BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 38(1), 115-127

On the stability of θ-derivations on JB^*-triples

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Abstract. We introduce the concept of θ-derivations on JB^*-triples and prove the Hyers–Ulam-Rassias stability of θ-derivations on JB^*-triples. We deal with the Hyers-Ulam-Rassias stability that was first introduced by Th.M. Rassias in the paper "On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300".

Keywords: Hyers–Ulam–Rassias stability, θ-derivation, JB^*-triple.

Mathematical subject classification: 39B52, 39B82, 47B48, 17Cxx.

1 Introduction

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [28]). Let $\mathcal{H}$ be a complex Hilbert space, regarded as the "state space" of a quantum mechanical system. Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product*

$$x \circ y := \frac{xy + yx}{2}\,.$$

This is a typical example of a (special) Jordan algebra. A commutative algebra $\mathcal{B}$ with product $x \circ y$ (not necessarily given by an anticommutator) is called a *Jordan algebra* if $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ holds for all $x, y \in \mathcal{B}$.

A complex Jordan algebra $\mathcal{B}$ with a product $x \circ y$, and a conjugate-linear algebra involution $x \mapsto x^*$ is called a JB^*-algebra if $\mathcal{B}$ carries a Banach space norm $\|\cdot\|$ satisfying $\|x\| = \|x^*\|$, $\|x \circ y\| \le \|x\| \cdot \|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$

Received 11 October 2005.
The first author was supported by Korea Research Foundation Grant KRF-2005-070-C00009.

for all $x, y \in \mathcal{B}$. Here $\{xyz\} := (x \circ y) \circ z + (y \circ z) \circ x - (x \circ z) \circ y$ denotes the *Jordan triple product* of $x, y, z \in \mathcal{B}$ (see [21, 22]).

The Jordan triple product of a JB^*-algebra leads us to a more general algebraic structure, the so-called JB^*-triple, which turns out to be appropriate for most applications to analysis. By a (complex) JB^*-triple we mean a complex Banach space $\mathcal{J}$ with a continuous triple product

$$\{\cdot, \cdot, \cdot\} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to \mathcal{J}$$

which is linear in the outer variables and conjugate linear in the middle variable, and has the following properties:

(i) (commutativity) $\{x, y, z\} = \{z, y, x\}$;

(ii) (Jordan identity)

$$L(a, b)\{x, y, z\} = \{L(a, b)x, y, z\} - \{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$$

for all $a, b, x, y, z, \in \mathcal{J}$ in which $L(a, b)x := \{a, b, x\}$;

(iii) For all $a \in \mathcal{J}$ the operator $L(a, a)$ is hermitian, i.e. $\|e^{itL(a,a)}\| = 1$, and has positive spectrum in the Banach algebra $B(\mathcal{J})$;

(iv) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$.

The class of JB^*-triples contains all C^*-algebras via $\{x, y, z\} = \dfrac{xy^*z + zy^*x}{2}$. Every JB^*-algebra is a JB^*-triple under the triple product

$$\{x, y, z\} := (x \circ y^*) \circ z + (y^* \circ z) \circ x - (x \circ z) \circ y^*.$$

Conversely, every JB^*-triple $\mathcal{J}$ with an element e satisfying $\{e, e, z\} = z$ for all $z \in \mathcal{J}$, is a unital JB^*-algebra equipped with the product $x \circ y := \{x, e, y\}$ and the involution $x^* := \{e, x, e\}$; cf. [9, 20, 26].

The stability problem of functional equations originated from a question of S.M. Ulam [27] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

D.H. Hyers [10] gave a first affirmative answer to the question of Ulam in the context of Banach spaces: Let E_1 and E_2 be Banach spaces. Assume that $f : E_1 \to E_2$ satisfies $\| f(x+y) - f(x) - f(y)\| \leq \epsilon$ for all $x, y \in E_1$ and some $\epsilon \geq 0$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that $\| f(x) - T(x)\| \leq \varepsilon$ for all $x \in E_1$..

