Microsystem Technologies (2019) 25:3005–3015
https://doi.org/10.1007/s00542-018-4244-0

TECHNICAL PAPER

Experimental investigation on size-dependent higher-mode vibration of cantilever microbeams

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Received: 4 September 2018 / Accepted: 24 November 2018 / Published online: 3 December 2018

Abstract
A new perspective is presented to study size-dependent elasticity experimentally within micron scale utilizing dynamic approach. The size-dependent vibration of cantilever microbeams made of nickel is studied for the first three transverse modes. The normalized natural frequency of the first mode manifests strong size effect as reported. Remarkably, the normalized natural frequencies of the second and third mode also increase to 1.9 times as the thickness of microbeams decreases from 15 to 2.1 μm. Similarly the normalized bending rigidity increases to about 3.5 times. It is the first time that elastic size effect is observed in vibration of higher modes. Moreover, the size-dependent vibration of the first mode is interpreted in light of the modified couple stress theory, and the theoretical prediction also fit the experimental results of high modes very well. Hence it is confirmed that modified couple stress theory is valid for vibration of higher modes too.

1 Introduction

Microbeams have been widely used in Micro/Nano-electro-mechanical systems (MEMS/NEMS). Previous research reported a substantial amount of potential applications such as atomic force microscopes (AFMs), micro-actuators, micro-switches and those energy harvesting devices have taken microbeams as the core element. Experimental investigations started since the 1990s have confirmed conspicuous size effects in both metals and polymers. Studies focused on the size effects of plastic behaviors at first, some representative works were carried out by Fleck et al. (1994), Stölken and Evans (1998), Chong and Lam (2011), Chong et al. (2011), and Liu et al. (2012). As examples, in torsion tests of polycrystalline Cu wires, Fleck et al. (1994) found the normalized torque increased drastically with the decreasing wire diameter. Chong et al. (2011) discovered the presence of strain gradient increased the torque by three to nine times at the same twist in torsion experiments.

Compared with abundant experimental researches in size-dependent plasticity, experiments on size effects in elasticity within micron scale, seem to be rare and start relatively late. The earliest work might be due to Lam et al. (2003) performed bending tests on cantilever microbeams made of epoxy. As a result, when the thickness of the microbeam reduced from 115 to 20 μm, the normalized bending rigidity increased by around 2.4 times. Later on, Andrew and Jonathan (2005) conducted similar experiments on cantilever microbeams made of polypropylene, the stiffness was at least four times larger than those predicted by the classical beam theory. Recent years, Tang and Alici (2011) evaluated the size effects in micro- and nano-sized silicon cantilevers by using atomic force microscopy, conclusions were made that the length-scale factor should be considered in accurately estimating the tip deflection. Similarly, Liebold and Müller (2016) performed AFM tests on microbeams made of epoxy and polymer SU-8, positive size effects depending on thickness were observed and quantified for microbeams. Lei et al. (2016) investigated the dynamic characteristics of cantilever microbeams made of nickel. Results reveal that the normalized natural frequency of the first mode is 2.1 times of the classical solution as the thickness decreases from 15 to 2.1 μm. More recently, Li et al. (2018) continued the study and proposed an experimental methodology to determine the material length scale parameter based on modified couple stress theory. Vibration of the first mode was investigated

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for copper and titanium cantilever microbeams, and material length scale parameter was obtained for the two material.

On the other hand, several higher-order continuum theories involving additional material length scale parameters are developed to interpret the size-dependent elasticity as well as match experimental results. The classical couple stress elasticity theory (Mindlin 1964; Toupin 1962) was proposed in the 1960s with two higher-order material length scale parameters. Yang et al. (2002) and Park and Gao (2006) improved the theory and only one material length scale parameter remained in the modified couple stress theory. Lam et al. (2003) introduced a new strain gradient theory containing three length scale parameters l_0, l_1, l_2, corresponding to the dilatation gradient vector, the deviatoric stretch gradient tensor and the curvature tensor. It's noted that strain gradient theory can be reduced to modified couple stress theory if l_0, l_1 are ignored.

In the light of those higher-order continuum theories, a bountiful amount of research has been carried out on the performance of microbeams. Either theoretical (Akgöz and Civalek 2013; Kong et al. 2008; Lei et al. 2015; Li and Hu 2015; Mercan and Civalek 2017) or numerical (Akgöz and Civalek 2011; Kahrobaiyan et al. 2013; Mercan and Civalek 2016; Zhang et al. 2013a, 2014), static (Ansari et al. 2015b; Chen et al. 2015; Civalek and Demir 2011; Mohammad-Abadi and Daneshmehr 2014; Zhang et al. 2013b) or dynamic (Chen and Li 2013; Dehrouyeh-Semnani et al. 2015; Kong et al. 2009; Lei et al. 2013; Shafiei et al. 2016; Sourki and Hoseini 2016; Su et al. 2018; Thai et al. 2015; Wang et al. 2013, 2015), linear (Al-Basyouni et al. 2015; Chen and Li 2013; Fang et al. 2018; Karamanlı and Vo 2018; Tao and Fu 2017; Thai et al. 2018) or nonlinear (Asghari et al. 2010; Baltacıoglu et al. 2010; Dai and Wang 2017; Dai et al. 2015; Ghayesh and Farokhi 2018; Krysko et al. 2017; Xia et al. 2010) investigations are employed. Akgöz and Civalek (2012) provided an analytical solution for microbeams bending for four kinds of boundary conditions respectively employing strain gradient theory and modified couple stress theory. Kong et al. (2008, 2009) analyzed the static and dynamic response of microbeams utilizing strain gradient elasticity theory and the modified couple stress theory respectively. Tao and Fu (2017) investigated the thermal buckling and postbuckling of composite laminated microbeams considering geometric nonlinear theory and Timoshenko beam hypothesis. Analytical and numerical solutions were employed to solve the problem. Yang et al. (2017) proposed a size-dependent composite laminated microbeam model based on a new modified couple stress theory which was suitable for anisotropic constitutive relation. Fang et al. (2018) established a size-dependent three-dimensional dynamic model for rotating functionally graded (FG) microbeams incorporating modified couple stress theory and von Kármán geometric nonlinearity, and free vibration was studied. Recently, interests are concentrated on the nonlinear behavior of microbeams. Dai and Wang (2017) proposed a new nonlinear model based on the modified couple stress theory for cantilever microbeams under electrical force to explore the nonlinear dynamics considering geometric and inertial nonlinearities. Ansari et al. conducted a series of research on size-dependent nonlinear structures (Gholami and Ansari 2018), both static (Ansari and Gholami 2016; Gholami and Ansari 2017) and dynamics (Ansari et al. 2015a, 2016a; Gholami and Ansari 2016) were investigated. For example, they (Ansari et al. 2016b; Gholami et al. 2015) respectively developed a nonlinear microstructure-dependent third-order shear deformable beam model for functionally graded microbeams and microswitches based on strain gradient elasticity theory. Recently Ghayesh (2018) presented a size-dependent model for the coupled nonlinear dynamics of extensible functionally graded microbeams with viscoelastic properties.

