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A Riemannian version of Korn's inequality

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Abstract. We prove a Korn type inequality for vector fields on a Riemann manifold. This inequality includes the special cases proved in the literature for domains in $\mathbb{R}^3$. If the domain is convex, we can considerably weaken the needed assumption on the boundary values.

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1. Introduction

Korn's inequality is the following integral inequality:

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega$ of class $C^{1,1}$. For $u \in H^{1,2}(\Omega, \mathbb{R}^n)$, we put

$$\varepsilon_{ij}(u) = \frac{1}{2}\Big(D_i u^j + D_j u^i\Big).$$

Then there exists a constant C such that

$$\int_\Omega |Du|^2 dx \le C\Big(\int_\Omega |u|^2 dx + \int_\Omega \sum_{ij} |\varepsilon_{ij}(u)|^2 dx\Big).$$

Korn's inequality is the basic tool for the existence of solutions of linearized displacement-traction equations in elasticity. The essential content of Korn's inequality is that the tensor $(\varepsilon_{ij})_{i,j=1,\cdots,n}$ incorporates only those components of the Jacobian tensor $Du = (D^i u_j)_{i,j=1,\cdots,n}$ of u that are orthogonal to infinitesimal rotations. Thus, the L^2-norm of the Jacobian tensor Du that measures the deformation of u is globally controlled by the norm of u itself and those components of Du that do not correspond to rigid motions of u, but to "real" deformations of the shape of u. See [4] for details.

This geometric interpretation suggests that such an inequality should also hold in the more general context of Riemannian geometry. It is the purpose of this note to derive such a Korn inequality on a Riemannian manifold. On one hand, this sheds new light on the geometric context of the original inequality. On the other hand it can

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be used for a linearized elasticity theory in the context of a Riemannian geometry theory, e.g. modeling a gravitational field. Our inequality includes some special versions given previously in [5] for domains in Euclidean $3-$space in curvilinear coordinates. Our geometric approach, however, allows to weaken the assumption on the set where the vector field has to vanish for a Korn inequality to hold for it.

Let $(\mathcal{M}, g)$ be an oriented Riemann manifold. The inner product on the tangent bundle $T\mathcal{M}$ induces norms on all tensor spaces. For example, if

$$T = T_{ij} dx^i \otimes dx^j \in T^*\mathcal{M} \otimes T^*\mathcal{M},$$

its pointwise norm is

$$|T|^2 = T_{ij} T_{st} g^{is} g^{jt}$$

where $\{g_{ij}\}$ is the matrix of the metric g and $\{g^{ij}\}$ its inverse.

Let X be a vector field on an open set $\Omega \subset \mathcal{M}$, T and $\mathcal{L}$ be tensor fields depending on the vector field X in some way. More precisely, if $X = u^l \dfrac{\partial}{\partial x^l}$ in a coordinate chart (U, x), we assume

$$T(X) = \Big(g_{jl} \frac{\partial u^l}{\partial x^i} + C_{ijl} u^l \Big) dx^i \otimes dx^j \tag{1.1}$$

and

$$\mathcal{L}(X) = \Big(g_{jl} \frac{\partial u^l}{\partial x^i} + g_{li} \frac{\partial u^l}{\partial x^j} + C_{ijl} u^l \Big) dx^i \otimes dx^j \tag{1.2}$$

where the C_{ijl} are smooth functions depending only on the metric g. In the sequel, the constant C and the functions C_{ijl} may be different from line to line but should be independent of the vector X.

With these notations we can state:

Theorem 1.1. *Let $\Omega \subset \mathcal{M}$ be an open set with boundary $\partial\Omega$ of $C^{1,1}$, the tensors T, $\mathcal{L}$ satisfying (1.1) and (1.2), then there is a positive constant C such that*

$$\int_\Omega |T(X)|^2 dvol \le C \left(\int_\Omega |X|^2 dvol + \int_\Omega |\mathcal{L}(X)|^2 dvol \right) \tag{1.3}$$

where $dvol$ is the volume form of the metric g.

Notice that if $g_{ij} = \delta_{ij}$ and $C_{ijl} = 0$, one get Korn's inequality from Theorem 1.1 again.

A tensor will be called of $T-$type or $\mathcal{L}-$type if it satisfies the conditions (1.1) or (1.2) respectively. The basic example of an $\mathcal{L}-$type tensor will be the Lie derivative $L_X g = \Big(g_{jl} \dfrac{\partial u^l}{\partial x^i} + g_{li} \dfrac{\partial u^l}{\partial x^j} + g_{ij,l} u^l \Big) dx^i \otimes dx^j$ of a metric g. Let ∇ be the Levi-Civita connection of the metric g, then $\nabla X \in T^*\mathcal{M} \otimes T\mathcal{M}$. By duality we get a $T-$type tensor $\widetilde{\nabla X} \in T^*\mathcal{M} \otimes T^*\mathcal{M}$. The norms of the tensors ∇X and $\widetilde{\nabla X}$ are the same. Therefore we obtain

Theorem 1.2. *Let $\Omega \subset \mathcal{M}$ be an open set with boundary $\partial\Omega$ of $C^{1,1}$, X be a vector field on the Riemann manifold $\mathcal{M}$, then there is a positive constant C such that*

$$\int_\Omega |\nabla X|^2 dvol \le C \left(\int_\Omega |X|^2 dvol + \int_\Omega |L_X g|^2 dvol \right). \tag{1.4}$$

Consider the differential system of first order

$$\mathcal{L}(X) = 0. \tag{1.5}$$

This is an elliptic system of first order with the strong unique continuation property. Therefore we have a Korn inequality with interior vanishing condition in the domain Ω.