Now assume that E_1 and E_2 are real normed spaces with E_2 complete, $f : E_1 \to E_2$ is a mapping such that for each fixed $x \in E_1$, the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\varepsilon \geq 0$ and $p \neq 1$ such that

$$\| f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$.

It was shown by Th. M. Rassias [23] for $p \in [0, 1)$ (and indeed $p < 1$) and by Z. Gajda [7] following the same approach as in [23] for $p > 1$ that there exists a unique linear map $T : E_1 \to E_2$ such that

$$\| f(x) - T(x)\| \leq \frac{2\epsilon}{|2^p - 2|}\|x\|^p$$

for all $x \in E_1$. It is shown that there is no analogue of Th.M. Rassias result for $p = 1$ (see [7, 25])

The inequality $\| f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$ has provided a lot of influence in the development of what is now known as *Hyers–Ulam–Rassias stability* of functional equations; cf. [5, 6, 11, 13, 24].

In 1992, Găvruta [8] proved the following.

Theorem 1.1. *Let G be an abelian group and X be a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that*

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to X$ is a mapping satisfying

$$\| f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$\| f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$.

It is easy to see that Theorem 1.1 is still valid if

$$\widetilde{\varphi}(x, y) = \sum_{j=1}^{\infty} 2^{-j} \varphi(2^{-j}x, 2^{-j}y) < \infty$$

(see also [11]).

Since then the topic of approximate mappings or the stability of functional equations was studied by several mathematicians; [2, 3, 15] and references therein. In particular, Jun and Lee proved the following theorem; cf. [12, Theorems 1 & 6].

Theorem 1.2. *Denote by* $\varphi : X \times X \to [0, \infty)$ *a function such that*

$$\widetilde{\varphi}(x, y) \;=\; \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty$$

$$\Big(resp.\ \widetilde{\varphi}(x, y) \;=\; \sum_{j=0}^{\infty} 3^j \varphi(3^{-j} x, 3^{-j} y) < \infty\Big)$$

for all $x, y \in X$. *Suppose that* $f : X \to Y$ *is a mapping with* $f(0) = 0$ *satisfying*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \varphi(x, y)$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$\| f(x) - T(x) \| \le \frac{1}{3}\big(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x)\big)$$

$$\left(resp.\ \| f(x) - T(x) \| \le \widetilde{\varphi}\left(\frac{x}{3}, \frac{-x}{3}\right) + \widetilde{\varphi}\left(\frac{-x}{3}, x\right), \right)$$

for all $x \in X$.

There are several various generalizations of the notion of derivation. It seems that they are first appeared in the framework of pure algebra (see [1]). Recently they have been treated in the Banach algebra theory (see [14]). In addition, the stability of these derivations is extensively studied by the present authors and others; see [4, 16, 18, 19] and references therein.

In this paper, using some ideas from [21], we introduce the notion of θ-derivations on JB^*-algebras as a generalization of derivations on JB^*-triples [9] and prove the Hyers–Ulam–Rassais stability of θ-derivations on JB^*-triples. Our result may be considered as a generalization of those of [20].

2 Stability of θ-derivations

Throughout this section, let $\mathcal{J}$ be a complex JB^*-triple with norm $\|\cdot\|$.

Definition 2.1. *Let $\theta : \mathcal{J} \to \mathcal{J}$ be a $\mathbb{C}$-linear mapping. A $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ is called a θ-derivation on $\mathcal{J}$ if*

$$D(\{xyz\}) = \{D(x)\theta(y)\theta(z)\} + \{\theta(x)D(y)\theta(z)\} + \{\theta(x)\theta(y)D(z)\}$$

for all $x, y, z \in \mathcal{J}$.

In particular, $D := \frac{1}{3}\theta$ gives rise a JB^*-homomorphism on $\mathcal{J}$. Hence our results can be regarded as an extension of those of [20]. Note that if D is a derivation on a JB^*-algebra then every derivation D can be represented as $D_1 + iD_2$ where D_1 and D_2 are $*$-preserving derivations.

