^2(\varepsilon) \times \{0\}$ is*

$$\sqrt{\frac{K(I)+1}{K(I)}}\arctan\left(\frac{1}{\sqrt{K(I)}}\right) \qquad \text{if } \varepsilon = -1, \tag{3.23}$$

$$\sqrt{\frac{K(I)-1}{K(I)}}\ln\left(\frac{\sqrt{K(I)}+1}{\sqrt{K(I)-1}}\right) \qquad \text{if } \varepsilon = 1. \tag{3.24}$$

Moreover, the equality holds if, and only if, Σ is the hemisphere of a complete $K(I)$-surface.

Proof. Observe that, since $K(I)$ is positive and greater than ε, the extrinsic curvature K is also positive, from (2.1) and (2.5). We can assume, without loss of generality, that Σ lies over the slice $\mathbb{M}^2(\varepsilon) \times \{0\}$ and so $\nu \leq 0$ everywhere because of the chosen orientation.

Let us consider the functions

$$f = \sqrt{\frac{K(I)+1}{K(I)}} \arctan\left(\frac{\nu}{\sqrt{K(I)}}\right) \qquad \text{if } \varepsilon = -1$$

$$f = -\sqrt{\frac{K(I)-1}{K(I)}} \ln\left(\frac{\sqrt{K(I)}-\nu}{\sqrt{K(I)-\nu^2}}\right) \qquad \text{if } \varepsilon = 1$$

and $\phi = h + f$ on S. Observe that $\phi = f \leq 0$ on ∂S. Our aim is to show that $\phi_{z\bar{z}} \geq 0$ on S, because under these conditions the maximum principle assures that $\phi \leq 0$ on S. From this last fact our estimate for the maximum height follows easily, that is, if $\phi \leq 0$ then (3.23) and (3.24) hold. Observe that these bounds are optimal, because they are achieved for the hemispheres of the complete $K(I)$-surfaces.

Let us see that $\phi_{z\bar{z}} \geq 0$. From (3.1), (3.5), (3.9) and (3.12), we have

$$f_{z\bar{z}} = \frac{\sqrt{K(I)-\varepsilon}}{K^2}\left(K\nu_{z\bar{z}} + 2\varepsilon\nu\,|\nu_z|^2\right) = \frac{-\sqrt{K(I)-\varepsilon}\,\nu}{K}\left(\frac{\varepsilon\,|\alpha|^2}{D} + KF\right).$$

Using (3.3) and (3.5), we obtain

$$KF + \frac{\varepsilon\,|\alpha|^2}{D} = KF + \varepsilon(|h_z|^2 - F\,\|T\|^2) = F(K(I)-\varepsilon) + \varepsilon\,|h_z|^2\,.$$

Thus, using (3.11) we get

$$\phi_{z\bar{z}} = \frac{(K(I)-\varepsilon)\,\nu}{K}\left(\rho - \sqrt{K(I)-\varepsilon}\,(F + \frac{\varepsilon|h_z|^2}{K(I)-\varepsilon})\right). \qquad (3.25)$$

From [AEG, Lemma 1], we have

$$K(I) - \varepsilon = \frac{\rho^2}{\left(F + \frac{\varepsilon|h_z|^2}{K(I)-\varepsilon}\right)^2 - \frac{|Q|^2}{(K(I)-\varepsilon)^2}}$$

or equivalently

$$\rho^2 - (K(I)-\varepsilon)\left(F + \frac{\varepsilon\,|h_z|^2}{K(I)-\varepsilon}\right)^2 = -\frac{|Q|^2}{K(I)-\varepsilon}. \qquad (3.26)$$

Since $F + \frac{\varepsilon|h_z|^2}{K(I)-\varepsilon}$ is the $(1,1)-$part of the metric A with respect to II, then it must be positive. Therefore, from (3.25) and (3.26), $\phi_{z\bar{z}}$ is greater than or equal to zero.

Finally, note that if the extreme value is achieved, then $\phi_{z\bar{z}} \equiv 0$. That is, Q or ν vanish identically on S. But, S cannot lie on a cylinder because $K(I) \neq 0$, that is, $\nu \not\equiv 0$. Therefore, $Q \equiv 0$ and the result follows from Theorem 3.1. □

Remark 3.1. From the Bonnet-Myers theorem, it can be deduced that every minimizing geodesic in a surface $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ with positive constant Gaussian curvature $K(I)$ has length less than or equal to $2\pi/\sqrt{K(I)}$. Hence, there always exists height estimates for such a surface, although these estimates are far from optimal. The bounds we provide in the above theorem are, as we have seen, the optimal ones.

