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On the existence of H-surfaces into Riemannian manifolds

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Abstract. This paper considers the existence of a local minimizer of a conformally invariant functional defined on a space of maps of a closed Riemann surface into a compact Riemannian manifold N. The functional is defined for a given tensor H on N of type (1,2) and we call its extremal an H-surface. In fact, we prove that there exists a local minimizer of the functional in a given homotopy class under certain conditions on N, H and the minimum of the Dirichlet integral of maps of the homotopy class.

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0. Introduction

Let Σ be a two dimensional compact Riemannian manifold without boundary and N an n-dimensional compact Riemannian manifold isometrically embedded into $\mathbb{R}^l$. For a smooth 2-form ω on N, we define the functional

$$I_\omega(u) := \frac{1}{2}\int_\Sigma |\nabla u|^2 dV_\Sigma + 2\int_\Sigma u^*\omega \tag{0.1}$$

for u $\in H^{1,2}(\Sigma; N)$. We note that functional (0.1) is invariant under an arbitrary conformal reparametrization of the domain. In fact, any conformally invariant functional satisfying a certain assumption can be written in the form of (0.1). (cf. [Gr] or [J];Theorem1.2.1). We call (smooth) extremals of functional (0.1) H-surfaces. The Euler-Lagrange equation of functional (0.1) is written as

$$trace(\nabla du) = 2H(u)(\nabla u \wedge \nabla u) \tag{0.2}$$

where H is the skew symmetric tensor of type(2,1) on N defined by

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$$d\omega_p(U,V,W) := \langle U, H(p)(V,W)\rangle \quad for \quad p \in N, \quad U,V,W \in T_pN$$

($\langle , \rangle$ denotes the metric tensor of N and the right hand side of (0.2) stands for

$$H(u)(\nabla u \wedge \nabla u) := \sigma^{-2} H(u)(u_*(\frac{\partial}{\partial x^1}), u_*(\frac{\partial}{\partial x^2}))$$

where $z = x^1 + \sqrt{-1}x^2$ denotes an isothermal coordinate and the metric tensor of Σ is written as $\sigma^2((dx^1)^2 + (dx^2)^2)$.

Some well-known equations are special cases of (0.2).

(1) If $d\omega = 0$, equation (0.2) is called the equation of harmonic maps.
(2) If a solution u of equation (0.2) is conformal, u parametrize a surface of prescribed mean curvature $H(u)$ as a submanifold at regular points.
(3) If $N = \mathbb{R}^3$, equation (0.2) is usually called the equation of surfaces of prescribed mean curvature. (But unless a solution is conformal, it does not parametrize surfaces of prescribed mean curvature H as a submanifold even at regular point.) In this case, functional (0.1) and equation (0.2) are usually written in the form;

(0.3) $$I_\omega := \frac{1}{2}\int |\nabla u|^2 + \frac{4}{3}Q(u)(u_{x_1} \wedge u_{x_2})dx$$

(0.4) $$\Delta u = 2H(u)u_{x_1} \wedge u_{u_2} (\text{where } div\, Q(u) := 3H(u)).$$

We refer to [J] Chapter1 and Chapter2 for more informations about basic results on extremals of the functional I_ω.

In this paper, we study the existence of a local minimizer of functional (0.1). Our fundamental problem can be stated as follows.

Problem($\star$). Does there exist an extremal or a (local) minimizer of functional I_ω, defined in (0.1), in a given homotopy class $\alpha \in [\Sigma, N]$?

Our main theorem below is an answer to Problem ($\star$). For $\Omega \subset \Sigma$, set

$$D(u;\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx$$

Main Theorem. *Let Σ be a closed Riemann surface and N a compact Riemannian manifold. Then there exists an absolute constant C such that; if there exists $u_0 \in H^{1,2}(\Sigma, N)$ and a (smooth) 2-form $\tilde{\omega}$ on $\mathbb{R}^l$ which is an extension of 2-form ω on N with*

$$|d\widetilde{\omega}| \cdot D(u_0;\Sigma) < C,$$

then, there exists a local minimizer of I_ω in the free homotopy class $[u_0]$ induced by u_0.

This theorem is a generalizations of a theorem of Sacks-Uhlenbeck [SaU] for harmonic maps and a theorem of Steffen [Ste] for surfaces of prescribed mean curvature. Let us recall these two theorems.

