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The Jacobian and the Ginzburg-Landau energy

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Abstract. We study the Ginzburg-Landau functional

$$I_\epsilon(u) := \frac{1}{\ln(1/\epsilon)} \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}\ (1-|u|^2)^2\ dx\ ,$$

for $u \in H^1(U;\mathbb{R}^2)$, where U is a bounded, open subset of $\mathbb{R}^2$. We show that if a sequence of functions u^ϵ satisfies $\sup I_\epsilon(u^\epsilon) < \infty$, then their Jacobians Ju^ϵ are precompact in the dual of $C_c^{0,\alpha}$ for every $\alpha \in (0,1]$. Moreover, any limiting measure is a sum of point masses. We also characterize the Γ-limit $I(\cdot)$ of the functionals $I_\epsilon(\cdot)$, in terms of the function space $B2V$ introduced by the authors in [16,17]: we show that $I(u)$ is finite if and only if $u \in B2V(U;S^1)$, and for $u \in B2V(U;S^1)$, $I(u)$ is equal to the total variation of the Jacobian measure Ju. When the domain U has dimension greater than two, we prove if $I_\epsilon(u^\epsilon) \le C$ then the Jacobians Ju^ϵ are again precompact in $\left(C_c^{0,\alpha}\right)^*$ for all $\alpha \in (0,1]$, and moreover we show that any limiting measure must be integer multiplicity rectifiable. We also show that the total variation of the Jacobian measure is a lower bound for the Γ limit of the Ginzburg-Landau functional.

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1 Introduction

The chief goal of this paper is to establish a connection between the Jacobian and the Ginzburg-Landau energy of a sequence of functions. Our main result is that, for a sequence of functions $u^\epsilon : \mathbb{R}^m \supset U \to \mathbb{R}^2$, $m \ge 2$, with uniformly bounded Ginzburg-Landau energy, the Jacobians Ju^ϵ are precompact in the dual space $\left(C^{0,\alpha}\right)^*$ for every $\alpha \in (0,1]$. We also characterize all possible weak limits of the Jacobians, and we prove a Γ-limit result for the Ginzburg-Landau functional:

$$I_\epsilon(u^\epsilon) := \frac{1}{\ln(1/\epsilon)} \int_U \frac{1}{2}|\nabla u^\epsilon|^2 + \frac{1}{4\epsilon^2}\ (1-|u^\epsilon|^2)^2\ dx\ ,$$

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where $\epsilon > 0$ is a small parameter. This functional is related to the Ginzburg-Landau model for superconductivity and it also serves as a model problem in which the singularities concentrate on sets of codimension two.

When the dimension of the domain is equal to two, the behavior of minimizers with given boundary data $g : \partial U \to S^1$ is the subject of the book by Bethuel, Brezis and Hélein [6]. If $\deg(g; \partial U) = d$ then in the limit, the energy of minimizers u^ϵ concentrates on $|d|$ points, called vortices. The limiting vortex configuration minimizes a renormalized energy that is explicitly given in [6]. Later alternate proofs were given by Lin [21] and Struwe [32]. One important technical step in [6] is a lower energy estimate in terms of the degree of the function around a given zero. A local version of these estimates as proved by the first author in [15] are key to our approach. Similar techniques were introduced independently in Sandier [27]. The asymptotic behavior of the minimizers in higher dimensions was first studied by Rivière [25,26]. He established a connection between the asymptotic behavior of the minimizer and the singular set of the limiting S^1 valued function.

In this paper, we study the Γ-limit of I_ϵ and related compactness properties. The corresponding problem for scalar-valued functions, or more generally for potentials with two or more equal minima, is completely understood due to work of Modica and Mortola [23,24], Modica [22], Sternberg [31], Kohn and Sternberg [20], Fonseca and Tartar [11], and Ambrosio [2]. In this setting, the singular set is a co-dimension one rectifiable set and the Γ-limit is proportional to the perimeter of this set. Since the definition of perimeter relies on the notion of a BV function, the space of BV functions plays a crucial role in the analysis of this problem.

Motivated by the analysis of I_ϵ and the central role of BV in the scalar case, the authors introduced and studied a class of functions called BnV in [16]; a short summary is provided in [17]. A function $u \in W^{1,n-1}(U; \mathbb{R}^n)$, for $U \subset \mathbb{R}^m, m \geq n$, is said to belong to BnV if the weak determinants of all n by n submatrices of the gradient matrix ∇u are signed Radon measures. For instance, if $U \subset \mathbb{R}^2$ and $u \in W^{1,1} \cap L^\infty(U; \mathbb{R}^2)$, set

$$j(u) := u \times \nabla u = (u \times u_{x_1}, u \times u_{x_2}) \tag{1.1}$$

where for $v = (v^1, v^2)$ and $w = (w^1, w^2)$ we write $v \times w := v^1 w^2 - v^2 w^1$. We then define

$$Ju = \frac{1}{2} \nabla \times j(u) = \frac{1}{2} \left((u \times u_{x_2})_{x_1} - (u \times u_{x_1})_{x_2} \right). \tag{1.2}$$

A priori Ju is only a distribution; we say that $u \in B2V$ if it happens to be a measure. For $U \subset \mathbb{R}^m$, $m \geq 3$, the definition of $B2V(U; \mathbb{R}^2)$ is similar and is given in Sect. 5.

The class BnV is very closely related to the *Cartesian Currents* of Giaquinta, Modica and Soucek [12,13]. This connection is discussed in detail in [16].

In [16] it is shown that if $u \in B2V(\mathbb{R}^m; S^1)$, then the Jacobian measures Ju is supported on an $m-2$ dimensional rectifiable set. In particular, if $u \in B2V(U; S^1)$ and $U \subset \mathbb{R}^2$, then there are $\{a_i\} \subset U$ and integers k_i such that

$$Ju = \pi \sum_i k_i \, \delta_{a_i} \,. \tag{1.3}$$

This is interpreted as encoding the location and degree of the topological singularities on u. Let $|Ju|$ be the total variation measure associated with Ju, i.e.,

$$|Ju|(B) = \pi \sum_i |k_i| \, \delta_{a_i}(B) \, . \tag{1.4}$$

Set

$$I(u) = \begin{cases} |J|(U) & \text{if } u \in B2V(U;S^1) \\ \infty & \text{otherwise.} \end{cases} \tag{1.5}$$

In Sect. 4, Theorem 4.1, we will prove that the Γ-limit of I_ϵ in the topology of $W^{1,1}(U;\mathbb{R}^2)$ is equal to $I(u)$:

Theorem 4.1 *Suppose $U \subset \mathbb{R}^2$. The Γ limit of I_ϵ in the topology of $W^{1,1}(U;\mathbb{R}^2)$ is equal to I, i.e., for every sequence u^ϵ converging to u in $W^{1,1}(U;\mathbb{R}^2)$,*

$$\liminf_{\epsilon \to 0} I_\epsilon(u^\epsilon) \geq I(u) \, ,$$

and for every $u \in B2V(U;S^1)$, there exist functions u^ϵ converging to u in $W^{1,1}(U;\mathbb{R}^2)$ satisfying

$$\liminf_{\epsilon \to 0} I_\epsilon(u^\epsilon) = I(u) \, .$$

When U is higher dimensional, we prove that the Jacobian is a lower bound for the Γ limit; see Theorem 5.2 below.

For the scalar Γ limit problem, the crucial observation is the following elementary inequality:

$$|\nabla h(u)| \; = \; \frac{1}{\sqrt{2}} |\nabla u||1-u^2| \; \leq \; \epsilon E^\epsilon(u) \; := \; \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{4\epsilon}(1-|u|^2)^2 \, ,$$

where $h(u) := [u - u^3/3]/\sqrt{2}$. A kind of vector generalization for this step was recently provided by Jin and Kohn [14]. They consider the same functional as above but for $u = \nabla \varphi$ for some scalar valued function φ. They obtain a lower bound for the energy of the form

$$\epsilon E^\epsilon(\nabla\varphi) + \text{ null Lagrangian } \geq \text{div } \Sigma(\nabla\varphi), \tag{1.6}$$

for a suitable chosen function Σ. This estimate was later used by Ambrosio, DeLellis and Mantegazza [4], and independently by DeSimone, Kohn, Müller and Otto [10] to prove a compactness result again in the case when u is a gradient, i.e., if

$$\sup_\epsilon \int_U \epsilon E^\epsilon(\nabla\varphi^\epsilon) \, dx < \infty \, ,$$

then, the sequence $\{\nabla\varphi^\epsilon\}$ is precompact in certain L^p spaces. These results are valid only in two dimensions.

When u is not a gradient the situation is completely different. The leading term in the energy now comes from the divergence-free part of u, and the natural scaling is

$$\int_U E^\epsilon(u^\epsilon) \, dx \sim \ln\left(\frac{1}{\epsilon}\right) \tag{1.7}$$

Under this assumption one cannot expect any compactness for $\{u^\epsilon\}$ in any L^p space. For example, if we let $u^\epsilon(x,y) = e^{ix\sqrt{\ln\epsilon}}$, then the sequence $\{u^\epsilon\}_{\epsilon\in(0,1]}$ satisfies (1.7), but is precompact only in the L^∞ weak-* (and weaker) topologies, and the weak limit $u = 0$ does not give much information about the behavior of the sequence $\{u^\epsilon\}$.

However, one expects that control over the energy should provide control over the limiting number and degree of singular points, as recorded in the limiting behavior of the Jacobians Ju^ϵ. To explain the main idea, let us suppose that $u^\epsilon \in C_c^\infty(u;\mathbb{R}^2)$ is a function that is non zero everywhere except at one point $a \in U$ with $u^\epsilon(a) = 0$. Then lower bounds of [15] or [28] imply that

$$I_\epsilon(u^\epsilon) = \frac{1}{\ln(1/\epsilon)} \int_U E^\epsilon(u^\epsilon)\, dx \geq |d|\,\pi\, + o(1)\,, \tag{1.8}$$

where d is the degree of u^ϵ around the point a.

Let ϕ be a nonnegative, smooth test function compactly supported in U. Using the Definition (1.2) of Ju, integration by parts and the co-area formula

$$\begin{aligned}\int_U \phi\, Ju^\epsilon\, dx &= \frac{1}{2}\int_U \nabla\times\phi\cdot j(u^\epsilon)\, dx\\ &= \frac{1}{2}\int_U \frac{\nabla\times\phi}{|\nabla\phi|}\cdot j(u^\epsilon)\,|\nabla\phi|\, dx\\ &= \frac{1}{2}\int_0^\infty \int_{\partial\Omega(s)} j(u^\epsilon)\cdot\mathbf{t}\, d\mathcal{H}^1\, ds \end{aligned}\tag{1.9}$$

Here we are writing $\Omega(s) := \{x \in U : \phi(x) > s\}$ and $\mathbf{t} := (\nabla\times\phi)/|\nabla\phi|$. Note that if s is a regular value of ϕ then $\mathbf{t}$ is an oriented unit tangent vector field along $\partial\Omega(s)$. Set $v^\epsilon := u^\epsilon/|u^\epsilon|$. Then $j(v^\epsilon) = j(u^\epsilon)/|u^\epsilon|^2$, and one can check that for any Jordan curve Γ enclosing a,

$$\int_\Gamma j(v^\epsilon)\cdot\mathbf{t}\, d\mathcal{H}^1 = 2\pi\,\mathrm{deg}(u^\epsilon;\Gamma) = 2\pi\, d\,.$$

Thus

$$\begin{aligned}\int_U \phi\, Ju^\epsilon\, dx = \pi &\int_0^\infty \mathrm{deg}(u^\epsilon;\partial\Omega(s))\, ds\\ &+\frac{1}{2}\int_0^\infty \int_{\partial\Omega(s)} [j(u^\epsilon) - j(v^\epsilon)]\cdot\mathbf{t}\, d\mathcal{H}^1\, ds\\ = \pi &\int_0^\infty \mathrm{deg}(u^\epsilon;\partial\Omega(s))\, ds\\ &+\int_0^\infty \int_{\partial\Omega(s)} \left[\frac{|u^\epsilon|^2-1}{2|u^\epsilon|^2}\right] j(u^\epsilon)\cdot\mathbf{t}\, d\mathcal{H}^1\, ds\,.\end{aligned}$$

Here we used the fact $(\nabla\times\phi)/|\nabla\phi|$ is equal to the unit tangent vector $\mathbf{t}$ of the level set of ϕ. Since by the Sard's Theorem, $\partial\Omega(t)$ is regular for almost every t, and since $\partial\Omega(t)$ encloses a if and only if $\phi(a) > t$,

$$\mathrm{deg}(u^\epsilon;\partial\Omega(t)) = \begin{cases} d & \text{if } t \in (0,\phi(a))\,,\\ 0 & \text{if } t > \phi(a)\,.\end{cases}$$

Hence,

$$\int_U \phi \, Ju^\epsilon \, dx = \pi \, d \, \phi(a) + B_\phi(u^\epsilon) \, .$$

We will show that the error term $B_\phi(u^\epsilon)$ can be controlled by the energy. Also, as seen in (1.8), the degree d is controlled by $I_\epsilon(u^\epsilon)$. Hence, the Jacobian measure Ju^ϵ is formally controlled by $I_\epsilon(u^\epsilon)$. This argument is made rigorous in Theorem 2.1; a sharper result along the same lines is given in Corollary 2.5. A key ingredient in both is an improved version of the degree type lower bound (1.8), valid under much weaker hypotheses about u^ϵ. The proof of this lower bound is deferred to the final section.

The estimates described above, combined with rather soft arguments, easily imply that Ju^ϵ is precompact in certain weak topologies, if the Ginzburg-Landau energies $I^\epsilon(u^\epsilon)$ are uniformly bounded. With a little extra work we establish in Sect. 3 the following result, which also characterizes all possible weak limits:

Theorem 3.1 *Suppose that $U \subset \mathbb{R}^2$, and let u^ϵ be a sequence of smooth functions satisfying*

$$K_U := \sup_{\epsilon \in (0,1]} I_\epsilon(u^\epsilon) \; < \; \infty.$$

Then there exists a subsequence ϵ_n converging to zero and a signed Radon measure J such that Ju^{ϵ_n} converges to J in the dual norm $C_c^{0,\alpha}(U)^*$ for every $\alpha \in (0,1]$. Moreover, there are $\{a_i\}_{i=1}^N \subset U$ and integers k_i such that

$$J = \pi \sum_{i=1}^N k_i \, \delta_{a_i} \, , \qquad \textit{and} \qquad |J|(U) = \pi \sum_i |k_i| \le K_U \, .$$

Finally, if the Ginzburg-Landau energy measure μ^ϵ converges weakly to a limit μ, then $J \ll \mu$, and $\frac{dJ}{d\mu} \le 1$, μ almost everywhere.

The energy measure μ^ϵ is defined in (1.11) below.

By a slicing argument, we use the two-dimensional compactness result and the estimates from Theorem 2.1 to extend the compactness result to higher dimensions.

Theorem 5.2 *Let $U \subset \mathbb{R}^m$, and suppose that $\{u^\epsilon\}_{\epsilon \in (0,1]}$ is a collection of smooth functions such that $K_U := \sup_{\epsilon \in (0,1]} I_\epsilon(u^\epsilon) < \infty$. Then there exists a subsequence $\epsilon_n \to 0$ and a Radon measure $\bar{J}$ such that*

(i): Ju^{ϵ_n} converges to a limit $\bar{J}$ in the $(C^{0,\alpha})^*$ norm for every $\alpha > 0$;
(ii): $\bar{J}/\pi$ is $(m-2)$-dimensional integer multiplicity rectifiable; and
(iii): If $\bar{\mu}$ is any weak limit of a subsequence of the Ginzburg-Landau energy measure μ^{ϵ_n}, then $|\bar{J}| \ll \bar{\mu}$, and $\frac{d|\bar{J}|}{d\bar{\mu}} \le 1$. In particular, $|\bar{J}|(U) \le K_U$.

Notice that the last assertion is the lower bound part of the Γ limit result in higher dimensions.

The definition of integer multiplicity rectifiable is given in Sect. 6. Informally, (ii) asserts that one can think of $\bar{J}$ as being supported in a Lipschitz submanifold of dimension $m-2$.

We close this introduction with a brief review of some related problems. The basic estimates in this paper come from lower bounds of the type introduced in [15, 27]. Similar lower bounds have played a central role in results about singular limits of evolution equations associated with the Ginzburg-Landau functional. These include the analysis by the authors [18] of dynamics of point vortices in Ginzburg-Landau heat flow, and analogous work by Colliander and Jerrard [7] dealing with the Ginzburg-Landau Schrödinger equation. The latter work also explicitly exploits the connection between the Ginzburg-Landau energy and the Jacobian, in a spirit similar to some of the results in this paper. In higher dimensional evolutionary problems, the energy and the Jacobian concentrates on sets with codimension two and these sets flow by the mean curvature flow; see for instance [19].

The same kind of lower bounds have been a basic ingredient in a series of papers by Sandier and Serfaty on the asymptotic behavior minimizers of the full Ginzburg-Landau model for superconductivity, see for example [29] among other works, and in recent work by Sandier [28] describing the limiting singular set of minimizers of general Ginzburg-Landau type functional in higher dimensions.

An independent forthcoming paper of Alberti, Baldo and Orlandi [1] also studies the asymptotic behavior of the functional I_ϵ.