Theorem 1.3. *Let $\Omega \subset \mathcal{M}$ be an open set with boundary $\partial\Omega$ of $C^{1,1}$, X be a vector field on the Riemann manifold $\mathcal{M}$, $e \subset \Omega$ be a subset with Hausdorff dimension larger than $n-2$, then there is a positive constant C such that*

$$\int_\Omega |T(X)|^2 dvol \leq C \int_\Omega |\mathcal{L}(X)|^2 dvol \tag{1.6}$$

holds for every vector field on Ω with $X|_e = 0$.

The Korn inequality with boundary condition usually requires that the vector field X vanishes on a set $e \subset \partial\Omega$ of positive $(n-1)$-Hausdorff measure. In the case of the Lie derivative and if Ω is convex, we have a weaker condition.

Theorem 1.4. *Let $\Omega \subset \mathcal{M}$ be a convex set with boundary $\partial\Omega$ of $C^{1,1}$, $\gamma \subset \partial\Omega$ with $\dim_H(\gamma) > n-2$, then there are positive constants δ and C such that the inequality*

$$\int_\Omega |T(X)|^2 dvol \leq C \int_\Omega |L_X g|^2 dvol \tag{1.7}$$

holds for any $X \in H^{1,2}(\Omega, T\Omega)$ that vanishes on γ.

Several schemes of proof have been introduced for the original Korn inequality. We shall partially employ here the strategy of Duvaut - Lions [6].

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2. Special cases

If the vector field X has compact support or the manifold has no boundary, the Korn inequality (1.3) is easy to deduce. In this section we will give an elementary proof under those conditions.

Let (U, x) be a coordinate chart, $X = u^l \dfrac{\partial}{\partial x^l}$, then

$$|\mathcal{L}|^2 = 2g_{kl}g^{jt}\frac{\partial u^k}{\partial x^j}\frac{\partial u^l}{\partial x^t} + 2\frac{\partial u^k}{\partial x^j}\frac{\partial u^j}{\partial x^k} + C_{ijk}u^k\frac{\partial u^j}{\partial x^i} + C_{kl}u^k u^l \tag{2.1}$$

and

$$|T|^2 = g_{kl}g^{jt}\frac{\partial u^k}{\partial x^j}\frac{\partial u^l}{\partial x^t} + C_{ijk}u^k\frac{\partial u^j}{\partial x^i} + C_{kl}u^k u^l. \tag{2.2}$$

Lemma 2.1.

$$g_{kl}g^{jt}\frac{\partial u^k}{\partial x^j}\frac{\partial u^l}{\partial x^t} \geq \frac{\lambda}{\Lambda}\sum_{i,j}\left|\frac{\partial u^j}{\partial x^i}\right|^2 \tag{2.3}$$

where $\Lambda \geq \lambda$ are positive numbers with

$$\lambda|\xi|^2 \leq g_{kl}\xi^k\xi^l \leq \Lambda|\xi|^2.$$

Proof. Trivial.

For any fixed $\varepsilon > 0$, by Lemma 2.1, we can find a positive constant C_ε which depends on the metric g and the choice of the coordinate chart (U, x) such that

$$|T(X)|^2 \leq \frac{(1+\varepsilon)}{2}|\mathcal{L}(X)|^2 + C_\varepsilon|X|^2 - (1+\varepsilon)\frac{\partial u^k}{\partial x^j}\frac{\partial u^j}{\partial x^k}. \tag{2.4}$$

So a crucial step in the proof of Theorem 1.1 should be to estimate the integral

$$\int_{U_\alpha}\frac{\partial u^k}{\partial x^j}\frac{\partial u^j}{\partial x^k}dvol.$$

It is easy to see that

$$\frac{\partial u^k}{\partial x^j}\frac{\partial u^j}{\partial x^k} - \frac{\partial u^k}{\partial x^k}\frac{\partial u^j}{\partial x^j} = \frac{\partial}{\partial x^k}\left\{u^j\frac{\partial u^k}{\partial x^j} - u^k\frac{\partial u^j}{\partial x^j}\right\}. \tag{2.5}$$

Lemma 2.2. *Let (V, Y) be another coordinate chart with $U \cap V \neq 0$, $X = v^j\frac{\partial}{\partial y^j}$, then*

$$\begin{aligned}&\frac{\partial}{\partial x^k}\left\{u^j\frac{\partial u^k}{\partial x^j} - u^k\frac{\partial u^j}{\partial x^j}\right\} \\ &= \frac{\partial}{\partial y^k}\left\{v^j\frac{\partial v^k}{\partial y^j} - v^k\frac{\partial v^j}{\partial y^j}\right\} + C_{ijk}v^k\frac{\partial v^j}{\partial y^i} + C_{kl}v^kv^l.\end{aligned} \tag{2.6}$$