Theorem 2.2. *Let $f, h : \mathcal{J} \to \mathcal{J}$ be mappings with $f(0) = h(0) = 0$ for which there exists a function $\varphi : \mathcal{J}^3 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) < \infty, \tag{2.1}$$

$$\|f(\mu x + y) - \mu f(x) - f(y)\| \le \varphi(x, y, 0), \tag{2.2}$$

$$\|h(\mu x + y) - \mu h(x) - h(y)\| \le \varphi(x, y, 0), \tag{2.3}$$

$$\begin{aligned}\|f(\{xyz\}) - \{f(x)h(y)h(z)\} - \{h(x)f(y)h(z)\}\\ -\{h(x)h(y)f(z)\}\| \le \varphi(x, y, z),\end{aligned} \tag{2.4}$$

for all $x, y, z \in \mathcal{J}$ and all $\mu \in S^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then there exist unique $\mathbb{C}$-linear mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ such that

$$\|f(x) - D(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x, 0), \tag{2.5}$$

$$\|h(x) - \theta(x)\| \le \frac{1}{2}\widetilde{\varphi}(x, x, 0) \tag{2.6}$$

for all $x \in \mathcal{J}$. Moreover, $D : \mathcal{J} \to \mathcal{J}$ is a θ-derivation on $\mathcal{J}$.

Proof. Let $\mu = 1 \in S^1$ and $z = 0$ in (2.2) and (2.3). It follows from Theorem 1.1 that there exist unique additive mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ satisfying (2.5) and

(2.6). The additive mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ are given by

$$D(x) = \lim_{l\to\infty} \frac{1}{2^l} f(2^l x), \tag{2.7}$$

$$\theta(x) = \lim_{l\to\infty} \frac{1}{2^l} h(2^l x) \tag{2.8}$$

for all $x \in \mathcal{J}$.

Let $\mu \in S^1$. Set $y = 0$ in (2.2). Then

$$\| f(\mu x) - \mu f(x)\| \le \varphi(x, 0, 0),$$

for all $x \in \mathcal{J}$. So that

$$2^{-l}(f(\mu 2^l x) - \mu f(2^l x))\| \le 2^{-l}\varphi(2^l x, 0, 0),$$

for all $x \in \mathcal{J}$. Since the right hand side tends to zero as $n \to \infty$, we have

$$D(\mu x) = \lim_{l\to\infty} \frac{f(2^l \mu x)}{2^l} = \lim_{l\to\infty} \frac{\mu f(2^l x)}{2^l} = \mu D(x)$$

for all $\mu \in \S^1$ and all $x \in \mathcal{J}$. Obviously, $D(0x) = 0 = 0D(x)$.

Next, let $\lambda = \alpha_1 + i\alpha_2 \in \mathbb{C}$, where $\alpha_1, \alpha_2 \in \mathbb{R}$. Let $\gamma_1 = \alpha_1 - \lfloor\alpha_1\rfloor, \gamma_2 = \alpha_2 - \lfloor\alpha_2\rfloor$, in which $\lfloor r\rfloor$ denotes the greatest integer less than or equal to the number r. Then $0 \le \gamma_i < 1$, $(1 \le i \le 2)$ and by using Remark 2.2.2 of [17] one can represent γ_i as

$$\gamma_i = \frac{\mu_{i,1} + \mu_{i,2}}{2}$$

in which $\mu_{i,j} \in S^1$, $(1 \le i, j \le 2)$. Since D is additive we infer that

$$\begin{aligned} D(\lambda x) &= D(\alpha_1 x) + iD(\alpha_2 x) \\ &= \lfloor\alpha_1\rfloor D(x) + D(\gamma_1 x) + i\big(\lfloor\alpha_2\rfloor D(x) + D(\gamma_2 x)\big) \\ &= \left(\lfloor\alpha_1\rfloor D(x) + \frac{1}{2} D(\mu_{1,1}x + \mu_{1,2}x)\right) \\ &\quad + i\left(\lfloor\alpha_2\rfloor D(x) + \frac{1}{2} D(\mu_{2,1}x + \mu_{2,2}x)\right) \\ &= \left(\lfloor\alpha_1\rfloor D(x) + \frac{1}{2}\mu_{1,1} D(x) + \frac{1}{2}\mu_{1,2} D(x)\right) \end{aligned}$$