The paper is organized as follows: In Sect. 2, we prove the Jacobian estimate in terms of the normalized Ginzburg-Landau energy. Using these estimates, we prove a compactness result in Sect. 3. Then, we prove the Γ-limit result in Sect. 4. Compactness in higher dimensions is proved in Sect. 5. The final section contains an appendix in which we establish some estimates used in Sect. 2.

Acknowledgement. After the completion of this manuscript we have learned that a sequence of smooth functions $u\epsilon$ converging to u and also whose renormalized Ginzburg-Landau energy converges to $\|Ju\|$ is constructed in the forthcoming paper of Alberti, Baldo and Orlandi [1]. This completes the Gamma convergence result in dimensions greater than two.

Notation

Given a function $u \in H^1(U;\mathbb{R}^2)$ we define the energy density

$$E^\epsilon(u) := \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1-|u|^2)^2 \tag{1.10}$$

and the energy measure

$$\mu_u^\epsilon(B) := \frac{1}{\ln(1/\epsilon)} \int_B E^\epsilon(u)\, dx. \tag{1.11}$$

We will typically write μ^ϵ instead of μ_u^ϵ when no ambiguity can result.

The distributional Jacobian Ju for a function $u : \mathbb{R}^2 \supset U \to \mathbb{R}^2$ is defined in (1.2). A definition valid when the domain has arbitrary dimension $m \geq 2$ is given in (5.2).

We write $B_r(x)$ to denote the *closed* ball $\{y \in \mathbb{R}^m \ : \ |x-y| \leq r\}$. We do not explicitly display the dimension m in our notation because it is normally clear from the context.

If $A \subset \mathbb{R}$, we typically use the notation $|A|$ to denote the 1-dimensional Lebesgue measure of A.

2 Jacobian estimate

The chief result of this section is the following estimate of the Jacobian in terms of the Ginzburg Landau energy. This estimate will be the main ingredient in the compactness result. We give a more precise version of the estimate at the end of the section.

Theorem 2.1. *Suppose* $\phi \in C_c^{0,1}(U)$ *and* $u \in H^1(U;\mathbb{R}^2)$*. For any* $\lambda \in (1,2]$*, and* $\epsilon \in (0,1]$,

$$\left| \int_U \phi \; Ju \; dx \right| \leq \pi d_\lambda \|\phi\|_\infty + \|\phi\|_{C^{0,1}} h^\epsilon(\phi, u, \lambda) \tag{2.1}$$

where

$$d_\lambda = \left\lfloor \frac{\lambda}{\pi} \mu_u^\epsilon(spt(\phi)) \right\rfloor , \tag{2.2}$$

$\lfloor x \rfloor$ *denotes the greatest integer less than or equal to* x*,*

$$h^\epsilon(\phi, u, \lambda) \leq C\epsilon^{\alpha(\lambda)}(1 + \mu_u^\epsilon(spt(\phi)))(1 + Leb^2(spt(\phi))) \; , \tag{2.3}$$

$\alpha(\lambda) = \frac{\lambda-1}{12\lambda}$*, and* C *is a constant independent of* u*,* ϕ*,* ϵ*,* λ *and* U*.*

Note that h^ϵ depends on ϕ only through the support of ϕ, and on u only through its (linear) dependence on $\mu_u^\epsilon(\mathrm{spt}(\phi))$.

It suffices to consider nonnegative test functions, since we can decompose an arbitrary function ϕ into its positive and negative parts. So we will assume that $\phi \geq 0$.

By an approximation argument, we may also assume that u is smooth.

Throughout this section we will use the notation

$$T \;=\; \|\phi\|_\infty \;=\; \max_U \phi(x). \tag{2.4}$$

As discussed in the Introduction, the main idea behind the above estimate is the following identity, which relies on the co-area formula, integration by parts, and the identity $Ju = \nabla \times j(u)/2$:

$$\int_U \phi \; Ju \; dx = \frac{1}{2} \int_0^T \int_{\partial\Omega(t)} j(u) \cdot \mathbf{t} \; d\mathcal{H}^1 \; dt \; , \tag{2.5}$$

where

$$\Omega(t) = \{ \; x \in U \mid \phi(x) > t \; \} \; , \tag{2.6}$$

$$\mathbf{t} = \text{ unit tangent to } \partial\Omega(t) \;=\; \frac{\nabla \times \phi}{|\nabla \times \phi|}.$$

The proof shows that

$$\int_{\partial\Omega(t)} j(u) \cdot \mathbf{t} \; d\mathcal{H}^1 \;\approx\; 2\pi \deg(u; \partial\Omega(t))$$

for most values of t. The other main point is then to prove that the set of t such that $\deg(u;\partial\Omega(t)) > d_\lambda$ has Leb^1 measure that can be controlled by $\mu^\epsilon(\mathrm{spt}(\phi))$. This last point is similar in spirit to results established in [7,15,27] for example. The details of the proof are given in Sect. 6.

Given $\phi \in C_c^{0,1}(U)$ we use the notation

$$\begin{aligned}\mathrm{Reg}(\phi) := \big\{ t \in [0,T] : \partial\Omega(t) = \phi^{-1}(t), \partial\Omega(t) \text{ is rectifiable},\\ \mathcal{H}^1(\partial\Omega(t)) < \infty \big\} .\end{aligned} \tag{2.7}$$

The coarea formula implies that $\mathrm{Reg}(\phi)$ is a set of full measure. For every $t \in \mathrm{Reg}(\phi)$, $\partial\Omega(t)$ is a union of finite Jordan curves $\Gamma_i(t)$, i.e.,

$$\partial\Omega(t) = \cup_i \, \Gamma_i(t) , \qquad \forall \, t \in \mathrm{Reg}(\phi).$$

In particular this holds for almost every t. For $t \in \mathrm{Reg}(\phi)$ we define

$$\Gamma(t) = \cup \left\{ \text{components } \Gamma_i(t) \text{ of } \partial\Omega(t) \mid \min_{x \in \Gamma_i(t)} |u(x)| > 1/2 \right\} . \tag{2.8}$$

We also define $\gamma(t) = \partial\Omega(t) \setminus \Gamma(t)$,

$$\gamma(t) = \cup \left\{ \text{components } \Gamma_i(t) \text{ of } \partial\Omega(t) \mid \min_{x \in \Gamma_i(t)} |u(x)| \le 1/2 \right\} . \tag{2.9}$$

When we want to indicate explicitly the dependence of $\Gamma(t)$ on ϕ and u, we will write $\Gamma_{\phi,u}(t)$.

We start the proof of Theorem 2.1 with two simple estimates.

Lemma 2.2. *For any set A,*

$$\int_A \left| \int_{\partial\Omega(t)} j(u) \cdot \mathbf{t} \, d\mathcal{H}^1 \right| \, dt \le \frac{|A|}{2} \int_{spt(\phi)} E^\epsilon(u) \, dx . \tag{2.10}$$

For any nonnegative function f,

$$\int_0^T \int_{\partial\Omega(t)} f(x) \, d\mathcal{H}^1 \, dt \le \|\nabla\phi\|_\infty \int_{spt(\phi)} f(x) \, dx . \tag{2.11}$$

Proof. For any $t \in \mathrm{Reg}(\phi)$, Stokes' Theorem yields

$$\int_{\partial\Omega(t)} j(u) \cdot \mathbf{t} \, d\mathcal{H}^1 = \frac{1}{2} \int_{\Omega(t)} Ju \, dx .$$

Since $|Ju| \le \frac{1}{2}|\nabla u|^2 \le E^\epsilon(u)$, (2.10) follows from the above identity.

For (2.11), we calculate by using the coarea formula,

$$\begin{aligned}\int_0^T \int_{\partial\Omega(t)} f \, d\mathcal{H}^1 \, dt &= \int_{spt(\phi)} f \, |\nabla\phi| \, dx \\ &\le \|\nabla\phi\|_\infty \int_{spt(\phi)} f \, dx.\end{aligned}$$

□

By definition $|u|$ falls below $1/2$ on $\gamma(t)$ and so we expect the Ginzburg-Landau energy to be large on $\gamma(t)$. The following technical lemma proves this under the assumption that $\gamma(t)$ is sufficiently large.

Lemma 2.3. *Suppose that*

$$\mathcal{H}^1(\gamma(t)) \geq \epsilon \, .$$

Then

$$\int_{\partial\Omega(t)} E^\epsilon(u) \, d\mathcal{H}^1 \geq \frac{1}{25\epsilon} \, .$$

Proof. This is very similar to Lemma 2.3 in [15]. Fix a connected component $\Gamma_i(t)$ of $\gamma(t)$ and set $\rho := |u|$ and

$$\gamma_i := \int_{\Gamma_i(t)} \frac{1}{2} |\nabla\rho|^2 \, d\mathcal{H}^1 \, .$$

By the Definition (2.9) of $\gamma(t)$ there is a point $x_{min} \in \Gamma_i(t)$ such that $\rho(x_{\min}) \leq 1/2$. Parametrize $\Gamma_i(t)$ by arclength so that

$$\Gamma_i(t) = \{ \, x(s) \mid s \in [0, G_i] \} \, , \qquad G_i := \mathcal{H}^1(\Gamma_i(t))$$

with $x_{\min} = x(0) = x(G_i)$. Then since $|\dot{x}(s)| = 1$,

$$\begin{aligned} \rho(x(s)) &= \rho(x(0)) + \int_0^s \nabla\rho(x(r)) \cdot \dot{x}(r) \, dr \\ &\leq \frac{1}{2} + s^{1/2} \left(\int_0^s |\nabla\rho(x(r))|^2 dr \right)^{1/2} \\ &\leq \frac{1}{2} + \sqrt{\gamma_i \, s} \leq \frac{3}{4} \, , \end{aligned}$$

provided that $s \leq \sigma_i := [G_i \wedge (1/16\gamma_i)]$. Then, for $s \in [0, \sigma_i]$, $(1 - \rho^2(x(s)))^2/4 \geq 1/25$. Therefore,

$$\begin{aligned} \int_{\Gamma_i(t)} E^\epsilon(u) \, d\mathcal{H}^1 &\geq \gamma_i + \int_{\Gamma_i(t)} \frac{1}{4\epsilon^2} (1-\rho^2)^2 \, d\mathcal{H}^1 \\ &\geq \gamma_i + \frac{\sigma_i}{25 \, \epsilon^2} \, . \end{aligned}$$

By calculus,

$$\gamma_i + \frac{\sigma_i}{25 \, \epsilon^2} = \gamma_i + \frac{G_i \wedge (1/16\gamma_i)}{25 \, \epsilon^2} \geq \frac{1}{25\epsilon} \left[\frac{G_i}{\epsilon} \wedge \frac{5}{2} \right] \, .$$

Thus

$$\int_{\Gamma_i(t)} E^\epsilon(u) \, d\mathcal{H}^1 \geq \frac{1}{25\epsilon} \left[\frac{G_i}{\epsilon} \wedge \frac{5}{2} \right] \, .$$

Since

$$\mathcal{H}^1(\gamma(t)) = \sum_{\{i | \Gamma_i(t) \text{ is a component of } \gamma(t)\}} \mathcal{H}^1(\Gamma_i(t)) = \sum_i G_i \geq \epsilon \, ,$$

we can sum over components $\Gamma_i(t)$ of $\gamma(t)$ to conclude that

$$\int_{\partial\Omega(t)} E^\epsilon(u)\, d\mathcal{H}^1 \ge \frac{1}{25\epsilon}\ .$$

□

We are now in a position to prove Theorem 2.1. In the proof we repeatedly absorb logarithmic factors by using the fact that if $\beta < \alpha$ then

$$\epsilon^\alpha \ln(1/\epsilon) \le C\epsilon^\beta$$

for some $C = C(\alpha, \beta)$ independent of $\epsilon \in (0, 1]$.

Proof of Theorem 2.1.
1. Recall that we are writing $T = \|\phi\|_\infty$. Fix $\lambda \in (1, 2]$ and define $d_\lambda := \lfloor \frac{\lambda}{\pi}\mu^\epsilon(\mathrm{spt}(\phi))\rfloor$. We define sets $A, B \subset [0, T]$ by

$$B := \{t \in \mathrm{Reg}(\phi)\ :\ |\mathrm{deg}(u; \Gamma(t))| \ge d_\lambda + 1 \text{ or } \mathcal{H}^1(\gamma(t)) \ge \epsilon\}, \tag{2.12}$$

$$A = \mathrm{Reg}(\phi) \setminus B. \tag{2.13}$$

Because almost every t belongs to $A \cup B = \mathrm{Reg}(\phi)$, (2.5) implies that

$$\begin{aligned} \int_U \phi\ Ju\ dx &= \frac{1}{2}\int_A \int_{\Gamma(t)} j(u)\cdot \mathbf{t}\ d\mathcal{H}^1\ dt \\ &\quad + \frac{1}{2}\int_A \int_{\gamma(t)} j(u)\cdot \mathbf{t}\ d\mathcal{H}^1\ dt + \frac{1}{2}\int_B \int_{\partial\Omega(t)} j(u)\cdot \mathbf{t}\ d\mathcal{H}^1\ dt \\ &= I_{A,\Gamma} + I_{A,\gamma} + I_B. \end{aligned} \tag{2.14}$$

2. Estimate of $I_{A,\Gamma}$
Suppose $t \in A$. On $\Gamma(t)$, $|u| \ge 1/2$ by the Definition (2.8), and we set $v := u/|u|$, so that $j(v) = j(u)/|u|^2$, and

$$\int_{\Gamma(t)} j(v)\cdot \mathbf{t}\ d\mathcal{H}^1 = 2\pi\ \mathrm{deg}(u; \Gamma(t)).$$

Then

$$\int_{\Gamma(t)} j(u)\cdot \mathbf{t}\ d\mathcal{H}^1 = 2\pi\ \mathrm{deg}(u; \Gamma(t)) + \int_{\Gamma(t)} j(u)\ \frac{|u|^2 - 1}{|u|^2}\cdot \mathbf{t}\ d\mathcal{H}^1\ .$$

Since $|j(u)| \le |u|\ |\nabla u|$, Cauchy's inequality and (2.11) imply that

$$\begin{aligned} \int_A \left| \int_{\Gamma(t)} j(u)\cdot \mathbf{t}\ d\mathcal{H}^1\ - 2\pi \mathrm{deg}(u; \Gamma(t)) \right|\ dt &\le \int_A \int_{\Gamma(t)} |\nabla u|\ \ \left|\frac{|u|^2 - 1}{|u|}\right|\ d\mathcal{H}^1 \\ &\le 4\epsilon \int_A \int_{\Gamma(t)} E^\epsilon(u)\ d\mathcal{H}^1 \\ &\le 4\epsilon \ln(1/\epsilon)\ \|\nabla\phi\|_\infty\ \mu^\epsilon(spt(\phi))\ . \end{aligned} \tag{2.15}$$

Clearly $A \subset [0,T]$ has measure less than $T = \|\phi\|_\infty$. Also, by the definition of A, if $t \in A$ and $\Gamma(t)$ is nonempty, then $|\deg(u;\Gamma(t))| \le d_\lambda$. It follows that

$$|I_{A,\Gamma}| \le \pi\|\phi\|_\infty d_\lambda + C\epsilon^{1/2}\|\nabla\phi\|_\infty \,\mu^\epsilon(spt(\phi)). \tag{2.16}$$

3. Estimate of $I_{A,\gamma}$

Using Cauchy's inequality and the elementary fact that $x \le \frac{1}{b}(1-x)^2 + (1+\frac{b}{4})$ for all $x \in \mathbb{R}$ and $b > 0$, we have

$$\begin{aligned}|j(u)| \le |u||\nabla u| &\le \frac{a}{2}\left(|\nabla u|^2 + \frac{1}{a^2}|u|^2\right)\\ &\le \frac{a}{2}\left(|\nabla u|^2 + \frac{(1-|u|^2)^2}{a^2 b}\right) + \frac{1}{2a}(1+\frac{b}{4})\end{aligned}$$

for every $a, b > 0$. We select $a = \epsilon^\alpha$ for $\alpha \in (0,1)$ and $b = \epsilon^{2-2\alpha}$ to find

$$|j(u)| \le C\epsilon^\alpha E^\epsilon(u) + C\epsilon^{-\alpha} \tag{2.17}$$

The Definition (2.13) of A implies that $|A| \le T = \|\phi\|_\infty$ and that $\mathcal{H}^1(\gamma(t)) < \epsilon$ for every $t \in A$, so we can take $\alpha = 1/2$ and use (2.11) to find

$$\begin{aligned}|I_{A,\gamma}| &\le C\int_A\int_{\gamma(t)} \sqrt{\epsilon}E^\epsilon(u)d\mathcal{H}^1(x)dt \;\; + C\int_A\int_{\gamma(t)}\frac{C}{\sqrt{\epsilon}}d\mathcal{H}^1(dx)dt\\ &\le C\epsilon^{1/3}\mu^\epsilon(\text{spt}(\phi))\|\nabla\phi\|_\infty \;\; + \; C\sqrt{\epsilon}\|\phi\|_\infty.\end{aligned}$$

4. Estimate of I_B

To estimate I_B we prove that B has small measure. Toward this end we define

$$B_1 := \{t \in \text{Reg}(\phi) \;:\mathcal{H}^1(\gamma(t)) \ge \epsilon\}$$

$$B_2 := \{t \in \text{Reg}(\phi) \;:\; \Gamma(t) \text{ is nonempty, and } |\deg(u;\Gamma(t))| \ge d_\lambda + 1\}. \tag{2.18}$$

The estimate of B_2 is deferred to Sect. 6, where we prove

Proposition 2.4. *For every $\lambda \in (1,2]$, $\epsilon \in (0,1]$, smooth $u : U \to \mathbb{R}^2$, and nonnegative test function $\phi \in C_c^{0,1}(U)$,*

$$|B_2| \;\le\; C\epsilon^{1-\frac{1}{\lambda}}\|\nabla\phi\|_\infty(d_\lambda+1) \;\le\; C\epsilon^{1-\frac{1}{\lambda}}\|\nabla\phi\|_\infty(1+\mu^\epsilon(spt(\phi)). \tag{2.19}$$

For the time being we assume this fact and use it to complete the proof of the theorem.