Proof. The transformation rule for tangent vectors gives

$$u^k = v^i\frac{\partial x^k}{\partial y^i}.$$

We have then

$$
\begin{aligned}
&\frac{\partial}{\partial x^k}\left\{u^j\frac{\partial u^k}{\partial x^j}-u^k\frac{\partial u^j}{\partial x^j}\right\}=\frac{\partial}{\partial x^k}\left\{u^j\frac{\partial}{\partial x^j}\left[v^i\frac{\partial x^k}{\partial y^i}\right]-u^k\frac{\partial}{\partial x^j}\left[v^i\frac{\partial x^j}{\partial y^i}\right]\right\}\\
&=\frac{\partial}{\partial x^k}\left\{u^j\frac{\partial v^i}{\partial x^j}\frac{\partial x^k}{\partial y^i}-u^k\frac{\partial v^i}{\partial x^j}\frac{\partial x^j}{\partial y^i}\right\}+C_{ijk}v^k\frac{\partial v^j}{\partial y^i}+C_{kl}v^kv^l\\
&=\frac{\partial}{\partial x^k}\left\{v^l\frac{\partial x^j}{\partial y^l}\frac{\partial v^i}{\partial x^j}\frac{\partial x^k}{\partial y^i}-v^l\frac{\partial x^k}{\partial y^l}\frac{\partial v^i}{\partial x^j}\frac{\partial x^j}{\partial y^i}\right\}+C_{ijk}v^k\frac{\partial v^j}{\partial y^i}+C_{kl}v^kv^l\\
&=\frac{\partial}{\partial x^k}\left\{v^l\frac{\partial v^j}{\partial y^l}\frac{\partial x^k}{\partial y^j}-v^l\frac{\partial v^j}{\partial y^j}\frac{\partial x^k}{\partial y^l}\right\}+C_{ijk}v^k\frac{\partial v^j}{\partial y^i}+C_{kl}v^kv^l\\
&=\frac{\partial}{\partial x^k}\left\{v^l\frac{\partial v^j}{\partial y^l}\right\}\frac{\partial x^k}{\partial y^j}-\frac{\partial}{\partial x^k}\left\{v^l\frac{\partial v^j}{\partial y^j}\right\}\frac{\partial x^k}{\partial y^l}+C_{ijk}v^k\frac{\partial v^j}{\partial y^i}+C_{kl}v^kv^l\\
&=\frac{\partial}{\partial y^j}\left\{v^l\frac{\partial v^j}{\partial y^l}\right\}-\frac{\partial}{\partial y^l}\left\{v^l\frac{\partial v^j}{\partial y^j}\right\}+C_{ijk}v^k\frac{\partial v^j}{\partial y^i}+C_{kl}v^kv^l\\
&=\frac{\partial}{\partial y^k}\left\{v^j\frac{\partial v^k}{\partial y^j}-v^k\frac{\partial v^j}{\partial y^j}\right\}+C_{ijk}v^k\frac{\partial v^j}{\partial y^i}+C_{kl}v^kv^l.
\end{aligned}
$$

Proposition 2.1. *Let $\Omega \subset \mathcal{M}$ be an open paracompact set with boundary $\partial\Omega$ of $C^{1,1}$. If the vector field X has compact support included in Ω and the tensors T,$\mathcal{L}$ satisfy (1.1) and (1.2) respectively, for any $\varepsilon > 0$ there is then a positive constant C_ε such that*

$$
\int_\Omega |T(X)|^2 dvol \le C_\varepsilon \int_\Omega |X|^2 dvol + (\frac{1}{2}+\varepsilon)\int_\Omega |\mathcal{L}(X)|^2 dvol \tag{2.7}
$$

where $dvol$ is the volume form of the metric g.

Proof. Let $\{U_\alpha\}$ be a finite collection of open sets of $\mathcal{M}$ such that

1. $U_\alpha \cap U_\beta = \emptyset$;
2. $\bigcup \overline{U}_\alpha = \overline{\Omega}$;
3. U_α has $C^{1,1}$ boundary ∂U_α.

We assume further more that for every U_α there is a coordinate representation x_α. By (2.4) and (2.5) we have

$$
\begin{aligned}
&\int_\Omega |T(X)|^2 dvol\\
&\le \sum_\alpha \int_{U_\alpha}\left\{\frac{(1+\varepsilon)}{2}|\mathcal{L}(X)|^2 + C_\varepsilon|X|^2 - \frac{1+\varepsilon}{2}\frac{\partial u_\alpha^k}{\partial x_\alpha^j}\frac{\partial u_\alpha^j}{\partial x_\alpha^k}\right\} dvol\\
&\quad\le C_\varepsilon \int_\Omega |X|^2 dvol + \frac{1+\varepsilon}{2}\int_\Omega |\mathcal{L}(X)|^2 dvol \qquad (2.8)\\
&\quad -\frac{1+\varepsilon}{2}\sum_\alpha \int_{U_\alpha}\frac{\partial}{\partial x_\alpha^k}\left[u_\alpha^j\frac{\partial u_\alpha^k}{\partial x_\alpha^j} - u_\alpha^k\frac{\partial u_\alpha^j}{\partial x_\alpha^j}\right] dvol.
\end{aligned}
$$

