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# A characterization of Clifford tori with constant scalar curvature one by the first stability eigenvalue

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*— Dedicated to the memory of Prof. José F. Escobar, Chepe*

**Abstract.** Let $M$ be a compact hypersurface with constant scalar curvature one immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. As is well-known, such hypersurfaces can be characterized variationally as critical points of the integral $\int_M H\,dv$. In this paper we derive a sharp upper bound for the first eigenvalue of the corresponding Jacobi operator in terms of the mean curvature of the hypersurface. Moreover, we prove that this bound is achieved only for the Clifford tori in $\mathbb{S}^{n+1}$ with scalar curvature one.

**Keywords:** constant scalar curvature, Clifford torus, Jacobi operator, first eigenvalue.

**Mathematical subject classification:** Primary 53C42, Secondary 53A10.

## 1 Introduction

Let $\psi : M^n \to \mathbb{S}^{n+1}$ be an immersed orientable hypersurface of the unit Euclidean sphere $\mathbb{S}^{n+1}$. We will denote by $A$ its second fundamental form (with respect to a globally defined normal unit vector field $N$), with principal curvatures $\kappa_1, \dots, \kappa_n$, and by $H$ the mean curvature of the hypersurface, $H = (1/n)S_1$, where $S_1 = \mathrm{tr}(A) = \sum_{i=1}^n \kappa_i$ is the first elementary symmetric function of the principal curvatures. We will also use the second elementary symmetric function of the principal curvatures, denoted by $S_2$, which is related to the (normalized) scalar curvature $R$ of the hypersurface by the Gauss equation

$$n(n-1)(R-1) = 2S_2 = 2\sum_{i<j=1}^{n} \kappa_i \kappa_j. \tag{1}$$

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The first Newton transformation of the hypersurface is given by

$$P_1 = S_1 I - A = nHI - A,$$

where $I$ stands for the identity operator on $\mathcal{X}(M)$. Observe that $P_1$ is also a self-adjoint linear operator which commutes with $A$, and $\mathrm{tr}(P_1) = (n-1)S_1 = n(n-1)H$. In a recent paper, Alencar, do Carmo and Santos [2], using the first Newton transformation, have obtained the following gap theorem for closed (compact without boundary) hypersurfaces of the sphere with constant scalar curvature $R = 1$ (equivalently $S_2 = 0$).

**Theorem 1.** [2, Theorem 1]. *Let $M^n$ be a closed orientable hypersurface with constant scalar curvature $R = 1$ isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. Assume that $S_1$ does not change sign and choose the orientation such that $S_1 \geq 0$. Assume further that*

$$|\sqrt{P_1}A|^2 \leq \mathrm{tr}(P_1) = (n-1)S_1.$$

*Then*

(i) $|\sqrt{P_1}A|^2 = (n-1)S_1$*;*

(ii) *$M$ is either totally geodesic or $M^n = \mathbb{S}^{n_1}(r_1) \times \mathbb{S}^{n_2}(r_2) \subset \mathbb{S}^{n+1}$, where $n_1 + n_2 = n$, $r_1^2 + r_2^2 = 1$, and $\beta = (r_2/r_1)^2$ satisfies the quadratic equation*

$$n_1(n_1-1)\beta^2 - 2n_1n_2\beta + n_2(n_2-1) = 0.$$

As explained by the authors, Theorem 1 above was inspired by a well known similar result on minimal hypersurfaces in the Euclidean sphere first proved by Simons [11] (part (i)), and later completed, simultaneous and independently, by Chern, do Carmo and Kobayashi [5] and Lawson [7].