The measure of B_1 is easily estimated: using (2.11) and Lemma 2.3,

$$\begin{aligned}\frac{1}{25\epsilon}|B_1| &\le \int_{t\in B_1}\int_{\partial\Omega(t)} E^\epsilon(u)d\mathcal{H}^1 dt\\ &\le \|\nabla\phi\|_\infty \ln(\frac{1}{\epsilon})\mu^\epsilon(\text{spt}(\phi)).\end{aligned} \tag{2.20}$$

Clearly $|B| \leq |B_1| + |B_2|$, so by combining (2.20) and (2.19) we obtain

$$|B| \leq C\epsilon^{\frac{\lambda-1}{2\lambda}} \|\nabla\phi\|_\infty (1 + \mu^\epsilon(\text{spt}(\phi))). \tag{2.21}$$

Finally, we use (2.10) to estimate

$$|I_B| \ \leq \ C\epsilon^{\frac{\lambda-1}{3\lambda}} \|\nabla\phi\|_\infty (1 + \mu^\epsilon(\text{spt}(\phi)))\mu^\epsilon(\text{spt}(\phi)). \tag{2.22}$$

5. The previous three steps imply that

$$\left| \int \phi Ju \, dx \right| \leq d_\lambda \|\phi\|_\infty + \|\phi\|_{C^1} h_0^\epsilon(\phi, u, \lambda)$$

for

$$h_0^\epsilon(\phi, u, \lambda) \leq C\epsilon^{4\alpha(\lambda)} \left(1 + \mu^\epsilon(\text{spt}(\phi)) + (\mu^\epsilon(\text{spt}(\phi)))^2\right), \qquad \alpha(\lambda) = \frac{\lambda - 1}{12\lambda}.$$

To complete the proof of the Theorem, note that by (2.10) and (2.17) (with $\alpha = 2\alpha(\lambda)$)

$$\begin{aligned}
\left| \int \phi Ju \, dx \right| &\leq \int_0^T \int_{\partial\Omega(t)} |j(u)| d\mathcal{H}^1 dt \\
&\leq C\|\nabla\phi\|_\infty \int_{\text{spt}(\phi)} \epsilon^{2\alpha(\lambda)} E^\epsilon(u) + \epsilon^{-2\alpha(\lambda)} \, dx \\
&\leq C\|\phi\|_{C^1} h_1^\epsilon(\phi, u, \lambda),
\end{aligned}$$

for $h_1^\epsilon = \epsilon^{\alpha(\lambda)}\mu^\epsilon(\text{spt}(\phi)) + \epsilon^{-2\alpha(\lambda)}\text{Leb}^2(\text{spt}(\phi))$. We define $h^\epsilon(\phi, u, \lambda) := \min\{h_0^\epsilon, h_1^\epsilon\}$, so that (2.1) clearly holds. It thus suffices to verify that (2.3) holds, that is,

$$h^\epsilon(\phi, u, \lambda) = \min\{h_0^\epsilon, h_1^\epsilon\} \leq C\epsilon^{\alpha(\lambda)}(1 + \mu^\epsilon(\text{spt}(\phi))(1 + \text{Leb}^2(\text{spt}(\phi)))$$

for some appropriately large constant C. This follows immediately from the definition of h_0^ϵ if $\mu^\epsilon(\text{spt}(\phi)) \leq \epsilon^{-3\alpha(\lambda)}$, and if not, it follows directly from the definition of h_1^ϵ. $\square$

Note that the result we have proved is in fact somewhat sharper than Theorem 2.1 as stated, in that it not only provides an upper bound for $\int \phi Ju$, but in fact gives an approximate value for the integral. The following corollary states a small technical modification of this sharper estimate.

Corollary 2.5. *Let U be a bounded, open subset of $\mathbb{R}^2$, and suppose that $\phi \in C_c^{0,1}(U)$ and $u \in H^1(U; \mathbb{R}^2)$. Define Reg$(\phi)$, $\Gamma(t)$ and $\gamma(t)$ as in (2.7), (2.8) and (2.9) respectively.*

Then for any $\lambda \in (1, 2]$ and $\epsilon \in (0, 1]$, there exists a set $A = A(\phi, u, \lambda, \epsilon) \subset (0, \|\phi\|_\infty)$ such that

$$|A| \ \geq \ \|\phi\|_\infty - C\epsilon^{\alpha(\lambda)}\|\nabla\phi\|_\infty(1 + \mu^\epsilon(\mathit{spt}(\phi))); \tag{2.23}$$

$$\Gamma(t) \text{ is nonempty, and } |\mathit{deg}(u; \Gamma(t))| \leq d_\lambda \qquad \forall t \in A; \text{ and} \tag{2.24}$$

$$\left| \int \phi Ju \ - \ \pi \int_{t \in A} \mathit{deg}(u; \Gamma(t)) \, dt \right| \ \leq \ \|\phi\|_{C^1} h^\epsilon(\phi, u, \lambda), \tag{2.25}$$

where h^ϵ is defined in (2.3) and d_λ is defined in (2.2).

Proof. We cannot take A to be the set defined in (2.13), as we have now imposed the additional condition that $\Gamma(t) \neq \emptyset$ for $t \in A$. So we let $\tilde{A}$ be the set formerly known as A, defined in (2.13), and we define

$$A = \{t \in \tilde{A} \ : \ \Gamma(t) \text{ is nonempty.}\}$$

Then (2.24) follows from the definition of $\tilde{A}$, and (2.25) follows from (2.15). We claim moreover that $\tilde{A} \setminus A$ has measure at most $\epsilon\|\nabla\phi\|_\infty$. In view of (2.21) and (2.13), this will suffice to establish (2.23), and thus to complete the proof of the Corollary.

To prove our claim, note first that for every $t \in \tilde{A}$, $\mathcal{H}^1(\gamma(t)) < \epsilon$. If $t \in \tilde{A} \setminus A$, then $\Gamma(t)$ is empty, and so $\mathcal{H}^1(\phi^{-1}(t)) = \mathcal{H}^1(\gamma(t)) < \epsilon$ for all $t \in \tilde{A} \setminus A$. On the other hand, let $x_0 \in U$ be a point such that $\phi(x_0) = \|\phi\|_\infty$. If $|y - x_0| \leq \epsilon$ then $\phi(y) \ \geq \ \|\phi\|_\infty - \epsilon \, \|\nabla\phi\|_\infty$. It follows that $B_\epsilon(x_0) \subset \Omega(t)$ for all $t < \|\phi\|_\infty - \epsilon \, \|\nabla\phi\|_\infty$. Thus the isoperimetric inequality implies that $\mathcal{H}^1(\phi^{-1})(t) \geq 2\pi\epsilon$.

We conclude that if $t \in \tilde{A} \setminus A$, then $t \geq \|\phi\|_\infty - \epsilon\|\nabla\phi\|_\infty$, which proves the claim. □

3 Compactness in two dimensions

In this section we consider a sequence of functions $u^\epsilon \in H^1(U;\mathbb{R}^2)$, where U is a bounded open subset of $\mathbb{R}^2$ and the renormalized Ginzburg-Landau energy is uniformly bounded:

$$K_U := \sup_{\epsilon\in(0,1]} \mu^\epsilon(U) < \infty \ , \qquad \mu^\epsilon := \mu^\epsilon_{u^\epsilon} . \tag{3.1}$$

As discussed in the Introduction we will show that under this assumption, the Jacobian is compact in the dual norm $(C^{0,\beta})^*$ for every $\beta \in (0,1]$. Compactness in higher dimensions will be the subject of Sect. 5.

We introduce the Jacobian (signed) measure

$$Ju^\epsilon(E) := \int_E \det(\nabla u^\epsilon) \ dx \ , \qquad E \subset U.$$

Since $\det(\nabla u^\epsilon) = \frac{1}{2}\nabla \times j(u^\epsilon)$ for $j(u^\epsilon) := u^\epsilon \times \nabla u^\epsilon$,

$$\int_{\mathbb{R}^2} \phi \, dJu^\epsilon = \frac{1}{2} \int_{\mathbb{R}^2} \nabla \times \phi(x) \cdot j(u^\epsilon)(x) \, dx \ , \qquad \forall \phi \in C^1_c(U),$$

where for a scalar function ϕ, we write $\nabla \times \phi := (\phi_{x_2}, -\phi_{x_1})$.

Theorem 3.1. *Let $\{u^\epsilon\} \subset H^1(U;\mathbb{R}^2)$ satisfy (3.1). Then there exists a subsequence ϵ_n converging to zero and a signed Radon measure J such that Ju^{ϵ_n} converges to J in the dual norm $\left(C_c^{0,\beta}\right)^*$ for every $\beta \in (0,1]$. Moreover, there are $\{a_i\}_{i=1}^N \subset U$ and integers k_i such that*

$$J = \pi \sum_{i=1}^N k_i \, \delta_{a_i} \ , \qquad \textit{and} \qquad |J|(U) = \pi \sum_i |k_i| \leq K_U \ .$$

Finally, if μ^ϵ converges weakly to a limit μ, then $J \ll \mu$, and $\frac{dJ}{d\mu}(x) \leq 1$ for μ almost every x.

We will first prove

Proposition 3.2. *Assume (3.1). Then, Ju^ϵ can be written in the form*

$$Ju^\epsilon = J_0^\epsilon + J_1^\epsilon$$

where J_0^ϵ and J_1^ϵ are signed measures such that

$$\|J_0^\epsilon\|_{(C^0)^*} \le C, \text{ and } \qquad \|J_1^\epsilon\|_{(C_c^{0,1})^*} \le C\epsilon^\alpha \tag{3.2}$$

for some $\alpha > 0$ and a constant C depending only on the constant K_U in (3.1).

Proof. **1.** In light of the assumption $\mu^\epsilon(U) \le K$, Theorem 2.1 (with $\lambda = 2$ and $\alpha = 1/24$, for example) implies that

$$\int \phi Ju^\epsilon \le C\|\phi\|_\infty + C\epsilon^\alpha \|\nabla\phi\|_\infty \qquad \text{for all } \phi \in C_c^{0,1}(U). \tag{3.3}$$

We write $\delta = \epsilon^\alpha$, and we define $U_\delta = \{x \in U : \text{dist}(x, \partial U) > \delta\}$. Let

$$\chi_\delta = \begin{cases} 1 & \text{if } x \in U_{2\delta} \\ 0 & \text{if not.} \end{cases}$$

We define $J_0^\epsilon := \chi_\delta(\eta^\delta * Ju^\epsilon)$, where η^δ is a standard mollifier with support in $B_\delta(0)$. We then define $J_1^\epsilon := Ju^\epsilon - J_0^\epsilon$.

Suppose that ϕ is a C^1 test function vanishing on ∂U, and note that

$$\int \phi\, J_0^\epsilon dx = \int \eta^\delta * (\chi_\delta \phi) Ju^\epsilon dx.$$

We write $\phi^\delta := \eta^\delta * (\chi_\delta\phi)$. It is clear that ϕ^δ is compactly supported in U, and one easily checks that

$$\|\phi^\delta\|_\infty \le \|\chi_\delta\phi\|_\infty \le \|\phi\|_\infty, \qquad \|\nabla\phi^\delta\|_\infty \le \frac{C}{\delta}\|\chi_\delta\phi\|_\infty \le \frac{C}{\delta}\|\phi\|_\infty.$$

Since $\delta = \varepsilon^\alpha$, (3.3) implies that

$$\int \phi\, J_0^\epsilon dx \le C\|\phi\|_\infty.$$

2. We now estimate J_1^ϵ. Given $\phi \in C_0^1(U)$, write

$$\phi_1 := \min\{\phi, 2\delta\|\nabla\phi\|_\infty\}, \qquad \phi_2 := \phi - \phi_1.$$

It is clear that $\phi \le 2\delta\|\nabla\phi\|_\infty$ in $U \setminus U_{2\delta}$, so ϕ_2 is supported in $U_{2\delta}$.

From the definitions,

$$\int \phi_1 J_1^\epsilon\, dx = \int (\phi_1 - \eta^\delta * (\chi_\delta\phi_1)) Ju^\epsilon\, dx.$$

It is clear that

$$\|\phi_1\|_\infty \le 2\delta\|\nabla\phi\|_\infty, \qquad \|\nabla\phi_1\|_\infty \le \|\nabla\phi\|_\infty.$$

Similarly, $\eta^\delta * (\chi_\delta \phi_1)$ satisfies

$$\|\eta^\delta * (\chi_\delta \phi_1)\|_\infty \le 2\delta \|\nabla \phi\|_\infty, \qquad \|\nabla \eta^\delta * (\chi_\delta \phi_1)\|_\infty \le \frac{C}{\delta}\|\phi_1\|_\infty \le C\|\nabla\phi\|_\infty.$$

So (3.3) implies that

$$\int \phi_1 J_1^\epsilon \, dx \le C\delta \|\nabla\phi\|_\infty = C\epsilon^\alpha \|\nabla \phi\|_\infty.$$

Finally, since ϕ_2 is supported in $U_{2\delta}$,

$$\int \phi_2 J_1^\epsilon \, dx = \int (\phi_2 - \eta^\delta * (\chi_\delta \phi_2)) Ju^\epsilon \, dx = \int (\phi_2 - \eta^\delta * \phi_2) Ju^\epsilon \, dx.$$

It is easy to check that

$$\|\phi_2 - \eta^\delta * \phi_2\|_\infty \le C\delta\|\nabla\phi\|_\infty, \qquad \|\nabla(\phi_2 - \eta^\delta * \phi_2)\|_\infty \le C\|\nabla\phi\|_\infty.$$

So we again use (3.3) to conclude

$$\int \phi_2 J_1^\epsilon \, dx \le C\delta \|\nabla\phi\|_\infty = C\epsilon^\alpha \|\nabla \phi\|_\infty.$$

□

Once we have the above decomposition, the compactness of the sequence Ju^ϵ follows from soft arguments.

Lemma 3.3. *If ν is a Radon measure on U, then*

$$\|\nu\|_{(C_c^{0,\alpha})^*} \le C\|\nu\|^\alpha_{(C_c^{0,1})^*}\|\nu\|^{1-\alpha}_{(C_c^0)^*}\,. \tag{3.4}$$

Proof. Since U is bounded and we are considering compactly supported functions, the Hölder seminorm is in fact a norm and is topologically equivalent to the usual $C^{0,\alpha}$ norm. So for this lemma we set

$$\|\phi\|_{C_c^{0,\alpha}(U)} \; := \; [u]_{C^{0,\alpha}} \; = \; \sup_{x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|^\alpha}, \qquad \alpha \in (0,1].$$

Fix $\phi \in C_c^{0,\alpha}$, and let $\tilde\phi^\epsilon = \eta^\epsilon * \phi$, where η^ϵ is a smoothing kernel and ϵ will be chosen later. Then one easily checks that

$$\|\tilde\phi^\epsilon\|_{C^{0,1}} \le C\epsilon^{\alpha-1}\|\phi\|_{C^{0,\alpha}} := M_\epsilon, \qquad \|\phi - \tilde\phi^\epsilon\|_{C^0} \le C\epsilon^\alpha \|\phi\|_{C^{0,\alpha}} \tag{3.5}$$

In particular, $|\tilde\phi^\epsilon| \le C\epsilon^\alpha\|\phi\|_{C^{0,\alpha}}$ on ∂U.

We next modify $\tilde\phi^\epsilon$ so that it vanishes on ∂U while continuing to satisfy the above estimates. Let

$$u(x) = \sup_{y\in\partial U}\left(\tilde\phi^\epsilon(y) - M_\epsilon|x-y|\right)^+, \quad v(x) = \sup_{y\in\partial U}\left(\tilde\phi^\epsilon(y) + M_\epsilon|x-y|\right)^-.$$

Then one easily checks that $\tilde\phi^\epsilon = u - v$ on ∂U. Moreover, if we define $\phi^\epsilon := \tilde\phi^\epsilon - u + v$, then ϕ^ϵ satisfies the estimates in (3.5) and also vanishes on ∂U.

