Calc. Var. 13, 459–468 (2001) **Calculus of Variations**
DOI (Digital Object Identifier) 10.1007/s005260100082

Min-Chun Hong

On the minimality of the p-harmonic map $\frac{x}{|x|} : B^n \to S^{n-1}$

Received: 8 November 2000 / Accepted: 4 December 2000 /
Published online: 4 May 2001 –

Abstract. We prove that for any real number p with $1 < p \leq n-1$, the map $\frac{x}{|x|} : B^n \to S^{n-1}$ is the unique minimizer of the p-energy functional $\int_{B^n} |\nabla u|^p \, dx$ among all maps in $W^{1,p}(B^n, S^{n-1})$ with boundary value x on ∂B^n.

1. Introduction

Let B^n be the unit ball in $\mathbb{R}^n$ with boundary $\partial B^n = S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. For any $p > 1$, denote

$$W^{1,p}(B^n, S^{n-1}) = \{u \in W^{1,p}(B^n, \mathbb{R}^n) : \quad |u| = 1 \text{ a.e. }\}.$$

We define the p-energy of a map $u \in W^{1,p}(B^n, S^{n-1})$ with $p > 1$ by

$$E_p(u) = \int_{B^n} |\nabla u|^p \, dx. \tag{1}$$

We say that a map u is "p-harmonic" if $u \in W^{1,p}(B^n, S^{n-1})$ satisfies

$$\int_{B^n} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \int_{B^n} |\nabla u|^p u \cdot \phi \, dx$$

for all $\phi \in W_0^{1,p}(B^n, \mathbb{R}^n) \cap L^\infty(B^n, \mathbb{R}^n)$.

Hildebrandt, Kaul and Widman in [HKW] found that the map $\frac{x}{|x|}$ from B^n to S^{n-1} is a weakly harmonic map (obviously also a p-harmonic map for $1 < p < n$). The question of whether the map $\frac{x}{|x|}$ is a minimizer of E_p for $1 < p < n$ has aroused great interests. When $p = 2$ and $n \geq 7$, Jäger and Kaul in [JK] first proved that the map $\frac{x}{|x|}$ is a minimizer of E_2. Brezis, Coron and Lieb in [BCL] proved the minimality for $p = 2$ and $n = 3$ (with another proof in [ABL]). Lin in [L] proved the minimality for $p = 2$ and $n \geq 3$. When $n \geq 3$, Coron and Gulliver in [CG] proved that the map $\frac{x}{|x|}$ is a minimizer of the p-energy E_p for all integers $p \in \{1, 2, 3, ..., n-1\}$ (with a simple proof in [AL]). The question of the p-energy

M.-C. Hong: School of Mathematical Sciences, The Australian National University, Canberra, ACT 0200, Australia

Research of the author was supported by the Australian Research Council

minimality of $\frac{x}{|x|}$ for any non-integer p remained open. It had no progress until recently Hardt, Lin and Wang in [HLW] proved that $\frac{x}{|x|}$ is a minimizer of the p-energy functional E_p for $p \in [n-1, n)$. For $n \geq 2 + p + 2\sqrt{p}$ and $p \geq 2$, the p-energy minimality of the map $\frac{x}{|x|}$ was proved independently by Wang in [W] and the author in [Ho]. However, the question is still open for other non-integer cases (see [H, section 8]). In this paper, we give an answer to the question in the following:

Theorem A. *When $n \geq 3$ and $1 < p \leq n-1$, we have*

$$E_p\left(\frac{x}{|x|}\right) \leq E_p(u)$$

for any $u \in W^{1,p}(B^n, S^{n-1})$ with $u = x$ on ∂B^n.