On the other hand, it is well known that hypersurfaces of the sphere with constant scalar curvature $R = 1$ can be characterized variationally as critical points of the integral $\int_M H\mathrm{d}v$, where $\mathrm{d}v$ stands for the volume element of $M$ (for the details see, for instance, [9, 10, 3]). The Jacobi equation of this variational problem is given by

$$T_1 f = L_1 f + |\sqrt{P_1}A|^2 f + \mathrm{tr}(P_1) f = L_1 f + |\sqrt{P_1}A|^2 f + n(n-1)Hf.$$

Here $f \in C^\infty(M)$ and $L_1$ is a second order differential operator defined by

$$L_1 f = \mathrm{div}(P_1(\nabla f)),$$

where $\nabla f$ is the gradient of $f$.

In general, the operator $L_1$ is not elliptic. It is clear from the definition that $L_1$ is elliptic if and only if $P_1$ is positive definite (or negative definite). In our case, for hypersurfaces of the sphere $\mathbb{S}^{n+1}$ with constant scalar curvature $R = 1$ (equivalently, $S_2 = 0$), $L_1$ is elliptic if and only if $n \geq 3$ and the third elementary symmetric function of the principal curvatures, denoted by $S_3$, does not vanish on $M$ (see [6, Proposition 1.5] and [2, Theorem 2.1]). When the operator $L_1$, and hence the Jacobi operator $T_1$, are elliptic, we may always choose the orientation such that $P_1$ is positive definite, $H > 0$ and $S_3 < 0$ on $M$. In that case, we can use the min-max characterization of the first eigenvalue of $T_1$ as

$$\lambda_1^{T_1} = min \left\{ \frac{-\int_M f T_1(f) \mathrm{d}v}{\int_M f^2 \mathrm{d}v}; \quad f \in C^\infty(M), \, f \not\equiv 0 \right\}. \tag{2}$$

Observe that with our criterion, a real number $\lambda \in \mathrm{Spec}(T_1)$ if and only if

$$T_1 f + \lambda f = 0$$

for some smooth function $f \in C^\infty(M)$, $f \not\equiv 0$. Using $f \equiv 1$ as a test function, it easily follows from (2) that

$$\lambda_1^{T_1} \leq -\frac{1}{\mathrm{vol}(M)} \left( \int_M (|\sqrt{P_1} A|^2 + n(n-1)H) \mathrm{d}v \right) < 0.$$

where $\mathrm{vol}(M)$ is the $n$-dimensional volume of $M$.

Appart from the totally geodesic equators, the easiest hypersurfaces in $\mathbb{S}^{n+1}$ with constant scalar curvature one belong to the family of the Clifford tori. A Clifford torus in $\mathbb{S}^{n+1}$ is obtained by considering the standard immersions $\mathbb{S}^m(r) \hookrightarrow \mathbb{R}^{m+1}$ and $\mathbb{S}^{n-m}(\sqrt{1-r^2}) \hookrightarrow \mathbb{R}^{n-m+1}$, for a given radius $0 < r < 1$ and integer $m \in \{1, \dots, n-1\}$, and taking the product immersion $\mathbb{S}^m(r) \times \mathbb{S}^{n-m}(\sqrt{1-r^2}) \hookrightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. Its principal curvatures are given by

$$\kappa_1 = \cdots = \kappa_m = -\frac{\sqrt{1-r^2}}{r}, \quad \kappa_{m+1} = \cdots = \kappa_n = \frac{r}{\sqrt{1-r^2}},$$

and its constant mean curvature $H = H(r)$ and constant (normalized) scalar curvature $R = R(r)$ are given by

$$nH(r) = S_1 = \frac{nr^2 - m}{r\sqrt{1-r^2}}$$

and

$$n(n-1)(R(r)-1) = 2S_2 = \frac{n(n-1)r^4 - 2(n-1)mr^2 + m(m-1)}{r^2(1-r^2)}.$$

In particular, $R(r) = 1$ if and only if $X = r^2$ satisfies the following quadratic equation

