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Energy identity of harmonic map flows from surfaces at finite singular time

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1. Introduction

Let (M, g) be a compact Riemannian surface, and $(N, h) \subset R^K$ be a Riemannian submanifold. Recall that a heat flow of harmonic map from M to N is given by

$$u_t = \Delta_g u + g^{ij} A(u)(D_i u, D_j u), \tag{1.1}$$

where A is the 2nd fundamental form of N in R^K (for simplicity we will omit g henceforth). Let $u : M \times (0, \infty) \to N$ be a global weak solution to (1.1), which is smooth away from a finite number of singular points $\{(x_i, t_i)\} \subset M \times (0, \infty)$. The existence of such a u was obtained by Struwe [St], which was a natural extension of [SaU]. Let (x_0, T_0) be a singular point of u and B be a small neighborhood of x_0, it is easy to show that, as $t \uparrow T_0$, $u(\cdot, t) \to u(\cdot, T_0)$ in $H^1 \cap C^\infty(B \setminus \{x_0\}, N)$ locally, but not in $H^1(B, N)$. Moreover, near x_0, by suitably rescaling $u(\cdot, t_i)$ for $t_i \uparrow T_0$, one can show there are finite many nonconstant harmonic maps $\omega_i : S^2 \to N$ $(1 \leq i \leq m)$, referred as *bubbles*, associated with $u(\cdot, t_i)$. It is clear that

$$\lim_{t_i \uparrow T_0} E(u(\cdot, t_i), B) \geq E(u(\cdot, T_0), B) + \sum_{i=1}^{m} E(\omega_i, S^2). \tag{$*$}$$

Here E denotes the energy on the respective sets. It is widely believed that the above inequality should be equality (cf. [J]). Indeed, recently there were many interesting and remarkable results related to this issue. Parker [P] proved both the energy identity and bubble tree convergence for sequences of harmonic maps from surfaces. More recently, people have considered bubbling phenomena for approximated harmonic maps or Palais-Smale sequences of controlled tension

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fields (say, bounded in L^2), which has not only its own interest but also important applications to heat flows (1.1). The energy identity for such Palais-Smale sequences was proved by Qing [Q] in the case N is the standard sphere, by Ding-Tian [DT] and, independently, Wang [W] in the general case. Most recently, the bubble tree convergence for such Palais-Smale sequences has been proved by Qing-Tian [QT] (cf. Chen-Tian [CT] for related results). There are also some results for high dimensional bubble phenomena due to Mou-Wang [MW].

When considering approximated harmonic maps $\{u_n\}$, Qing-Tian [QT] proved that if u_n have their tension fields bounded in L^2, then bubbles and the weak limit are connected together without necks. In particular, the image of u_n converges pointwise to the image of the limit bubble tree maps.

For solutions to (1.1), the energy inequality (cf. [St]) implies that there exist $t_n \uparrow \infty$ such that

$$\sup_n \|\partial_t u(\cdot, t_n)\|_{L^2(M)} < \infty. \tag{1.3}$$

In particular, Qing-Tian [QT] obatined

Theorem I. *There exist a harmonic map $u_\infty : M \to N$ and a finite number of bubbles $\{\omega_i\}_{i=1}^m$, $\{a_n^i\}_{i=1}^m \subset M$, and $\{\lambda_n^i\}_{i=1}^m \subset R_+$ such that*

$$\|u(\cdot, t_n) - u_\infty(\cdot) - \sum_{i=1}^m \omega_n^i(\cdot)\|_{L^\infty(M)} \to 0, \tag{1.4}$$

where $\omega_n^i(\cdot) = \omega_i(\frac{\cdot - a_n^i}{\lambda_n^i}) - \omega_i(\infty)$.

It is also very interesting to ask whether the above weak limits u_∞ is unique (i.e. independent of subsequences of t_n), and there was some progress made by Topping [T].