For the proof of Theorem A, we consider it in two different cases. The first case is $2 \leq p \leq n-1$. Applying the Coarea formula of [F] and the important approximation result for mappings into $(n-1)$-sphere of $W^{1,n-1}$ in [CG], we first prove Theorem 4. Then we improve the idea [CG] that the modified 2-energy of any map $u : B^n \to S^{n-1}$ is a constant times the average of $\int r^{2-p}|\nabla(\pi_Y \circ u)|^2$ over all 3-planes Y in $\mathbb{R}^n$ where π_Y maps $s \in S^{n-1}$ to the nearest point in the 2-sphere (see Lemma 5). Another key point is to introduce a polar coordinate in B^n so that we cut B^n into three dimensional cones which are essentially equivalent to the three dimensional ball B^3. Through Theorem 4 and Lemma 5, we prove Theorem 6 which is slightly stronger than Theorem A for $2 \leq p \leq n-1$. We would like to point out that our proof for even integers $p > 2$ here is simpler than the one in [CG] for integers $p \in \{3, ..., n-1\}$. The second case is to consider $p > 1$. In this case, we improve the sophisticated technique in [CG] of estimating the modified 1-energy of a map $u \in W^{1,p}(B^n, S^{n-1})$ by averaging a related functional of the composition of u with all nearest-point projections π_Y of S^{n-1} onto its geodesic 1-spheres (see Lemma 7). Theorem A is proved by Lemma 7 together with several observations from the case one. Finally, we prove in Theorem 9 that the minimizer $\frac{x}{|x|}$ is the unique minimizer of E_p in $W^{1,p}(B^n, S^{n-1})$ with boundary value x on ∂B^n.

Acknowledgement. The author would like to thank Ben Andrews for many useful discussions and comments, Chang-You Wang and Xu-Jia Wang for some comments and Jean-Michel Coron and Robert Gulliver for their interests.

2. The proof of Theorem A

Case I. $2 \leq p \leq n-1$

For any $u \in W^{1,p}(B^n, S^{n-1})$, we denote

$$J(u) = \left[\det(\nabla u \nabla u^T)\right]^{1/2}.$$

Lemma 1. *(Coarea formula) If $v \in C^{0,1}(\Omega, S^{n-1})$ for an open set $\Omega \subset B^n$, then*

$$\int_\Omega fJ(v)\,dx = \int_{S^{n-1}} \int_{v^{-1}(s)} f\,dH^1\,dA_{S^{n-1}} \tag{2}$$

for every integrable $\bar{\mathbb{R}}$ valued function f on Ω.

Proof. See [F], Theorem 3.2.22. □

Lemma 2. *For each $s \in \partial B^n$, let $M(s)$ be a Lipschitz curve inside B^n joining the s to $-s$, and let $f(t) \geq 0$ be a non-decreasing and continuous function in $t \in [0,1]$. Then*

$$\int_{M(s)} f(|x|)\,d\mathcal{H}^1(x) \geq 2\int_0^1 f(r)dr. \tag{$*$}$$

Proof. Without loss of generality, set $s = (1,0,...,0)$ and $-s = (-1,0,...,0)$. We split $M(s)$ into two Lipschitz curves $M^+(s)$ and $M^-(s)$ where $M^+(s)$ is a Lipschitz curve inside the domain $\{x = (x_1,...x_n) \in B^n : x_1 \geq 0\}$ and $M^-(s)$ is another Lipschitz curve in the domain $\{x = (x_1,...x_n) \in B^n : x_1 \leq 0\}$. For each $r_i \geq 0$, we consider the $(n-1)$-plane $\mathbb{R}^{n-1}_{r_i} := \{x = (x_1,...,x_n) \in \mathbb{R}^n : x_1 = r_i\}$. Let r_i, r_{i+1} be two positive numbers with $0 \leq r_i < r_{i+1} \leq 1$. Then we have two $n-1$-dimensional planes $\mathbb{R}^{n-1}_{r_i}$ and $\mathbb{R}^{n-1}_{r_{i+1}}$ corresponding to r_i and r_{i+1}. Then we see that $M^+(s)$ must across at least one point a_i on the plane $\mathbb{R}^{n-1}_{r_i}$ and another point a_{i+1} on the plane $\mathbb{R}^{n-1}_{r_{i+1}}$. Without loss of generality, we may assume that the curve $M^+(s)$ from a_i to a_{i+1} inside the domain between two planes $\mathbb{R}^{n-1}_{r_i}$ and $\mathbb{R}^{n-1}_{r_i+1}$. The length of the curve of M^+_s between a_i and a_{i+1} must be larger than or equal to $|r_{i+1} - r_i|$ which is the distance of the two planes $\mathbb{R}^{n-1}_{r_i}$ and $\mathbb{R}^{n-1}_{r_{i+1}}$. For each r_i, we have $|a_i| \geq r_i$. Since $f(t)$ is non-decreasing in t, we have