An evident fact could be found that comparing with massive theoretical and numerical discussions about size-dependent microbeams, experiments proving the evidence of size effect, especially in elasticity is definitely limited. All the related size-dependent experiments within micron scale are listed in the second paragraph above. Apparently, one can see the experimental methodology is monotonous, size effects could be only observed by static bending, except one kind of study with vibration tests [see (Lei et al. 2016; Li et al. 2018)]. Hence it is essential to seek for more experimental proofs of elastic size effects as well as explore different manifestations of size effects. To that end, an experimental study is carried out on the linear vibration of metallic microbeams exploring the natural frequency of higher modes. Indeed, attempts have been made to test higher modes natural frequencies in macro scale (Banerjee et al. 2007; Jaworski and Dowell 2008; Raville et al. 1961; Wu et al. 2014) in the past decades. Banerjee et al. (2007) studied the natural frequencies of the first, second and third mode for various sandwich cantilever beams. Jaworski and Dowell (2008) investigated the vibration of flexural-free cantilevered beams with multiple cross-section steps and the first and second flapwise (out-of-plane) bending mode frequencies as well as the first chordwise (in-plane) bending mode frequency were obtained.

In the present paper, elastic size effect within micron scale is confirmed in a new aspect, i.e. vibration of higher modes in microbeams. Natural frequencies of cantilever microbeams are measured for the first three modes based on a set of self-assembled vibration measurement system. The cantilever microbeams, made of nickel, are employed in the experiments with thickness ranging from 15 to

2.1 μm. Results reveal that for natural frequency of the first mode, the normalized frequency grows around 1.9 times as the thickness of microbeams decrease to 2.1 μm, which accords well with results in the previous work (Lei et al. 2016). For natural frequencies of the second mode and third mode, remarkably, the normalized frequencies both also increase to about 1.9 times with the thickness decreasing to 2.1 μm. That would be significant evidence of the existence of elastic size effect in the vibration of higher modes. Then the observations are interpreted based on modified couple stress theory (Lam et al. 2003) and the two are in full agreement.

2 Experiment setup

Nickel (Ni) foils (99.99% purity) with respective thickness of 2.1, 3.2, 5.2, 10.0 and 15.0 μm are employed in the experiment. The Young's modulus (E), Poisson's ratio (v) and mass density (ρ) of Ni are 207 GPa, 0.29 and 8900 kg/m^3 respectively.

The geometric size of a cantilever microbeam is prescribed as 1 mm wide and 5 mm long. It is decided for consideration that too high an aspect ratio of a cantilever microbeam could easily lead to buckling while too low an aspect ratio is not suitable for a beam theory. Also too low an aspect ratio would cause difficulties in assembling as well as achieving resonance. Firstly Ni foils of each thickness were cut into rectangular strips with width 1 mm and length 10 mm. The samples are then annealed at 300 °C for 5 h to eliminate the machining residual stress. Otherwise, the resultant experimental data would be too scattered to be analyzed due to the effects of residual stress in the samples induced in fabrication process. Finally one end of each strip is glued to a rigid plate with 5 mm, while the other end extends out of the plate edge. Thus cantilever microbeams with width 1 mm and length 5 mm are prepared and subsequently mounted on the non-contact dynamic vibration measurement system.

The non-contact dynamic vibration measurement system consists of a laser Doppler vibrometer (LDV), a direct digital frequency synthesis (DDS) signal generator with minimum resolution of 0.01 Hz, a loudspeaker with power of 2 W, a computer, a three-dimensional (3D) translation stage and a conical shell, as illustrated in Fig. 1. The LDV probe with laser beam diameter of 50 μm at the focused point is used for vibration detection of the microbeam. The maximum sampling rate of LDV is as high as 20 MHz, which is capable of capturing the vibration response of cantilever microbeams accurately. And then the vibration response is displayed on the computer connected to LDV. The DDS signal generator generates continuous sinusoidal acoustic wave to excite the cantilever microbeam by the aid of loudspeaker. As for the conical shell, it is mounted on the loudspeaker, aiming at gathering energy. When the excitation frequency reaches a relatively high level (e.g. more than 2000 Hz), the vibration from the loudspeaker will become so weak as to be not able to excite the microbeams. In other words, the resulting vibration response in the microbeams at that time would be too subtle to be detected. Therefore conical shell will be mounted only if the vibration response cannot be captured.