$$+ \ i\left(\lfloor\alpha_2\rfloor D(x) + \frac{1}{2}\mu_{2,1}D(x) + \frac{1}{2}\mu_{2,2}D(x)\right)$$
$$= \alpha_1 D(x) + i\alpha_2 D(x)$$
$$= \lambda D(x).$$

for all $x \in \mathcal{J}$. So that the additive mappings $D : \mathcal{J} \to \mathcal{J}$ is $\mathbb{C}$-linear. A similar argument shows that θ is $\mathbb{C}$-linear.

It follows from (2.4) that

$$\frac{1}{2^{3l}}\| f(2^{3l}\{xyz\}) - \{f(2^l x)h(2^l y)h(2^l z)\} - \{h(2^l x) f(2^l y)h(2^l z)\}$$
$$-\{h(2^l x)h(2^l y) f(2^l z)\}\| \le \frac{1}{2^{3l}}\varphi(2^l x, 2^l y, 2^l z) \le \frac{1}{2^l}\varphi(2^l x, 2^l y, 2^l z),$$

which tends to zero as $l \to \infty$ for all $x, y, z \in \mathcal{J}$ by (2.1). By (2.7) and (2.8),

$$D(\{xyz\}) = \{D(x)\theta(y)\theta(z)\} + \{\theta(x)D(y)\theta(z)\} + \{\theta(x)\theta(y)D(z)\}$$

for all $x, y, z \in \mathcal{J}$. So the additive mapping $D\colon \mathcal{J} \to \mathcal{J}$ is a θ-derivation on $\mathcal{J}$.□

Remark. It is easy to verify that the theorem is true if

$$\widetilde{\varphi}(x, y) := \sum_{j=1}^{\infty} 2^{-j}\varphi(2^{-j}x, 2^{-j}y) < \infty.$$

Corollary 2.3. *Let $f, h : \mathcal{J} \to \mathcal{J}$ be mappings with $f(0) = h(0) = 0$ for which there exist constants $\epsilon \ge 0$ and $p \ne 1$ such that*

$$\| f(\mu x + y) - \mu f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p),$$

$$\|h(\mu x + y) - \mu h(x) - h(y)\| \le \epsilon(\|x\|^p + \|y\|^p),$$

$$\| f(\{xyz\}) - \{f(x)h(y)h(z)\} - \{h(x) f(y)h(z)\}$$
$$-\{h(x)h(y) f(z)\}\| \le \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{J}$ and all $\mu \in S^1$. Then there exist unique $\mathbb{C}$-linear mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ such that

$$\| f(x) - D(x)\| \le \frac{2\epsilon}{|2 - 2^p|}\|x\|^p,$$

$$\|h(x) - \theta(x)\| \leq \frac{2\epsilon}{|2 - 2^p|}\|x\|^p$$

for all $x \in \mathcal{J}$. Moreover, $D : \mathcal{J} \to \mathcal{J}$ is a θ-derivation on $\mathcal{J}$.

Proof. Define $\varphi(x, y, z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.1 and the remark following the theorem. □