$$f(|a_i|) \geq f(r_i).$$

Let r_i, $i = 1,...,k$, be k different points of $[0,1]$ with $0 = r_1 < r_2 < ... < r_k = 1$. Then we have k corresponding points a_{r_i} in $M_+(s)$, and

$$\sum_{i=1}^{k-1} f(|a_i|)|a_{i+1} - a_i|_{M^+(s)} \geq \sum_{i=1}^{k-1} f(r_i)|r_{i+1} - r_i|.$$

Let t_k be the maximum $|r_{i+1} - r_i|$ for $i = 1,...,k-1$. Letting $t_k \to 0$, we have

$$\int_{M^+(s)} f(|x|)d\mathcal{H}^1 \geq \int_0^1 f(r)dr$$

so the inequality $(*)$ is proved. □

Now we need an approximation result for maps in $W^{1,n-1}(B^n, S^{n-1})$. As in [CG], we define $\mathcal{R}$ to be the class of maps $v \in W^{1,n-1}(B^n; S^{n-1})$ such that

(i) $v = \frac{x}{|x|}$ on a neighbourhood of ∂B^n;
(ii) v is locally Lipschitz on $B^n \backslash \Sigma$, for a set Σ of finitely many points; and

(iii) for a.e. $s \in S^{n-1}$, $v^{-1}(s) \cup v^{-1}(-s) \cup \Sigma$ is a regular, oriented 1-dimensional Lipschitz manifold of B^n having boundary only in ∂B^n.

From [CG, Theorem 3.2], we have

Proposition 3. *$\mathcal{R}$ is dense in*

$$\mathcal{E}_x^{n-1} = \{u \in W^{1,n-1}(B^n, S^{n-1}) : u = x \text{ on } \partial B^n\}.$$

Then we have

Theorem 4. *Let $f(r) \geq 0$ be a non-decreasing and continuous function in $r \in [0,1]$. Then we have*

$$\int_{B^n} f(r)|\nabla \frac{x}{|x|}|^{n-1}\, dx \leq \int_{B^n} f(r)|\nabla u|^{n-1}\, dx \tag{3}$$

for every map $u \in W^{1,n-1}(B^n, S^{n-1})$ with $u = x$ on ∂B^n, where $r = |x|$ in B^n.

Proof. According to Proposition 3, it is enough to consider all maps v in the class $\mathcal{R}$. For each $v \in \mathcal{R}$, we denote $\Omega = B^n \backslash \Sigma$ and $M(s) = v^{-1}(s) \cap v^{-1}(-s) \cap \Sigma$.

Applying the Coarea formula (Lemma 1), we obtain

$$\begin{aligned} \int_\Omega f(|x|)J(v)\, dx &= \int_{S^{n-1}} \int_{v^{-1}(s)} f(|x|)\, d\mathcal{H}^1\, dA_{S^{n-1}} \\ &= \frac{1}{2} \int_{S^{n-1}} \int_{M(s)} f(|x|)\, d\mathcal{H}^1\, dA_{S^{n-1}}, \end{aligned} \tag{4}$$

where $M(s) := v^{-1}(s) \cup v^{-1}(-s) \cup \Sigma$ is a regular, oriented Lipschitz 1-dimensional manifold having boundary in ∂B^n.