Before the experiment the distance and angular between the Laser probe and microbeam are elaborately adjusted to ensure the laser beam is focused on the free end of the microbeam. The cantilever microbeam is mounted uprightly on the 3-D translation stage, in that way the laser beam transmitted from the LDV could be reflected the same path right back to the laser probe, which guarantees a good reflection rate. The vibration response of the cantilever microbeam is then detected and displayed on the computer connected to the LDV.

The first three vibration modes of cantilever microbeams are depicted in Fig. 2. To detect the response accurately, the laser spot should be focused on the place where the deflection is relatively large. As an example, the laser spot for natural frequency of the first mode should be focused on the free end of the microbeam, as shown in Fig. 2, which has been completed in the preparation stage before.

The natural frequency of the first mode is measured at first. The excitation frequency of the sinusoidal signal starts from a relatively low level less than the natural frequency and is increased gradually. Each time after the increase, wait for 5 s to allow the response to stay stable, and then the amplitude of the vibration is recorded. When the excitation frequency approaches the resonant frequency, adjust the excitation frequency at the step of 1 Hz for sake of accuracy. The process continues until the recorded amplitude no longer increases but reversely decreases to a relatively low level, which indicates the excitation frequency has surpassed the resonant frequency. Thus a complete frequency–response curve near the resonant frequency is obtained. From the frequency–response curve, one can read the resonant frequency, and the error is within 0.5 Hz.

For natural frequencies of the second and third modes, the conical shell will be mounted on the loudspeaker if the excitation frequency increases to more than 2000 Hz. And the laser spot should be adjusted to the corresponding location as shown in Fig. 2. Identical to the former, each time after regulation, wait for 5 s to allow the response to stay stable, and then the amplitude of response is recorded. Some typical amplitude-response curves (A–F curve) of the three modes are illustrated in Fig. 3. One can see that the maximum amplitude of the microbeam is comparable to the respective thickness, which means the deflection is

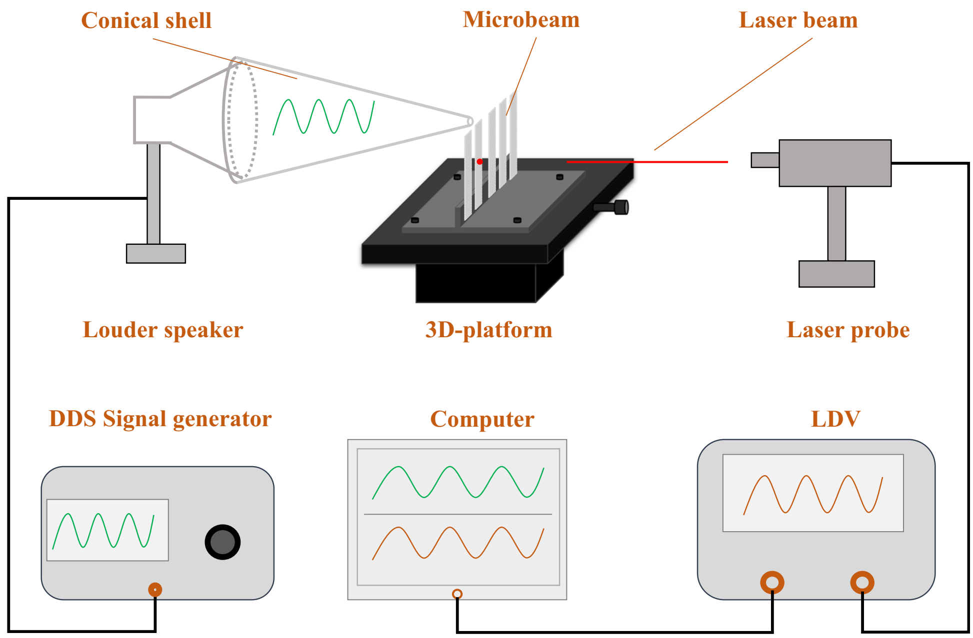

Fig. 1 Schematic diagram of vibration measurement system

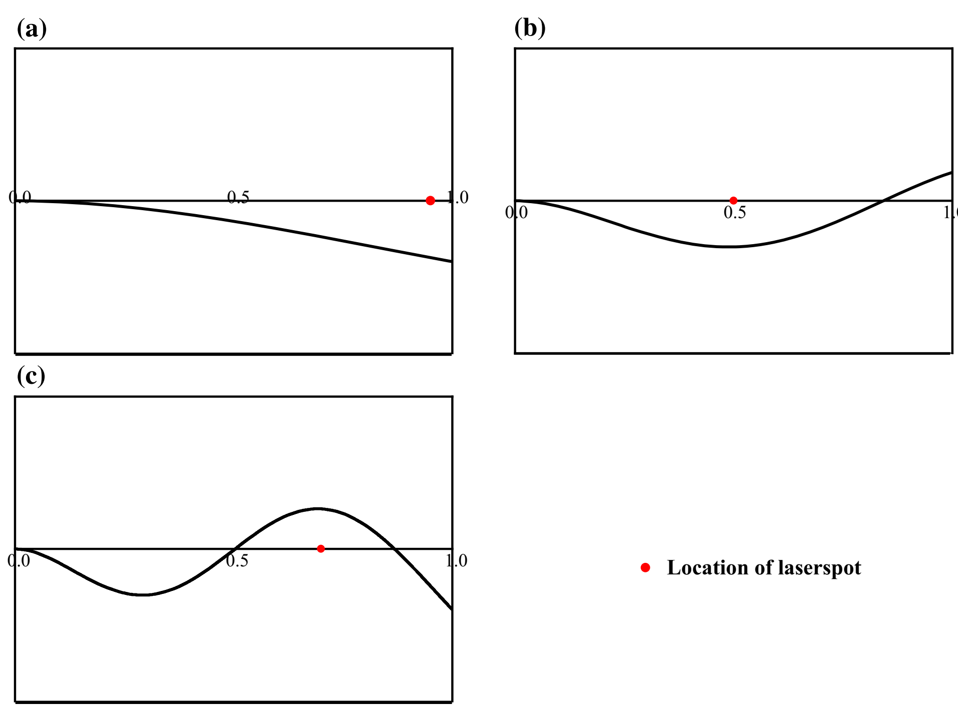

Fig. 2 Modal shapes of cantilever microbeam for the first three modes: **a** the first mode, **b** the second mode, **c** the third mode

infinitesimal. More importantly, the frequency–response curve is symmetric near the peak of the curve, which indicates the vibration of each mode is in the linear elastic range.