For the last sum we can apply Green's formula on every domain U_α and obtain a sum, $I = \sum_\alpha \int_{\partial U_\alpha}$, of the integrals on the boundaries ∂U_α. By the assumption that the vector field X has compact support in Ω, if one part of the boundary of U_α appears in the summation I, it will appear twice. By Lemma 2.2 the terms

$$\int_{\partial U_\alpha} u_\alpha^j \frac{\partial u_\alpha^k}{\partial x_\alpha^j}$$

cancel because of the orientability of the manifold $\mathcal{M}$. Hence

$$\left|\sum_\alpha \int_{U_\alpha} \frac{\partial}{\partial x_\alpha^k}[u_\alpha^j \frac{\partial u_\alpha^k}{\partial x_\alpha^j} - u_\alpha^k \frac{\partial u_\alpha^j}{\partial x_\alpha^j}]dvol\right| \leq C\sum_\alpha \int_{U_\alpha} \left|u_\alpha^j \frac{\partial u_\alpha^k}{\partial x_\alpha^j}\right| dvol.$$

By Lemma 2.1 we can then find a constant C_ε such that

$$\begin{aligned}&\frac{1+\varepsilon}{2}\left|\sum_\alpha \int_{U_\alpha} \frac{\partial}{\partial x_\alpha^k}[u_\alpha^j \frac{\partial u_\alpha^k}{\partial x_\alpha^j} - u_\alpha^k \frac{\partial u_\alpha^j}{\partial x_\alpha^j}]dvol\right| \\ &\leq \frac{\varepsilon}{2}\int_\Omega |T(X)|^2 dvol + C_\varepsilon \int_\Omega |X|^2 dvol.\end{aligned}$$

Therefore the inequality (2.8) implies that

$$\int_\Omega |T(X)|^2 dvol \leq C_\varepsilon \int_\Omega |X|^2 dvol + \frac{1+\varepsilon}{2-\varepsilon}\int_\Omega |\mathcal{L}(X)|^2 dvol. \tag{2.9}$$

This proves Proposition 2.1.

3. Sobolev norms

Let $\{U_\alpha\}$ be a finite collection of open sets in Ω. We choose the collection as a cover of Ω and require that there is a coordinate x_α on every open set U_α. For a suitable choice of the cover $\{U_\alpha\}$, one then has for every tensor $\mathcal{L} = \mathcal{L}_{ij} dx^i \otimes dx^j$

$$\int_{U_\alpha} |\mathcal{L}|^2 dvol \simeq \sum_{ij} \int_{U_\alpha} |\mathcal{L}_{ij}|^2 dx_\alpha \tag{3.1}$$

where $dvol$ is the volume form on the Riemann manifold and $dx_\alpha = dx_\alpha^1 \wedge dx_\alpha^2 \wedge \cdots \wedge dx_\alpha^n$.

For an open set U of the Euclidean space $\mathbb{R}^n$, we put

$$B = \{v \in H_0^{1,2}(U); \|v\|_{H^{1,2}} \leq 1\}.$$

For $f \in C^\infty(U)$, we have

$$\|f\|_{H^{-1,2}(U)} = \sup_{v \in B} \int_U f v dx.$$

The Hilbert space $H^{-1,2}(U)$ is the closure of C^∞ w.r.t. the norm $\|\cdot\|_{H^{-1,2}(U)}$. $H^{-1,2}(U)$ may be defined as the dual space of $H_0^{1,2}(U)$ with respect to the L^2-product. The equivalence of the two definitions may be verified by standard mollification arguments. Obviously, $L^2 \subset H^{-1,2}(U)$. We also note

Lemma 3.1. *For $f \in L^2(U)$, we have $D_j f \in H^{-1,2}(U)$, where $D_j f$ has to be interpreted as a distributional derivative.*

Lemma 3.2. *For $f \in L^2(U)$, we put*

$$\|f\|_* := \|f\|_{H^{-1,2}(U)} + \sum_{j=1}^{n} \|D_j f\|_{H^{-1,2}(U)}.$$

Then there exist constants C_1, C_2 with

$$C_1 \|f\|_{L^2(U)} \leq \|f\|_* \leq \|f\|_{L^2(U)}. \tag{3.2}$$

Lemma 3.3. *Let $T = T_{ij} dx^i \otimes dx^j$ and $\mathcal{L} = \mathcal{L}_{ij} dx^i \otimes dx^j$ be a $T-$type and $\mathcal{L}-$type tensor respectively, then*

$$\partial_k T_{ij} = \frac{1}{2} \left[\partial_k \mathcal{L}_{ij} + \partial_i \mathcal{L}_{kj} - \partial_j \mathcal{L}_{ik} \right] + C_l u^l + C_{kl} \partial_k u^l. \tag{3.3}$$

Proof. We just need to write down the second order derivative of the component u^l.

$$\begin{aligned}
\partial_k T_{ij} &= g_{jl} \frac{\partial^2 u^l}{\partial x^k \partial x^i} + \cdots \\
\partial_k \mathcal{L}_{ij} &= g_{jl} \frac{\partial^2 u^l}{\partial x^k \partial x^i} + g_{il} \frac{\partial^2 u^l}{\partial x^k \partial x^j} + \cdots \\
\partial_i \mathcal{L}_{kj} &= g_{jl} \frac{\partial^2 u^l}{\partial x^k \partial x^i} + g_{kl} \frac{\partial^2 u^l}{\partial x^i \partial x^j} + \cdots \\
\partial_j \mathcal{L}_{ik} &= g_{kl} \frac{\partial^2 u^l}{\partial x^j \partial x^i} + g_{il} \frac{\partial^2 u^l}{\partial x^k \partial x^j} + \cdots .
\end{aligned}$$

Hence the identity (3.3) follows easily.