By Lemma 2, we have

$$\int_{M(s)} f(|x|)\, d\mathcal{H}^1 \geq 2 \int_0^1 f(r) dr.$$

Combining this with (4), we have

$$\int_{B^n} f(|x|)J(v)\, dx \geq \int_{B^n} f(|x|)J(\frac{x}{|x|})\, dx. \tag{5}$$

By applying the arithmetic geometric mean inequality to the eigenvalues of the symmetric non-negative definite $(n-1) \times (n-1)$ matrix $\nabla v \nabla v^T$, we have

$$|\nabla v|^{n-1}(x) \geq (n-1)^{\frac{n-1}{2}} J(v(x)) \tag{6}$$

where equality holds for $v = \frac{x}{|x|}$. By (5)–(6), we obtain

$$\int_{B^n} f(|x|)|\nabla v|^{n-1}\, dx \geq \int_{B^n} f(|x|)|\nabla \frac{x}{|x|}|^{n-1}\, dx \tag{7}$$

for any $v \in \mathcal{R}$. Theorem 4 follows. □

Given a 3-plane $Y \subset \mathbb{R}^n$, we define

$$\pi_Y : S^{n-1} \to S^{n-1} \cap Y$$

by $\pi_Y(u) = u'/|u'|$, where u' is the orthogonal projection of u onto Y. The singular set of π_Y is the $(n-4)$-sphere $S^{n-1} \cap Y$.

We need to modify a lemma of [CG, Lemma 1.2] in the following:

Lemma 5. *For any p with $2 \le p < n$, there is a constant $c = c(n) > 0$ such that for any $u \in W^{1,p}(B^n, S^{n-1})$*

$$c \int_{B^n} r^{2-p} |\nabla u|^2 \, dx = \int_{Y \in G_3(\mathbb{R}^n)} \int_{B^n} r^{2-p} |\nabla(\pi_Y \circ u)|^2 \, dx \, dG(Y) \tag{8}$$

where dG is the bi-invariant volume form on the Grassmann manifold $G_3(\mathbb{R}^n)$.

Proof. As in [CG], for any tangent vector V to S^{n-1}, we have

$$c|V|^2 = \int_{Y \in G_3(\mathbb{R}^n)} |D\pi_Y(V)|^2 \, dG(Y).$$

Then

$$c r^{2-p} |\nabla u|^2 = \int_{Y \in G_3(\mathbb{R}^n)} r^{2-p} |\nabla(\pi_Y \circ u)|^2 \, dG(Y). \tag{9}$$

Integrating both sides of (9) over B^n, we obtain (8) by Fubini's theorem. □

Then we have

Theorem 6. *For $2 \le p < n$, we have*

$$\int_{B^n} r^{2-p} |\nabla \frac{x}{|x|}|^2 \, dx \le \int_{B^n} r^{2-p} |\nabla u|^2 \, dx \tag{10}$$

for any $u \in W^{1,p}(B^n, S^{n-1})$ with $u = x$ on ∂B^n.

Proof. Let Y be a 3-plane $\subset \mathbb{R}^n$. After a rotation, we may assume that $Y = \mathbb{R}^3$. We write $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $y = (x_1, x_2, x_3) \in \mathbb{R}^3$, $z = (x_4, ..., x_n) \in \mathbb{R}^{n-3}$ and $r^2 = |y|^2 + |z|^2$. We also denote $v = \pi_Y \circ u$ and $v_0 = \pi_Y \circ u_0 = \frac{y}{|y|}$ with $u_0 = \frac{x}{|x|}$. We claim

$$\int_{B^n} r^{2-p} |\nabla v|^2 \, dx \ge \int_{B^n} r^{2-p} |\nabla v_0|^2 \, dx. \tag{**}$$

To prove this, we introduce polar coordinates $(r, \phi_1, ..., \phi_{n-1})$ in B^n as follows:

$$\begin{aligned}
x_1 &= r \sin\phi_1 \sin\phi_2 ... \sin\phi_{n-2} \sin\phi_{n-1} \\
x_2 &= r \cos\phi_1 \sin\phi_2 ... \sin\phi_{n-2} \sin\phi_{n-1} \\
x_3 &= r \cos\phi_2 ... \sin\phi_{n-2} \sin\phi_{n-1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x_{n-1} &= r \cos\phi_{n-2} \sin\phi_{n-1} \\
x_n &= r \cos\phi_{n-1}.
\end{aligned}$$

Then the Jacobian in these polar coordinates is

$$J=\frac{\partial(x_1,x_2,...,x_n)}{\partial(r,\phi_1,..,\phi_{n-1})}=r^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}...\sin\phi_2.$$