So

$$\begin{aligned}\int \phi d\nu &= \int \phi^\epsilon \, d\mu + \int (\phi - \phi^\epsilon) \, d\nu \\ &\le \|\phi^\epsilon\|_{C^{0,1}} \|\nu\|_{(C_c^{0,1})^*} + \|\phi - \phi^\epsilon\|_{C^0} \|\nu\|_{(C^0)^*} \\ &\le C \|\phi\|_{C^{0,\alpha}} \left(\epsilon^{\alpha-1} \|\nu\|_{(C_c^{0,1})^*} + \epsilon^\alpha \|\nu\|_{(C^0)^*} \right).\end{aligned}$$

Taking $\epsilon = \|\nu\|_{(C_c^{0,1})^*} / \|\nu\|_{(C^0)^*}$ gives the conclusion of the lemma. □

Lemma 3.4. *If* $\alpha > 0$, *then* $(C^0)^* \subset\subset (C^{0,\alpha})^*$.

Proof. The Arzela-Ascoli Theorem implies that any sequence that is bounded on $C^{0,\alpha}$ is precompact in C^0. The lemma follows by duality.

More concretely: given a sequence of measures bounded in $(C^0)^*$, we can extract a subsequence, say μ_n that converges to a limit μ in the weak-* topology. We must show that this sequence converges in norm in $(C^{0,\alpha})^*$. If not, then we can find a sequence of functions ψ_n with $\|\psi_n\|_{C^{0,\alpha}} \le 1$ such that

$$\int \psi_n d(\mu_n - \mu) \ge c_0 > 0 \tag{3.6}$$

for all n. However, the Arzela-Ascoli theorem implies that, upon passing to a subsequence, ψ_n converges to some limit ψ uniformly, whence (3.6) is impossible. □

We now prove

Theorem 3.5. *Assume (3.1). Then* Ju^ϵ *is strongly precompact in* $(C^{0,\beta})^*$ *for all* $\beta > 0$.

Proof. By Proposition 3.2 we can write $Ju^\epsilon = J_0^\epsilon + J_1^\epsilon$, where the two measures on the right-hand side satisfy (3.2).

Fix any $\beta \in (0,1]$. Lemma 3.4 implies that $\{J_0^\epsilon\}$ is precompact in $(C^{0,\beta})^* \subset (C_c^{0,\beta})^*$.

Also, it is clear from the definitions that

$$\|J_1^\epsilon\|_{(C^0)^*} \le \|Ju^\epsilon\|_{L^1} + \|J_0^\epsilon\|_{(C^0)^*} \le C\|\nabla u^\epsilon\|_{L^2}^2 + C \le K \ln(\frac{1}{\epsilon}).$$

So together with (3.2) and the interpolation inequality (3.4) this implies that $\|J_1^\epsilon\|_{(C_c^{0,\beta})^*} \to 0$ as $\epsilon \to 0$. □

Remark 3.6. The above result is sharp in the sense that Ju^ϵ need not be precompact, or even weakly precompact, in $(C^0)^*$. To see this, consider the sequence of functions

$$u^\epsilon(x,y) = (1,0) + \epsilon^2 (\ln(\frac{1}{\epsilon}))^{1/2} (\cos(\frac{x}{\epsilon^2}), \sin(\frac{y}{\epsilon^2}))$$

on the open unit disk D in the plane. One easily verifies that $\mu^\epsilon(D) \le C$, and that $\|Ju^\epsilon\|_{(C^0)^*} = \|Ju^\epsilon\|_{L^1} \ge c^{-1} \ln(\frac{1}{\epsilon})$. In particular, since $\|Ju^\epsilon\|_{(C^0)^*}$ is unbounded, the Uniform Boundedness Principle implies that the sequence cannot converge weakly in $(C^0)^*$.

Remark 3.7. Suppose ν^ϵ is any sequence of measures on a bounded open set $U \subset \mathbb{R}^m$, and that

$$|\nu^\epsilon|(U) \le K \ln(\frac{1}{\epsilon}), \qquad \int \phi d\nu^\epsilon \le C\|\phi\|_\infty + C\epsilon^\alpha \|\nabla\phi\|_\infty$$

for some $\alpha > 0$. The arguments given above then show, with essentially no change, that $\{\nu^\epsilon\}$ is precompact in $(C^{0,\beta})^*$ for all $\beta \in (0,1]$.

We are now in a position to give the

Proof of Theorem 3.1. Suppose $\{u^\epsilon\}_{\epsilon\in(0,1]} \subset H^1(U;\mathbb{R}^2)$ is a sequence satisfying (3.1). By an approximation argument, we may assume that in fact each u^ϵ is smooth. In view of Theorem 3.5, we can find a measure J and a subsequence ϵ_n such that $Ju^{\epsilon_n} \to J$ in $(C_c^{0,\beta})^*$ for every $\beta \in (0,1]$.
1. Since μ^{ϵ_n} is a sequence of uniformly bounded, nonnegative Radon measures, we may assume upon passing to a further subsequence (still labeled ϵ_n) that there is a Radon measure μ such that

$$\mu_n := \mu^{\epsilon_n} \stackrel{*}{\rightharpoonup} \mu \ ,$$

in the weak* topology of Radon measures in U. For $x \in U$, set

$$\Theta(x) := \lim_{r\downarrow 0} \ \mu\left(B_r(x) \cap U\right) \ .$$

We first claim that J is supported only on the points with $\Theta(x) \ge \pi$.

Indeed, suppose that $\Theta(x_0) < \pi$ at some $x_0 \in U$. Then there exists some $r_0 > 0$ and a number $\alpha < \pi$ such that

$$\mu_n(B_{r_0}(x_0)) \ \le \ \alpha < \pi$$

for all sufficiently large n. Then Theorem 2.1 with $\lambda = (\alpha+\pi)/\alpha > 1$ immediately implies that

$$\int \phi \, dJ(x) = \lim_{n\to\infty} \int \phi \ Ju^{\epsilon_n} dx \ = 0$$

for all smooth ϕ with support in $B_{r_0}(x_0)$, since $d_\lambda = 0$ for such ϕ. Thus $x_0 \notin \mathrm{spt}(J)$.

Since μ is bounded on U, there are finitely many points $\{a_i\}_i \subset U$ such that

$$\Theta(a_i) \ge \pi \ .$$

Therefore there are constants c_i such that the limit measure J satisfies

$$J = \pi \sum_i \ c_i \ \delta_{a_i} \ .$$

We need to prove that c_i's are integers and that $\pi|c_i| \le \Theta(a_i)$ for all i; this will immediately imply all the remaining conclusions of Theorem 3.1.

2. Choose $r_1 \le 1$ so that $B_{r_1}(a_1)$ does not intersect $\{a_i\}_{i>1} \cup \partial U$. We may also assume, taking r_1 smaller if necessary, that there exists some $\lambda > 1$ and an integer N_0 such that

$$d_\lambda := \lfloor \frac{\lambda}{\pi} \mu_n(B_{r_1}(a_1)) \rfloor \le \frac{1}{\pi} \Theta(a_1) \qquad \forall n \ge N_0. \tag{3.7}$$

We first apply Corollary 2.5 to the function $\phi(x) := (r_1 - |x - a_1|)^+$, which is supported in $B_{r_1}(a_1)$. Let $A^n = A(\phi, u^{\epsilon_n}, \lambda, \epsilon_n)$ be the set whose existence is asserted in Corollary 2.5. Note that if $t \in A^n$, then $\Gamma_{\phi, u^{\epsilon_n}}(t)$ is nonempty, which is to say that there is a component of $\phi^{-1}(t)$ on which $\min |u| \ge 1/2$. However, $\phi^{-1}(t) = \partial B_{r_1 - t}(a_1)$ is connected, so in fact $\Gamma_{\phi, u^{\epsilon_n}}(t) = \partial B_{r_1 - t}(a_1)$ for all $t \in A^n$. So for every $t \in A^n$ and $n \ge N_0$, Corollary 2.5 and the choice of λ imply that

$$\min_{x \in \partial B_{r_1 - t}(a_1)} |u^{\epsilon_n}| \ge \frac{1}{2}, \qquad |\mathrm{deg}(u^{\epsilon_n}; \partial B_{r_1 - t}(a_1))| \le d_\lambda \le \frac{1}{\pi} \Theta(a_1).$$

It follows that for all such n there is an integer $d(n) d_\lambda$ such that the set

$$S_n^{d(n)} := \{ r \in [0, r_1] \ : \ \min_{\partial B_r(a_i)} |u^{\epsilon_n}| > \frac{1}{2}, \ \mathrm{deg}(u^{\epsilon_n}; \partial B_r) = d(n) \}$$

has measure at least $k_0 := \frac{r_1}{3 d_\lambda}$. Note also that $S_n^{d(n)}$ is open, since u^{ϵ_n} is by assumption continuous (indeed, smooth). We can therefore find an open set $\Sigma_n \subset S_n^{d(n)}$ such that $|\Sigma_n| = k_0$.

3. We now define new test functions ψ^n as follows. First let

$$f^n(r) = |[r, r_1] \cap \Sigma_n|$$

We then define $\psi^n(x) = f^n(|x - a_1|)$. One can then check that t is a regular value of ψ^n if and only if

$$(\psi^n)^{-1}(t) = \partial B_r(a_1) \quad \text{for some } r \in \Sigma_n.$$

In particular, $\mathrm{deg}(u; (\psi^n)^{-1}(t)) = d(n)$ for a.e. $0 < t < \|\psi^n\|_\infty = k_0$.

One can then easily check, using Corollary 2.5, that

$$\int \psi^n J u^{\epsilon_n} dx = \pi d(n) k_0 + O(\epsilon^\alpha).$$

On the other hand, since the functions ψ^n are uniformly bounded in $C_c^{0,1}$ and since $J u^{\epsilon_n} \to J = \pi \sum c_i \delta_{a_i}$ in $C_c^{0,1}(U)^*$

$$0 = \lim_n \left| \int \psi^n J u^{\epsilon_n} dx - \pi c_1 \psi^n(a_1) \right| = \lim_n \left| \int \psi^n J u^{\epsilon_n} dx - \pi c_1 k_0 \right| .$$

Comparing the last two equations, we find that $d(n) = c_1$ for all sufficiently large n. In particular, c_1 is an integer and $|c_1| \le d_\lambda \le \frac{1}{\pi} \Theta(a_1)$, which is what we needed to show. $\square$

4 Gamma limit

Let U be an open bounded subset of $\mathbb{R}^2$ with a smooth boundary.

Recall that in the Introduction we have defined the function space $B2V$. Results in [16] discussed in the Introduction show that if $u \in B2V(U;S^1)$, then the Jacobian measure Ju has the form

$$Ju = \pi \sum_j k_j \, \delta_{a_j} \ , \tag{4.1}$$

for finite collections of points $\{a_j\} \subset U$ and integers k_j.

In this section we study the Γ limit of the functionals

$$I_\epsilon(u) := \frac{1}{\ln(1/\epsilon)} \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1-|u|^2)^2 \, dx \ ,$$

as ϵ tends to zero and show that the limiting functional is

$$I(u) := \begin{cases} |Ju|(U) = \pi \ \sum_i |k_i| \ , & \text{if } u \in B2V(U;S^1) \ , \\ +\infty \ , & \text{if } u \notin B2V(U;S^1) \ . \end{cases}$$

We refer the reader to the book of Dal Maso [8] for more information on Γ limits.

Theorem 4.1. *The Γ limit of I_ϵ in the topology of $W^{1,1}(U;\mathbb{R}^2)$ is equal to I, i.e., for every sequence u^ϵ converging to u in $W^{1,1}(U;\mathbb{R}^2)$,*

$$\liminf_{\epsilon \to 0} I_\epsilon(u^\epsilon) \geq I(u) \ , \tag{4.2}$$

and for every $u \in B2V(U;S^1)$, there exist functions u^ϵ converging to u in $W^{1,1}(U;\mathbb{R}^2)$ satisfying

$$\liminf_{\epsilon \to 0} I_\epsilon(u^\epsilon) = I(u) \ . \tag{4.3}$$

In the next section, we will prove (4.2) in higher dimensions. We believe that (4.3) holds in higher dimensions as well.

Proof. We start with the proof of (4.2). Suppose that u^ϵ converges to u in $W^{1,1}(U;\mathbb{R}^2)$. We assume that

$$\liminf_{\epsilon} I_\epsilon(u^\epsilon) < \infty \ ,$$

as there would be nothing to prove otherwise.

1. By the Compactness Theorem 3.1, there exists a subsequence ϵ_n converging to zero such that the Jacobian measure Ju^{ϵ_n} converges to a Radon measure J in $(C_c^{0,\beta})^*$ for all $\beta > 0$. We claim that $J = Ju$. In particular this will show that $u \in B2V(U;S^1)$.

To simplify the notation, set $u_n := u^{\epsilon_n}$.

2. We directly estimate that

$$\left|j(u_n) - \frac{j(u_n)}{|u_n|^2 \wedge 1}\right| \leq |u_n||\nabla u_n| \left|\frac{|u_n|^2 \wedge 1 - 1}{|u_n|^2 \wedge 1}\right|$$
$$= |\nabla u_n| \; \frac{|1-|u_n|^2|}{|u_n|} \; \chi_{|u_n| \geq 1}$$
$$\leq \epsilon_n \left[\frac{1}{2}|\nabla u_n|^2 + \frac{1}{2\epsilon_n^2}(1-|u_n|^2)^2\right] \; .$$

Hence,

$$\lim_{n\to\infty} \int_U \left|j(u_n) - \frac{j(u_n)}{|u_n|^2 \wedge 1}\right| \; dx = 0 \; .$$

3. Set $v_n := u_n/(|u_n|^2 \wedge 1)$ so that

$$\frac{1}{|u_n|^2 \wedge 1} \; j(u_n) - j(u) = v_n \times \nabla u_n - u \times \nabla u$$
$$= v_n \times (\nabla u_n - \nabla u) + (v_n - u) \times \nabla u \; .$$

Hence

$$\left|\frac{1}{|u_n|^2 \wedge 1} \; j(u_n) - j(u)\right| \leq |v_n||\nabla u_n - \nabla u| + |v_n - u||\nabla u|$$
$$\leq |\nabla u_n - \nabla u| + |v_n - u||\nabla u| \; .$$

Since u_n converges to u in $W^{1,1}(U;\mathbb{R}^2)$, there exists a subsequence, denoted by n again, so that u_n converges to u almost everywhere. Hence $|v_n - u||\nabla u|$ converges to zero almost everywhere and also it is less than $2|\nabla u|$. So we may use the dominated convergence theorem to conclude that

$$\lim_{n\to\infty} \int_U \left|\frac{1}{|u_n|^2 \wedge 1} \; j(u_n) - j(u)\right| \; dx = 0 \; .$$

4. Steps 2 and 3 imply that on a subsequence $j(u_n)$ converges to $j(u)$ in L^1. Hence, Ju^{ϵ_n} converges to Ju in the sense of distributions. This implies that $J = Ju$. Since by Theorem 3.1, J is a Radon measure, so is Ju and therefore $u \in B2V(U;\mathbb{R}^2)$. It is also clear that $|u| = 1$ almost everywhere. Hence, $u \in B2V(U;S^1)$.

5. The Jacobian estimate (2.1) implies that

$$\left|\int_U \; \phi \; Ju(dx)\right| = \lim_{n\to\infty} \left|\int_U \; \phi \; Ju_n(dx)\right|$$
$$\leq \lambda \|\phi\|_\infty \liminf_{n\to\infty} I^{\epsilon_n}(u_n) \; ,$$

for every $\lambda > 1$. Hence,

$$\liminf_{n\to\infty} I^{\epsilon_n}(u_n) \geq \sup \left\{ \left|\int_U \; \phi \; Ju(dx)\right| \; : \; \|\phi\|_\infty \leq 1 \right\}$$
$$= |Ju|(U)$$
$$= I(u) \; .$$

This proves (4.2).

6. We continue by proving the Γ-limit upper bound (4.3). Fix $u \in B2V(U;S^1)$. As remarked above, it is shown in [16] that Ju must have the form

$$Ju = \pi \sum_j k_j \ \delta_{a_j} \ ,$$

It suffices to show that, given any sufficiently small $\delta > 0$, there exists a sequence of functions $\{v^\epsilon\} \subset H^1(U;\mathbb{R}^2)$ such that

$$I_\epsilon(v^\epsilon) \to \pi \sum |k_j|, \qquad \limsup_\epsilon \|v^\epsilon - u\|_{W^{1,1}(U)} \le C\delta.$$

To do this, fix some small $\delta > 0$. Let $r_0 > 0$ be a number such that the balls $\{B_{2r}(a_j)\}$ are pairwise disjoint and do not intersect ∂U, whenever $r \le r_0$, and select some $r > 0$ such that

$$\sum_j \int_{B_{2r}(a_j)} |\nabla u| \, dx \ \le \ \delta, \qquad r \le \min\{r_0, \delta\}. \tag{4.4}$$

For any $s > 0$, let U_s denote $U \setminus \cup_j B_s(a_j)$. Demengel [9] proves that if V is an open subset of $\mathbb{R}^2$, then smooth functions taking values in S^1 are dense in the subspace $\{w \in W^{1,1}(V;S^1) \ : \ Jw = 0\}$. Since $Ju = 0$ on U_r, this implies that there exists a function $v \in C^\infty(U_r, S^1)$ such that

$$\|u - v\|_{W^{1,1}(U_r)} \le \delta. \tag{4.5}$$

Demengel's proof in fact shows that we may also assume that

$$\|j(u) - j(v)\|_{L^1(U_r)} \le \delta. \tag{4.6}$$

7. Clearly (4.4) and (4.5) imply that

$$\sum_j \int_r^{2r} \int_{\partial B_s(a_j)} |\nabla v(x)| \, d\mathcal{H}^1(x) ds \ = \ \sum_j \int_{B_{2r}\setminus B_r(a_j)} |\nabla v| \, dx \ \le \ 2\delta.$$

So for each j we can find some number $r_j \in [r, 2r]$ such that

$$\int_{\partial B_{r_j}(a_j)} |\nabla v(x)| \, d\mathcal{H}^1(x) \le \frac{2\delta}{r} \tag{4.7}$$

We also claim that

$$\deg(v; \partial B_{r_j}(a_j)) = k_j \tag{4.8}$$

if δ is sufficiently small. Indeed, since v is smooth and S^1-valued it is clear that $s \mapsto \deg(v; \partial B_s(a_j))$ is constant for $s \in [r, 2r_0]$, so we only need to verify that this constant must equal k_j. To do this, note that if ϕ is any function of the form $\phi(x) = \bar\phi(|x - a_j|)$ that is constant on $B_r(a_j)$ and has its support in $B_{2r_0}(a_j)$, then

$$\frac{1}{2} \int \nabla \times \phi \cdot j(v) \, dx \ = \ \frac{1}{2} \int_0^\infty \deg(v; \phi^{-1}(s)) \, ds \ = \ \pi\phi(a_j)\deg(v; \partial B_{r_j}(a_j))$$

and

$$\frac{1}{2}\int \nabla\times\phi\cdot j(u)\,dx \;=\; \int \phi\,dJu \;=\; \pi\phi(a_j)k_j.$$

If δ is small enough, (4.8) follows from these two identities and (4.6), since $\nabla\times\phi$ is supported in U_r.