Now we give a proof of Theorem 1.1 by a closed graph argument.
Let $X = u^l \frac{\partial}{\partial x^l_\alpha}$, $\mathcal{L} = \mathcal{L}_{ij} dx^i_\alpha \otimes dx^j_\alpha$ on the chart (U_α, x_α) and

$$\int_{U_\alpha} |X|^2 dvol + \int_{U_\alpha} |\mathcal{L}(X)|^2 dvol$$

be finite, by (3.1), we have then

$$u^l \in L^2(U_\alpha), \quad \mathcal{L}_{ij} \in L^2(U_\alpha).$$

Lemma 3.1 implies that

$$\partial_k u^l \in H^{-1,2}(U_\alpha), \;\; \partial_k \mathcal{L}_{ij} \in H^{-1,2}(U_\alpha).$$

Hence $T_{ij} \in H^{-1,2}(U_\alpha)$ and moreover $\partial_k T_{ij} \in H^{-1,2}(U_\alpha)$. Therefore

$$\int_{U_\alpha} |T_{ij}|^2 dx_\alpha < \infty.$$

What we have obtained is that

$$\int_{U_\alpha} |X|^2 dvol + \int_{U_\alpha} |\mathcal{L}(X)|^2 dvol < \infty \Rightarrow \int_{U_\alpha} |T_{ij}|^2 dx_\alpha < \infty.$$

Hörmander's comparision theorem([8], Ch.2 §6) about operators on Banach spaces concludes that there is a constant C with

$$\int_{U_\alpha} |T_{ij}|^2 dx_\alpha \leq C \left\{ \int_{U_\alpha} |X|^2 dvol + \int_{U_\alpha} |\mathcal{L}|^2 dvol \right\}.$$

Using (3.1) again, we obtain

$$\int_{U_\alpha} |T(X)|^2 dvol \leq C \left\{ \int_{U_\alpha} |X|^2 dvol + \int_{U_\alpha} |\mathcal{L}(X)|^2 dvol \right\}.$$

If the cover $\{U_\alpha\}$ is locally finite, we conclude then that

$$\int_{U} |T(X)|^2 dvol \leq C \left\{ \int_{U} |X|^2 dvol + \int_{U} |\mathcal{L}(X)|^2 dvol \right\}.$$

This ends the proof of Theorem 1.1.

Remark. We can deduce some special kinds of Korn's inequality without boundary conditions. For example we have

Corollary 3.4. *Let Ω be an domain in R^2 and let $\theta \in C^3(\Omega, R^3)$ be an injective mapping such that the two vectors $a_\alpha = \partial_\alpha \theta$, $\alpha = 1, 2$ are linearly independent at all points of $\overline{\Omega}$. Given $\eta = (\eta_i) \in H^{1,2}(\Omega) \times H^{1,2}(\Omega) \times H^{2,2}(\Omega)$. Let*

$$\gamma_{\alpha\beta}(\eta) := \tfrac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta} - b_{\alpha\beta}\eta_3,$$
$$\rho_{\alpha\beta}(\eta) := \partial_{\alpha\beta}\eta_3 + R(\eta, \nabla\eta), \quad \alpha, \beta = 1, 2$$

with R a linear function of η and $\nabla\eta$[1]*, then there exists a constant $C = C(\Omega, \theta)$ such that*

$$\sum_{\alpha=1}^{2} |\eta_\alpha|_{H^{1,2}(\Omega)} + |\eta_3|_{H^{2,2}(\Omega)} \leq C \left(\sum_{\alpha=1}^{2} |\eta_\alpha|_{L^2(\Omega)} + |\eta_3|_{H^{1,2}(\Omega)} + \sum_{\alpha,\beta=1}^{2} |\gamma_{\alpha\beta}(\eta)|_{L^2(\Omega)} + \sum_{\alpha,\beta=1}^{2} |\rho_{\alpha\beta}(\eta)|_{L^2(\Omega)} \right). \tag{3.8}$$

Proof. Set $g_{ij} = \delta_{ij}$ in Theorem 1.1 and notice that

$$|\eta_3|_{H^{2,2}(\Omega)} \leq \sum_{\alpha,\beta=1}^{2} |\rho_{\alpha\beta}(\eta)|_{L^2(\Omega)} + |\eta_3|_{H^{1,2}(\Omega)}.$$

The inequality (3.4) is deduced from (1.3).

[1] For the precise formulation of R, Γ and $b_{\alpha\beta}$ in terms of the mapping θ see [5] Ch.2.