To reduce experimental error, the natural frequency of one specimen is measured at least three times for each mode and the experimental data are averaged. The exact length and width of the cantilever microbeams are examined in an optical microscope to ensure the tolerance is within 0.05 mm before the experiments, if not, the specimen is discarded. Furthermore, the actual thickness is measured by cutting a square hole vertically in the center of the specimen with focused ion beam (FIB). Besides, the surface roughness of each sample is tested in a laser microscopy. The measured roughness is small enough to perform the experiment. Some more details about the experiment are referred to Lei et al. (2016).

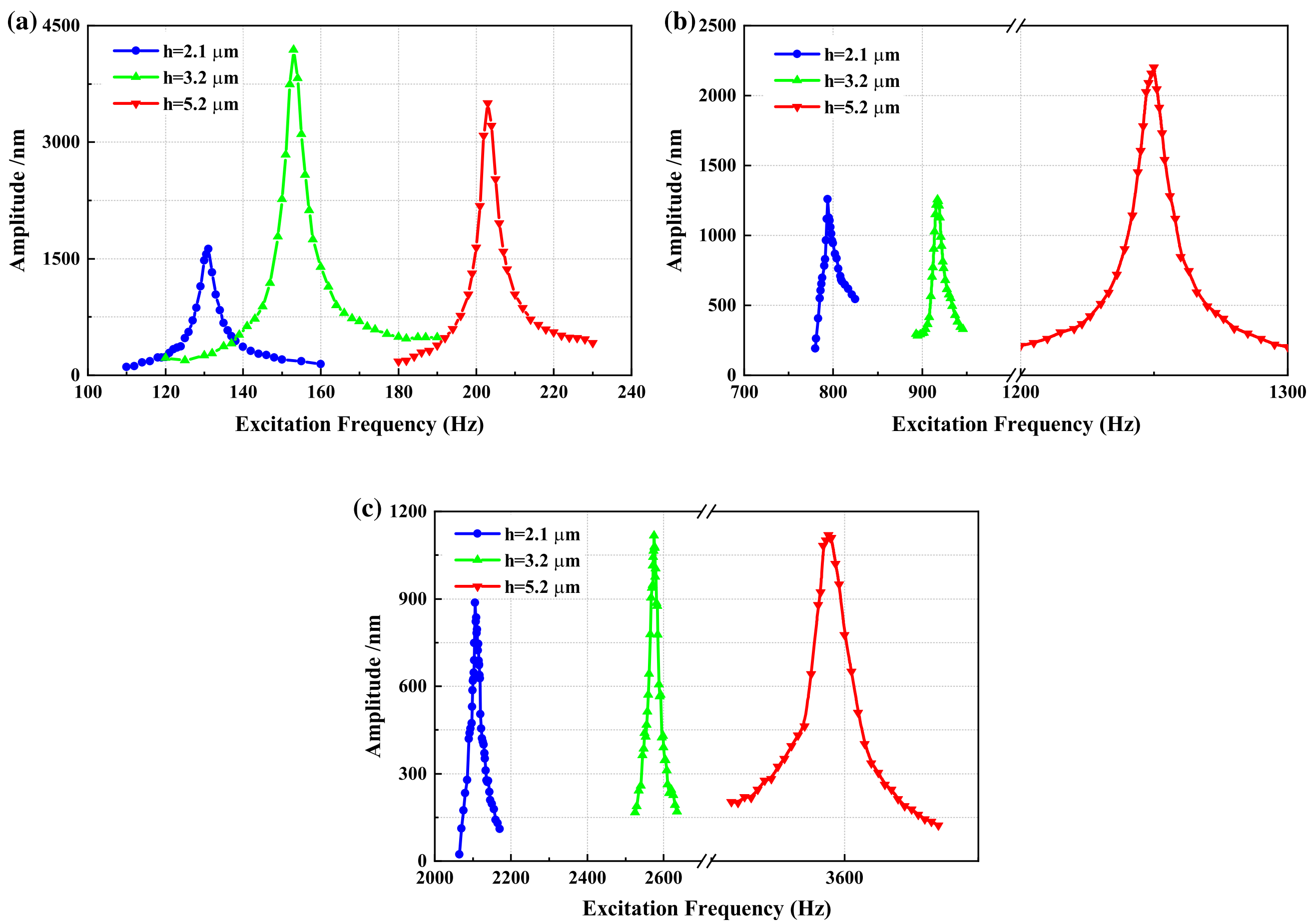

Fig. 3 Amplitude–frequency curves of different thicknesses and modes: **a** the first mode, **b** the second mode, **c** the third mode

3 Results and discussion

The obtained natural frequencies for the second and third modes are listed in Tables 1 and 2. The averaged natural frequencies of the first mode in Fig. 6 show a full agreement with the results in the literature (Lei et al. 2016). The normalized natural frequency is defined as the ratio of experimental frequency to classical solution of frequency, i.e.