8. We claim that for each j there exists smooth functions v_j^ϵ, defined in $B_{r_j}(a_j)$ such that $v_j^\epsilon(x) = v(x)$ for $x \in \partial B_{r_j}(a_j)$,

$$\text{(4.9)}\quad \int_{B_{r_j}(a_j)} |\nabla v_j^\epsilon| dx \le C\delta, \text{ and } \lim_{\epsilon\to 0}\frac{1}{|\ln\epsilon|}\int_{B_{r_j}(a_j)} E^\epsilon(v_j^\epsilon)dx = \pi|k_j|.$$

To see this, fix some j. We may assume without loss of generality that $a_j = 0$, and due to (4.8) we can write

$$v(x) = \exp[i(k_j\theta + \alpha_j + \psi(x))] \qquad \text{for } x \in \partial B_{r_j}$$

where α_j is a constant, ψ is a smooth, single-valued function on ∂B_{r_j}, and θ as usual satisfies $\frac{x}{|x|} = (\cos\theta, \sin\theta)$. We are identifying $\mathbb{R}^2 \cong \mathbb{C}$ in the usual way. We extend ψ to be homogeneous of degree zero on $\mathbb{R}^2 \setminus \{0\}$, and we define

$$v_j^\epsilon(x) \;=\; \exp[i(k_j\theta + \alpha_j + \frac{2|x| - r_j}{r_j}\psi(x))] \quad \text{if} \quad \frac{1}{2}r_j \le |x| \le r_j.$$

For $|x| \le \frac{1}{2}r_j$ we define $v_j^\epsilon(x)$ to be a minimizer of

$$\int_{B_{r_j/2}} E^\epsilon(w)\,dx$$

subject to the boundary conditions $w = \exp[i(k_j\theta + \alpha_j)]$ on $\partial B_{r_j/2}$.

Since v_j^ϵ restricted to the annulus $B_{r_j} \setminus B_{r_j/2}$ is just a fixed smooth function of unit modulus, independent of ϵ, it is clear that

$$\lim_\epsilon \frac{1}{|\ln\epsilon|}\int_{B_{r_j}\setminus B_{r_j/2}} E^\epsilon(v_j^\epsilon)\,dx \;=\; \lim_\epsilon \frac{1}{|\ln\epsilon|}\int_{B_{r_j}\setminus B_{r_j/2}} \frac{1}{2}|\nabla v_j^\epsilon|^2\,dx \;=\; 0.$$

Also, using (4.7) one can check that

$$\int_{B_{r_j}\setminus B_{r_j/2}(a_j)} |\nabla v_j^\epsilon| dx \;\le\; C\delta.$$

Finally, the book of Bethuel, Brezis, and Hélein gives a detailed description of the asymptotics of Ginzburg-Landau energy-minimizers, and their results imply that

$$\lim_\epsilon \frac{1}{|\ln\epsilon|}\int_{B_{r_j/2}} E^\epsilon(v_j^\epsilon)\,dx \;=\; \pi|k_j|, \quad \limsup_\epsilon \int_{B_{r_j/2}} |\nabla v_j^\epsilon|\,dx \;\le\; Cr_j \le C\delta.$$

Putting these facts together we find that the sequence $\{v_j^\epsilon\}$ has the properties specified in (4.9).

9. Finally we define

$$v^\epsilon(x) = \begin{cases} v(x) & \text{if } x \in U \setminus \big(\cup_j B_{r_j}(a_j)\big) \\ v_j^\epsilon(x) & \text{if } x \in B_{r_j}(a_j) \end{cases}$$

Since v is a fixed smooth function and $|v| \equiv 1$, $\frac{1}{|\ln \epsilon|} E^\epsilon(v) = \frac{1}{|\ln \epsilon|} |\nabla v|^2$ tends to zero uniformly as $\epsilon \to 0$. Thus it is clear from (4.9) that

$$\lim_{\epsilon \to 0} \frac{1}{|\ln \epsilon|} \int_U E^\epsilon(v^\epsilon) dx \;=\; \sum_j \lim_{\epsilon \to 0} \frac{1}{|\ln \epsilon|} \int_{B_{r_j}(a_j)} E^\epsilon(v_j^\epsilon) dx = \pi \sum_j |k_j|.$$

Also,

$$\begin{aligned} \|u - v^\epsilon\|_{W^{1,1}(U)} &\le \|u - v\|_{W^{1,1}(U_r)} \\ &+ \sum_j \Big(\|u\|_{W^{1,1}(B_{r_j}(a_j))} + \|v_j^\epsilon\|_{W^{1,1}(B_{r_j}(a_j))} \Big) \le C\delta \end{aligned}$$

by (4.4), (4.5), and (4.9). So the sequence $\{v^\epsilon\}$ has all the required properties. □

5 Compactness in higher dimensions

Now suppose U is a bounded, open subset of $\mathbb{R}^m$ with $m \ge 3$.

In this section we will show that if $\{u^\epsilon\}_{\epsilon \in (0,1]} \subset H^1(U; \mathbb{R}^2)$ is a sequence of functions such that the normalized Ginzburg-Landau energy measure $\mu^\epsilon(U)$ is uniformly bounded, then the Jacobians Ju^ϵ are precompact in $(C^{0,\beta})^*$ for all $\beta > 0$, and any limit is rectifiable. In addition, we prove that

$$|\bar{J}|(U) \le \liminf \mu^\epsilon(U).$$

This is not a full Γ-convergence result, but it shows that the mass of the Jacobian is a reasonable candidate for the Γ-limit. We also believe that the compactness result and the upper bound for the Jacobian (ie, lower bound for the energy) are interesting and will be useful in other contexts.

We start by defining some of the terms used above. We remark that good general references for this material include Giaquinta *et. al* [12] and Simon [30].

For $u : \mathbb{R}^m \subset U \to \mathbb{R}^2$ with $m \ge 2$ we view the Jacobian as a measure taking values in the exterior algebra $\Lambda^2 \mathbb{R}^m$. For every n (and in particular for $n = 2$) we endow $\Lambda^n \mathbb{R}^m$ with the natural inner product structure, which we denote $(\cdot, \cdot)$, and for a multivector $v \in \Lambda^n \mathbb{R}^m$ we write $|v| = (v, v)^{1/2}$. If $u \in W^{1,1}(U; \mathbb{R}^2)$ we define

$$j(u) = \sum_{i=1}^m u \times u_{x_i}\, dx^i \,, \tag{5.1}$$

and if $j(u) \in L^1_{\text{loc}}$, we define

$$Ju = \frac{1}{2}\, d\, j(u) \qquad \text{in the sense of distributions} \tag{5.2}$$

where d is the exterior derivative. Thus if $u \in H^1_{\text{loc}}$, then

$$Ju = \sum_{i<j} J^{ij} u \, dx^i \wedge dx^j = \frac{1}{2} \sum_{i,j} J^{ij} u \, dx^i \wedge dx^j,$$

where $J^{ij}u = -J^{ji}u = u_{x_i} \times u_{x_j} = \det(u_{x_i}, u_{x_j})$. For sufficiently differentiable $u : \mathbb{R}^m \to \mathbb{R}^n$ one can define in a similar way Ju as a measure taking values in $\Lambda^n \mathbb{R}^m$. We omit the most general definition as we will not need it here.

A set $M \subset \mathbb{R}^m$ is said to be a k-dimensional rectifiable set if there are Lipschitz functions $f_j : \mathbb{R}^k \to \mathbb{R}^m$ and measurable subsets A_j of $\mathbb{R}^k$ such that

$$M = M_0 \cup \left(\cup_{j=1}^{\infty} f_j(A_j) \right), \qquad \mathcal{H}^k(M_0) = 0.$$

Thus, in a precise measure theoretic sense, a k-dimensional rectifiable set is not much worse than a k-dimensional Lipschitz submanifold. Rectifiable sets can also be characterized by the fact that they have k-dimensional approximate tangent spaces $\mathcal{H}^k$ almost everywhere; see [30] or [12].

Suppose that M is an oriented, rectifiable $(m-n)$-dimensional subset of $\mathbb{R}^m$, and for $\mathcal{H}^{m-n}$ almost every $x \in M$, let $\nu(x) \in \Lambda^n \mathbb{R}^m$ be the unit n-vector representing the appropriately oriented normal space to M. (It is more convenient for our purposes to work with normal spaces rather than tangent spaces.) Suppose also that $\theta : M \to \mathbb{N}$ is a $\mathcal{H}^{m-n}$-integrable function. One can define a measure J taking values in $\Lambda^n \mathbb{R}^m$ by

$$\int \phi(x) J(dx) = \int_M \phi(x) \cdot \nu(x) \theta(x) \mathcal{H}^{m-n}(dx) \quad \forall \phi \in C^0(\mathbb{R}^m; \Lambda^n \mathbb{R}^m). \tag{5.3}$$

We say that a measure J taking values in $\Lambda^n \mathbb{R}^m$ is $(m-n)$-dimensional integer multiplicity rectifiable (or more briefly, integer multiplicity rectifiable) if it has the form (5.3) for some rectifiable set M and an integer-valued function θ as above.

The class of functions for which Ju is a measure is denoted $BnV(U, \mathbb{R}^n)$ and was defined and studied in [16]. In particular we prove there that if $u \in BnV(\mathbb{R}^m, S^{n-1})$ then $\frac{1}{\omega_n} Ju$ is integer multiplicity rectifiable, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. This is deduced as a consequence of a more general rectifiability criterion which we recall here, as we will need it later.

Let J be a measure on a subset $U \subset \mathbb{R}^m$ taking values in $\Lambda^n \mathbb{R}^m$, where $n \le m$. We can write J in the form $J = \nu |J|$, where $|J|$ is a nonnegative Radon measure, and ν is a $|J|$-measurable function taking values in $\Lambda^n \mathbb{R}^m$ such that $|\nu(x)| = 1$ at $|J|$-a.e. $x \in U$.

Suppose that $e_1, ..., e_m$ is an orthonormal basis for $\mathbb{R}^m$. Given any point $x \in \mathbb{R}^m$, we write $y_i = x \cdot e_i$ if $i = 1, ..., m-n$; and $z_i = x \cdot e_{m-n+i}$ if $i \in 1, \ldots, n$. We write $\mathbb{R}^{m-n}_y$ to denote the span of $\{e_i\}_{i=1}^{m-n}$. Similarly, $\mathbb{R}^n_z = \text{span}\{e_i\}_{i=m-n+1}^m$. Thus we identify points $x \in \mathbb{R}^m$ with corresponding $(y,z) \in \mathbb{R}^{m-n}_y \times \mathbb{R}^n_z$. Let $dz := dz^1 \wedge \ldots \wedge dz^n$, and let J^z denote the scalar signed measure defined by $J^z := (dz, \nu)|J|$.

We say that J^z is locally represented by slices $J_y(dz)$ if, given any open set $O \subset U$ of the form $O = O_y \times O_z$, with $O_y \subset \mathbb{R}^{m-n}_y$ and $O_z \subset \mathbb{R}^n_z$, there exist signed Radon measures $J_y(dz)$ on O_z for a.e. $y \in O_y$, such that

$$\int \phi J^z = \int_{O_y} \int_{O_z} \phi(y,z) J_y(dz)\, dy \tag{5.4}$$

for all continuous ϕ with compact support in O.

We say that a statement holds for $a.e.\ J_y(dz)$ if, for every open set O as above, it is valid for a.e. $y \in O_y$.

In [16] we prove the following

Theorem 5.1. *Suppose that J is a Radon measure on $U \subset \mathbb{R}^m$ taking values in $\Lambda^n \mathbb{R}^m$, and also that $dJ = 0$ in the sense of distributions. Suppose also that for every choice on an orthonormal basis $\{e_i\}_{i=1}^m$ (determining a decomposition of $\mathbb{R}^m$ into $\mathbb{R}^{m-n}_y \times \mathbb{R}^n_z$) J^z is represented locally by slices, and that for $a.e.\ y \in O_y$ these slices have the form*

$$J_y(dz) = \sum_{i=1}^{K} d_i \delta_{a_i}(dz)$$

for an integers K and d_i, and points $a_i \in O_z$.

Then J is rectifiable.

A much more general version of this result was later established by Ambrosio and Kirchheim [3]. A similar theorem in somewhat different and very general setting was proved independently (and slightly earlier) by White [33].

We will need Theorem 5.1 to prove

Theorem 5.2. *Let $U \subset \mathbb{R}^m$, and suppose that $\{u^\epsilon\}_{\epsilon \in (0,1]}$ is a collection of functions in $W^{1,2}(U;\mathbb{R}^2)$ such that $\mu^\epsilon(U) \le K_U < \infty$ for all ϵ. Then there exists a subsequence $\epsilon_n \to 0$ such that*

(i): Ju^{ϵ_n} *converges to a limit $\bar{J}$ in the $(C^{0,\alpha})^*$ norm for every $\alpha > 0$;*
(ii): *For any choice of basis $\{e_i\}$ for $\mathbb{R}^m$ (determining a decomposition of $\mathbb{R}^m$ into $\mathbb{R}^{m-2}_y \times \mathbb{R}^2_z$), $\bar{J}^z$ is represented locally by slices $\bar{J}_y(dz)$, and for a.e. y these slices have the form $\bar{J}_y(dz) = \pi \sum_{i=1}^K d_i \delta_{a_i}$, with $d_i \in \mathbb{Z}$ for all i.*
(iii): $d\bar{J} = 0$ *in the sense of distributions, and $\frac{1}{\pi}\bar{J}$ is integer multiplicity rectifiable;*
(iv): *Finally, if $\bar{\mu}$ is any weak limit of a subsequence of μ^{ϵ_n}, then $|\bar{J}| \ll \bar{\mu}$, and $\frac{d|\bar{J}|}{d\bar{\mu}} \le 1$. In particular, $|\bar{J}|(U) \le K_U$.*

Remark 5.3. For any $\bar{J}$ as above, (iii) and the definition of rectifiability imply that a lower density bound:

$$\liminf_{r \to 0} \frac{|\bar{J}|(B_r(x))}{\mathcal{H}^{m-2}(B_r(x))} \ge \pi$$

for $|\bar{J}|$ almost every x. Also, if $\bar{\mu}$ is as in (iv), then clearly the $m-2$-dimensional density of μ is greater than $m-2$-dimensional density of $\bar{J}$. In particular,

$$\liminf_{r \to 0} \frac{\bar{\mu}(B_r(x))}{\mathcal{H}^{m-2}(B_r(x))} \ge \pi$$

for $|\bar{J}|$ almost every x.

The basic idea of the proof is to decompose a component of Ju^ϵ, for example $J^{m-1,m}u^\epsilon$, into two-dimensional slices, say $J_y^\epsilon(dz)$, and to use the two-dimensional estimates on each slice. Arguing in this fashion, it is quite easy to obtain uniform estimates for $J^{m,m-1}u^\epsilon$ in certain weak spaces, and these imply (i) by results of Sect. 3.

To prove (ii), it is convenient to view the sliced measures $J_y^\epsilon(dz)$ as constituting a function mapping $\mathbb{R}_y^{m-2}$ into $C_c^1(\mathbb{R}_z^2)^*$; the latter is a space that contains measures on $\mathbb{R}_z^2$ and is endowed with a rather weak topology. Claim (i) can be seen as assertion that the function $y \mapsto J_y^\epsilon(dz)$ is precompact in some weak sense. What one would like to do is to show that in fact $y \mapsto J_y^\epsilon(dz)$ is precompact in some stronger sense, for example in $L^1(\mathbb{R}_y^{m-2};(C_c^1(\mathbb{R}_z^2)^*)$, so that one can extract a subsequence that converges to some limiting function $y \mapsto \bar{J}_y(dz)$ in L^1. In particular, after passing to a further subsequence we could then assume that $J_y^{\epsilon_n}(dz) \to \bar{J}_y(dz)$ for almost every y. In addition, by our two-dimensional results, for almost every y, one can find a subsequence $\epsilon_{n_m} \to 0$ (in general depending on y) such that $J_y^{\epsilon_{n_m}}(dz)$ converges to some limit that has the form sought in (ii). By combining these results one can hope to show that in fact $\frac{1}{\pi}\bar{J}_y(dz)$ is a sum of point masses with integer multiplicities.