4. Deformations

Given a vector field X on the manifold $\mathcal{M}$, there is a local 1-parameter group φ_t of diffeomorphisms of the manifold $\mathcal{M}$ generated by the systems of ODEs

$$\begin{cases} \frac{d\varphi_t}{dt} = X(\varphi_t); \\ \varphi_t|_{t=0} = x. \end{cases} \tag{4.1}$$

In general, a local 1-parameter group may not be a group, i.e. the equation (4.1) may have only a short time solution. However if the vector field X has compact support, we have a 1-parameter group φ_t of diffeomorphisms (See [7]). Define the deformation of the diffeomorphism φ_t as $G^t = \nabla\varphi^T g(\varphi_t)\nabla\varphi$. The deformation G^t is also a metric on the manifold $\mathcal{M}$. In the coordinate chart (U, x), the metric G^t has the form

$$G^t = \frac{\partial\varphi_t^k}{\partial x_i} g_{kl}(\varphi_t) \frac{\partial\varphi_t^l}{\partial x_j}. \tag{4.2}$$

Hence

$$\frac{dG^t}{dt}\Big|_{t=0} = \left(g_{jl}\frac{\partial u^l}{\partial x^i} + g_{li}\frac{\partial u^l}{\partial x^j} + g_{ij,l}u^l \right) dx^i \otimes dx^j. \tag{4.3}$$

So, $\frac{dG^t}{dt}\Big|_{t=0} = L_X g$, the Lie derivative of the metric g along the vector field X. On the other hand, for the Levi-Civita connection ∇ of the metric g, one has

$$\nabla X = \left(\frac{\partial u^j}{\partial x_i} + \Gamma_{il}^j u^l \right) dx^i \otimes \frac{\partial}{\partial x^j}. \tag{4.4}$$

Define a dual tensor $\widetilde{\nabla X}$ of ∇X as

$$\widetilde{\nabla X} = (\frac{\partial u^k}{\partial x_i} + \Gamma_{il}^k u^l) g_{jk} dx^i \otimes dx^j.$$

The tensor $\widetilde{\nabla X}$ is then a $T-$type tensor. Obviously,

$$|\nabla X| = |\widetilde{\nabla X}|.$$

Hence

$$\int_\Omega |\nabla X|^2 dvol \le C \left(\int_\Omega |X|^2 dvol + \int_\Omega |L_X g|^2 dvol \right). \tag{4.5}$$

This proves Theorem 1.2.

As an illustration of Korn's inequality, we can see the well known fact that the vector space of Killing fields on a finite dimensional compact Riemannian manifold $\mathcal{M}$ is finite-dimensional itself. (Passing to a two-sheeted cover, we may assume that $\mathcal{M}$ is oriented.) A Killing vector field X is by definition an infinitesimal isometry and thus satisfies

$$L_X g = 0.$$

Thus, by Theorem 1.2,

$$\int_\mathcal{M} |\nabla X|^2 dvol \le C \int_\mathcal{M} |X|^2 dvol. \tag{4.6}$$

Combining the Korn inequality with the compactness theorem of Rellich, we conclude that the unit ball in the space

$$\left\{ X \text{ vector field on } \mathcal{M} : L_X g = 0, \int_{\mathcal{M}} |X|^2 dvol < \infty \right\}$$

is compact, hence of finite dimension.

5. Ellipticity and boundary conditions

We can view the tensor $\mathcal{L}$ as a map from the tangent space $T\mathcal{M}$ to the tensor space $T^*\mathcal{M} \otimes T^*\mathcal{M}$. This is a linear differential operator of first order. It is interesting that the map is elliptic.

Fixed $P \in \mathcal{M}, \xi \in T^*_P\mathcal{M}, u \in T_P\mathcal{M}$. Take $f \in C^\infty(\mathcal{M}), X \in C^\infty(\mathcal{M}, T\mathcal{M})$ such that

$$\begin{array}{l} df(P) = \xi, \quad f(P) = 0; \\ X(P) = u, \end{array}$$

the relation

$$\sigma_{\mathcal{L}}(\xi)u = \mathcal{L}(fX)(P)$$

then determines a map $\sigma_{\mathcal{L}}(\xi) : T_P\mathcal{M} \to T^*_P\mathcal{M} \otimes T^*_P\mathcal{M}$. $\sigma_{\mathcal{L}}$ is the principal symbol of the differential $\mathcal{L}$. Trivially,

$$\sigma_{\mathcal{L}}(\xi)u = \Big(g_{jl}\xi_i + g_{il}\xi_j\Big)u^l dx^i \otimes dx^j.$$

And we have

$$|\sigma_{\mathcal{L}}(\xi)u|^2 = 2|\xi|^2|u|^2 + 2(<\xi, u>)^2. \tag{5.1}$$

Hence the principal symbol $\sigma_{\mathcal{L}}(\xi)$ has maximal rank on $T_P\mathcal{M}$ for every $\xi \in T^*(\mathcal{M}) \setminus \{0\}$. This just means the ellipticity of the differential operator $\mathcal{L}$.

For a vector field $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ we denote its zero set by

$$\mathcal{N}(X) = \{P \in \mathcal{M}; X(P) = 0\}.$$

If X is a solution of the equation

$$\mathcal{L}(X) = 0. \tag{5.2}$$

we have a differential inequality by (3.3)

$$|\nabla^2 X|^2 \leq C(|\nabla X|^2 + |X|^2).$$

Hence the solution of the equation (5.2) has the strong unique continuation property. That is to say that if a solution of the equation satisfies

$$\lim_{P \to P_0} \rho(P, P_0)^{-N} X(P) = 0, \quad \forall N > 0$$

with $\rho(\cdot, \cdot)$ the metric on the manifold $\mathcal{M}$, X will vanish on $\mathcal{M}$ everywhere. See [2] for details. In fact we can prove that a nontrivial solution of the equation (5.2) has only zero points of first order. We will give a direct proof for this fact in the following proposition.