Despite these serious efforts, it is still a difficult open problem to understand the behavior of solutions to (1.1) near the singular points at finite time, whose existence was proved by Chang-Ding-Ye [CDY]. In the effort to understand this, we discover a different but simpler proof of the above Theorem I. Recall that the main idea of [QT] is follows. First, they showed the tangential energy of the sequence in the neck region decays exponentially by using a special case of the three circles theorem due to Simon [Sl] for the perturbated system and comparsions of the energy with piece-wise linear functions (i.e., geodesics in the flat metric). Then they used the L^1 estimates of the Hopf differentials to control the radial energy by the tangential energy. Both steps are somewhat involved. Here, for heat flows, we calculated the second order derivative of the tangential energy directly and found the so-called almost convexity property (cf. Lemma 2.1 below) which, in turn, implies the exponential decay property. Then we use the Pohozaev inequality, which is an easy adoption of that for harmonic maps to the approximated harmonic maps, to control the radial energy (cf. Lemma 2.4 below).

For a singular time $T_0 < \infty$, it does not seem possible to choose a sequence $t_n \uparrow T_0$ such that $u(\cdot, t_n)$ satisfies (1.4). However we observe that the energy density behaves like δ mass near the singular point so that if we rescale u by suitable scales going to zero then (1.4) holds for the rescaled ones (cf. Lemma 4.1). Energy identity accounting for the δ mass by finite many bubbles can then be proved by applying the above method to the rescaled maps and energy identity result for sequences of harmonic maps from S^2 (cf. [J] [P]). Therefore, we can prove

Theorem II. *For $T_0 < \infty$, let $u \in C^\infty(M \times (0, T_0), N)$ be a solution to (1.1) with T_0 as its singular time. Then there exist a finite many bubbles $\{\omega_i\}_{i=1}^l$ such that*

$$\lim_{t \uparrow T_0} E(u(\cdot, t), M) = E(u(\cdot, T_0), M) + \sum_{i=1}^{l} E(\omega_i, S^2). \tag{1.5}$$

We remark that the method here actually implies that if there are multiple bubbles at a point then there is no necks between bubbles but there may have a neck between bubbles and the weak limit $u(\cdot, T_0)$. However, we believe that $u(\cdot, T_0)$ is still continuous.

2. Preliminary estimates

The first Lemma is inspired by Parker [P].

Lemma 2.1. *There exists $\epsilon_0 > 0$ such that if $u \in C^\infty([T_1, T_2] \times S^1, N)$ satisfies*

$$u_{tt} + u_{\theta\theta} = A(u)(Du, Du) + F, \tag{2.1}$$

and $\sup_{[T_1,T_2]\times S^1} |Du| \le \epsilon_0$. *Then for $t \in [T_1, T_2]$,*

$$\frac{d^2}{dt^2} \int_{S^1} |u_\theta|^2 \ge \int_{S^1} |u_\theta|^2 - C \int_{S^1} |F|^2, \tag{2.2}$$

for some $C > 0$.

Proof. Direct computation, integration by parts, and substitution of (2.1) give

$$\begin{aligned} \frac{d^2}{dt^2} \int_{S^1} |u_\theta|^2 &= 2 \int_{S^1} |u_{\theta t}|^2 + 2 \int_{S^1} u_\theta u_{\theta tt} \\ &= 2 \int_{S^1} |u_{\theta t}|^2 - 2 \int_{S^1} u_{\theta\theta} u_{tt} \\ &= 2 \int_{S^1} |u_{\theta t}|^2 + 2 \int_{S^1} |u_{\theta\theta}|^2 \\ &\quad - 2 \int_{S^1} u_{\theta\theta}(A(u)(Du, Du) + F) \\ &= I + II + III. \end{aligned}$$

Now we estimate III as follows.

$$\begin{aligned} III &= 2\int_{S^1} u_\theta (A(u)(Du, Du))_\theta - 2\int_{S^1} u_{\theta\theta} F \\ &= 2\int_{S^1} u_\theta (DA(u)(Du, Du) u_\theta + 2A(u)(u_{\theta\theta}, u_\theta) \\ &\quad + 2A(u)(u_{\theta t}, u_t)) - 2\int_{S^1} u_{\theta\theta} F. \end{aligned}$$