The key point is then to establish some sort of strong compactness of the sequence of functions $y \mapsto J_y^\epsilon(dz)$ as $\epsilon \to 0$. We do this using the observation from [16,17] that the total variation of $y \mapsto J_y^\epsilon(dz)$ in the $(C_c^{0,1})^*$ norm can be estimated by controlling "orthogonal" components of Ju^ϵ, which is already done in the proof of (i). Using this one can argue that the functions $y \mapsto J_y^\epsilon(dz)$ have uniformly bounded variation in $(C_c^{0,1})^*$, modulo terms that vanish in still weaker norms, and this gives the necessary strong convergence. (The terms involving weaker norms force us to work with test functions that are C^2 instead of C^1 in much of the proof.)

The remaining points follow quite directly from (ii) and the rectifiability criterion of Theorem 5.1, and from the two-dimensional results.

Proof.
1. To prove compactness, it suffices to show that any component $J^{ij}u^\epsilon$ is precompact. Without loss of generality we consider $J^{m-1,m}u^\epsilon$. We write $x \cong (y,z) \in \mathbb{R}^{m-2} \times \mathbb{R}^2$, so that $y_i = x_i$ for $i \le m-2$, and $z_i = x_{m-2+i}$ for $i = 1,2$. We also write $J^{z,\epsilon}$ as a shorthand for $J^{m-1,m}u^\epsilon$. Note that $J^{z,\epsilon}$ is just the Jacobian of u^ϵ in the z variables: $J^{z,\epsilon} = u^\epsilon_{x_{m-1}} \times u^\epsilon_{x_m} = u^\epsilon_{z_1} \times u^\epsilon_{z_2}$.

For any $\phi \in C_c^1(U)$,

$$\int \phi J^{z,\epsilon}\, dx = \int_{\mathbb{R}_y^{m-2}} \int_{\{z\in\mathbb{R}^2:(y,z)\in U\}} \phi(y,z) \det(u^\epsilon_{z_1}, u^\epsilon_{z_2})\, dz\, dy. \tag{5.5}$$

Note that $\phi(y,\cdot)$ is C^1 and compactly supported in $\{z \in \mathbb{R}^2 : (y,z) \in U\}$, and that $\text{Leb}^2(spt(\phi(y,\cdot))$ is bounded uniformly for $y \in \mathbb{R}^{m-2}$. Thus Theorem 2.1 implies that

$$\left| \int_{\{z\in\mathbb{R}^2:(y,z)\in U\}} \phi(y,z) \det(u_{z_1}, u_{z_2})\, dz \right|$$

$$(5.6) \quad \leq \; C\left(\|\phi(y,\cdot)\|_\infty + \epsilon^\alpha \|\phi(y,\cdot)\|_{C^{0,1}}\right) \frac{1}{|\ln\epsilon|} \int_{\{z\in\mathbb{R}^2:(y,z)\in U\}} E^\epsilon(u^\epsilon)dz$$

where C is a constant depending only on spt(ϕ). Integrating over $y \in \mathbb{R}^{m-2}$ we obtain

$$(5.7) \qquad \left| \int \phi J^{z,\epsilon}\, dx \right| \leq C\left(\|\phi\|_\infty + \epsilon^\alpha \|\nabla_z \phi\|_\infty\right) \mu^\epsilon(U).$$

Here $\nabla_z\phi$ denotes the gradient with respect to the z variables only, reflecting the fact that only z derivatives appear in the term $\|\phi(y,\cdot)\|_{C^{0,1}}$ on the right-hand side of (5.6). Since $\mu^\epsilon(U)$ is bounded by assumption, we deduce that

$$(5.8) \qquad \left| \int \phi J^{z,\epsilon}\, dx \right| \leq C\left(\|\phi\|_\infty + \epsilon^\alpha \|\phi\|_{C^{0,1}}\right).$$

It is also clear that

$$(5.9) \qquad \int_U |J^{z,\epsilon}| dx \leq C \int_U |\nabla u^\epsilon|^2 dx \leq C \ln(\frac{1}{\epsilon}),$$

so Remark 3.7 implies that $\{J^{z,\epsilon}\}$ is precompact in $(C^{0,\alpha})^*$ for all $\alpha > 0$.

The main part of the proof is to show that $\bar{J}^z$ is locally represented by slices. This will be done in the next four steps.

2. Let ϵ_n be a subsequence such that Ju^{ϵ_n} converges to a limit $\bar{J}$ in $(C^{0,\alpha})^*$ for all $\alpha > 0$. Fix an arbitrary orthonormal basis $\{e_i\}_{i=1}^m$ for $\mathbb{R}^m$. Using this basis we write $x \cong (y,z)$ and $\mathbb{R}^m = \mathbb{R}^{m-2}_y \times \mathbb{R}^2_z$ as above. We write $\bar{J}^{ij} = \text{wk}\lim J^{ij}u^{\epsilon_n}$, so that $\bar{J} = \sum_{i<j} \bar{J}^{ij} dx^i \wedge dx^j$. We also write $\bar{J}^z = \bar{J}^{m,m-1} = \text{wk}\lim J^{z,\epsilon_n}$.

Let $O = O_y \times O_z$ be subset of U, For each n and each $y \in O_y$ let $J^n_y(dz)$ be the measure on O_z whose density with respect to Lebesgue measure is $u^{\epsilon_n}_{z_1} \times u^{\epsilon_n}_{z_2}(y,z)$. We also write $J^{z,n}$ for J^{z,ϵ_n}. From the definitions it is clear that $J^{z,n}(dx)$ is locally represented by the slices $J^n_y(dz)$; this assertion is simply the obvious identity (5.5).

Fix any $\psi \in C^2_c(O)$, and for each n define

$$(5.10) \qquad \tilde{\psi}^n(y) = \int_{O_z} \psi(y,z) J^n_y(dz), \qquad y \in O_y.$$

In this step we show that $\tilde{\psi}^n$ is precompact in $L^1(O_y)$, with the aid of a lemma whose proof appears somewhat later.

For any $v \in C^1_c(O_y)$, let $\phi(x) = \phi(y,z) = v(y)\psi(y,z)$. Then (5.5) and (5.8) imply that

$$\int_{O_y} v(y)\tilde{\psi}^n(y)dy = \int_{O_y}\int_{O_z} v(y)\psi(y,z)J^n_y(dz)dy$$

$$(5.11) \qquad = \int_O \phi(x) J^{z,n}(dx) \leq C(\|\phi\|_\infty + \|\nabla_z\phi\|_\infty) = C\|v\|_\infty$$

for some constant depending on $\|\psi\|_{C^1}$ but independent of n and v.

Next fix any $k \in \{1, \ldots m-2\}$ and any $w \in C_c^1(O_y)$ and compute

$$\begin{aligned} \int_{O_y} w_{y_k}(y)\tilde{\psi}^n(y)dy &= \int_{O_y}\int_{O_z} w_{y_k}(y)\psi(y,z)J_y^n(dz)dy \\ &= \int_O \left[(w\psi)_{y_k} - w\psi_{y_k}\right] J^{z,n}(dx). \end{aligned} \tag{5.12}$$

We deduce from (5.7) that

$$|\int_{O_y} w(y)\psi_{y_k}(y,z)J^{z,n}(dx)| \le C\|w\|_\infty \|\psi\|_{C^2}. \tag{5.13}$$

To estimate the remaining term, note that for any $i, j, k \in \{1, \ldots, m\}$

$$(J^{ij}u^{\epsilon_n})_{x_k} + (J^{jk}u^{\epsilon_n})_{x_i} + (J^{ki}u^{\epsilon_n})_{x_j} = 0$$

in the sense of distributions. Take $i = m-1$, $j = m$; then $J^{ij}u^{\epsilon_n} = J^{z,n}$, and also $x_i = z_1, x_j = z_2, x_k = y_k$. Thus

$$\begin{aligned} \int_O (w\psi)_{y_k} J^{z,n}(dx) = &- \int_O (w\psi)_{z_1} J^{m,k} u^{\epsilon_n}(dx) \\ &- \int_O (w\psi)_{z_2} J^{k,m-1} u^{\epsilon_n}(dx) \end{aligned} \tag{5.14}$$

Note that $(w\psi)_{z_i} = w\psi_{z_i}$, since w depends only on y. Thus we can use (5.8) to estimate the right-hand side of the above equation, and combine with (5.12) and (5.13) to conclude that

$$\int_{O_y} w_{y_k}(y)\tilde{\psi}^n(y)dy \le C(\|w\|_\infty + \epsilon_n^\alpha \|\nabla w\|) \tag{5.15}$$

for some C depending on $\|\psi\|_{C^2}$ but independent of n. Also, if we estimate the right-hand side of (5.14) using (5.9) instead of (5.8) we easily find that

$$\int_{O_y} w_{y_k}(y)\tilde{\psi}^n(y)dy \le C\left(\ln(\frac{1}{\epsilon}) + 1\right) \|w\|_\infty. \tag{5.16}$$

We have shown that (5.11), (5.15), and (5.16) hold for all $w, v \in C_c^1(O_y)$. According to Lemma 5.4, proven below, this is sufficient to establish that $\{\tilde{\psi}^n\}_n$ is precompact in L^1.

If ψ is not compactly supported in O but $\psi(y,\cdot)$ is compactly supported in O_z for every $y \in O_y$, we can apply the above arguments to the functions $\Psi(y,z) = \chi(y)\psi(y,z)$, where $\chi \in C_c^\infty(O_y)$ and $\chi \equiv 1$ on some open subset $V \subset\subset O_y$, to find that $\{\tilde{\Psi}^n\}$ is precompact in L^1, and hence that $\{\tilde{\psi}^n\}$ is precompact in L^1_{loc}.

3. Now fix a countable dense subset $\{\psi_k\}$ of $C_c^2(O_z)$, and for each ψ_k and each n, define

$$\tilde{\psi}_k^n(y) = \int_{O_z} \psi^k(z)J_y^n(dz) \qquad y \in O_y.$$

The results of Step 2 imply that $\{\tilde\psi_k^n\}_{n=1}^\infty$ is precompact in $L^1_{\rm loc}(O_y)$ for each k, so using a diagonal argument and passing to a subsequence (which we still label ϵ_n) we may arrange that there is a set $A_1 \subset O_y$ of measure zero such that for every k, $\tilde\psi_k^n(y)$ converges to a finite limit $\tilde\psi_k(y)$ for all $y \in O_y \setminus A_1$, as $n \to \infty$.

Next define

$$\mu_y^{\epsilon_n}(O_z) := |\ln \epsilon_n|^{-1} \int_{O_z} E^{\epsilon_n}(y,z)dz$$

Clearly $\int_{O_y} \mu_y^{\epsilon_n}(O_z)dy = \mu^{\epsilon_n}(O) \le K_U$.

It follows that there exists a set $A_2 \subset O_y$ of measure zero such that $\liminf_n \mu_y^{\epsilon_n}(O_z) < \infty$ for all $y \in O_y \setminus A_2$.

Let $A = A_1 \cup A_2$, and note that A has measure zero.

4. Fix some $y \in O_y \setminus A$, consider any subsequence ϵ_{n_m} such that

$$\mu_y^{\epsilon_{n_m}}(O_z) \le C. \tag{5.17}$$

Such subsequences exist by virtue of the definition of A. The two-dimensional compactness results imply that $J_y^{n_m}(dz)$ is precompact in $\cup_{\alpha>0}(C^{0,\alpha})^*$. Let $\bar J_y$ be any limit. From Step 3 it is clear that $\int \psi_k(y,z)\bar J_y(dz) = \tilde\psi_k(y)$ for every ψ_k in the dense subset $\{\psi_k\}_k$ of $C_c^2(O_z)$. This implies that $\bar J_y$ is uniquely determined, independent of the choice of a subsequence ϵ_{n_m}, and hence that any limit of $J_y^{n'_m}(dz)$ for any sequence n'_m satisfying (5.17) must equal $\bar J_y(dz)$.

This defines $\bar J_y$ for every $y \in O_y \setminus A$. Note that, as a consequence of the 2-dimensional results, $\bar J_y(dz)$ has the form $\pi \sum_i d_i \delta_{a_i}(dz)$ for almost every y, for integers d_i and points $a_i \in O_z$ that of course depend on y.

Returning to the subsequence obtained in Step 3, we see that if $y \in O_y \setminus A$, then

$$\lim_n \int \psi(z) J_y^n(dz) = \lim_m \int \psi(z) J_y^{n_m}(dz) = \int \psi(z)\bar J_y(dz)$$

for any continuous ψ, whenever the left-most limit exists. The two-dimensional results also imply that if B is any open subset of O_z, then

$$\liminf \mu_y^{\epsilon_n}(B) \ge \bar J_y(B). \tag{5.18}$$

5. We now show that $\bar J^z$ is represented locally by the slices $\bar J_y(dz)$, where (we recall) $\bar J^z(dx) = \text{wk lim } J^{z,n}(dx) \ = \ \text{wk lim } J^{m-1,m}u^{\epsilon_n}\ dx$.

To do this, fix any $\psi \in C_c^2(O)$ and compute

$$\begin{aligned}\int \psi(x)\bar J^z(dx) &= \lim_n \int_O \psi(x) J^{z,n}(dx)\\ &= \lim_n \int_{O_y}\int_{O_z} \psi(y,z) J_y^n(dz)dy = \lim_n \int_{O_y} \tilde\psi^n(y)dy,\end{aligned}$$

for $\tilde\psi^n$ as in (5.10). We know from Step 2 that $\{\tilde\psi^n\}$ is precompact in L^1. Fix a convergent subsequence, and let $\tilde\psi$ denote the limit. We may assume upon passing to a further subsequence, still labeled $\tilde\psi^n$, that $\tilde\psi^n(y) \to \tilde\psi(y)$ at all y in $O_y \setminus A_3$, where A_3 is some set of measure zero.

Also, if $y \in O_y \setminus (A \cup A_3)$, Step 4 implies that

$$\tilde{\psi}(y) = \lim_n \tilde{\psi}^n(y) = \lim_n \int_{O_z} \psi(y,z) J_y^n(dz) = \int_{O_z} \psi(y,z)\bar{J}_y(dz).$$

Thus

$$\int_O \psi(x)\bar{J}^z(dx) = \int_{O_y}\int_{O_z} \psi(y,z)\bar{J}_y(dz)$$

as claimed. Since C^2 is dense in C^0. this holds for all continuous ψ. This completes the proof of (ii).

6. It is clear that if $\phi \in C_c^1(U; \Lambda^2\mathbb{R}^m)$, then

$$\int \phi \cdot d\bar{J} = \lim_n \int \phi \cdot dJu^{\epsilon_n} = \lim_n \int \phi \cdot d^2 \frac{1}{2} j(u^{\epsilon_n}) = 0.$$

Thus $d\bar{J} = 0$ in the sense of distributions.

In view of Theorem 5.1 and (ii), this shows that $\frac{1}{\pi}\bar{J}$ is rectifiable. Thus we have established (iii).

7. It remains to prove (iv).

We write $\bar{J}$ in the form $\bar{\nu}|\bar{J}|$, where $|\bar{J}|$ is a nonnegative Radon measure, and $\bar{\nu}$ is a $|\bar{J}|$-measurable function taking values in $\Lambda^2\mathbb{R}^m$, such that $|\bar{\nu}(x)| = 1$ for $|\bar{J}|$- a.e. $x \in \mathbb{R}^m$. Since $\bar{J}$ is rectifiable, $\bar{\nu}$ is simple at $|\bar{J}|$ a.e. x, that is, it has the form $\bar{\nu} = \nu^1 \wedge \nu^2$ for orthogonal unit vectors $\nu^i \in \Lambda^1\mathbb{R}^m$. General theorems on differentiation of measures imply that

$$\lim_r \frac{1}{|\bar{J}|(B_r(x))} \int_{B_r(x)} |\bar{\nu}(x') - \bar{\nu}(x)|\,|\bar{J}|(dx') = 0 \tag{5.19}$$

at $|\bar{J}|$-a.e. $x \in \mathbb{R}^m$.

Let $\bar{\mu}$ be a weak* limit of μ^ϵ. It suffices to show that

$$|\bar{J}|(B_r(x)) \le (1 + o_r(1))\bar{\mu}(B_r(x)) \tag{5.20}$$

at every point x where (5.19) holds and $\bar{\nu}(x)$ is simple.