$$n(n-1)X^2 - 2m(n-1)X + m(m-1) = 0, \quad 0 < X < 1.$$

Observe that for each fixed dimension $n$, $n \geq 3$, there are exactly (up to congruences) $n-2$ Clifford tori in $\mathbb{S}^{n+1}$ with constant scalar curvature one (equivalently, $S_2 = 0$), which are of the form

$$\mathbb{S}^m(r_m) \times \mathbb{S}^{n-m}(\sqrt{1-r_m^2}), \quad m = 1, \dots, n-2,$$

with

$$r_m^2 = \frac{(n-1)m + \sqrt{(n-1)m(n-m)}}{n(n-1)},$$

and all of them satisfy (under the appropriate orientation) $3S_3 = -(n-1)S_1 = -n(n-1)H < 0$ (for the details, we refer the reader to [2, Section 2.2]). As observed by Alencar, do Carmo and Santos in [2], for these Clifford tori with constant scalar curvature one, the Jacobi stability operator $T_1$ is elliptic and reduces to

$$T_1 = L_1 + 2(n-1)S_1 = L_1 + 2n(n-1)H, \quad H = \text{positive constant},$$

and

$$\lambda_1^{T_1} = -2(n-1)S_1 = -2n(n-1)H < 0.$$

Motivated by the value of $\lambda_1^{T_1}$ for these Clifford tori, in this paper we will prove the following result.

**Theorem 2.** *Let $M^n$ be a closed orientable hypersurface with constant scalar curvature $R = 1$ (equivalently, $S_2 = 0$) isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. Assume that $n \geq 3$ and $S_3$ does not vanishes on $M$, and choose the orientation such that $H > 0$. Let $\lambda_1^{T_1}$ stands for the first eigenvalue of the Jacobi stability operator*

$$T_1 = L_1 + |\sqrt{P_1}A|^2 + \mathrm{tr}(P_1) = L_1 + |\sqrt{P_1}A|^2 + n(n-1)H.$$

*Then*

$$\lambda_1^{T_1} \leq -2n(n-1) \min H \tag{3}$$

*and equality holds if and only if $M$ is a Clifford torus with constant scalar curvature one; that is, up to a congruence,*

$$M = \mathbb{S}^m(r_m) \times \mathbb{S}^{n-m}(\sqrt{1-r_m^2}) \subset \mathbb{S}^{n+1}, \quad m = 1, \dots, n-2,$$

*with*

$$r_m^2 = \frac{(n-1)m + \sqrt{(n-1)m(n-m)}}{n(n-1)}.$$

Theorem 2 was motivated by a similar result for minimal hypersurfaces of the sphere, recently obtained by Perdomo [8] (see also the previous papers by Simons [11] and Wu [12]). Specifically, that result states that if $M^n$ is a closed orientable minimal hypersurface of the sphere $\mathbb{S}^{n+1}$ which is not totally geodesic, and $\lambda_1^J$ stands for the first eigenvalue of its stability operator $J = -\Delta - |A|^2 - n$, then $\lambda_1^J \leq -2n$, with equality if and only if $M$ is a minimal Clifford torus $\mathbb{S}^{n-m}(\sqrt{(n-m)/n}) \times \mathbb{S}^m(\sqrt{m/n}) \subset \mathbb{S}^{n+1}$.

Our bound (3) is sharp and achieved only for the Clifford tori in $\mathbb{S}^{n+1}$ with scalar curvature one, $\mathbb{S}^m(r_m) \times \mathbb{S}^{n-m}(\sqrt{1-r_m^2}) \subset \mathbb{S}^{n+1}$ with $m = 1, \dots, n-2$. However, our bound does not depend only on the dimension $n$ of the manifold, but also on its mean curvature $H$. It would be very interesting to find a bound $c(n)$ which would depend only on the dimension $n$. A natural candidate for $c(n)$ would be the maximum value of $\lambda_1^{T_1}$ over the Clifford tori

$$\mathbb{S}^m(r_m) \times \mathbb{S}^{n-m}\left(\sqrt{1-r_m^2}\right) \subset \mathbb{S}^{n+1}$$