Hence, by Cauchy inequality,

$$\begin{aligned} |III| &\leq 2\|DA\|_{L^\infty(N)} \sup_{[T_1,T_2]\times S^1} |Du|^2 \int_{S^1} |u_\theta|^2 \\ &\quad + 4\|A\|_{L^\infty(N)} \int_{S^1} |u_{\theta\theta}||u_\theta|^2 \\ &\quad + 4\|A\|_{L^\infty(N)} \int_{S^1} |u_{\theta t}||u_\theta||u_t| + 2\int_{S^1} |u_{\theta\theta}||F| \\ &\leq (\frac{1}{2} + C\epsilon_0^2)\int_{S^1} |u_{\theta\theta}|^2 + C\epsilon_0^2 \int_{S^1} |u_\theta|^2 + C\epsilon_0^2 \int_{S^1} |u_{\theta t}|^2 + C\int_{S^1} |F|^2. \end{aligned}$$

Therefore, if we choose ϵ_0 sufficently small, then

$$\frac{d^2}{dt^2}\int_{S^1} |u_\theta|^2 \geq \frac{17}{16}\int_{S^1} |u_{\theta\theta}|^2 - \frac{1}{16}\int_{S^1} |u_\theta|^2 - C\int_{S^1} |F|^2.$$

On the other hand, the Poincaré inequality of S^1 gives

$$\int_{S^1} |u_\theta|^2 \leq \int_{S^1} |u_{\theta\theta}|^2,$$

therefore

$$\frac{d^2}{dt^2}\int_{S^1} |u_\theta|^2 \geq \int_{S^1} |u_\theta|^2 - C\int_{S^1} |F|^2.$$

This gives (2.2).

Now we analyze the solutions to the following 2nd ODE.

$$P_1'' - P_1 = -G(t), T_1 \leq t \leq T_2, \tag{2.3}$$

$$P_1(T_1) = \epsilon_1, \tag{2.4}$$

$$P_1(T_2) = \epsilon_2. \tag{2.5}$$

Here $G(\geq 0) \in L^1([T_1, T_2])$ is given , $\epsilon_1 = \int_{S^1\times\{T_1\}} |u_\theta|^2$, and $\epsilon_2 = \int_{S^1\times\{T_2\}} |u_\theta|^2$. In fact, we can solve (2.3)–(2.5) explictly and get

Lemma 2.2. *Let $P_1 : [T_1, T_2] \to R$ be a solution to (2.3)–(2.5), then*

$$P_1(t) = Ae^t + Be^{-t} - \frac{1}{2}\int_t^{T_2} G(s)(e^{s-t} - e^{t-s})\, ds, \tag{2.6}$$

where

$$A=\frac{e^{T_2}\epsilon_2-e^{T_1}\epsilon_1+\frac{1}{2}\int_{T_1}^{T_2}G(s)(e^s-e^{2T_1-s})\,ds}{e^{2T_2}-e^{2T_1}},\tag{2.7}$$

$$B=\frac{e^{T_1+2T_2}\epsilon_1-e^{2T_1+T_2}\epsilon_2}{e^{2T_2}-e^{2T_1}}-\frac{1}{2}e^{2T_2}\frac{\int_{T_1}^{T_2}G(s)(e^s-e^{2T_1-s})\,ds}{e^{2T_2}-e^{2T_1}}.\tag{2.8}$$

Denote $P(t)=\int_{S^1\times\{t\}}|u_\theta|^2$. Then the maximum principle implies

$$P(t)\le P_1(t),\forall t\in[T_1,T_2].$$

Hence we obtain, by direct calculation,

Lemma 2.3. *Under the same conditions of Lemma 2.2. Assume $G(t)=e^{-2t}H(t)$ with $H\in L^1([T_1,T_2])$ and $0<T_1<<T_2<\infty$. Then*

$$\begin{aligned}\int_{T_1}^{T_2}|P(t)|^{\frac{1}{2}}\,dt &\le |A|^{\frac{1}{2}}(e^{\frac{T_2}{2}}-e^{\frac{T_1}{2}})+|B|^{\frac{1}{2}}(e^{-\frac{T_1}{2}}-e^{-\frac{T_2}{2}})\\ &\quad+(e^{-\frac{T_1}{2}}-e^{-\frac{T_2}{2}})(\int_{T_1}^{T_2}|H(t)|\,dt)^{\frac{1}{2}}\\ &\le C(\sqrt{\epsilon_1}+\sqrt{\epsilon_2})+C(\int_{T_1}^{T_2}|H(t)|\,dt)^{\frac{1}{2}}.\end{aligned}\tag{2.9}$$

Now we drive the Pohozaev inequality for two dimensional approximated harmonic maps.