$$\bar{\varpi} = \frac{\omega_e}{\omega_c} \tag{1}$$

where ω_c is given by

$$\omega_{cn} = \beta_n^2 \sqrt{\frac{EI}{\rho A L^4}}, \quad \beta_n = 1.875, 4.694, 7.855, \ldots \quad (n = 1, 2, 3, \ldots) \tag{2}$$

where EI is the bending rigidity, ρ, A, L are the mass density, cross section area and length of a cantilever beam. Subscript n stands for the nth mode. Hence the normalized bending rigidity is subsequently defined as

$$\xi = \left(\frac{\omega_e}{\omega_c}\right)^2 \tag{3}$$

The normalized frequency and normalized bending rigidity are depicted in Figs. 4 and 5. Figure 4 a shows that

Table 1 Natural frequencies of the second mode

Thickness (μm)	Frequency (Hz)					Average
	No. 01	No. 02	No. 03	No. 04	No. 05	
2.1	799	815	734	767	729	769
3.2	960	912	901	978	976	945
5.2	1229	1158	1477	1288	1143	1259
10.0	2218	2115	2082	2123	2051	2118
15.0	2884	2898	3025	2890	2816	2903

Table 2 Natural frequencies of the third mode

Thickness (μm)	Frequency (Hz)					Average
	No. 01	No. 02	No. 03	No. 04	No. 05	
2.1	2101	2217	2079	2162	2036	2119
3.2	2598	2569	2787	2447	2908	2662
5.2	3230	3220	3290	3807	3245	3358
10.0	6290	6000	5840	5965	5760	5971
15.0	8070	7925	8530	8075	7910	8102

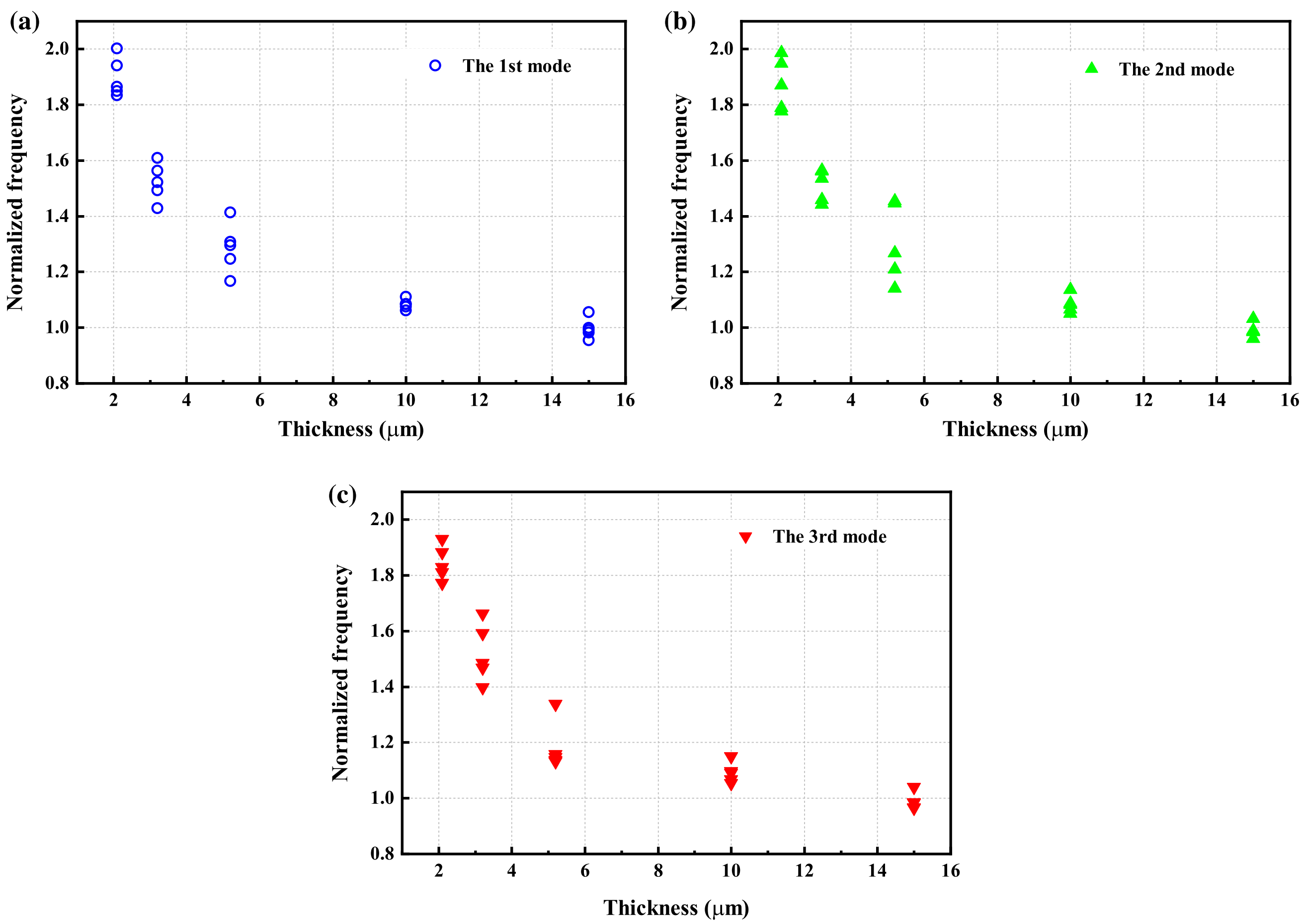

Fig. 4 Normalized frequencies of three modes: **a** the first mode, **b** the second mode, **c** the third mode

as thickness decreases, the normalized frequency of the first mode increases nearly once higher. Similarly, the normalized bending rigidity grows to about four times. This observation is in good accordance with the former experiment (Lei et al. 2016). More significantly, it is clearly demonstrated in Fig. 4b, c that with decreasing thickness, the normalized frequencies of the second and third mode increase to about twice. Correspondingly, the normalized bending rigidity in Fig. 5b, c grows nearly three times higher. This is the substantial evidence revealing that vibration of higher modes is size-dependent, which is observed for the first time.