with $m = 1, \dots, n-2$. If we denote by $\lambda_1^{T_1}(n,m)$ the value of $\lambda_1^{T_1}$ for $\mathbb{S}^m(r_m) \times \mathbb{S}^{n-m}(\sqrt{1-r_m^2})$, a direct computation shows that

$$\begin{aligned} \lambda_1^{T_1}(n,m) &= -2n(n-1)H(r_m) \\ &= \frac{-2n(n-1)\sqrt{m(n-m)}}{\sqrt{(n-2)m(n-m) + (n-2m)\sqrt{(n-1)m(n-m)}}} \end{aligned}$$

and

$$\lambda_1^{T_1}(n, n-2) < \cdots < \lambda_1^{T_1}(n,2) < \lambda_1^{T_1}(n,1) = -n(n-1)\sqrt{\frac{2}{n-2}}.$$

Therefore, one could expect that, under the hypothesis of our Theorem 2, it holds that

$$\lambda_1^{T_1} \leq -n(n-1)\sqrt{\frac{2}{n-2}},$$

with equality if and only if $M$ is, up to a congruence, the Clifford torus $\mathbb{S}^1(\sqrt{2/n}) \times \mathbb{S}^{n-1}(\sqrt{(n-2)/n})$. Unfortunately, as far as we know, our technique does not allow us to conclude so.

This paper was finished while the first author was visiting the Instituto de Matemática Pura e Aplicada (IMPA) at Rio de Janeiro, Brazil, in July 2003. There, he had the ocassion to speak to Professors do Carmo and Santos about Theorem 2, and he was informed that they have also obtained, simultaneous and independently, the following related result [4], which can be seen as a consequence of our Theorem 2.

**Corollary 3.** [4] *Let $M^n$ be a closed orientable hypersurface with constant scalar curvature $R = 1$ (equivalently, $S_2 = 0$) isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. Assume that $n \geq 3$ and $S_3$ does not vanishes on $M$, and choose the orientation such that $H > 0$. If $\lambda_1^{T_1} \geq -2n(n-1)H$, then $M$ is a Clifford torus with constant scalar curvature one.*

## 2 Proof of Theorem 2

The estimative (3) in our Theorem 2 will be an application of the following result concerning the first eigenvalue of the Jacobi stability operator $T_1$.

**Proposition 4.** *Let $M^n$ be a closed orientable hypersurface with constant scalar curvature $R = 1$ (equivalently, $S_2 = 0$) isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. Assume that $n \geq 3$ and $S_3$ does not vanishes on $M$, and choose the orientation such that $H > 0$. If $\lambda_1^{T_1}$ stands for the first eigenvalue of the Jacobi stability operator $T_1$, then*

$$\lambda_1^{T_1} \leq -2n(n-1)\frac{\int_M H^3 \mathrm{d}v}{\int_M H^2 \mathrm{d}v}$$

*where $\mathrm{d}v$ stands for the volume element of $M^n$.*

**Proof.** Since $H > 0$ on $M$, we can use $H$ as a test function in (2) to estimate $\lambda_1^{T_1}$. Let us recall the following Simons type formula for $L_1(H)$, which for the case of hypersurfaces with scalar curvature $R = 1$ immersed into $\mathbb{S}^{n+1}$ reads as follows

$$\begin{aligned} L_1(H) &= \frac{1}{n}|\nabla A|^2 - n|\nabla H|^2 + n(n-1)H^2 + 3HS_3 \\ &= \frac{1}{n}|\nabla A|^2 - n|\nabla H|^2 + n(n-1)H^2 - |\sqrt{P_1}A|^2 H, \end{aligned} \tag{4}$$