We would like to point out that the hypothesis $K(I) \geq 1$ in $\mathbb{S}^2 \times \mathbb{R}$ can be changed for $K(I) > 0$, $K(I) \neq 1$. In fact, if $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ is a surface with Gaussian curvature $K(I)$, $0 < K(I) < 1$, then there does not exist a point p where the height function attains a maximum or minimum, because from the Gauss equation it would be $k_1(p)k_2(p) < 0$ and so in any neighborhood of p there would be points of greater and less height than p. In particular, we answer a question posed by the authors in [AEG].

Proposition 3.2. *There does not exist a complete surface with constant Gaussian curvature $K(I)$, $0 < K(I) < 1$, in $\mathbb{S}^2 \times \mathbb{R}$.*

As a consequence of the classical Alexandrov reflection principle for slices in $\mathbb{M}^2(\varepsilon) \times \mathbb{R}$, we get the following

Corollary 3.2. *Let $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ be an embedding of a compact surface with positive constant Gaussian curvature $K(I) > \varepsilon$, such that its boundary $\psi(\partial S)$ (possibly empty) is contained in a slice $\mathbb{M}^2(\varepsilon) \times \{t_0\}$. Then the height difference between its upper point and its lower point is less than or equal to*

$$2\sqrt{\frac{K(I)+1}{K(I)}}\,\arctan\left(\frac{1}{\sqrt{K(I)}}\right) \qquad \text{if } \varepsilon = -1$$

$$2\sqrt{\frac{K(I)-1}{K(I)}}\,\ln\left(\frac{\sqrt{K(I)}+1}{\sqrt{K(I)-1}}\right) \qquad \text{if } \varepsilon = 1.$$

Moreover, if such a difference is attained, $\psi(S)$ is, up to isometries, the complete $K(I)$-surface.

4 Representation of $K(I)$-surfaces with $K > 0$

Let $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ be a $K(I)$-surface with positive extrinsic curvature and assume that II is positive definite. We will also assume that the surface is transverse to each slice $\mathbb{M}^2(\varepsilon) \times \{t\}$ for all $t \in \mathbb{R}$, that is, T never vanishes on

S, or equivalently, $\nu^2 < 1$ on S. We will refer to such a surface as a *transverse surface*.

Let us observe that the points where $\nu^2 = 1$ is an isolated set because $K > 0$ (that is, II is positive definite).

From (3.3) and (3.10) one has

$$\frac{\alpha h_z + \overline{\alpha} h_{\bar z}}{1-\nu^2} = D = |E|^2 - F^2 = \left|\frac{\alpha + F h_{\bar z}}{h_z}\right|^2 - F^2,$$

that is,

$$F = -\frac{|\alpha|^2}{\alpha h_z + \bar\alpha h_{\bar z}} + \frac{|h_z|^2}{1-\nu^2}.$$

Hence, by using (3.3) again

$$E = \frac{\bar\alpha^2}{\alpha h_z + \bar\alpha h_{\bar z}} + \frac{h_z^2}{1-\nu^2}.$$

On the other hand, from (3.5) and (3.9) one gets

$$\alpha = \frac{\rho\, \nu_{\bar z}}{K(I) - \varepsilon \nu^2}. \tag{4.1}$$

In addition, from (3.1), (3.5) and (3.10)

$$\rho^2 = -D\,K = -\frac{K(I)-\varepsilon\nu^2}{1-\nu^2}(\alpha h_z + \bar\alpha h_{\bar z}).$$

With all of this, E, F can be written in terms of h_z, ν, ν_z and $K(I)$ as

$$\begin{aligned} E &= \frac{h_z^2}{1-\nu^2} - \frac{\nu_z^2}{(1-\nu^2)(K(I)-\varepsilon\nu^2)}, \\ F &= \frac{|h_z|^2}{1-\nu^2} + \frac{|\nu_z|^2}{(1-\nu^2)(K(I)-\varepsilon\nu^2)}. \end{aligned}$$