Using the above polar coordinates, we have

$$\begin{aligned}&\int_{B^n} r^{2-p}|\nabla_y v|^2\,dx\\ &=\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}\int_0^1|\nabla_y v|^2 r^{n-1+2-p}\sin^{n-2}\phi_{n-1}\ldots\sin\phi_2\,dr\,d\phi_1\ldots\,d\phi_{n-1}\\ &=\int_0^\pi\cdots\int_0^\pi G_{\phi_3,\ldots,\phi_{n-1}}(v,p)\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin^2\phi_3\,d\phi_3\ldots\,d\phi_{n-1}\end{aligned}$$

where

$$G_{\phi_3,\ldots,\phi_{n-1}}(v,p):=\int_0^\pi\int_0^{2\pi}\int_0^1 r^{n-1-p}|\nabla_y v|^2 r^2\sin\phi_2\,dr\,d\phi_1\,d\phi_2.$$

For any fixed pair $(\phi_3^0,...,\phi_{n-1}^0)$, $(r,\phi_1,\phi_2,\phi_3^0,...\phi_{n-1}^0)$ are coordinates in a three dimensional cone $\mathcal{C}_0$ in B^n with boundary $(1,\phi_1,\phi_2,\phi_3^0,...\phi_{n-1}^0)$. Through a transformation, the cone $\mathcal{C}_0$ is essentially equivalent to a 3-dimensional ball B^3. We define polar coordinates in B^3 by

$$\begin{aligned}\tilde{x}_1&=r\sin\phi_1\sin\phi_2\\ \tilde{x}_2&=r\cos\phi_1\sin\phi_2\\ \tilde{x}_3&=r\cos\phi_2.\end{aligned}$$

Then from the polar coordinates in the cone $\mathcal{C}_0$, we know

$$\begin{aligned}x_1&=\tilde{x}_1\sin\phi_3^0...\sin\phi_{n-2}^0\sin\phi_{n-1}^0\\ x_2&=\tilde{x}_2\sin\phi_3^0...\sin\phi_{n-2}^0\sin\phi_{n-1}^0\\ x_3&=\tilde{x}_3\sin\phi_3^0...\sin\phi_{n-2}^0\sin\phi_{n-1}^0.\end{aligned}$$

By Theorem 4 (with n=3), we have

$$\begin{aligned}G_{\phi_3^0,\ldots,\phi_{n-1}^0}(v,p)&=c_1\int_{B^3}|\tilde{x}|^{n-1-p}|\nabla_{\tilde{x}}v|^2\,d\tilde{x}\\ &\geq c_1\int_{B^3}|\tilde{x}|^{n-1-p}|\nabla_{\tilde{x}}v_0|^2\,d\tilde{x}=G_{\phi_3^0,\ldots,\phi_{n-1}^0}(v_0,p)\end{aligned}$$

due to the fact that $v|_{\partial B^3}=v_0|_{\partial B^3}=\frac{\tilde{x}}{|\tilde{x}|}=\frac{y}{|y|}$ where c_1 is a constant depending only on $\phi_3^0,...,\phi_{n-1}^0$. This implies

$$\begin{aligned}\int_{B^n} r^{2-p}|\nabla v|^2\,dx&\geq\int_{B^n} r^{2-p}|\nabla_y v|^2\,dx\\ &\geq\int_{B^n} r^{2-p}|\nabla_y v_0|^2\,dx=\int_{B^n} r^{2-p}|\nabla v_0|^2\,dx.\end{aligned}$$

This proves our claim ($**$). Combining (8) with the claim ($**$) yields (10). □

As a consequence of Theorem 4, we give a proof of Theorem A for the case of $2 \le p \le n-1$:

Proof of Theorem A for case one. For $x \in B^n$, we have

$$|\nabla \frac{x}{|x|}|^2 = \frac{n-1}{r^2}.$$

For any $u \in W^{1,p}(B^n, S^{n-1})$ with $u = x$, we have from (10)

$$\int_{B^n} |\nabla \frac{x}{|x|}|^p \, dx \le \int_{B^n} |\nabla \frac{x}{|x|}|^{p-2} |\nabla u|^2 \, dx.$$