Lemma 2.4. *Let $u\in C^\infty(B_1^2,N)$ be a solution to*

$$\Delta u+A(u)(Du,Du)=h,\tag{2.10}$$

with $h\in L^2(B_1^2)$. Then

$$\int_{\partial B_R}|u_r|^2\le R^{-2}\int_{\partial B_R}|u_\theta|^2+2\int_{B_R}|h||Du|,\tag{2.11}$$

for any $0<R<1$.

Proof. Multiplying both sides of (2.10) by xDu and integrating it over B_R, we get

$$\int_{B_R}|Du|^2-R\int_{\partial B_R}|u_r|^2+\frac{1}{2}\int_{B_R}xD(|Du|^2)=-\int_{B_R}h\cdot xDu.$$

Note also that

$$\frac{1}{2}\int_{B_R}xD(|Du|^2)=-\int_{B_R}|Du|^2+\frac{1}{2}R\int_{\partial B_R}|Du|^2.$$

Hence,

$$\frac{1}{2}\int_{\partial B_R}|Du|^2-\int_{\partial B_R}|u_r|^2=-R^{-1}\int_{B_R}h\cdot xDu,$$

which implies (2.11), if we write $|Du|^2=|u_r|^2+\frac{1}{r^2}|u_\theta|^2$.

Lemma 2.5. *Let $u\in C^\infty(B_1^2\times(0,t_0),N)$ be a solution to (1.1). Then, for $0<t\le s<t_0$ and $0<R\le\frac{1}{2}$,*

$$\int_{B_R}|Du|^2(x,s)\,dx\le\int_{B_{2R}}|Du|^2(x,t)\,dx+C(s-t)R^{-2}E_0,\tag{2.12}$$

and

$$\int_{B_R}|Du|^2(x,t)\,dx\le\int_{B_{2R}}|Du|^2(x,s)\,dx+C\int_t^s\int_{B_1}|\partial_t u|^2+C(s-t)R^{-2}E_0.\tag{2.13}$$

Here $E_0=E(u(\cdot,0),M)$.

Proof. Let $\phi\in C_0^\infty(B_1^2)$ be such that $0\le\phi\le 1$, $\phi=1$ on B_R, and $\phi=0$ outside B_{2R}. Multiplying (1.1) by $\phi^2\partial_t u$, we get

$$\begin{aligned}&-2\int_{B_1^2}|Du|^2|D\phi|^2-\frac{1}{2}\int_{B_1^2}|\partial_t u|^2\phi^2\\ &\le\int_{B_1^2}|\partial_t u|^2\phi^2+\frac{d}{dt}(\frac{1}{2}\int_{B_1^2}|Du|^2\phi^2)\\ &\le 2\int_{B_1^2}|Du|^2|D\phi|^2+\frac{1}{2}\int_{B_1^2}|\partial_t u|^2\phi^2.\end{aligned}$$

Integrating these inequalities from t to s, one get (2.12) and (2.13).

For $R>0$ and $(x,t)\in R^2\times R_-$, denote $P_R(x,t)=\{(y,s)\in R^2\times R_-:|y-x|\le R, t-R^2\le s\le t\}$. Now we can state the small energy regularity estimates (cf. [St1] for proofs).

Lemma 2.6. *There exist $\epsilon_0>0$ and $C>0$ such that if $u\in C^\infty(R^2\times R_-,N)$ is a solution to (1.1) satisfying $R^{-2}\int_{P_R(x,t)}|Du|^2\le\epsilon_0^2$ for some $(x,t)\in R^2\times R_-$, then*

$$R^2\sup_{P_{\frac{R}{2}}(x,t)}|Du|^2\le CR^{-2}\int_{P_R(x,t)}|Du|^2,\tag{2.14}$$

and

$$R^4\sup_{P_{\frac{R}{2}}(x,t)}|\partial_t u|^2\le C(\epsilon_0,E_0).\tag{2.15}$$