Thus, from (3.3), (3.9) and (4.1)

$$\begin{aligned} \alpha &= -\frac{\nu_z h_{\bar z} + \nu_{\bar z} h_z}{(1-\nu^2)(K(I)-\varepsilon\nu^2)}\nu_{\bar z} \\ \rho &= -\frac{\nu_z h_{\bar z} + \nu_{\bar z} h_z}{1-\nu^2} > 0. \end{aligned}$$

At this point it would be interesting to observe that the induced metric and the second fundamental form can be recovered for any transverse surface with positive extrinsic curvature in terms of h, ν, $K(I)$ and the conformal structure given by II. In fact, the compatibility equations for the existence of an immersion from a simply connected Riemann surface with fixed functions $h : S \longrightarrow \mathbb{R}$ and $\nu : S \longrightarrow (-1, 1)$ and fixed Gaussian curvature are given by Lemma 3.1.

The compatibility equations are much more simple for a surface with constant Gaussian curvature:

Theorem 4.1. *Let S be a surface and $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ a transverse $K(I)$-immersion such that $K > 0$. Then, given a local conformal parameter z for II, the first and second fundamental forms of ψ can be recovered in terms of its height function h, ν and the constant $K(I)$ as*

$$E = \frac{h_z^2}{1-\nu^2} - \frac{\nu_z^2}{(1-\nu^2)(K(I)-\varepsilon\nu^2)} \tag{4.2}$$

$$F = \frac{|h_z|^2}{1-\nu^2} + \frac{|\nu_z|^2}{(1-\nu^2)(K(I)-\varepsilon\nu^2)} \tag{4.3}$$

$$\rho = -\frac{\nu_z h_{\bar{z}} + \nu_{\bar{z}} h_z}{1-\nu^2} > 0. \tag{4.4}$$

In addition h and ν verify

$$h_{z\bar{z}} = -\frac{(K(I)-\varepsilon)\nu}{(1-\nu^2)(K(I)-\varepsilon\nu^2)}(\nu_z h_{\bar{z}} + \nu_{\bar{z}} h_z) \tag{4.5}$$

$$\nu_{z\bar{z}} = -\frac{\nu}{(1-\nu^2)(K(I)-\varepsilon\nu^2)}\Big((K(I)-2\varepsilon\nu^2+\varepsilon)|\nu_z|^2 + (K(I)-\varepsilon\nu^2)^2|h_z|^2\Big). \tag{4.6}$$

Conversely, let $K(I) \in \mathbb{R}$ and $\varepsilon \in \{-1, 1\}$ be constants, $h : S \longrightarrow \mathbb{R}$ and $\nu : S \longrightarrow (-1, 1)$ functions on a simply-connected Riemann surface S satisfying (4.5), (4.6), *$K(I) - \varepsilon\nu^2 > 0$ and $\nu_z h_{\bar{z}} + \nu_{\bar{z}} h_z < 0$. Then, there exists an immersion $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ such that S is a transverse $K(I)$-surface whose first and second fundamental forms are given by*

$$I = E dz^2 + 2F|dz|^2 + \bar{E} d\bar{z}^2, \qquad II = 2\rho|dz|^2$$

where E, F and ρ are defined by (4.2), (4.3) *and* (4.4) *respectively, whose height function and the fourth coordinate of its normal are given by h and ν respectively, and the structure given by its second fundamental form is the one of S. Moreover, this immersion is unique up to an isometry of $\mathbb{M}^2(\varepsilon) \times \mathbb{R}$.*

Proof. For the first part we observe that (4.2), (4.3) and (4.4) are already computed. In addition, (4.5) and (4.6) are equivalent to (3.11) and (3.12) when the previous expressions of E, F and ρ are used.

The second part is a tedious but straightforward computation bearing in mind Lemma 3.1. □

This representation has a special interest for transverse surfaces with Gaussian curvature $K(I) = 1$ in $\mathbb{S}^2 \times \mathbb{R}$. In this case, it is not known if the slices are the only complete surfaces with constant Gaussian curvature $K(I) = 1$ in $\mathbb{S}^2 \times \mathbb{R}$ [AEG]. In addition, the compatibility equations will be reduced to the elliptic sinh-Gordon equation.