(for a proof, see [1, Lemma 3.7], taking into account that in our case $S_2 = 0$ and $|A| = S_1 = nH$, and $|\sqrt{P_1}A|^2 = -3S_3 > 0$). Moreover, we also know (see [1, Lemma 4.1]) that

$$|\nabla A|^2 \geq n^2|\nabla H|^2, \tag{5}$$

so that

$$L_1(H) \geq (n(n-1)H - |\sqrt{P_1}A|^2)H, \tag{6}$$

with equality if and only if $|\nabla A|^2 = n^2|\nabla H|^2$. Therefore,

$$HT_1(H) = HL_1H + |\sqrt{P_1}A|^2H^2 + n(n-1)H^2 \geq 2n(n-1)H^3,$$

Now, using $f = H$ in (3), we conclude from here that

$$\lambda_1^{T_1} \leq \frac{-\int_M HT_1(H)\mathrm{d}v}{\int_M H^2\mathrm{d}v} \leq -2n(n-1)\frac{\int_M H^3\mathrm{d}v}{\int_M H^2\mathrm{d}v},$$

which completes the proof of Proposition 4. $\square$

Now we are ready to prove our Theorem 2. From Proposition 4, we easily see that

$$\lambda_1^{T_1} \leq -2n(n-1)\frac{\int_M H^3\mathrm{d}v}{\int_M H^2\mathrm{d}v} \leq -2n(n-1)\min H. \tag{7}$$

Moreover, if $\lambda_1^{T_1} = -2n(n-1)\min H$, then, from the proof of Proposition 4, equality also holds in (5), which gives that

$$|\nabla A|^2 = n^2|\nabla H|^2. \tag{8}$$

On the other hand, when $\lambda_1^{T_1} = -2n(n-1)\min H$ we also get from (7) that

$$\int_M H^3\mathrm{d}v = \left(\int_M H^2\mathrm{d}v\right)\min H,$$

that is,

$$\int_M H^2(H - \min H)\mathrm{d}v = 0.$$

But $H^2 > 0$ and $(H - \min H) \geq 0$ on $M$, so that $H \equiv \min H$ is constant on $M$. By (8), this implies that $\nabla A = 0$, that is, the second fundamental form is parallel. Finally, we apply a result of Lawson [7, Theorem 4] (see also [5, Lemma 3]) to conclude that $M$ is a Clifford torus with constant scalar curvature $R = 1$; that is, $M$ is, up to congruences, a Clifford torus of the form

$$M = \mathbb{S}^m(r_m) \times \mathbb{S}^{n-m}(\sqrt{1 - r_m^2}), \quad m = 1, \ldots, n-2,$$

with

$$r_m^2 = \frac{(n-1)m + \sqrt{(n-1)m(n-m)}}{n(n-1)}.$$

Conversely, we already know that for every Clifford torus with constant scalar curvature one, the Jacobi stability operator $T_1$ is elliptic and reduces to

$$T_1 = L_1 + 2(n-1)S_1 = L_1 + 2n(n-1)H, \quad H = \text{positive constant},$$

and the first stability eigenvalue is given by

$$\lambda_1^{T_1} = -2n(n-1)H = -2n(n-1)\min H.$$

## 3 Another proof of the case of equality in Theorem 2

In this section we would like to show how Perdomo's technique in [8] also works to characterize the Clifford tori with scalar curvature one as the only closed hypersurfaces with constant scalar curvature $R = 1$ in $\mathbb{S}^{n+1}$ whose first stability eigenvalue $\lambda_1^{T_1}$ satisfies $\lambda_1^{T_1} = -2n(n-1)\min H$, under the assumption that the Jacobi operator $T_1$ is elliptic (equivalently, $n \geq 3$ and $S_3$ does not vanishes on $M$).