Theorem (Sacks-Uhlenbeck). *Let Σ be a closed Riemann surface and N a compact Riemannian manifold with $\pi_2(N) = 0$. Then, in any homotopy class $\alpha \in [\Sigma, N]$, there exists an energy minimizing harmonic map.*

Theorem (Steffen). *Let Q and H be as in (0.3) and (0.4). If there exists $u_0 \in H^{1,2}(\Omega; \mathbb{R}^3)(\Omega \subset \mathbb{R}^2)$ with*

$$H_0^2 \cdot D(u_0; \Omega) < \frac{2}{3}\pi,$$

where $H_0 := \sup_{u \in \mathbb{R}^3} |H(u)|$, then there exists a local minimizer of functional (0.3) in $\{u_0\} + H_0^{1,2}(\Omega; \mathbb{R}^3)$.

Remark.

(1) Theorem 0.2 was reproved by Struwe [Str1] as a corollary of his theorem on the heat flow of harmonic maps. Our basic ideas for the arguments in Section 4 come from his method of heat flow of surfaces of constant mean curvature in [Str2].
(2) The result similar to Theorem 0.3 was also proved by Steffen [Ste] for the Plateau problem of disk-type. A weaker version of Theorem 0.3 is obtained by Wente [W] previously.

Now we shall outline the contents of this paper briefly. In the first section, we fix the notations and derive the Euler-Lagrange equation for the functional I_ω. In section 2, we recall notations and theorems from geometric measure theory which are needed to estimate the second term $\int_\Sigma u^*\omega$ in functional (0.1). Section 3 describes the convergence properties of any sequence of solutions of Euler-Lagrange equation with bounded Dirichlet integrals. Section 4 is devoted to the study of evolution problems corresponding to our variational problem based on [Str2]. Finally, in the last section, we prove our existence theorems for closed domain.

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1. Notations

$C^{k,\alpha}, L^p, H^{k,p}$ denote the usual Hölder, Lebesgue, Sobolev space. When we distinguish the time variable from the space variables, we use the notation

$$\begin{aligned} &S^{k,\alpha}(\Omega \times [T_1, T_2]) \\ &\quad := \{u \in C(\Omega \times [T_1, T_2]); \partial_t^r \partial_x^s u \in C^\alpha(\Omega \times [T_1, T_2]) \\ &\quad \text{if} \quad 2r + |s| \le k\} \\ &L^{k,p}(\Omega \times [T_1, T_2]) \\ &\quad := \{u \in L^p(\Omega \times [T_1, T_2]); \partial_t^r \partial_x^s u \in L^p(\Omega \times [T_1, T_2]) \\ &\quad \text{if} \quad 2r + |s| \le k\} \end{aligned}$$

for $\Omega \subset \mathbb{R}^n$. Here, s denotes the multi index, i.e.

$$s = (s_1, \ldots, s_n) \in \mathbb{N} \times \ldots \times \mathbb{N} \quad , |s| := s_1 + \ldots + s_n,$$
$$\partial_x^s u := \left(\frac{\partial}{\partial x_1}\right)^{s_1} \cdot \cdots \cdot \left(\frac{\partial}{\partial x_n}\right)^{s_n} u.$$

We define

$$H^{k,p}(\Sigma, N) := \left\{u \in H^{k,p}(\Sigma; \mathbb{R}^l); u(x) \in N \text{ for almost every } x \in \Sigma\right\}$$

for a manifold N embedded in $\mathbb{R}^l$. $L^{k,p}(\Sigma; N)$ is defined in the same manner. Mainly, we work in $H^{1,2}(\Sigma; N)$. Note that $u \in H^{1,2}(\Sigma; N)$ induces free homotopy class since $dim\,\Sigma = 2$ (see [ScU]). In the sequel, we fix a 2-form on $\mathbb{R}^l$ which is an extension of 2-form ω on N and denote it again by ω. Adapting the usual Einstein's summation convention with respect to the coordinate in $\mathbb{R}^l$, set

$$\omega(u) = \frac{1}{2} b_{ij}(u) du^i \wedge du^j \quad (1 \le i, j \le l),$$

where b_{ij} is skew-symmetric. We also define the tensor field H of type (1,2) on $\mathbb{R}^l$ by