Proposition 5.1. *Let the vector field X satisfy the equation (5.2) on the domain Ω and vanish at a point $P \in \Omega$ of second order, i.e., $X(P) = 0$ and $\nabla X(P) = 0$, then $X \equiv 0$.*

Proof. Let γ be a geodesic starting from the point P, $\gamma(0) = P$. From (3.3) we have a differential inequality

$$\frac{d}{dt}|\nabla X|^2(\gamma(t)) \leq C_1|\nabla X|^2(\gamma(t)) + C_2|X|^2(\gamma(t)). \tag{5.3}$$

Multiplying this inequality by $\delta - t$ and integrate w.r.t. t on the interval $(0, \delta)$ then gives by the condition $|\nabla X|(\gamma(0)) = 0$ that

$$\begin{aligned} &\int_0^\delta |\nabla X|^2(\gamma(t))dt \\ &\leq C_1 \int_0^\delta (\delta - t)|\nabla X|^2(\gamma(t))dt + C_2 \int_0^\delta (\delta - t)|X|^2(\gamma(t))dt. \end{aligned} \tag{5.4}$$

We also have the elementary inequality

$$\int_0^\delta (\delta - t)|X|^2(\gamma(t))dt \leq \int_0^\delta (\delta - t)^2|\nabla X| \cdot |X|(\gamma(t))dt \tag{5.5}$$

because $|X|(\gamma(0)) = 0$. Combining (5.5) and the Schwarz inequality implies

$$\int_0^\delta (\delta - t)|X|^2(\gamma(t))dt \leq 2 \int_0^\delta (\delta - t)^2|\nabla X|^2(\gamma(t))dt$$

for $\delta \leq \frac{1}{2}$. Inserting this into (5.4) we see that

$$\int_0^\delta |\nabla X|^2(\gamma(t))dt \leq C \int_0^\delta (\delta - t)|\nabla X|^2(\gamma(t))dt. \tag{5.6}$$

Therefore $|\nabla X|(\gamma(t)) = 0$, and then $X(\gamma(t)) = 0$ on $[0, \delta_\gamma)$ provided $\delta_\gamma C < 1$. From the above procedure we see that the constant δ_γ depends only on those quantities that appear in (3.3) and the connection ∇. So there is a neighbourhood $B(P, \delta)$ of the point $P \in \Omega$ on which the vector field X vanishes. The fact that the radius δ is independent of the point P gives us the chance to extend the above argument step by step to every point in the domain Ω. Therefore the vector field X vanishes everywhere on the domain Ω. This ends the proof of Proposition 5.1.

Let $\mathcal{N}_{fin}(X)$ be the zero sets of finite order. A theorem of C. Bär [3] says that the set $\mathcal{N}_{fin}(X)$ has Hausdorff dimension at last $n - 2$ provided X is a nontrivial solution of an elliptic system. By this result and Proposition 5.1 we get immediately

Corollary 5.2. *Let $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ be a nontrivial solution of the equation (5.2), then the Hausdorff dimension of the zero set of the vector field X is at least of codimension two, i.e.,*

$$\dim_H(\mathcal{N}(X)) \leq n - 2. \tag{5.7}$$

We use Corollary 5.2 to deduce a new kind of Korn type inequality with interior vanishing condition in the domain Ω.

Theorem 5.3. *Let $e \subset \Omega \subset \mathcal{M}$ with $\dim_H(e) > n-2$, then there exists a constant C such that inequality*

$$\int_\Omega |T(X)|^2 dvol \leq C \int_\Omega |\mathcal{L}(X)|^2 dvol \tag{5.8}$$

holds for any $X \in H^{1,2}(\Omega, T\Omega)$ with $X|_e = 0$.

Proof. The proof is a standard contradiction argument. If the announced inequality were false, there would exist a sequence $\{X_j\} \subset H^{1.2}(\Omega, T\Omega)$ with

$$\begin{array}{l} \int_\Omega |T(X_j)|^2 = 1, \\ \int_\Omega |\mathcal{L}(X_j)|^2 \to 0. \end{array} \tag{5.9}$$

Combining the Korn inequality with the compactness of Rellich, we may assume that this sequence converges to X in $H^{1,2}(\Omega, T\Omega)$ strongly. Hence

$$\mathcal{L}(X) = 0.$$

By (3.3) we have $X \in H^{2,2}(\Omega, T\Omega)$ and then for all integral k, $X \in H^{k,2}(\Omega, T\Omega)$. Therefore $X \in C^\infty(\Omega, T\Omega)$. What we want to deduce is that

$$X|_e = 0. \tag{5.10}$$

We derive it by an extended version of Egorov's theorem which says that a strongly convergent sequence in $H^{1,2}(\Omega, T\Omega)$ has a pointwise convergent subsequence outside a set of Hausdorff dimension at least $n-2$. See [1]and [9]. By our assumption of $\dim_H(e) > n-2$, the vector field X will vanish on the set e and then vanishes on the whole domain Ω by Corollary 5.2. This is a contradiction with (5.9).

The last result in this note will concern the boundary conditions for the Korn inequality. The special feature in the Korn inequality is that one only needs partial information on the boundary Ω.