According to Kong et al. (2008) and Lei et al. (2016) the size-dependent vibration of the first mode is interpreted by modified couple stress theory (MCS theory). In the essence of modified couple stress theory, the governing equation is written as.

$$\rho A\ddot{W} + (EI + \mu A l^2)W^{(4)} = 0 \tag{4}$$

Hence the natural frequency is derived as.

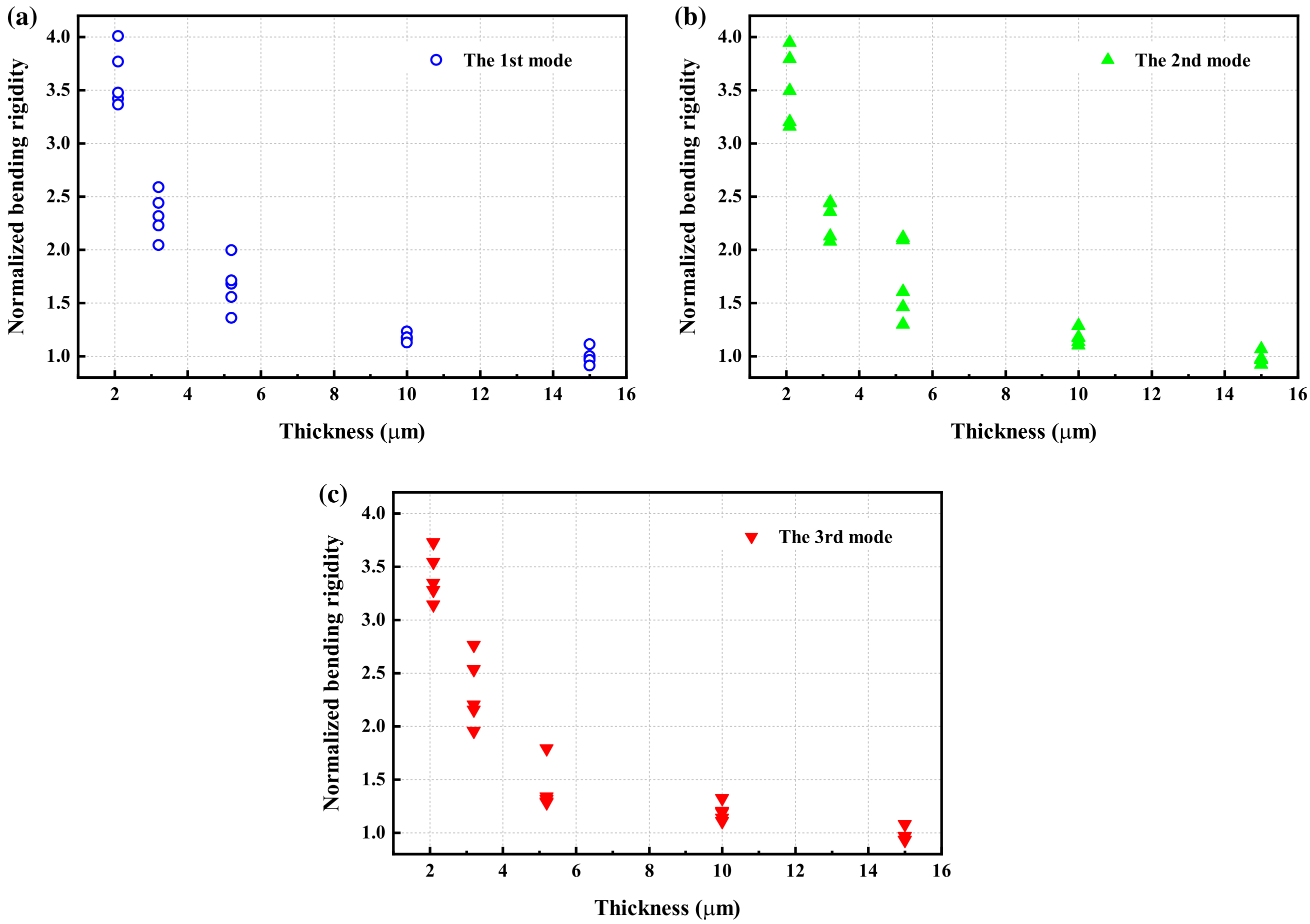

Fig. 5 Normalized bending rigidity of three modes: **a** the first mode, **b** the second mode, **c** the third mode

$$\omega_{tn} = \beta_n^2 \sqrt{\frac{EI + \mu A l^2}{\rho A L^4}}, \qquad (n = 1,\ 2,\ 3, \ldots) \tag{5}$$

where μ is the shear modulus and connected to Young's modulus with Poissson's ratio.

$$\mu = \frac{E}{2(1+v)} \tag{6}$$

Therefore the normalized frequency is derived combining Eqs. (2) and (5)

$$\bar{\omega} = \sqrt{1 + 6\left(\frac{l}{h}\right)^2} \tag{7}$$

where h is the thickness of a cantilever microbeam, and the Poisson's ratio is neglected for slender beams with large aspect ratio, since the Poisson's effect is of less importance. Lei et al. (2016) fitted the curve with Eq. (7) and experimental data, the obtained length scale parameter l is 1.553 μm, meanwhile they also employed modified strain gradient theory and set the three parameters equal to each other, the corresponding length scale parameter was obtained as $l_0 = l_1 = l_2 = 0.843\mu$m. The fitted curves with experimental data in the two experiments are illustrated in Fig. 6. One can see that average data of the present experimesnt agree well with the results by the former study.