$$\langle H(p)(V, W), U\rangle = d\omega(U, V, W),$$

where $U, V, W \in T_p\mathbb{R}^l$ and $\langle , \rangle$ denotes the canonical inner product in $\mathbb{R}^l$. In terms of the coordinate in $\mathbb{R}^l$,

$$H^i(p)(V, W) = H^i_{jk}(p) V^j W^k$$

where

$$H^i_{jk}(p) = \frac{1}{4}\left(\frac{\partial b_{ij}}{\partial u^k}(p) + \frac{\partial b_{jk}}{\partial u^i}(p) + \frac{\partial b_{ki}}{\partial u^j}(p)\right)$$

Now we derive the Euler-Lagrange equation of I_ω in terms of the coordinate in $\mathbb{R}^l$. For given $\varphi \in C_0^\infty(\Sigma; \mathbb{R}^l)$ we can define the variation through $u \in C^1(\Sigma, N)$ by $u_t := \pi(u + t\varphi)$ for sufficiently small t. Then we have the first variational formula of I_ω;

$$\begin{aligned}
\langle DI_\omega, \dot{u}_0\rangle &= \frac{\partial}{\partial t}\Big|_{t=0} I_\omega(u_t) \\
&= \frac{\partial}{\partial t}\Big|_{t=0}\left[\frac{1}{2}\int_\Sigma |\nabla u_t|^2 dV\right] + \frac{\partial}{\partial t}\Big|_{t=0}\left[2\int_\Sigma u^*\omega\right] \\
&= \int_\Sigma g_{\alpha\beta}\left\{D_\alpha u^i D_\beta \varphi^i + D_{jk}\pi^i(u) D_\alpha u^j D_\beta u^k \varphi^i\right\}\sqrt{|g|}dx^1 dx^2 \\
&\quad + \int_\sigma 2H^i_{jk}\det(Du^j, Du^k) D_l \pi^i \varphi^l dx^1 dx^2 \\
&= \int_\Sigma \left\{\langle \nabla u, \nabla\varphi\rangle + \langle D^2\pi(u)(\nabla u, \nabla u), \varphi\rangle\right. \\
&\quad \left. + 2\langle H(u)(\nabla u \wedge \nabla u), D\pi(u)\cdot\varphi\rangle\right\} dV,
\end{aligned}$$

where $(g_{\alpha\beta})$ denotes the metric tensor of Σ and $D^2\pi(\nabla u, \nabla u)$ and $H^i(u)(\nabla u \wedge \nabla u)$ are defined by

$$\begin{aligned} D^2\pi(\nabla u, \nabla u) &:= g^{\alpha\beta} D_{jk}\pi(u) D_\alpha u^j D_\beta u^k \\ H^i(u)(\nabla u \wedge \nabla u) &:= \frac{1}{\sqrt{|g|}} H^i_{jk}(u) \det(Du^j, Du^k). \end{aligned}$$

Hence the Euler-Lagrange equation of I_ω is written as

$$\Delta_\Sigma u = D^2\pi(u)(\nabla u, \nabla u) + 2D\pi \cdot H(u)(\nabla u \wedge \nabla u) \tag{1.1}$$

where Δ_Σ stands for the Laplace-Beltrami operator on Σ.

Very often our arguments do not depend on the special structure of the non-linear term and valid for more general equation of the following type:

$$\Delta_\Sigma u = \Gamma(u)(\nabla u, \nabla u). \tag{1.2}$$

where $\Gamma(u)(\nabla u, \nabla u)$ is defined for a given symmetric tensor A of type (1, 2) on $\mathbb{R}^l$ and a given skew-symmetric tensor B of type (1, 2) on $\mathbb{R}^l$ by

$$\Gamma(u)(\nabla u, \nabla u) := trace(u^* A) + B(u)(\nabla u \wedge \nabla u)$$

($B(u)(\nabla u \wedge \nabla u)$ is defined in the same manner as $H(u)(\nabla u \wedge \nabla u)$).

2. Isoperimetric inequalities and volume functionals

For $u \in H^{1,2}(\Sigma; N)$, we set

$$V(u)[\omega] = V_\omega(u) := \int_\Sigma u^*\omega \tag{2.1}$$

We call $V(u)[\omega] = V_\omega(u)$ the volume functinal. The notation $V(u)[\cdot]$ is used when we think the volume functional as a current, while we use $V_\omega(\cdot)$ when we think it as a functional.