Theorem 5.5. *Let $\Omega \subset \mathcal{M}$ be an open set with boundary $\partial\Omega$ of $C^{1,1}$, then there are positive constants δ and C such that the inequality*

$$\int_\Omega |T(X)|^2 dvol \leq C \int_\Omega |\mathcal{L}(X)|^2 dvol \tag{5.11}$$

holds for any $X \in H^{1,2}(\Omega, T\Omega)$ that vanishes on the intersection of $\partial\Omega$ with some ball $B(P, \delta)$, $P \in \partial\Omega$.

Proof. Let $X \in H^{1,2}(\Omega, T\mathcal{M})$ be a vector field with $\mathcal{L}(X) = 0$ and $X|_\gamma = 0$. It can be proved by (3.3) that $X \in H^{k,2}(\Omega, T\mathcal{M})$ for any k. Therefore $X \in C^{0,1}(\overline{\Omega}, T\mathcal{M})$.

We can take a positive constant δ with the property that at every point $P \in \partial\Omega$ there are coordinates around the point P such that $B(P, 2\delta) \cap \Omega = \{x : x^1 >$

$0\} \cap B(P, 2\delta)$. Let $\Omega_r = (0, r) \times \{B(P, \delta) \cap \partial\Omega\}$. Suppose the vector field X satisfies the condition in the Theorem. Multiplying the two sides of the inequality

$$\frac{\partial}{\partial x^1}|X|^2 \leq |\nabla_{\frac{\partial}{\partial x^1}} X|^2 + |X|^2$$

by $r - x^1$ and integrating on the domain Ω_r we will get

$$\int_{\Omega_r} |X|^2 dvol \leq r \int_{\Omega_r} \left(|\nabla_{\frac{\partial}{\partial x^1}} X|^2 + |X|^2\right) dvol.$$

We also have a constant C_1 which depends only on the metric g so that

$$|\nabla_{\frac{\partial}{\partial x^1}} X|^2 \leq C_1 \left(|\mathcal{L}(x)|^2 + \sum_{j\geq 2} |\nabla_{\frac{\partial}{\partial x^j}} X|^2\right).$$

Using Theorem 1.1 w.r.t. the variables $(x^2, \cdots, x^n)$ gives

$$\int_{\Omega_r} |X|^2 dvol \leq r \int_{\Omega_r} \left((1 + C)|X|^2 + C|\mathcal{L}(X)|^2\right) dvol \tag{5.12}$$

with a constant C independent of r. If the vector field X also satisfies the equation (5.2) on the domain Ω, we have then an inequality

$$\int_{\Omega_r} |X|^2 dvol \leq r(1 + C) \int_{\Omega_r} |X|^2 dvol. \tag{5.13}$$

This implies that $X = 0$ on the domain Ω_r provided $r(1+C) < 1$. Hence the vector field X vanishes at every point of the domain Ω by the strong unique continuation property. The same contradiction argument as in the proof of Theorem 5.3 gives the announced inequality (5.11).

In the case of the Lie derivative and if we assume that Ω is convex in the sense that the shortest geodesic between any two points in Ω is also contained in Ω, we can weaken the condition on the size of the set at which X vanishes. The key observation is the following Lemma.

Lemma 5.6. *Let $i(\mathcal{M})$ be the injectivity radius of the manifold $\mathcal{M}$, X a Killing field, i.e., $L_X g = 0$. If the Killing field X vanishes at the points P, Q with the distance between P and Q less than the injectivity radius $i(\mathcal{M})$, then it will also vanish on the shortest geodesic connecting the points P and Q.*

Proof. Let ψ_t be the $1-$parameter group generated by the vector fields X. This group consists of local isometries. It also has fixed points P and Q by the vanishing condition. Since the distance is less than the injectivity radius, the shortest geodesic between the points P and Q will be unique and therefore invariant under the isometry group ψ_t. Hence the Killing field X vanishes on the geodesic.

Corollary 5.7. *Let $\mathcal{M}$, $i(\mathcal{M})$ and X be as in the above Lemma, $\gamma \subset \overline{\Omega}$ with diameter $d(\gamma) = max\{dist(P, Q); P, Q \in \Omega\} < i(\mathcal{M})$. If the Killing field X vanishes at every point $P \in \gamma$ then it will also vanish on the convex envelope of γ. Furthermore, if $dim_H(\gamma) > n - 2$, then X vanishes identically on the set Ω.*

Proof. The first statement is a direct application of Lemma 5.6. Let e be the convex envelope of the set γ, then e, as a convex set, has Hausdorff dimension at least codimension 1 because $\dim_H(\gamma) > n-2$. If $\dim_H(e \cap \Omega) \leq n-2$, then there is a point $P \in \partial\Omega$ with $B(P,\delta) \cap \partial\Omega \subset e$ for some $\delta > 0$. An application of Theorem 5.5 gives the second statement in the Corollary 5.7. Otherwise, we use Corollary 5.2 to assure that X vanishes at every point of Ω.

Theorem 5.8. *Let $\Omega \subset \mathcal{M}$ be a convex set with boundary $\partial\Omega$ of $C^{1,1}$, $\gamma \subset \partial\Omega$ with $\dim_H(\gamma) > n-2$, then there are positive constants δ and C such that the inequality*

$$\int_\Omega |T(X)|^2 dvol \leq C \int_\Omega |L_X g|^2 dvol \tag{5.14}$$

holds for any $X \in H^{1,2}(\Omega, T\Omega)$ that vanishes on γ.

Proof. Using Corollary 5.7 and the same contradiction argument as above.

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