Thus, let $\psi : S \longrightarrow \mathbb{S}^2 \times \mathbb{R}$ be a transverse immersion with $K(I) = 1$ in the previous conditions. Since $\nu^2 < 1$, it follows from (3.5) that $K > 0$ and so we can assume that II is a Riemannian metric on S. Then, we are under the hypothesis of Theorem 4.1 and consequently the height function h is harmonic for II and ν verifies

$$\nu_{z\bar{z}} = -\nu \left(|h_z|^2 + 2\frac{|\nu_z|^2}{1-\nu^2} \right).$$

If we define $\omega = arctanh(\nu)$, this last equation becomes

$$\omega_{z\bar{z}} + |h_z|^2 \sinh(\omega)\cosh(\omega) = 0, \tag{4.7}$$

and we can reformulated Theorem 4.1 for these surfaces as follows:

Corollary 4.1. *Let S be a connected surface and $\psi : S \longrightarrow \mathbb{S}^2 \times \mathbb{R}$ a transverse immersion with $K(I) = 1$. Then, given a local conformal parameter z for II, the first and second fundamental forms of ψ can be recovered in terms of its height function h and $\omega = \operatorname{arctanh}(\nu)$ as*

$$\begin{aligned} E &= \cosh^2(\omega)h_z^2 - \omega_z^2 && (4.8)\\ F &= \cosh^2(\omega)|h_z|^2 + |\omega_z|^2 && (4.9)\\ \rho &= -(\omega_z h_{\bar{z}} + \omega_{\bar{z}} h_z) > 0 && (4.10)\end{aligned}$$

where ω verifies (4.7).

Conversely, let $h : S \longrightarrow \mathbb{R}$ be a harmonic function on a simply-connected Riemann surface S and $\omega : S \longrightarrow \mathbb{R}$ a function verifying $\omega_z h_{\bar{z}} + \omega_{\bar{z}} h_z < 0$ and (4.7). *Then, there exists an immersion $\psi : S \longrightarrow \mathbb{S}^2 \times \mathbb{R}$ such that S is transverse with constant Gaussian curvature $K(I) = 1$, whose first and second* *fundamental forms are given by*

$$I = Edz^2 + 2F|dz|^2 + \bar{E}d\bar{z}^2,$$
$$II = 2\rho|dz|^2$$

where E, F and ρ are defined by (4.8), (4.9) *and* (4.10) *respectively, whose height function and the fourth coordinate of its normal are given by h and $\nu = \tanh(\omega)$ respectively, and the structure given by its second fundamental form is the one on S. Moreover, this immersion is unique up to an isometry of $\mathbb{S}^2 \times \mathbb{R}$.*

Since $h_z \neq 0$ in a transverse immersion and h_z is a holomorphic function when $K(I) = 1$ in $\mathbb{S}^2 \times \mathbb{R}$, we can take a conformal parameter ζ such that $d\zeta = h_z dz$. Thus, (4.7) becomes the classical elliptic sinh-Gordon equation

$$\omega_{\zeta\bar{\zeta}} + \sinh(\omega)\cosh(\omega) = 0.$$

5 Representation of $K(I)$-surfaces with $K < 0$

Similar results to the ones given in the previous section can be obtained for transverse $K(I)$-surfaces with negative extrinsic curvature. In this case, II is a Lorentzian metric and S can be considered as a Lorentz surface with the induced conformal structure (see [We]).

So, we can take asymptotic coordinates (u, v) for a surface with $K < 0$ such that the first and second fundamental form can be written as

$$I = Edu^2 + 2Fdudv + Gdv^2$$
$$II = 2fdudv.$$

After a development similar in essence to the one in the case of positive extrinsic curvature we obtain some results which are enunciated without a proof in order to not repeat the same computations.

Theorem 5.1. *Let S be a connected surface and $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ a transverse $K(I)$-immersion such that $K < 0$. Then, given asymptotic coordinates (u, v) for II, the first and second fundamental forms of ψ can be recovered in terms of its height function h, ν and the constant $K(I)$ as*

$$E = \frac{h_u^2}{1-\nu^2} - \frac{\nu_u^2}{(1-\nu^2)(K_I - \varepsilon\nu^2)} \tag{5.1}$$

$$F = \frac{h_u h_v}{1-\nu^2} + \frac{\nu_u \nu_v}{(1-\nu^2)(K_I - \varepsilon\nu^2)} \tag{5.2}$$

$$G = \frac{h_v^2}{1-\nu^2} - \frac{\nu_v^2}{(1-\nu^2)(K_I - \varepsilon\nu^2)} \tag{5.3}$$

$$f = -\frac{\nu_u h_v + \nu_v h_u}{1-\nu^2} > 0. \tag{5.4}$$

In addition, h and ν verify

$$h_{uv} = -\frac{(K_I - \varepsilon)\nu}{(1-\nu^2)(K_I - \varepsilon\nu^2)}(\nu_u h_v + \nu_v h_u) \tag{5.5}$$

$$\nu_{uv} = -\frac{\nu}{(1-\nu^2)(K_I - \varepsilon\nu^2)}\big((K_I - 2\varepsilon\nu^2 + \varepsilon)\nu_u\nu_v + (K_I - \varepsilon\nu^2)^2 h_u h_v\big). \tag{5.6}$$