We shall recall basic definitions and notations of geometric measure theory. See e.g. [F], [H-S], [Mg], [Si] for more informations about geometric measure theory.

Forms and Currents

Let $\mathscr{D}^n(\mathbb{R}^{n+k})$ be the space of smooth (i.e. C^∞) n-forms on $\mathbb{R}^{n+k}$ with compact support with the usual topology, namely,
"$\{\alpha_i\} \subset \mathscr{D}^n(\mathbb{R}^{n+k})$ converges to $\alpha \in \mathscr{D}^n(\mathbb{R}^{n+k})$ iff the following two conditions hold,

(1) *supp* α_i is contained in some compact set in $\mathbb{R}^{n+k}$ independent of i.
(2) Every derivative of every coefficient of α_i converges uniformly to that of α."

Then, we define the space of currents $\mathscr{D}_n(\mathbb{R}^{n+k})$ as the topological dual of $\mathscr{D}^n(\mathbb{R}^{n+k})$. The support of a current $T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ is defined as the smallest closed subset $K \subset \mathbb{R}^{n+k}$ such that for any $\alpha \in \mathscr{D}^n(\mathbb{R}^{n+k})$ with *supp* $\alpha \cap K = \emptyset$, we have $T[\alpha] = 0$. We say $\{T_j\} \subset \mathscr{D}_n(\mathbb{R}^{n+k})$ converges weakly to $T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ iff $T_j(\alpha) \to T(\alpha)$ for any $\alpha \in \mathscr{D}^n(\mathbb{R}^{n+k})$.

Mass and Comass

For $\alpha \in (\wedge^n \mathbb{R}^{n+k})^*$ (i.e. α is a skew symmetric multilinear form), we define the comass $|\alpha|$ of α as

$$|\alpha| := \sup\left\{\alpha(x_1, ..., x_n); x_i \in \mathbb{R}^{n+k}, |x_i| < 1\right\}.$$

And we define the comass $|\alpha|_K$ of $\alpha \in \mathscr{D}^n(\mathbb{R}^{n+k})$ on $K \subset \mathbb{R}^{n+k}$ by

$$|\alpha|_K = \sup_{x \in K} |\alpha(x)|.$$

(If $K = \mathbb{R}^{n+k}$, we simply write $|\alpha|$.) Then, we can define the mass $\|T\|$ of current $T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ by

$$\|T\| := \sup\left\{T(\alpha); |\alpha| \leq 1, \alpha \in \mathscr{D}^n(\mathbb{R}^{n+k})\right\}.$$

If $\|T\| < \infty$, T is called a current with finite mass.

Boundary of Currents

The boundary $\partial T \in \mathscr{D}_{n+1}(\mathbb{R}^{n+k})$ of $T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ is defined by

$$\partial T(\alpha) := T(d\alpha).$$

$T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ is called a closed current iff $\partial T = 0$.

Rectifiable Set

$M \subset \mathbb{R}^{n+k}$ is called a countably n-rectifiable set, iff M is written in the form

$$M = M_0 \cup \left(\bigcup_{j=0}^{\infty} F_j(A_j)\right)$$

where $A_j \subset \mathbb{R}^n$, $F_j : A_j \to \mathbb{R}^{n+k}$ is a Lipschitz map for $j \geq 1$ and $\mathscr{H}^n(M_0) = 0$. ($\mathscr{H}^n$ denotes the n-dimensional Hausdorff measure.)

If M is countably n-rectifiable, we can define the approximate tangent space $T_x M$ for $\mathscr{H}^n$-almost every $x \in M$.

Integral Current

$T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ is called an integral current, iff T is written in the form

$$T(\alpha) = \int_M \langle \alpha(x), \xi(x) \rangle \Theta(x) d\mathscr{H}^n(x)$$

where M is a countably n-rectifiable set in $\mathbb{R}^{n+k}$ and $\Theta(x)$ an integer-valued $\mathscr{H}^n$-summable function in M. $\xi : M \mapsto \wedge(\mathbb{R}^{n+k})$ is a $\mathscr{H}^n$ measurable function such that for $\mathscr{H}^n$-almost every $x \in M$, ξ can be expressed as

$$\xi(x) = \tau_1 \wedge \cdots \wedge \tau_n$$

where $\{\tau_1, \cdot\cdot, \tau_n\}$ is an orthogonal basis of $T_x M$.