Fix such a point x. After a change of basis we can assume that $\bar{\nu}(x) = \nu^1(x) \wedge \nu^2(x) = dx^{m-1} \wedge dx^m$. As above we decompose $\mathbb{R}^m$ as $\mathbb{R}_y^{m-2} \times \mathbb{R}_z^2$, and we write $x = (y,z)$, $dz = dx^{m-1} \wedge dx^m$, $\bar{J}^z = (dz, \bar{\nu})|\bar{J}|$, and so on. First note that by (5.19),

$$\begin{aligned}\bar{J}^z(B_r(x)) &= \int_{B_r(x)} (dz, \bar{\nu}(x))\,|\bar{J}|(dx') + \int (dz, \bar{\nu}(x') - \bar{\nu}(x))\,|\bar{J}|(dx') \\ &= (1 + o_r(1))|\bar{J}|(B_r). \end{aligned}\tag{5.21}$$

Because $\bar{J}^z$ is represented by slices $\bar{J}_y(dz)$,

$$\bar{J}^z(B_r(x)) = \int_{\{y' \in \mathbb{R}^{m-2} : |y-y'| \le r\}} \bar{J}_{y'}(B_{r(y')}(z))\,dy' \tag{5.22}$$

for $r(y') = (r^2 - |y' - y|^2)^{1/2}$. Thus since B_r denotes a *closed* ball,

$$\begin{aligned}\bar\mu(B_r(x)) &\geq \liminf \mu^{\epsilon_n}(B_r(x)) \\ &= \liminf \int_{\{y'\in\mathbb{R}^{m-2}:|y-y'|\leq r\}} \mu^{\epsilon_n}_{y'}(B_{r(y')}(z))\,dy' \\ &\geq \int_{\{y'\in\mathbb{R}^{m-2}:|y-y'|\leq r\}} \bar J_{y'}(B_{r(y')}(z))\,dy'\end{aligned}$$

using (5.18) and Fatou's Lemma in the last line. The desired estimate (5.20) now follows from (5.21) and (5.22). □

We now prove the lemma used above.

Lemma 5.4. *Suppose that $U \subset \mathbb{R}^k$, that f^ϵ is a sequence of uniformly compactly supported functions in $BV(U)$, and that there exist positive constants C and α such that*

(5.23) $$\int_U v(y) f^\epsilon(y)\,dy \leq C\|v\|_\infty,$$

(5.24) $$\int_U w(y)\cdot\nabla f^\epsilon(y)\,dy \leq C(\|w\|_\infty + \epsilon^\alpha\|\nabla w\|_\infty)$$

(5.25) $$\int_U w(y)\cdot\nabla f^\epsilon(y)\,dy \leq C(1+\ln(\frac{1}{\epsilon}))\|w\|_\infty$$

for all $v \in C^1_c(U)$ and $w \in C^1_c(U;\mathbb{R}^k)$. Then $\{f^\epsilon\}$ is precompact in L^1.

Proof.
1. First note that (5.23) implies that $\|f^\epsilon\|_{L^1} \leq C$.

We write $\delta = \epsilon^\alpha$, $f^\epsilon_0 = \eta^\delta * f^\epsilon$, and $f^\epsilon_1 = f^\epsilon - f^\epsilon_0$. Clearly $\|f^\epsilon_i\|_{L^1} \leq C$ for $i = 0, 1$. Also, for ϵ sufficiently small, f^ϵ_i is compactly supported for $i = 1, 2$.

The proof of Proposition 3.2 shows that

(5.26) $$\|\nabla f^\epsilon_0\|_{(C^0)^*} \leq C, \qquad \|\nabla f^\epsilon_1\|_{(C^{0,1}_c)^*} \leq C\epsilon^\alpha.$$

Also, using (5.25) and the interpolation inequality (3.4), we find that for every $\beta \in (0,1)$ there exists some $\alpha' > 0$ such that

(5.27) $$\|\nabla f^\epsilon_1\|_{(C^{0,\beta}_c)^*} \leq C\epsilon^{\alpha'}.$$

The first estimate of (5.26) implies that $\{f^\epsilon_0\}$ is uniformly bounded in $BV(U)$, and hence precompact in L^1. So to prove the lemma, it suffices to show that f^ϵ_1 converges to zero in $L^1(U)$.

2. Let $\phi \in C^1_c(U)$, and compute

$$\begin{aligned}\int \phi f^\epsilon_1 &= \int\int \phi(x)\eta^\delta(y)\left(f^\epsilon(x) - f^\epsilon(x-y))\right)\,dxdy \\ &= \int_0^1\int\int \phi(x)\eta^\delta(y)\nabla f^\epsilon(x-sy)\cdot y\,dy\,dx\,ds.\end{aligned}$$

We make a change of variables in the dx integral to find that

$$\int \phi f_1^\epsilon = \int w^\delta \cdot \nabla f^\epsilon, \text{ for } w^\delta(x) = \int_0^1 \int_{B_\delta} \phi(x+sy) y \eta^\delta(y)\, dy\, ds. \tag{5.28}$$

Note that w^δ is compactly supported in U if ϵ is sufficiently small.

3. Let $\zeta^\delta(y) := y\eta^\delta(y) = \delta^{1-n}\zeta^1(\frac{y}{\delta})$. It is clear from the definition of w^δ that

$$\|w^\delta\|_\infty \ \le\ \|\zeta^\delta\|_1\ \|\phi\|_\infty \ \le\ C\delta\|\phi\|_\infty.$$

We now estimate a Hölder seminorm of w^δ. By another change of variable,

$$\begin{aligned} w^\delta(x) &= \int_0^1 \int_{B_{s\delta}(x)} \phi(y) s^{-n} \zeta^\delta(\frac{y-x}{s})\, dy\, ds \\ &= \int_0^1 \frac{1}{s} \int_{B_{s\delta}(x)} \phi(y) \zeta^{s\delta}(y-x)\, dy\, ds. \end{aligned}$$

Also, for any $x_1, x_2 \in U$ and any $\beta \in (0,1)$, $|\zeta^{s\delta}(x_1) - \zeta^{s\delta}(x_2)| \le C|x_1 - x_2|^\beta (s\delta)^{1-\beta-n}$, so one easily estimates from the above expression for w^δ that

$$|w^\delta(x_1) - w^\delta(x_2)| \le C\|\phi\|_\infty |x_1 - x_2|^\beta \frac{\delta^{1-\beta}}{1-\beta}$$

for all $\beta \in (0,1)$. In particular, $\|w^\delta\|_{C_c^{0,\beta}} \ \le\ C(\beta)\delta^{1-\beta}$.

4. Combining (5.26), (5.27), (5.28), and Step 3, we conclude there exists $C, \alpha' > 0$ such that if $\phi \in C_c^1(U)$ then

$$\begin{aligned} \left| \int \phi f_1^\epsilon \right| &\le \left| \int w^\delta \cdot \nabla f_0^\epsilon \right| + \left| \int w^\delta \cdot \nabla f_1^\epsilon \right| \\ &\le \|w^\delta\|_\infty \|\nabla f_0^\epsilon\|_{(C^0)^*} + \|w^\delta\|_{C_c^{0,\beta}} \|\nabla f_1^\epsilon\|_{(C_c^{0,\beta})^*} \\ &\le C\epsilon^{\alpha'} \|\phi\|_\infty \end{aligned}$$

for all ϵ sufficiently small (depending on the support of ϕ.) This clearly implies that $f_1^\epsilon \to 0$ in $L^1_{\text{loc}}(U)$. Since there exists some $V \subset\subset U$ such that $\text{spt}(f_1^\epsilon) \subset V$ for all ϵ sufficiently small, this in fact shows that $\{f_1^\epsilon\}$ is precompact in $L^1(U)$. □

6 Appendix

In this section we present the proof of Proposition 2.4. We follow very closely arguments introduced in [15].

In this section, U is a bounded open subset of $\mathbb{R}^2$, and $u \in H^1(U;\mathbb{R}^2)$ is a function that we have assumed (without loss of generality) to be smooth. In addition, ϕ is a nonnegative Lipschitz test function that vanishes on ∂U.

We will use notation from Sect. 2, in particular, $\Omega(t), \text{Reg}(\phi), \Gamma(t)$, and $\gamma(t)$, defined in (2.6) and (2.7)-(2.9).

We will use the notation

$$t_\epsilon := \epsilon\|\nabla\phi\|_\infty \tag{6.1}$$

For any positive integer d, let

(6.2) $D^\epsilon_d := \{t \in \mathrm{Reg}(\phi) : t \geq t_\epsilon, \Gamma(t) \text{ is nonempty, and } |\deg(u; \Gamma(t))| \geq d\}$.

Recall that for Proposition (2.4) we want to estimate the measure of a set $B_2 \subset \mathrm{Reg}(\phi)$, and from the Definition (2.18) of B_2 we see that

(6.3) $D^\epsilon_{d^*_\lambda} = B_2 \cap \{t : t \geq t_\epsilon\}$, for $d^*_\lambda := \lfloor \frac{\lambda}{\pi}\mu^\epsilon(\mathrm{spt}(\phi))\rfloor + 1 \geq \frac{\lambda}{\pi}\mu^\epsilon(\mathrm{spt}(\phi))$.

We further define

(6.4) $$\lambda^\epsilon(r) = \min_{m\in[0,1]}\left[\frac{m^2\pi}{r} + \frac{(1-m)^2}{c_0\epsilon}\right], \quad \Lambda^\epsilon(r) := \int_0^r \lambda^\epsilon(s) \wedge \frac{c_1}{\epsilon} ds$$

for certain constants c_0, c_1 whose choice is discussed below.

The main result of this section is the following theorem, from which we will easily obtain Proposition 2.4.

Theorem 6.1. *If $u : U \to \mathbb{R}^2$ is a smooth function and ϕ is a nonnegative Lipschitz function such that $\phi = 0$ on ∂U, then for any positive integer d,*

$$d\Lambda^\epsilon\left(\frac{|D^\epsilon_d|}{2d\|\nabla\phi\|_\infty}\right) \leq \int_{spt(\phi)} E^\epsilon(u) = \ln(\frac{1}{\epsilon})\mu^\epsilon(spt\phi).$$

Note that for any $t_2 > t_1$ the ratio $(t_2 - t_1)/\|\nabla\phi\|_\infty$ is a lower bound for the distance between $\partial\Omega(t_2)$ and $\partial\Omega(t_1)$. This explains the role of $\|\nabla\phi\|_\infty$ in the estimate.

Similar results were proven in [15] under more or less the assumption that D^ϵ_d is an interval; and in [7] in the case $d = 1$. Related results have also appeared in Sandier [27].

Note that the case covered in the statement of the theorem, $\{x : \phi(x) > 0\} \subset U$, can be reduced to the case $\{x : \phi(x) > 0\} = U$, if we replace U by $\tilde{U} := \{x \in U : \phi(x) > 0\}$. So we will henceforth assume for notational simplicity that this holds, so that $\mathrm{spt}(\phi) = \bar{U}$.

We introduce more notation and definitions, taken from [15].

We let S denote the set on which $|u|$ is small, that is,

(6.5) $$\{x \in U : |u(x)| \leq 1/2\}.$$

We define the *essential* part S_E of S to be

(6.6) $$S_E := \cup\{\text{components } S_i \text{ of } S : \deg(u; \partial S_i) \neq 0.\}.$$

We also define the *negligible* part S_N of S to be $S_N := S \setminus S_E$. For any subset $V \subset U$ such that $\partial V \cap S_E \neq \emptyset$, we define the generalized degree

(6.7) $$\mathrm{dg}(u; \partial V) := \sum\{\deg(u; \partial S_i) \mid \text{components } S_i \text{ of } S_E \text{ such that } S_i \subset\subset V\}.$$

In Proposition 3.2, [15], it is shown that the constants c_0, c_1 in (6.4) can be chosen such that if $x_0 \in U$, $\epsilon \le r_0 < r_1$, $B_{r_1}(x_1) \subset U$, and

$$[B_{r_1}(x_0) \setminus B_{r_0}(x_0)] \cap S_E = \emptyset\,, \quad |\mathrm{dg}(u; \partial B_\rho(x_0))| = d > 0\,, \qquad \forall\, \rho \in [r_0, r_1]$$

then

$$\int_{B_{r_1}(x_0)\setminus B_{r_0}(x_0)} E^\epsilon(u)dx \ge \; d\,[\Lambda^\epsilon(\frac{r_1}{d}) - \Lambda^\epsilon(\frac{r_0}{d})]\,. \tag{6.8}$$

The following elementary estimates are proved in Propositions 3.1 and 3.2 in [15]:

$$\Lambda^\epsilon(r_1 + r_2) \le \Lambda^\epsilon(r_1) + \Lambda^\epsilon(r_2)\,, \tag{6.9}$$

$$s \mapsto \frac{1}{s}\,\Lambda^\epsilon(s) \quad \text{is nonincreasing}\,, \qquad \frac{1}{s}\Lambda^\epsilon(s) \le \frac{c_1}{\epsilon} \quad \forall s \tag{6.10}$$

and $\Lambda^\epsilon(r) \ge \pi \ln(r/\epsilon) - c_2$ for some constant c_2. Also, clearly, $\lambda^\epsilon(r) \le \pi/r$, and therefore, by redefining c_2 if necessary,

$$|\Lambda^\epsilon(r) - \pi \ln(r/\epsilon)\,| \le c_2 \qquad \forall\, r \ge \epsilon\,. \tag{6.11}$$

We now use Theorem 6.1 and the above facts about Λ^ϵ to give the proof of Proposition 2.4. After this, the rest of this appendix is devoted to proving Theorem 6.1.

Proof of Proposition 2.4
We need to show that

$$|B_2| \;\le\; C\epsilon^{1-\frac{1}{\lambda}}\|\nabla\phi\|_\infty d^*_\lambda, \qquad d^*_\lambda := d_\lambda + 1.$$

Let $R := \frac{|D^\epsilon_{d^*_\lambda}|}{2\|\nabla\phi\|_\infty}$. From (6.3) and the Definition (6.1) of t_ϵ it suffices to show that

$$\frac{R}{d^*_\lambda} \le C\epsilon^{1-\frac{1}{\lambda}}.$$

We may assume that $\frac{R}{d^*_\lambda} \ge \epsilon$, as otherwise the conclusion is obvious. Then (6.11), Theorem 6.1, and the choice (6.3) of d^*_λ imply that

$$\begin{aligned}
\ln\left(\frac{R}{d^*_\lambda}\right) &= \frac{1}{\pi}\left[\pi\ln\left(\frac{R}{\epsilon d^*_\lambda}\right) - \pi\ln\left(\frac{1}{\epsilon}\right)\right] \\
&\le \frac{1}{\pi}\Lambda^\epsilon\left(\frac{R}{d^*_\lambda}\right) + C - \ln\left(\frac{1}{\epsilon}\right) \\
&\le \frac{1}{\pi d^*_\lambda}\ln\left(\frac{1}{\epsilon}\right)\mu^\epsilon(\mathrm{spt}(\phi)) + C - \ln\left(\frac{1}{\epsilon}\right) \\
&\le \left(\frac{1}{\lambda} - 1\right)\ln\left(\frac{1}{\epsilon}\right) + C.
\end{aligned}$$

. $\square$

For the proof of Theorem 6.1 we define

$$S^\epsilon_E := \cup\{\text{components } S_i \text{ of } S_E \;:\; S_i \subset \Omega(t_\epsilon)\}. \tag{6.12}$$

Note that if $x \in \Omega(t_\epsilon)$ and $y \in \partial U$, then

$$|x-y| \, \|\nabla\phi\|_\infty \;\geq\; |\phi(x)-\phi(y)| = |\phi(x)| \;\geq\; t_\epsilon \; = \epsilon\|\nabla\phi\|_\infty .$$

In particular,

(6.13) $$\text{dist}(x, \partial U) \geq \epsilon \qquad \text{for all } x \in S_E^\epsilon .$$

Note also that if $V \subset \Omega(t_\epsilon)$, then

$$\text{dg}(u;\partial V) := \sum \{\deg(u;\partial S_i) \mid \text{components } S_i \text{ of } S_E^\epsilon \text{ such that } S_i \subset\subset V\} .$$

In other words, for such sets V we can ignore $S_E \setminus S_E^\epsilon$ when computing dg$(u;\partial V)$. In the proof of Theorem 1 below we will always be concerned with subsets $V \subset \Omega(t_\epsilon)$, so this will always be the case.

Our strategy for proving Theorem 6.1 will be to find a collection of balls such we have a good lower bound for the Ginzburg-Landau energy on each ball. We then show that the sum of the radii of the balls is bounded below by $\frac{|D_d^\epsilon|}{2\|\nabla\phi\|_\infty}$, hence obtaining a lower bound for the total Ginzburg-Landau energy in terms of this quantity.

We find the collection of balls by starting from an initial collection of small balls that cover S_E^ϵ, then letting these balls grow by expanding them and combining them. The first step is thus to establish the existence of the initial collection of small balls. This is the content of

Proposition 6.2. *There is a collection of closed, pairwise disjoint balls $\{B_i^*\}_{i=1}^k$ with radii r_i^* such that*

(6.14) $$S_E^\epsilon \subset \cup_{i=1}^k B_i^* ,$$

(6.15) $$r_i^* \geq \epsilon \qquad \forall i.$$

(6.16) $$\int_{B_i^* \cap U} E^\epsilon(u)dx \;\geq\; \frac{c_0}{\epsilon} r_i^* \;\geq\; \Lambda^\epsilon(r_i^*)$$

This is essentially proved in Proposition 3.3 in [15]. The idea of the proof is as follows:

Given any component S_i of S_E, fix a point $x_i \in S_i$. Let ρ_i be the smallest number such that $S_i \subset B_{\rho_i}(x_i)$, and let $r_i = \max(\rho_i, \epsilon)$.