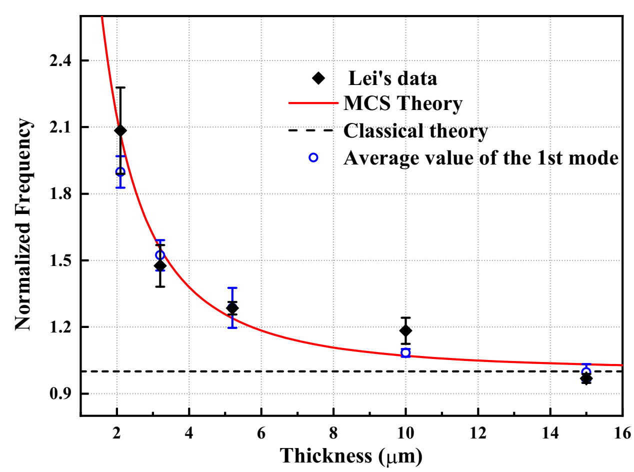

Fig. 6 The fitted curve and experimental data in literature (Lei et al. 2016) and data in the present study

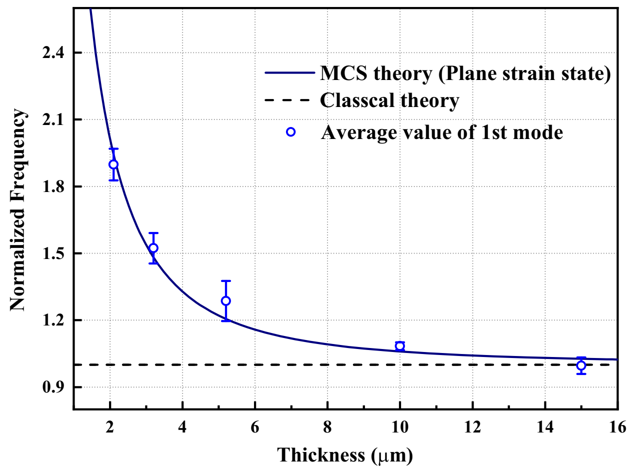

Fig. 7 Experimental results and size effect predicted by MCS theory considering plane strain state

However, one should keep in mind the premise of Eq. (7) is that the beams under consideration possess relatively large aspect ratio. Li et al. (2018) have discussed in detail the three different scenarios of a common beam, plane strain state and plane stress state. Under the experimental condition performed by Lei et al. (2016) and the present one, with aspect ratio $L/b = 5$ while b/h is more than 50, the displacement along the width direction could be ignored. Therefore it might be more appropriate to consider plane strain state in the fitting. Consequently bending rigidity EI is replaced by $EI/(1-v^2)$ in Eqs. (2) and (5), thus Eq. (7) is transformed to

$$\bar{\omega} = \sqrt{1 + 6(1-v)\left(\frac{l}{h}\right)^2} \tag{8}$$

Fitting the experimental data of the first mode with Eq. (8), one could have a more insightful view of size effect, as shown in Fig. 7.

Meanwhile, if the normalized frequencies of the second and third mode are put together with the fitted curve acquired by the data of the first mode, interestingly, one can find in Fig. 8 the high consistency within the two results. The high consistency indicates that the modified couple stress theory could be applied to interpret not only size-dependent vibration of the first mode, but also the validity is granted for vibration of higher modes.

Based on Eqs. (3) and (8), the normalized bending rigidity is related to thickness and length scale parameter by.

$$\xi = 1 + 6(1-v)\left(\frac{l}{h}\right)^2 \tag{9}$$

The normalized bending rigidity with changing thickness in three modes is depicted in Fig. 9. The normalized bending rigidity manifests the same trend in three modes and conforms to the tendency predicted by modified couple stress theory. According to the experiment by Lam et al. (2003), when the beam thickness reduced from 115 to 20 μm, the normalized bending rigidity increase to 2.4 times of the classical value. In this study, one could find out that the normalized bending rigidity increases to about 3.5 times in all three modes as thickness decreases from 15 to 2.1 μm. It may imply a stronger size effect in nickel than in epoxy.

4 Conclusions

Size-dependent vibration of higher modes in nickel cantilever microbeams is carried out to explore elastic size effect in micron scale in the present work. As a result,

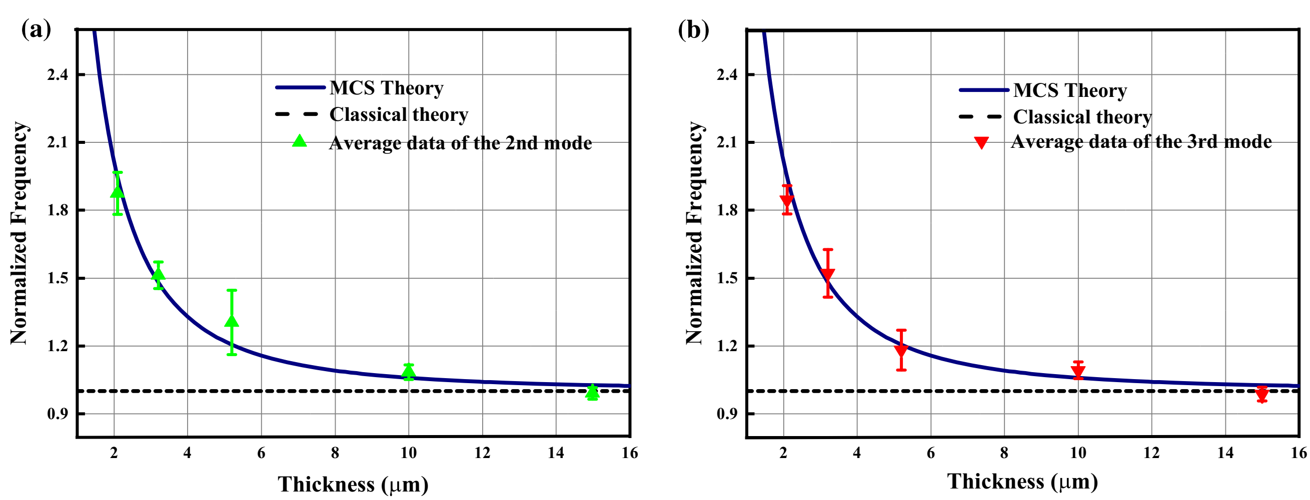

Fig. 8 The normalized frequency and result predicted by MCS theory: **a** the second mode, **b** the third mode