We need the isoperimetric inequality due to Federer-Fleming [F-F]. (The best constant, which is attained by the currents induced by spheres, is obtained by Almgren [Al]).

Theorem 2.1 (Federer-Fleming, Almgren). *If $T \in \mathscr{D}_n(\mathbb{R}^{n+k})$ is an integral current with $\partial T = 0$, then there exists an integral current $R \in \mathscr{D}_{n+1}(\mathbb{R}^{n+k})$ with $\partial R = T$ such that*

$$\|R\| \leq \gamma(n) \cdot \|T\|^{\frac{n+1}{n}}$$

where

$$\gamma(n) := \frac{1}{\alpha(n+1)^{\frac{1}{n}} \cdot (n+1)^{\frac{n+1}{n}}} = \frac{\Gamma(\frac{n+3}{2})}{((n+1)\pi^{\frac{1}{2}})^{\frac{n+1}{n}}}$$

$\alpha(n+1)$ denotes (n+1)-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{n+1}$. Moreover, if T is compactly supported, we can choose R with compact support.

We also need the following compactness theorem.(cf. [F]; 4.2.17 or [H-S]; Lecture 4 in Hardt's lecture.)

Theorem 2.2. *Suppose $\{T_j\} \in \mathscr{D}_n(\mathbb{R}^l)$, T_j and ∂T_j are integral currents for each j, and*

$$\sup_j \{\|T_j\| + \|\partial T_j\|\} < \infty.$$

Then a subsequence of $\{T_j\}$ converges weakly to an integral current T.

Lemma 2.3.

(1) For any $\alpha \in \mathscr{D}^2(\mathbb{R}^{\lessdot})$,

$$V_\alpha : H^{1,2}(\Sigma; N) \ni u \mapsto V(u)[\alpha] \in \mathbb{R}$$

is a continuous functional.
(2) For $u \in H^{1,2}(\Sigma; N)$,

$$V(u) : \mathscr{D}^2(\mathbb{R}^{\lessdot}) \ni \alpha \mapsto \int_\Sigma \approxeq^* \alpha \in \mathbb{R}$$

is a closed integral current with

$$\text{(2.2)} \qquad \|V(u)\| \leq \mathscr{A}(u) := \int_\Sigma \left(|\frac{\partial u}{\partial x^1}|^2 |\frac{\partial u}{\partial x^2}|^2 - \langle \frac{\partial u}{\partial x^1}, \frac{\partial u}{\partial x^2} \rangle^2 \right)^{\frac{1}{2}} dx^1 dx^2,$$

Proof. Take a sequence $\{u_\nu\}$ with $u_\nu \to u$ in $H^{1,2}(\Sigma; \mathbb{R}^{\lessdot})$. For any $\alpha \in \mathscr{D}^2(\mathbb{R}^{\lessdot})$, we have

$$\Big| \int_{\Sigma} u_\nu^* \alpha - \int_{\Sigma} u^* \alpha \Big| \leq \sum_k \Big\{ \Big| \int_{U_k} \{\alpha_{ij}(u) - \alpha_{ij}(u_\nu)\} \det(u^i_{x^1}, u^j_{x^2}) \Big|$$
$$+ \Big| \int_{U_k} \alpha_{ij}(u_\nu) \det(u^i_{x^1} - u^i_{\nu,x^1}, u^j_{x^2}) \Big| + \Big| \int_{U_k} \alpha_{ij}(u_\nu) \det(u^i_{\nu,x^1}, u^j_{x^2} - u^j_{\nu,x^2}) \Big| \Big\} \tag{2.3}$$

where U_k is a coordinate system covering Σ. The first term converges to 0 by Lebesgue's convergence theorem by choosing a suitable subsequence, the second and the third terms also converge to zero, since $|\nabla(u_\nu - u)|_{L^2} \to 0$. This implies assertion (1).

To prove (2), we first check the assertion for $u \in C^\infty(\Sigma, \mathbb{R}^k)$.