If $r_i = \epsilon$ then (6.16) holds for $B_{r_i}(x_i)$ because

$$\int_{S_i} |Du|^2 \, dx \geq C^{-1} \int_{S_i} |Ju| \, dx \geq C^{-1} \left| \int_{S_i} Judx \right|$$

(6.17) $$\geq C^{-1} |deg(u;\partial S_i)| \geq C^{-1} .$$

If $r_i > \epsilon$, then the definition of r_i implies that $\partial B_r(x_i) \cap S_i \neq \emptyset$ for every $r < r_i$. Also, (6.13) implies that $\mathcal{H}^1(\partial B_r \cap U) \geq \epsilon$ for every $r \geq \frac{\epsilon}{2\pi}$. So Lemma 2.3 implies that

(6.18) $$\int_{\partial B_r(x_i)} E^\epsilon(u) \, d\mathcal{H}^1 \geq \frac{1}{25\epsilon} \qquad \forall \, r \in [\frac{\epsilon}{2\pi}, r_i]$$

and this implies (6.16) for each ball $B_{r_i}(x_i)$.

If two or more of these balls intersect, they can be combined into larger balls, relabeling as necessary. One can use the Besicovitch Covering Theorem to control the overlap and show that the larger balls still satisfy (6.16). The details of this argument appear in [15]. □

Proposition 6.2 differs from Proposition 3.3 of [15] in that in the latter, S_E appears in place of S_E^ϵ in the counterpart of (6.14). Since (6.13) need not hold for all $x \in S_E$, this makes it a little harder to prove (6.18) and necessitates some assumptions about the smoothness of ∂U. Thus stating the conclusion in terms of S_E^ϵ rather than S_E simplifies the result a little and eliminates the need for assumptions about ∂U.

The lower bound (6.17) is useless if S_i has degree zero, which makes it impossible, in general, to cover S_N with balls satisfying the stated conditions. It is this fact that forces us to introduce the generalized degree dg.

Our next result is Lemma 3.1 in [15]. It is used below when we allow the small balls to grow and merge, to form large balls. For the sake of completeness, here we state it and give its short proof.

Lemma 6.3. *Given any finite collection of closed balls in $\mathbb{R}^k$, say $\{C_i\}_{i=1}^N$, we can find a collection $\{\tilde{C}_i\}_{i=1}^{\tilde{N}}$ of pairwise disjoint balls such that*

$$\bigcup_{i=1}^{N} C_i \subset \bigcup_{i=1}^{\tilde{N}} \tilde{C}_i,$$

$$\sum_{C_j \subset \tilde{C}_i} diam C_j = diam \tilde{C}_i,$$

$$\tilde{N} \leq N, \qquad \textit{with strict inequality unless } \{C_i\}_{i=1}^N \textit{ is pairwise disjoint.}$$

Proof. Replace pairs of intersecting balls C_i, C_j by larger single balls $\tilde{C}$ such that $C_i \cup C_j \subset \tilde{C}$ and diam $\tilde{C}$ = diam C_i + diam C_j, continuing until a pairwise disjoint collection is reached. This collection has the stated properties. □

We next show that, starting from the initial collection of balls, we can let them grow in such a way that each ball continues to satisfy a good lower bound. We follow the presentation of Sandier and Serfaty [29].

As above let $\{B_i^*\}$ denote the balls found in Proposition 6.2, with radii r_i^* and generalized degree $d_i^* := \mathrm{dg}(u; \partial B_i^*)$. Define

$$\sigma^* := \min\{ \frac{r_i^*}{|d_i^*|} \mid d_i^* \neq 0 \} .$$

Proposition 6.4. *For every $\sigma \geq \sigma^*$, there exists a collection of disjoint, closed balls $\mathcal{B}(\sigma) = \{B_k^\sigma\}_{k=1}^{k(\sigma)}$ satisfying $r_k^\sigma \geq \epsilon$,*

$$S_E^\epsilon \subset \cup_k B_k^\sigma , \tag{6.19}$$

$$\int_{U\cap B_k^\sigma} E^\epsilon(u)\,dx \geq \frac{r_k^\sigma}{\sigma}\,\Lambda^\epsilon(\sigma)\,, \tag{6.20}$$

$$r_k^\sigma \geq \sigma|d_k^\sigma| \qquad \textit{whenever } B_k^\sigma \cap \partial U = \emptyset\,, \tag{6.21}$$

where r_k^σ *is the radius and* d_k^σ *is the generalized degree.*

Proof. Let C be the set of all $\sigma \geq \sigma^*$ for which such a collection exists.

1. We first claim that $\sigma^* \in C$. Indeed $\{B_k^*\}$ be the collection of balls constructed in Proposition 6.2. Set $\mathcal{B}(\sigma^*) := \{B_k^*\}$. The Definition (6.4) of Λ^ϵ easily implies that $\Lambda^\epsilon(\sigma)/\sigma \leq c_1/\epsilon$ for all σ, so Proposition 6.2 implies that this collection satisfies (6.19) and (6.20). Also, (6.21) is satisfied due to the definition of σ^*.

2. In step we will show that C is closed. Let $\{\sigma^n\}_n$ be a sequence in C and suppose that σ^n converges to σ_0 as n tends to infinity. Since the balls are disjoint, and their radii are at least ϵ, the total number of balls $k(\sigma_n)$ is uniformly bounded in n. Therefore by passing to a subsequence we may assume that $k(\sigma_n)$ is equal to a constant k_0 independent of n. By passing to a further subsequence, we may assume that the radii $r_k^{\sigma_n}$ and the centers $a_k^{\sigma_n}$ converge to r_k^0 and a_k^0, respectively, for each $k \leq k_0$. Let $B_{k,0}$ be the closed ball centered at a_k^0 with radius r_k^0. It is clear that this collection of balls satisfies (6.19), (6.20), and (6.21). If the balls are disjoint , we set $\mathcal{B}(\sigma_0) := \{B_{k,0}\}_{k=1}^{k_0}$. If they are not disjoint, we apply the amalgamation process outlined in Lemma 6.3. Let $\{B_j^{\sigma_0}\}$ be the resulting balls and $r_j^{\sigma_0}$ be their radius. Then, by Lemma 6.3

$$r_j^{\sigma_0} \ = \sum_{B_{k,0}\subset B_j} r_{k,0} \geq \sum_{B_{k,0}\subset B_j} \sigma_0|d_{k,0}|. \tag{6.22}$$

Since $\{B_{k,0}\}$ satisfies (6.20), this implies that

$$\begin{aligned}\int_{U\cap B_j} E^\epsilon(u)\,dx &\geq \sum_{B_{k,0}\subset B_j} \int_{U\cap B_{k,0}} E^\epsilon(u)\,dx \\ &\geq \sum_{B_{k,0}\subset B_j} \frac{r_{k,0}}{\sigma_0}\,\Lambda^\epsilon(\sigma_0) \\ &\geq \frac{r_j^{\sigma_0}}{\sigma_0}\,\Lambda^\epsilon(\sigma_0)\ .\end{aligned}$$

Hence, $\mathcal{B}(\sigma_0) := \{B_j^{\sigma_0}\}$ satisfies (6.20). Moreover,

$$\left|d_j^{\sigma_0}\right| \ = \left|\sum_{B_{k,0}\subset B_j} d_{k,0}\right| \ \leq \sum_{B_{k,0}\subset B_j} |d_{k,0}|,$$

and this together with (6.22) implies that the balls in the collection $\mathcal{B}(\sigma_0)$ satisfy (6.21).

3. Suppose that $\sigma_1 \in C$. We will show that there is $\delta > 0$ such that $[\sigma_1, \sigma_1+\delta] \subset C$. Indeed, let K_1 be the set of indices k such that $B_k^{\sigma_1} \cap \partial U = \emptyset$ and set

$$s_1 := \min_{k\in K_1} \frac{r_k^{\sigma_1}}{|d_k^{\sigma_1}|}\ .$$

By (6.21), $\sigma_1 \leq s_1$. If this inequality is strict, we set $\mathcal{B}(\sigma) = \mathcal{B}(\sigma_1)$ for all $\sigma \in [\sigma_1, s_1]$. It is clear that this collection of balls satisfies (6.19), and (6.21). Also (6.20) follows from (6.10). So let us assume that $s_1 = \sigma_1$, and let $K_2 \subset K_1$ be the indices k which minimize the ratio $r_k^{\sigma_1}/d_k^{\sigma_1}$. For $\sigma \geq \sigma_1$, set

$$r_k^\sigma := \begin{cases} r_k^{\sigma_1} \ , & \text{if } k \notin K_2 \ , \\ \frac{\sigma}{\sigma_1} \, r_k^{\sigma_1} \ , & \text{if } k \in K_2 \ . \end{cases}$$

Let B_k^σ be the closed ball with radius r_k^σ with the same center as $B_k^{\sigma_1}$ and let $\mathcal{B}(\sigma)$ be the collection of these balls. Since $\{B_k^{\sigma_1}\}_k$ are disjoint closed sets, there is $\delta_1 > 0$ such that for all $\sigma \in [\sigma_1, \sigma_1 + \delta_1]$ B_k^σ's are disjoint and

$$K_1(\sigma) := \{ \ k \mid B_k^\sigma \cap \partial U = \emptyset \ \} = K_1 \ .$$

Then, for $k \in K_2$,

$$\frac{r_k^\sigma}{\sigma} = \frac{r_k^{\sigma_1}}{\sigma_1} = |d_k^{\sigma_1}| = |d_k^\sigma| \ ,$$

and for $k \notin K_2$,

$$\frac{r_k^\sigma}{\sigma} = \frac{\sigma_1}{\sigma} \, \frac{r_k^{\sigma_1}}{\sigma_1} \ .$$

Since for $k \notin K_2$, $r_k^{\sigma_1}/\sigma_1 > |d_k^{\sigma_1}|$, there is $0 < \delta \leq \delta_1$ such that (6.21) is satisfied by the collection $\mathcal{B}(\sigma)$. Since $r_k^\sigma \geq r_k^{\sigma_1}$, (6.19) is also satisfied.

To verify (6.20), we observe that for $k \notin K_2$, $B_k^\sigma = B_k^{\sigma_1}$ and (6.20) is satisfied in light of (6.10). If, however, $k \in K_2$, then

$$d_k^\sigma = d_k^{\sigma_1} \ , \qquad r_k^\sigma = \sigma |d_k^\sigma| \ , \tag{6.23}$$

and

$$[B_k^\sigma \setminus B_k^{\sigma_1}] \cap S^E = \emptyset \ .$$

Then by (6.8),

$$\begin{aligned} \int_{B_k^\sigma} E^\epsilon(u)\,dx &= \int_{B_k^{\sigma_1}} E^\epsilon(u)\,dx + \int_{B_k^\sigma \setminus B_k^{\sigma_1}} E^\epsilon(u)\,dx \\ &\geq \frac{r_k^{\sigma_1}}{\sigma_1} \, \Lambda^\epsilon(\sigma_1) + |d_k^{\sigma_1}| \left[\Lambda^\epsilon\left(\frac{r_k^\sigma}{|d_k^\sigma|}\right) - \Lambda^\epsilon\left(\frac{r_k^{\sigma_1}}{|d_k^{\sigma_1}|}\right) \right] \\ &= |d_k^{\sigma_1}| \, \Lambda^\epsilon\left(\frac{r_k^{\sigma_1}}{|d_k^{\sigma_1}|}\right) + |d_k^{\sigma_1}| \left[\Lambda^\epsilon\left(\frac{r_k^\sigma}{|d_k^\sigma|}\right) - \Lambda^\epsilon\left(\frac{r_k^{\sigma_1}}{|d_k^{\sigma_1}|}\right) \right] \\ &= |d_k^{\sigma_1}| \, \Lambda^\epsilon\left(\frac{r_k^\sigma}{|d_k^\sigma|}\right) \\ &= \frac{r_k^\sigma}{\sigma} \, \Lambda^\epsilon(\sigma) \ . \end{aligned}$$

Here we repeatedly used the identities (6.23) and the fact that $B_k^{\sigma_1}$ satisfies (6.20). Hence $\mathcal{B}(\sigma)$ also satisfies (6.20) for all $\sigma \in [\sigma_1, \sigma_1 + \delta]$.

4. We have shown that C is closed set including σ^* and for every $\sigma \in C$, there exists $\delta > 0$ such that $[\sigma, \sigma + \delta] \subset C$. Hence, $C = [\sigma^*, \infty)$. □

Remark 6.5. For $\sigma \geq \sigma^*$, set

$$R(\sigma) := \sum_{k=1}^{k(\sigma)} r_k^\sigma .$$

In the above construction $R(\cdot)$ is a nondecreasing and a continuous function on its domain.

We are now ready for the

Proof of Theorem 6.1 Set $R := |D_d^\epsilon|/(2\|\nabla\phi\|_\infty)$ and $\bar\sigma := R/d$. Let σ^* be as in the previous Lemma. We suppose that D_d^ϵ is nonempty as there is nothing to prove otherwise .

1. First suppose that $\bar\sigma < \sigma^*$. The opposite inequality will be treated later in the proof.

Consider the balls $\{B_k^*\}$ constructed in Proposition 6.2. By (6.16) and the definition of σ^*,

$$\begin{aligned}\int_U E^\epsilon(u)\,dx &\geq \sum_k \int_{U\cap B_k^*} E^\epsilon(u)\,dx \\ &\geq \sum_k \frac{c_1}{\epsilon}\, r_k^* \;\geq\; \frac{c_1\sigma^*}{\epsilon} \sum_k |d_k^*| \;\geq\; c_1 \frac{R}{d\epsilon} \sum_k |d_k^*| \,.\end{aligned}$$

Let $t_0 \in D_d^\epsilon$. Then the Definition (6.2) of D_d^ϵ implies that $d \leq |\deg(u;\Gamma(t_0))|$ and by Definition (2.8), $|u| > 1/2$ on $\Gamma(t_0)$. Hence $d \leq |\mathrm{dg}(u;\Gamma(t_0))|$. Moreover, by (6.7) and (6.14),

$$|dg(u;\Gamma(t_0))| \;\leq \sum_{\{k\,:\,B_k^*\cap\Omega(t_0)\neq\emptyset\}} |d_k^*| \;\leq\; \sum_k |d_k^*| \,.$$

Hence by (6.10),

$$\int_U E^\epsilon(u)\,dx \geq c_1 \frac{R}{d\epsilon}\, d \geq d\Lambda^\epsilon\left(\frac{R}{d}\right)$$

which is what we needed to prove.

2. We now assume that $\bar\sigma \geq \sigma^*$. Consider the collection of balls $\mathcal{B}(\bar\sigma)$ provided by Proposition 6.4. Assume towards a contradiction that

$$\sum_k r_k^{\bar\sigma} < R\,. \tag{6.24}$$

Set

$$C := \{\; t \in (0, \|\phi\|_\infty) \mid \Gamma(t) \cap [\cup_k B_k^{\bar\sigma}] \neq \emptyset \;\}\,.$$

The definitions imply that $C \;\subset\; \cup_k\; \phi(B_k^{\bar\sigma})$, and as a consequence

$$|C| \leq 2\|\nabla\phi\|_\infty \sum_k r_k^{\bar\sigma} < 2\|\nabla\phi\|_\infty R = |D_d^\epsilon|\,.$$

Hence $D_d^\epsilon \setminus C \neq \emptyset$.

3. Let $t_0 \in D_d^\epsilon \setminus C$. The definition of D_d^ϵ implies that $|\mathrm{dg}(u;\Gamma(t_0))| = |\deg(u;\Gamma(t_0))| \geq d$. On the other hand, the definition of C implies that $\Gamma(t_0) \cap (\cup_k B_k^{\bar\sigma}) = \emptyset$, so (6.19) and the additivity of the degree yield

$$\begin{aligned} d \ \leq \ |dg(u;\Gamma(t_0))| &\leq \sum_{\{k\,:\, B_k^{\bar\sigma}\subset\Omega(t_0)\,\}} |d_k^{\bar\sigma}| \\ &\leq \sum_{\{k\,:\, B_k^{\bar\sigma}\cap\partial U=\emptyset\,\}} |d_k^{\bar\sigma}| \\ &\leq \sum_{\{k\,:\, B_k^{\bar\sigma}\cap\partial U=\emptyset\,\}} \frac{r_k^{\bar\sigma}}{\bar\sigma}\ , \end{aligned}$$

by (6.21). On the other hand by (6.24),

$$d = \frac{R}{\bar\sigma} > \sum_k \frac{r_k^{\bar\sigma}}{\bar\sigma}\ .$$

Therefore we conclude that (6.24) is false.

4. By the previous step $\sum_k r_k^{\bar\sigma} \geq R = d\bar\sigma$. Hence by (6.20),

$$\begin{aligned} \int_U E^\epsilon(u)\,dx &\geq \sum_k \int_{U\cap B_k^{\bar\sigma}} E^\epsilon(u)\,dx \\ &\geq \sum_k r_k^{\bar\sigma}\,\frac{\Lambda^\epsilon(\bar\sigma)}{\bar\sigma} \\ &\geq d\,\Lambda^\epsilon(\bar\sigma)\ . \end{aligned}$$

□

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