Theorem 2.4. *Let $f, h : \mathcal{J} \to \mathcal{J}$ be mappings with $f(0) = h(0) = 0$ for which there exists a function $\varphi : \mathcal{J}^3 \to [0, \infty)$ satisfying (2.4) such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{3^j}\varphi(3^j x, 3^j y, 3^j z) < \infty,$$

$$\left\| 2f\left(\frac{\mu x + y}{2}\right) - \mu f(x) - f(y) \right\| \leq \varphi(x, y, 0), \tag{2.9}$$

$$\left\| 2h\left(\frac{\mu x + y}{2}\right) - \mu h(x) - h(y) \right\| \leq \varphi(x, y, 0) \tag{2.10}$$

for all $x, y, z \in \mathcal{J}$ and all $\mu \in S^1$. Then there exist unique $\mathbb{C}$-linear mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ such that

$$\|f(x) - D(x)\| \leq \frac{1}{3}\big(\widetilde{\varphi}(x, -x, 0) + \widetilde{\varphi}(-x, 3x, 0)\big), \tag{2.11}$$

$$\|h(x) - \theta(x)\| \leq \frac{1}{3}\big(\widetilde{\varphi}(x, -x, 0) + \widetilde{\varphi}(-x, 3x, 0)\big) \tag{2.12}$$

for all $x \in \mathcal{J}$. Moreover, $D : \mathcal{J} \to \mathcal{J}$ is a θ-derivation on $\mathcal{J}$.

Proof. Let $z = 0$ in (2.9) and (2.10). It follows from Theorem 1.2 that there exist unique additive mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ satisfying (2.11) and (2.12). The additive mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ are given by

$$D(x) = \lim_{l \to \infty} \frac{1}{3^l} f(3^l x),$$

$$\theta(x) = \lim_{l \to \infty} \frac{1}{3^l} h(3^l x)$$

for all $x \in \mathcal{J}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.5. *Let $f, h : \mathcal{J} \to \mathcal{J}$ be mappings with $f(0) = h(0) = 0$ for which there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that*

$$\left\| 2f\left(\frac{\mu x + y}{2}\right) - \mu f(x) - f(y) \right\| \leq \epsilon(\|x\|^p + \|y\|^p),$$

$$\left\| 2h\left(\frac{\mu x + y}{2}\right) - \mu h(x) - h(y) \right\| \leq \epsilon(\|x\|^p + \|y\|^p),$$

$$\begin{aligned} &\| f(\{xyz\}) - \{f(x)h(y)h(z)\} - \{h(x)f(y)h(z)\} \\ &-\{h(x)h(y)f(z)\}\| \leq \epsilon(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in \mathcal{J}$ and all $\mu \in S^1$. Then there exist unique $\mathbb{C}$-linear mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ such that

$$\| f(x) - D(x)\| \leq \frac{3 + 3^p}{3 - 3^p}\epsilon\|x\|^p,$$

$$\|h(x) - \theta(x)\| \leq \frac{3 + 3^p}{3 - 3^p}\epsilon\|x\|^p$$

for all $x \in \mathcal{J}$. Moreover, $D : \mathcal{J} \to \mathcal{J}$ is a θ-derivation on $\mathcal{J}$.

Proof. Define $\varphi(x, y, z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.3. □

Theorem 2.6. *Let $f, h : \mathcal{J} \to \mathcal{J}$ be mappings with $f(0) = h(0) = 0$ for which there exists a function $\varphi : \mathcal{J}^3 \to [0, \infty)$ satisfying* (2.9), (2.10) *and* (2.4) *such that*

$$\sum_{j=0}^{\infty} 3^{3j}\varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right) < \infty \tag{2.16}$$

for all $x, y, z \in \mathcal{J}$. Then there exist unique $\mathbb{C}$-linear mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ such that

$$\| f(x) - D(x)\| \leq \widetilde{\varphi}\left(\frac{x}{3}, -\frac{x}{3}, 0\right) + \widetilde{\varphi}\left(-\frac{x}{3}, x, 0\right), \tag{2.17}$$

$$\|h(x) - \theta(x)\| \leq \widetilde{\varphi}\left(\frac{x}{3}, -\frac{x}{3}, 0\right) + \widetilde{\varphi}\left(-\frac{x}{3}, x, 0\right) \tag{2.18}$$

for all $x \in \mathcal{J}$, where

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} 3^j\varphi\left(\frac{x}{3^j}, \frac{y}{3^j}, \frac{z}{3^j}\right)$$

for all $x, y, z \in \mathcal{J}$. Moreover, $D : \mathcal{J} \to \mathcal{J}$ is a θ-derivation on $\mathcal{J}$.