Corollary 2.7. *There exists $\epsilon_0>0$ such that if $u\in C^\infty(R^2\times R_-,N)$ is a solution to (1.1) satisfying $\int_{P_{R_0}(x_0,t_0)}|Du|^2\le\frac{\epsilon_0^2}{4}$ and $\int_{P_{R_0}(x_0,t_0)}|\partial_t u|^2\le\frac{\epsilon_0^2}{4}$, then*

$$R_0|Du|(x_0,t_0) \le C(E_0),\ R_0^2|\partial_t u|(x_0,t_0) \le C(\epsilon_0,E_0). \tag{2.16}$$

Proof. Let $\delta_0 = \min\{\frac{\epsilon_0}{\sqrt{4CE_0}}, \frac{1}{4}\}$. From (2.13), we have, for any $t \in [t_0-\delta_0^2R_0^2, t_0]$,

$$\int_{B_{\delta_0 R_0}(x_0)} |Du|^2(x,t)\,dx \le \frac{3\epsilon_0^2}{4}.$$

Therefore,

$$(\delta_0 R_0)^{-2}\int_{P_{\delta_0 R_0}(x_0,t_0)} |Du|^2\,dxdt \le \frac{3\epsilon_0^2}{4}, \tag{2.17}$$

and (2.14) of Lemma 2.6 implies

$$R_0^2|Du|^2(x_0,t_0) \le C\delta_0^{-2}\epsilon_0^2 \le C(E_0),$$

and (2.15) gives

$$R_0^4|\partial_t u|^2(x_0,t_0) \le C(\epsilon_0,E_0).$$

3. A new proof of Theorem I

We may assume $M = B_1^2$ henceforth. Let $t_n \uparrow \infty$ be such that

$$\lim_{n\to\infty}\int_{B_1^2} |\partial_t u|^2(\cdot,t_n) = 0,\ \lim_{n\to\infty}\int_{B_1^2\times[t_n-1,t_n]} |\partial_t u|^2 = 0. \tag{3.1}$$

Denote $u_n(\cdot) = u(\cdot,t_n)$. From the reduction procedure of bubbling illustrated by [DT] (cf. also [Q] [W]), theorem I follows from the following lemma, which deals with the single bubble case. To describe it more clearly, let's assume that $u_n = u(\cdot,t_n) \to u_\infty$ in $H^1(B_\delta \setminus \{0\}, N)$ locally but not in $H^1(B_\delta, N)$, here δ is given and small. Assume also that there only exists one bubble ω_1 such that for some $\lambda_n \to 0$ and $x_n \to 0$,

$$\tilde{u}_n(x) = u_n(x_n + \lambda_n x) \to \omega_1$$

in $H^1 \cap C^1(R^2, N)$ locally. For large $R > 0$, denote $A_n(\delta,R) = \{x \in R^2 : R\lambda_n \le |x - x_n| \le \delta\}$ and $\Sigma_n(\delta,R) = [|\log\delta|, |\log R\lambda_n|] \times S^1$. Therefore $f(r,\theta) = (e^{-r},\theta) : \Sigma_n(\delta,R) \to A_n(\Sigma,R)$ is conformal if $\Sigma_n(\delta,R)$ is equipped with the flat metric. Let $v_n : \Sigma_n(\delta,R) \to N$ be $v_n(r,\theta) = u_n(e^{-r},\theta)$. Then

$$\Delta v_n + A(v_n)(Dv_n, Dv_n) = \overline{h}_n,\ \text{ in } \Sigma_n(\delta,R), \tag{3.2}$$

where $\overline{h}_n(r,\theta) = e^{-2r}\partial_t u(e^{-r},\theta,t_n)$ and

$$\|\overline{h}_n\|_{L^2([r,\infty)\times S^1)} \le e^{-r}\|\partial_t u(\cdot,t_n)\|_{L^2(B_{e^{-r}})}. \tag{3.3}$$

Also the conformal invariance of E implies,

$$\int_{\Sigma_n(\delta,R)} |Dv_n|^2 = \int_{A_n(\delta,R)} |Du_n|^2. \tag{3.4}$$

From the assumption that there exists only one bubble ω_1, we know (cf. [DT])

$$\int_{B_{e^{-(r-2)}}\setminus B_{e^{-(r+2)}}} |Du_n|^2 = \int_{[r-2,r+2]\times S^1} |Dv_n|^2 \le \frac{1}{4}\epsilon_0^2, \tag{3.5}$$