By the Hölder inequality, we have

$$\int_{B^n} |\nabla \frac{x}{|x|}|^{p-2} |\nabla v|^2 \le \left[\int_{B^n} |\nabla \frac{x}{|x|}|^p \right]^{\frac{p-2}{p}} \left[\int_{B^n} |\nabla u|^p \right]^{\frac{2}{p}}$$

due to the fact that $p \ge 2$. Thus for $2 \le p \le n-1$,

$$\int_{B^n} |\nabla \frac{x}{|x|}|^p \, dx \le \int_{B^n} |\nabla u|^p \, dx$$

for any $u \in W^{1,p}(B^n, S^{n-1})$ with $u = x$ on ∂B^n. □

Case II. $1 < p \le n-1$

In this case, our proof is based on the techniques of [CG, Sect. 2].

Let Y be a 2-plane in $\mathbb{R}^n$ and consider a projection

$$\pi_Y : S^{n-1} \to S^{n-1} \cap Y$$

defined by $\pi_Y(u) = \frac{u'}{|u'|}$ where u' is the orthogoal projection of u onto Y.

Lemma 7. *For $n > p \ge 1$, there exists a constant $c = c(n)$ such that for any $u \in W^{1,p}(B^n, S^{n-1})$*

$$c \int_{B^n} r^{1-p} |\nabla u| \, dx \ge ⨍_{G_2(\mathbb{R}^n)} \int_{B^n} r^{1-p} J(\pi \circ u) \, dx \, dG(Y)$$

where $⨍_{G_2(\mathbb{R}^n)} := \frac{1}{|G_2(\mathbb{R}^n)|} \int_{G_2(\mathbb{R}^n)}$ denotes the average over the Grassmann manifold $G_2(\mathbb{R}^n)$. Moreover, equality holds if $u = \frac{x}{|x|}$ is horizontally conformal.

Proof. For a 2-plane Y and a map u, we write Z for the 1-plane in $T_{u(x)}S^{n-1}$ parallel to the subspace of Y orthogonal to $\pi_Y(u(x))$, where $T_{u(x)}S^{n-1}$ is the tangent plane of S^{n-1} at $u(x)$.

As in [CG, Lemma 2.2], we have

$$r^{1-p} |\nabla u| \ge (n-1)^{\frac{1}{2}} r^{1-p} ⨍_{G_1(T_u S^{n-1})} \left[\det \left(\pi_Z \nabla u \nabla u^T \pi_Z^T \right) \right]^{1/2} dG(Z). \tag{11}$$

Moreover,

$$\nabla(\pi_Y \circ u)(x) = \frac{\pi_Z \circ \nabla u(x)}{\cos d(u(x), Y)}$$

where $d(u(x), Y)$ is the distance in S^n from $u(x) \in S^n$ to $Y \cap S^n$.

Thus

$$(12) \quad \fint_{G_2(\mathbb{R}^n)} J(\pi_Y \circ u)(x)\, dG(Y) = c' \fint_{G_1(T_u S^{n-1})} \left[\det\left(\pi_Z \nabla u \nabla u^T \pi_Z^T\right)\right]^{1/2} dG(Z)$$

where $c' = c'(n)$ is independent of x and u. Using (11)–(12), Lemma 7 is proved with $c = (n-1)^{1/2} c'$ □

Now we complete the proof of Theorem A.

Proof of Theorem A. Let Y be a 2-plane in $\mathbb{R}^n$. We denote

$$v = \pi_Y \circ u : B^n \to S^{n-1} \cap Y$$

and $J(v) = [\det\,(\nabla v \nabla v^T)]^{1/2}$.