Conversely, let $K(I) \in \mathbb{R}$ *and* $\varepsilon \in \{-1, 1\}$ *be constants,* $h : S \longrightarrow \mathbb{R}$ *and* $\nu : S \longrightarrow (-1, 1)$ *functions on a simply-connected Lorentz surface S satisfying* (5.5), (5.6), $K(I) - \varepsilon\nu^2 < 0$ *and* $\nu_u h_v + \nu_v h_u < 0$. *Then, there exists an immersion* $\psi : S \longrightarrow \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ *such that S is a transverse* $K(I)$*-surface whose first and second fundamental forms are given by*

$$I = E du^2 + 2F du dv + G dv^2,$$
$$II = 2f du dv$$

where E, F, G and f are defined by (5.1), (5.2), (5.3) *and* (5.4) *respectively, whose height function and the fourth coordinate of its normal are given by h and ν respectively, and the structure given by its second fundamental form is the one on S. Moreover, this immersion is unique up to an isometry of* $\mathbb{M}^2(\varepsilon) \times \mathbb{R}$.

If $\psi : S \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ is a transverse immersion with $K(I) = -1$, it follows from (3.5) that $K < 0$ at every point. In addition, from Theorem 5.1, h is harmonic for II and ν verifies

$$\nu_{uv} = \nu\left(h_u h_v - 2\frac{\nu_u \nu_v}{1-\nu^2}\right).$$

By taking $\omega = arctanh(\nu)$, this last equation becomes

$$\omega_{uv} - h_u h_v \sinh(\omega)\cosh(\omega) = 0. \tag{5.7}$$

Corollary 5.1. *Let S be a connected surface and $\psi : S \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ a transverse immersion with $K(I) = -1$. Then, given local asymptotic coordinates (u, v) for II, the first and second fundamental forms of ψ can be recovered in terms of its height function h and $\omega = \operatorname{arctanh}(\nu)$ as*

$$E = \cosh^2(\omega) h_u^2 + \omega_u^2 \tag{5.8}$$
$$F = \cosh^2(\omega) h_u h_v - \omega_u \omega_v \tag{5.9}$$
$$G = \cosh^2(\omega) h_v^2 + \omega_v^2 \tag{5.10}$$
$$f = -(\omega_u h_v + \omega_v h_u) > 0 \tag{5.11}$$

where ω verifies (5.7).

Conversely, let $h : S \longrightarrow \mathbb{R}$ be a harmonic function on a simply-connected Lorentz surface S and $\omega : S \longrightarrow \mathbb{R}$ a function verifying $\omega_u h_v + \omega_v h_u < 0$ and (5.7)*. Then, there exists an immersion $\psi : S \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ such that S is transverse with constant Gaussian curvature $K(I) = -1$, whose first and second fundamental forms are given by*

$$I = E du^2 + 2F du dv + G dv^2,$$
$$II = 2f du dv$$

where E, F, G and f are defined by (5.8), (5.9), (5.10) *and* (5.11) *respectively, whose height function and the fourth coordinate of its normal are given by h and $\nu = \tanh(\omega)$ respectively, and the structure given by its second fundamental form is the one of S. Moreover, this immersion is unique up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$.*

From this result, it is easy to obtain complete surfaces with $K(I) = -1$ in $\mathbb{H}^2 \times \mathbb{R}$, for instance, if one takes $h_u \equiv 0$ and chooses a suitable $\omega(u, v) = \omega_1(u) + \omega_2(v)$.

Moreover, since $h_{uv} = 0$ if h_u and h_v do not vanish then taking the new parameters $(\bar{u}, \bar{v})$ such that $d\bar{u} = h_u du$, $d\bar{v} = h_v dv$ the previous equation (5.7) becomes the hyperbolic sinh-Gordon equation

$$\omega_{\bar{u}\bar{v}} - \sinh \omega \cosh \omega = 0.$$

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