$\forall r \in [|\log\delta|, |\log R\lambda_n|]$. With these preparations, we have

Lemma 3.1. *Assume u_n v_n are as above. Then*

$$\lim_{\delta\downarrow 0}\lim_{R\uparrow\infty}\lim_{n\to\infty}\int_{A_n(\delta,R)} |Du_n|^2 = 0, \tag{3.6}$$

and

$$\lim_{\delta\downarrow 0}\lim_{R\uparrow\infty}\lim_{n\to\infty} osc_{A_n(\delta,R)} u_n = 0. \tag{3.7}$$

Proof. From (3.1) and (3.5), one can apply Corollary 2.7 to get

$$\begin{aligned} |Dv_n|(r,\theta) &= e^{-r}|Du_n|(e^{-r},\theta) \le C(E_0),\\ \overline{h}_n(r,\theta) &= e^{-2r}|\partial_t u|(e^{-r},\theta,t_n) \le C(\epsilon_0,E_0). \end{aligned} \tag{3.8}$$

$\forall r \in [|\log\delta|, |\log R\lambda_n|]$. Let $G_n(r) = \int_{S^1\times\{r\}} |\overline{h}_n(r,\theta)|^2$. Then, by (3.1) and (3.3), we have

$$\int_{|\log\delta|}^{|\log R\lambda_n|} e^{2r} G_n(r)\,dr = \int_{A_n(\delta,R)} |\partial_t u(\cdot,t_n)|^2 \to 0. \tag{3.9}$$

Using (3.2), (3.8), (3.9), and $W^{2,4}$ interior estimates, we get

$$\begin{aligned} \|D^2 v_n\|_{L^4([r-1,r+1]\times S^1)} &\le C(\|Dv_n\|_{L^4([r-2,r+2]\times S^1)}\\ &\quad + \||Dv_n|^2\|_{L^4([r-2,r+2]\times S^1)} + \|\overline{h}_n\|_{L^4([r-2,r+2]\times S^1)})\\ &\le C[\|Dv_n\|^{\frac12}_{L^\infty(\Sigma_n(\delta,R))}\|Dv_n\|^{\frac12}_{L^2([r-2,r+2]\times S^1)}\\ &\quad + \|\overline{h}_n\|^{\frac12}_{L^\infty(\Sigma_n(\delta,R))}\|\overline{h}_n\|^{\frac12}_{L^2([r-2,r+2]\times S^1)}] \le C\epsilon_0. \end{aligned}$$

$\forall r \in [|\log\delta|, |\log R\lambda_n|]$.

Therefore, the Sobolev embedding theorem and (3.5) give

$$\|Dv_n\|_{L^\infty(\Sigma_n(\delta,R))} \le C\epsilon_0.$$

Hence we can apply Lemma 2.1– 2.3, with u, F, G, T_1, T_2, replaced by v_n, $\overline{h}_n$, G_n, $|\log\delta|$, $|\log R\lambda_n|$ respectively, to conclude

$$\int_{|\log\delta|}^{|\log R\lambda_n|}\Big(\int_{S^1}|(v_n)_\theta|^2\Big)^{\frac12} \le \Big(\int_{S^1\times\{|\log R\lambda_n|\}}|(v_n)_\theta|^2\Big)^{\frac12} + \Big(\int_{S^1\times\{|\log\delta|\}}|(v_n)_\theta|^2\Big)^{\frac12} + \sqrt{\delta}\Big(\int_{B_\delta}|\partial_t u(\cdot,t_n)|^2\Big)^{\frac12} \to 0. \tag{3.10}$$

Here we have used the fact that both $\int_{S^1\times\{|\log R\lambda_n|\}}|(v_n)_\theta|^2$ and $\int_{S^1\times\{|\log\delta|\}}|(v_n)_\theta|^2$ converge to zero. Applying Lemma 2.4, we have

$$\int_{S^1\times\{r\}}|(v_n)_r|^2 \le \int_{S^1\times\{r\}}|(v_n)_\theta|^2 + 2e^{-r}\int_{B_{e^{-r}}}|\partial_t u(\cdot,t_n)||Du_n|, \tag{3.11}$$