Without loss of generality, we may assume that $Y = \mathbb{R}^2$ and $S^{n-1} \cap Y = S^1$. We write

$$\nabla v = \begin{pmatrix} \frac{\partial v^1}{\partial x_1} & \cdots & \frac{\partial v^1}{\partial x_n} \\ \frac{\partial v^2}{\partial x_1} & \cdots & \frac{\partial v^2}{\partial x_n} \end{pmatrix}$$

Then we consider M_k to be any 2×2 matrix defined by

$$M_k := \begin{pmatrix} \frac{\partial v^1}{\partial x_{k_1}} & \frac{\partial v^1}{\partial x_{k_2}} \\ \frac{\partial v^2}{\partial x_{k_1}} & \frac{\partial v^2}{\partial x_{k_2}} \end{pmatrix}$$

where k_1 and k_2 are two different elements of $\{1, 2, ..., n\}$, i.e. $1 \le k_1 < k_2 \le n$. By Laplace's identity, we have

$$\det(\nabla v \nabla v^T) = \sum_{k=1}^{c_n^{(2)}} (\det M_k)(\det M_k^T) = \sum_{k=1}^{c_n^{(2)}} |\det M_k|^2$$

where $c_n^{(2)} = \frac{n!}{(n-2)!2!}$. We write $x = (y, z) \in \mathbb{R}^n$ with $y \in \mathbb{R}^2$ and $z \in \mathbb{R}^{n-2}$. Thus

$$J(v) \ge J_y(v) = [\det\,(\nabla_y v \nabla_y v^T)]^{1/2}.$$

Then we have

$$\int_{B^n} r^{1-p} J(v)\, dx \ge \int_{B^n} r^{1-p} J_y(v)\, dy$$

where equality holds if $v = v_0 = \pi \circ u_0$ with $u_0 = \frac{x}{|x|}$.

By (5), we have

$$\int_{B^2} r^{n-1-p} J_y(v)\, dy \ge \int_{B^2} r^{n-1-p} J_y\left(\frac{y}{|y|}\right) dy.$$

Then repeating a similar argument with polar coordinates in B^n as in Theorem 6, we have

$$\int_{B^n} r^{1-p} J(v)\, dx \geq \int_{B^n} r^{1-p} J(v_0)\, dx$$

where $v = \pi_Y u$ and $v_0 = \pi_y \circ u_0$.

By Lemma 7, we obtain

$$\int_{B^n} r^{1-p} |\nabla u|\, dx \geq \int_{B^n} r^{1-p} |\nabla u_0|\, dx \tag{13}$$

which implies

$$\int_{B^n} |\nabla u|^p\, dx \geq \int_{B^n} |\nabla u_0|^p\, dx$$

for all $u \in W^{1,p}(B^n, S^{n-1})$ with $u = x$ on ∂B^n. □

By Fubini's theorem, we have

Remark 8. *For $m, n \in \mathbb{Z}$ with $m > n$, write $x = (y, z) \in \mathbb{R}^m$ with $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^{m-n}$. Then the map $u_0(x) = \frac{y}{|y|} : B^m \to S^{n-1}$ is a minimizer of E_p in $W^{1,p}(B^m, S^{n-1})$ with boundary value u_0 on ∂B^n.*

Now we prove a uniqueness theorem in the following:

Theorem 9. *For $1 < p \leq n-1$, the map $u_0 = \frac{x}{|x|} : B^n \to S^{n-1}$ is the unique minimizer of E_p in $W^{1,p}(B^n, S^{n-1})$ with boundary value x on ∂B^n.*

Proof. By Theorem A, we know that u_0 is a minimizer of E_p in $W^{1,p}(B^n, S^{n-1})$. We assume that u_1 is another minimizer of E_p in $W^{1,p}(B^n, S^{n-1})$ with boundary value x on ∂B^n, i.e.

$$\int_{B^n} |\nabla u_1|^p\, dx = \int_{B^n} |\nabla u_0|^p\, dx. \tag{14}$$

When $1 < p \leq n-1$, By (13), we know

$$\int_{B^n} |\nabla u_0|^p\, dx \leq \int_{B^n} |\nabla u_0|^{p-1} |\nabla u|\, dx$$

for any $u \in W^{1,p}$ with $u = x$ on ∂B^n. By Young's inequality, we have

$$|\nabla u_0|^{p-1} |\nabla u_1| \leq \frac{p-1}{p} |\nabla u_0|^p + \frac{1}{p} |\nabla u_1|^p. \tag{15}$$