Proof. By Theorem 1.2, it follows from (2.16), (2.9) and (2.10) that there exist unique additive mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ satisfying (2.17) and (2.18). The additive mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ are given by

$$D(x) = \lim_{l\to\infty} 3^l f\left(\frac{x}{3^l}\right), \tag{2.19}$$

$$\theta(x) = \lim_{l\to\infty} 3^l h\left(\frac{x}{3^l}\right) \tag{2.20}$$

for all $x \in \mathcal{J}$.

By a similar method to the proof of Theorem 2.1, one can show that $D, \theta : \mathcal{J} \to \mathcal{J}$ are $\mathbb{C}$-linear mappings.

It follows from (2.4) that

$$3^{3l} \| f\left(\frac{\{xyz\}}{3^{3l}}\right) - \left\{f\left(\frac{x}{3^l}\right) h\left(\frac{y}{3^l}\right) h\left(\frac{z}{3^l}\right)\right\} - \left\{h\left(\frac{x}{3^l}\right) f\left(\frac{y}{3^l}\right) h\left(\frac{z}{3^l}\right)\right\}$$
$$- \left\{h\left(\frac{x}{3^l}\right) h\left(\frac{y}{3^l}\right) f\left(\frac{z}{3^l}\right)\right\} \| \le 3^{3l}\varphi\left(\frac{x}{3^l}, \frac{y}{3^l}, \frac{z}{3^l}\right),$$

which tends to zero as $l \to \infty$ for all $x, y, z \in \mathcal{J}$ by (2.16). By (2.19) and (2.20),

$$D(\{xyz\}) = \{D(x)\theta(y)\theta(z)\} + \{\theta(x)D(y)\theta(z)\} + \{\theta(x)\theta(y)D(z)\}$$

for all $x, y, z \in \mathcal{J}$. So the additive mapping $D : \mathcal{J} \to \mathcal{J}$ is a θ-derivation on $\mathcal{J}$.□

Corollary 2.7. *Let $f, h : \mathcal{J} \to \mathcal{J}$ be mappings with $f(0) = h(0) = 0$ for which there exist constants $\epsilon \ge 0$ and $p \in (3, \infty)$ such that*

$$\left\|2 f\left(\frac{\mu x + y}{2}\right) - \mu f(x) - f(y)\right\| \le \epsilon(\|x\|^p + \|y\|^p),$$

$$\left\|2h\left(\frac{\mu x + y}{2}\right) - \mu h(x) - h(y)\right\| \le \epsilon(\|x\|^p + \|y\|^p),$$

$$\| f(\{xyz\}) - \{f(x)h(y)h(z)\} - \{h(x) f(y)h(z)\}$$
$$-\{h(x)h(y) f(z)\}\| \le \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{J}$ and all $\mu \in S^1$. Then there exist unique $\mathbb{C}$-linear mappings $D, \theta : \mathcal{J} \to \mathcal{J}$ such that

$$\| f(x) - D(x)\| \le \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p,$$

$$\|h(x) - \theta(x)\| \leq \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p$$

for all $x \in \mathcal{J}$. Moreover, $D : \mathcal{J} \to \mathcal{J}$ is a θ-derivation on $\mathcal{J}$.

Proof. Define $\varphi(x, y, z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$, and apply Theorem 2.5. □

Definition 2.8. *Let $\theta : \mathcal{J} \to \mathcal{J}$ be a $\mathbb{C}$-linear mapping. A $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ is called a Jordan θ-derivation on $\mathcal{J}$ if*

$$D(\{xxx\}) = \{D(x)\theta(x)\theta(x)\} + \{\theta(x)D(x)\theta(x)\} + \{\theta(x)\theta(x)D(x)\}$$

holds for all $x \in \mathcal{J}$.

Problem 2.1. Is every Jordan θ-derivation a θ-derivation?

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