for any $r\in[|\log\delta|,|\log R\lambda_n|]$. In particular,

$$\begin{aligned}\int_{|\log\delta|}^{|\log R\lambda_n|}\Big(\int_{S^1}|(v_n)_r|^2\Big)^{\frac12} &\le \int_{|\log\delta|}^{|\log R\lambda_n|}\Big(\int_{S^1}|(v_n)_\theta|^2\Big)^{\frac12}\\ &\quad + 2\int_{|\log\delta|}^{|\log R\lambda_n|}e^{-\frac{r}{2}}\Big(\int_{B_{e^{-r}}}|\partial_t u(\cdot,t_n)||Du_n|\Big)^{\frac12}\\ &\le o(1)+2\Big(\int_{|\log\delta|}^{|\log R\lambda_n|}e^{-\frac{r}{2}}\Big)\Big(\int_{B_\delta}|\partial_t u(\cdot,t_n)|^2\Big)^{\frac14}\Big(\int_{B_\delta}|Du_n|^2\Big)^{\frac14}\\ &\le o(1)+2\sqrt{\delta}\Big(\int_{B_\delta}|\partial_t u(\cdot,t_n)|^2\Big)^{\frac14}\Big(\int_{B_\delta}|Du_n|^2\Big)^{\frac14}\to 0.\end{aligned} \tag{3.12}$$

Therefore,

$$\int_{\Sigma_n(\delta,R)}|Dv_n| \le (2\pi)^{\frac12}\Big(\int_{|\log\delta|}^{|\log R\lambda_n|}\Big[\Big(\int_{S^1}|(v_n)_r|^2\Big)^{\frac12}+\Big(\int_{S^1}|(v_n)_\theta|^2\Big)^{\frac12}\Big]dr \to 0.$$

This clearly implies (3.7). It is very easy to verify that (3.7) follows from (3.6).

4. Proof of Theorem II

In this section, we prove the energy identity (1.5). First, we observe

Lemma 4.1. *Let $u\in C^\infty(B_1^2\times(0,t_0),N)$ be a solution to (1.1) with $(0,t_0)$ being its only singular point. Then there exists a positive m such that*

$$|Du|^2(x,t)\,dx \to m\delta_0 + |Du|^2(x,t_0)\,dx, \tag{4.1}$$

for $t\uparrow t_0$, as Radon measures. Here δ_0 denotes the δ-mass at 0.

Proof. For any two $s_i\uparrow t_0$, $t_i\uparrow t_0$, according to Lemma 4.1 there exist $m>0$ and $m'>0$ such that, after taking subsequences,

$$\begin{aligned}|Du|^2(x,s_i)\,dx &\to m\delta_0+|Du|^2(x,t_0)\,dx\\ |Du|^2(x,t_i)\,dx &\to m'\delta_0+|Du|^2(x,t_0)\,dx,\end{aligned}$$

as Radon measures in B_1^2.

For any $\epsilon > 0$, there exists $\eta > 0$ such that $\int_{B_{2\eta}^2} |Du|^2(x, t_0) \le \epsilon$. Therefore, (2.12) and (2.13) imply

$$
\begin{aligned}
m &\ge \int_{B_{2\eta}^2} |Du|^2(x, s_i) - \epsilon \\
&\ge \int_{B_\eta} |Du|^2(x, t_i) - C\delta^{-2}|s_i - t_i|E_0 - \int_{s_i}^{t_i} \int_{B_1^2} |\partial_t u|^2 - \epsilon \\
&\ge \int_{B_\eta} |Du|^2(x, t_i) - 2\epsilon \ge m' - 2\epsilon.
\end{aligned}
$$

Hence $m \ge m'$. Similarly $m \le m'$.

Proof of Theorem II. Assume $T_0 = 0$, $M = B_1^2$, and $(0, 0)$ is the only singular point of u From (4.1), we know that there exists $t_n \uparrow 0$ and $\lambda_n \downarrow 0$ such that

$$\lim_{n\to\infty} \int_{B_{\lambda_n}} |Du|^2(x, t_n)\, dx = m. \tag{4.2}$$

Let $u_n(x, t) = u(\lambda_n x, t_n + \lambda_n^2 t)$. Then u_n satisfies (1.1) on $B_{\lambda_n^{-1}}^2 \times [-2, 0)$ and