By the assumption of minimizers u_0, we have

$$\begin{aligned} \int_{B^n} |\nabla u_0|^p\, dx &\leq \int_{B^n} |\nabla u_0|^{p-1} |\nabla u_1|\, dx \\ &\leq \int_{B^n} \left(\frac{p-1}{p} |\nabla u_0|^p + \frac{1}{p} |\nabla u_1|^p \right) dx = \int_{B^n} |\nabla u_0|^p\, dx. \end{aligned}$$

which implies

$$|\nabla u_1| = |\nabla u_0| \quad \text{for a.e. } x \in B^n. \tag{16}$$

Since u_1 is a minimizer, it is also stationary, i.e. it satisfies

$$\int_{B^n} [|\nabla u|^p \mathrm{div}\phi - p|\nabla u|^{p-2} u_{x_i} u_{x_k} \phi^k_{x_i}]\, dx = 0.$$

for all $\phi \in C^1_0(B^n; \mathbb{R}^n)$. It is well-known that u_1 also satisfies the following monotonicity formula:

$$\rho^{p-n} \int_{B_\rho} |\nabla u_1|^p\, dx - \sigma^{p-n} \int_{B_\sigma} |\nabla u_1|^p\, dx = p \int_{B_\rho \setminus B_\sigma} r^{p-n} |\frac{\partial u_1}{\partial r}|^2\, dx.$$

for two constants σ, ρ with $0 < \sigma < \rho \leq 1$. Since $|\nabla u_1| = |\nabla u_0| = \frac{(n-1)^{1/2}}{r}$, the right side of the above monotonicity identity is zero which implies

$$\int_{B_\rho \setminus B_\sigma} r^{p-n} |\frac{\partial u_1}{\partial r}|^2\, dx = 0.$$

This implies

$$\frac{\partial u_1}{\partial r} = 0, \quad \text{a.e. for } 0 < r \leq 1$$

Therefore $u_1 = u_0$ a.e. in B^n □

References

[ABL] F. J. Almgren, W. Browder and E.H. Lieb, Co-area, liquid crystals, and minimal surfaces, S. S. Chern, ed. Springer Lecture Notes in math. **1306** (1988), 1–12

[AL] M. Avellenada and F. H. Lin, Fonctions quasi affines et minimization de $\int |\nabla u|^p$, C. R. Acad. Sci. Paris **306** (1988), 355–358

[BCL] H. Brezis, J.-M. Coron and E.H. Lieb, Harmonic maps with defects, Commun. Math. Phys. **107** (1986), 649–705

[CG] J. -M. Coron and R. Gulliver, Minimizing p-harmonic maps into spheres, J. Reine Angew. Math. **401** (1989), 82–100

[F] H. Federer, Geometric Measure Theory, Berlin-Heidelberg-New York (1969)

[H] R. Hardt, Singularities of harmonic maps, Bull. Ameri. Math, Soci. **34** (1997), 15–34

[HL] R. Hardt and F. H. Lin, Mappings minimizing the L^p norm of the gradient, Comm. P. A. M. **15** (1987), 555–588

[HLW] R. Hardt, F. H. Lin and C. Y. Wang, The p-energy minimality of $x/|x|$, Commu. in Analysis and Geometry **6** (1998), 141–152

[HKW] S. Hilderandt, H. Kaul and K.-O. Widman, An existence theorem for harmonic mappping of Riemannian manifolds, Acta Math. **138** (1977), 1–16

[Ho] M.-C. Hong, On the Jäger Kaul theorem concerning harmonic maps, Ann. Inst. H. Poincare, Analyse non Liéare **17** (2000), 35–46

[JK] W. Jäger and H. Kaul, Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems, J. Reine Angew. Math. **343** (1983), 146–161

[L] F.-H. Lin, Une remarque sur L'application $x/|x|$, C. R. Acad. Sc. Paris **305** (1987), 529–531

[Lu] S. Luckhaus, Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold, Ind. Univ. Math. J. **37** (1988), 349–367

[W] C. Y. Wang, Minimality, stability, and perturbation for some p-harmonic maps, Ind. Univ. Math. **47** (1998), 725–740