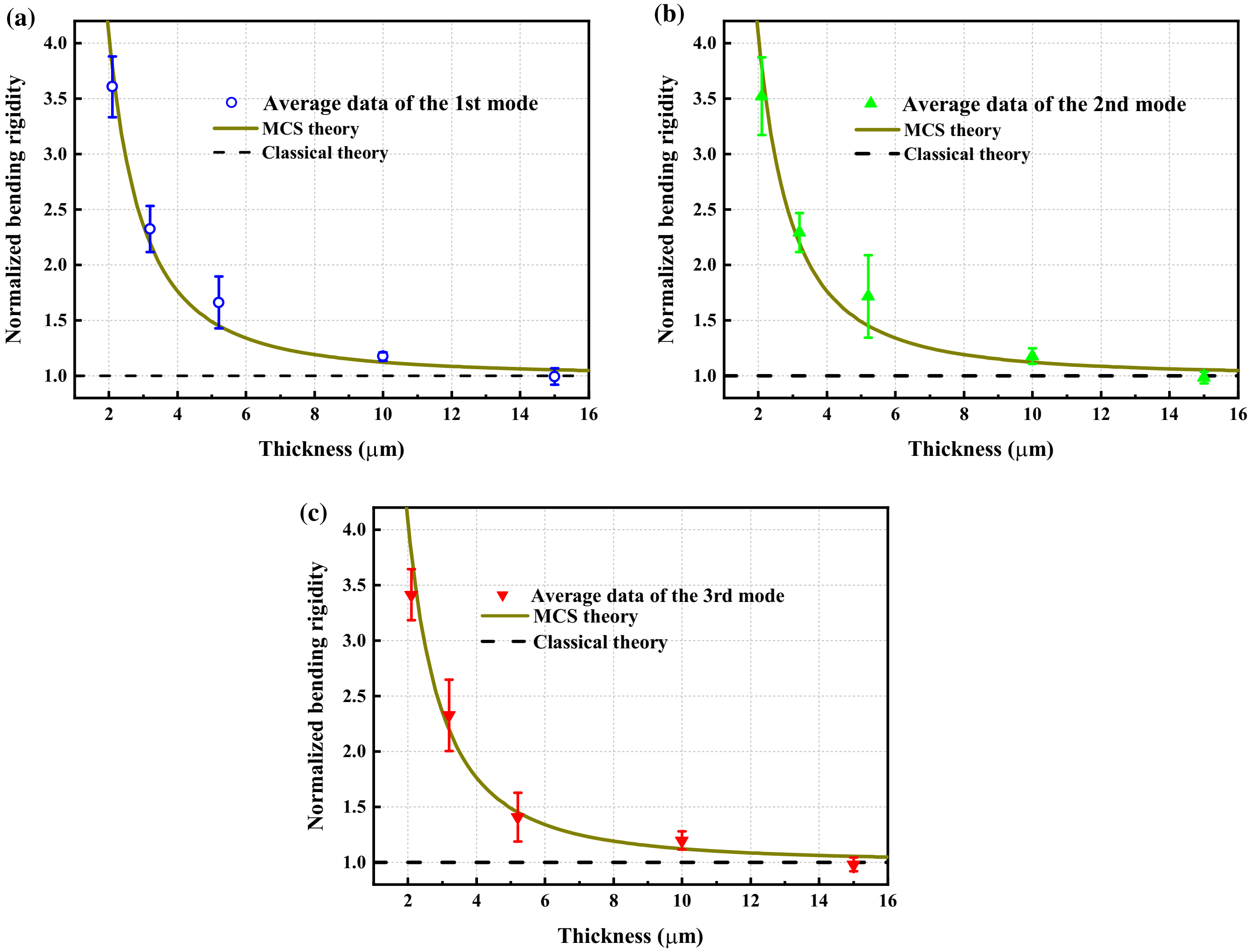

Fig. 9 The normalized bending rigidity and result predicted by MCS theory: **a** the first mode, **b** the second mode, **c** the third mode

positive size effect is observed not only in vibration of the first mode, but also of higher modes (the second mode and third mode), where the latter has not been reported in the open literature yet.

Specifically, the size effect has a significant effect on the vibration of higher modes as the thickness of microbeams approach the magnitude of length scale parameter. The normalized frequency, in all three modes, increased to 1.9 times when the thickness decreased from 15 to 2.1 μm. The normalized bending rigidity increased to 3.5 times correspondingly. In addition, the size-dependent vibration of the first mode agrees well with the theoretical explanation and the experimental results in the literature. Remarkably, the length scale parameter obtained by the first order is also in good agreement with vibration of second and third modes. That would be solid verification about the validity of modified couple stress theory in the size-dependent vibration of higher modes.

The employed methodology is more convenient and economical to realize compared with those employing AFMs and nanoindentation. Meanwhile the study provides rare and valuable experimental results on the vibration of higher modes in size-dependent microbeams, it would be useful for the further understanding of elastic size effects, and also of great help of for the design of MEMS/NEMS.

Acknowledgements This work was financially supported by the NSFC (No. 11772138) and the Ministry of Science and Technology of China (No. 2018YFF01014200). The authors are also grateful for the support of the NSFC (Nos. 11802101, 11702103) and the China Postdoctoral Science Foundation (No. 2018M642819).

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