$$\int_{-2}^{2} \int_{B_{\lambda_n^{-1}}^2} |\partial_t u_n|^2 = \int_{t_n - 2\lambda_n^2}^{t_n + 2\lambda_n^2} \int_{B_1^2} |\partial_t u|^2 \to 0, \tag{4.3}$$

as $n \to \infty$. Therefore, by Fubin's theorem, there exists $\eta_n \in (-1, -\frac{1}{2})$ such that

$$\int_{B_{\lambda_n^{-1}}^2} |\partial_t u_n|^2(\cdot, \eta_n) \to 0, \; \int_{B_{\lambda_n^{-1}}^2 \times (-2,2)} |\partial_t u_n|^2 \to 0. \tag{4.4}$$

Note also, from (2.12), that

$$\int_{B_R} |Du_n|^2(\cdot, \eta_n) \ge \int_{B_1} |Du_n|^2(\cdot, 0) - CR^{-2}E_0 \ge m - CR^{-2}E_0. \tag{4.5}$$

In particular,

$$\lim_{R\to\infty} \int_{B_R} |Du_n|^2(\cdot, \eta_n) \ge m. \tag{4.6}$$

In fact, by Lemma 4.1, we have

$$\lim_{R\to\infty} \int_{B_R} |Du_n|^2(\cdot, \eta_n) = m. \tag{4.7}$$

From (4.7), we know that for each $R > 0$ $u_n(\cdot, \eta_n)$ weakly converges to $v \in H^1(B_R, N)$. In fact, v is a constant map, since we can assume $|t_n| \le 2\lambda_n^2$ and observe

$$\int_{B_R} |u_n(\cdot,\eta_n) - u_n(\cdot, -t_n\lambda_n^{-2})|^2 \le 4\int_{-2}^{2}\int_{B_R} |\partial_t u_n|^2 \to 0,$$

and

$$\int_{B_R} |Du_n(\cdot, -t_n\lambda_n^{-2})|^2 = \int_{B_{R\lambda_n}} |Du|^2(\cdot,0) \to 0.$$

For each $R > 0$, we now apply the proof of theorem II (i.e., Sect. 3) to $u_n(\cdot,\eta_n)$ on B_R to conclude that there exist N_R bubbles $\{\omega_{i,R}\}_{i=1}^{N_R}$ such that

$$\lim_{n\to\infty}\int_{B_R} |Du_n|^2(\cdot,\eta_n) = \sum_{i=1}^{N_R} E(\omega_{i,R}, S^2). \tag{4.8}$$

Since there exists a universial $\epsilon_0 > 0$ such that any bubble $\omega : S^2 \to N$ has $E(\omega, S^2) \ge \epsilon_0$, we know that $1 \le N_R \le [\frac{m}{\epsilon_0}]$. Therefore, there are a $d \in [1, [\frac{m}{\epsilon_0}]]$ and a subsequence $R \uparrow \infty$ such that $N_R = d$ and

$$m = \lim_{R\uparrow\infty}\lim_{n\to\infty}\int_{B_R} |Du_n|^2(\cdot,\eta_n) = \lim_{R\uparrow\infty}\sum_{i=1}^{d} E(\omega_{i,R}, S^2). \tag{4.9}$$

Note that for $i = 1,\cdots,d$, $\{\omega_{i,R}\}$ are sequences of harmonic maps from S^2 to N whose energies are uniformly bounded. Hence we can apply the results of Jost [J] (cf. also Parker [P]) to conclude that for $i = 1,\cdots,d$, there exist $N_i \in [1, [\frac{m}{\epsilon_0}]]$ and N_i bubbles $\{\omega_{i,j}\}_{j=1}^{N_i}$ such that

$$\lim_{R\uparrow\infty} E(\omega_{i,R}, S^2) = \sum_{j=1}^{N_l} E(\omega_{i,j}, S^2). \tag{4.10}$$

Therefore, by (4.8), (4.9), (4.10), we have

$$m = \sum_{i=1}^{d}\sum_{j=1}^{N_i} E(\omega_{i,j}, S^2). \tag{4.11}$$

Here $\omega_{i,j}$ are bubbles for $1 \le i \le d, 1 \le j \le N_i$. It is easy to see that (4.11) and Lemma 4.1 imply (1.5). The proof is complete.

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