To see it, assume that $\lambda_1^{T_1} = -2n(n-1)\min H$. As is well known, since $T_1$ is assumed to be elliptic, then its first eigenvalue $\lambda_1^{T_1}$ is simple and its eigenspace is generated by a first positive eigenfunction $\rho \in C^\infty(M)$. Then

$$T_1\rho + \lambda_1^{T_1}\rho = 0$$

or, equivalently,

$$L_1(\rho) = -\big(\lambda_1^{T_1} + |\sqrt{P_1}A|^2 + n(n-1)H\big)\rho. \tag{9}$$

Observe that $\nabla\rho^{-1} = -\rho^{-2}\nabla\rho$ and

$$\begin{aligned} L_1(\rho^{-1}) &= \operatorname{div}(-\rho^{-2}P_1(\nabla\rho)) = -\rho^{-2}L_1(\rho) + 2\rho^{-3}\langle\nabla\rho, P_1(\nabla\rho)\rangle \\ &= (\lambda_1^{T_1} + |\sqrt{P_1}A|^2 + n(n-1)H)\rho^{-1} + 2\rho^{-3}\langle\nabla\rho, P_1(\nabla\rho)\rangle. \end{aligned}$$

Define $f = H\rho^{-1} \in C^\infty(M)$. Then

$$\nabla f = H\nabla\rho^{-1} + \rho^{-1}\nabla H = -H\rho^{-2}\nabla\rho + \rho^{-1}\nabla H,$$

and, using also the inequality (6), we can compute as follows

$$\begin{aligned} L_1 f &= L_1(H\rho^{-1}) = HL_1(\rho^{-1}) + \rho^{-1}L_1(H) + 2\langle\nabla H, P_1(\nabla\rho^{-1})\rangle \\ &= H(\lambda_1^{T_1} + |\sqrt{P_1}A|^2 + n(n-1)nH)\rho^{-1} + 2H\rho^{-3}\langle\nabla\rho, P_1(\nabla\rho)\rangle \\ &+ \rho^{-1}L_1(H) - 2\rho^{-2}\langle\nabla H, P_1(\nabla\rho)\rangle \\ &= f(\lambda_1^{T_1} + |\sqrt{P_1}A|^2 + n(n-1)H) \\ &+ \rho^{-1}L_1(H) - 2\rho^{-1}\langle\nabla f, P_1(\nabla\rho)\rangle \qquad (10) \\ &\geq f(\lambda_1^{T_1} + |\sqrt{P_1}A|^2 + n(n-1)H) + f(n(n-1)H - |\sqrt{P_1}A|^2) \\ &- 2\rho^{-1}\langle\nabla f, P_1(\nabla\rho)\rangle \\ &= 2n(n-1)f(H - \min H) - 2\rho^{-1}\langle\nabla f, P_1(\nabla\rho)\rangle \\ &\geq -2\rho^{-1}\langle\nabla f, P_1(\nabla\rho)\rangle. \end{aligned}$$

Summing up,

$$L_1 f + 2\rho^{-1}\langle\nabla f, P_1(\nabla\rho)\rangle \geq 0 \quad \text{on} \quad M. \qquad (11)$$

Let $p_0 \in M$ be the point where the positive function $f$ attains its maximum on $M$, and let $\Omega$ be a region around $p_0$ on which $f$ is greater than some positive constant. Since the maximum of $f$ in $\Omega$ is obtained in the interior of $\Omega$, by (11) and the maximum principle we deduce that $f$ is constant on $\Omega$. Since $M$ is connected, we conclude that $f$ is constant on all $M$. Therefore, $\nabla f = 0$, $L_1(f) = 0$, and equality trivially holds in (11). That means that both inequalities in the computation of (10) must be equalities. Observe now that the first inequality in (10) becomes an equality if and only if equality holds in (6), that is, if and only if $|\nabla A|^2 = n^2|\nabla H|^2$. Besides, the second inequality in (10) becomes an equality if and only if $H = \min H$ is constant on $M$. As a consequence, the second fundamental form $A$ is parallel, and we apply the rigidity result of Lawson to conclude that $M$ is a Clifford torus with scalar curvature $R = 1$.

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