(a) By Stokes's Theorem, it is clear that $\partial V(u) = 0, \quad \partial V(u, v) = 0$.

(b) By the definition of mass of currents, it can be easily checked that $V(u)$ satisfies inequality (2.2).

(c) $V(u)$ is an integral current, since $V(u)$ can be written in the form

$$V(u)[\alpha] = \int_{u(\Sigma)} \langle \alpha(x), \xi(x) \rangle \Theta(x) d\mathscr{H}^2(x)$$

where

$$\begin{aligned} \Theta(x) \quad &= \{\text{number of } u^{-1}(x) \text{ with multiplicity}\} \\ &= \sum_{u(p)=x} sign(Du(p)). \end{aligned}$$

Θ is integer-valued for almost every $x \in u(\Sigma)$ by Sard's Theorem and Θ is summable, since we have by the area formula

$$\int_{u(\Sigma)} |\Theta| d\mathscr{H}^2 \leq \mathscr{A}(u).$$

Thus for $u \in C^\infty(\Sigma, \mathbb{R}^k)$, $V(u)[\cdot]$ is a closed integral current with (2.2). To establish the assertion for $u \in H^{1,2}(\Sigma, N)$, choose a sequence $\{u_\nu\} \in C^\infty(\Sigma; \mathbb{R}^k)$ with $u_\nu \to u$ in $H^{1,2}(\Sigma; \mathbb{R}^k)$. Then, by assertion (1), we have

$$V(u_\nu)[\alpha] \to V(u)[\alpha] \text{ for any } \alpha \in \mathscr{D}^2(\mathbb{R}^k).$$

Hence $V(u)$ is the weak limit of sequences of closed integral currents with (2.2). Thus, $V(u)$ is also a closed integral current by Theorem 2.2 and satisfies inequality (2.2). Q.E.D.

Proposition 2.4. *Fix a smooth 2-form ω on $\mathbb{R}^k$ with $|d\omega| < \infty$. The functional defined by*

$$V_\omega : H^{1,2}(\Sigma; N) \ni u \mapsto \int_{\Sigma} u^* \omega \in \mathbb{R}$$

is continuous and there holds

$$|V_\omega(u)| \le \gamma(2)\mathscr{A}(u)^{\frac{3}{2}} \cdot |d\omega|. \tag{2.4}$$

Proof. Since the support of current $V(u)$ is contained in compact manifold N, we may assume that ω is compactly supported to prove the continuity of $V_\omega(\cdot)$. Hence, the continuity is an immediate consequence of Lemma 2.3.

Since $V(u)$ is a compactly supported closed integral current by Lemme 2.3, Theorem 2.1 implies that there exists compactly supported current $R \in \mathscr{D}_3(\mathbb{R}^\ell)$ with $\partial R = V(u)$ and $\|R\| \le \gamma(2)\|V(u)\|^{\frac{3}{2}}$.

Since $V(u)$ and R are compactly supported, we can choose $\alpha \in D^2(\mathbb{R}^\ell)$ with $\omega = \alpha$ in some neighborhood of $suppV(u) \cup suppR$. Then we have

$$|V_\omega(u)| = |V(u)[\alpha]| = |R(u)[d\alpha]| \le \gamma(2)\mathscr{A}(u)^{\frac{3}{2}} \cdot |d\alpha|_{suppR(u)} \le \gamma(2)\mathscr{A}(u)^{\frac{3}{2}} \cdot |d\omega|.$$

This proves inequality (2.4). Q.E.D.

3. Convergence of extremals

In this section, we shall obtain estimates for solutions of equation of type (1.2):

$$\Delta_\Sigma u = \Gamma(u)(\nabla u, \nabla u),$$

With respect to any isothermal coordinate (x_1, x_2), equation (1.2) is expressed as:

$$\Delta_0 u = \Gamma(u)(\nabla u, \nabla u) \tag{3.1}$$

where $\Delta_0 := (\frac{\partial}{\partial x^1})^2 + (\frac{\partial}{\partial x^2})^2$. Namely equation (1.2) is conformally invariant. From this observation, we have the following important fact: in order to obtain a local estimate for equation (1.2), we can assume that the domain is a domain in $\mathbb{R}^2$ with the flat metric by passing to an isothermal coordinate.

First, we define the homothetical transformation which is needed to observe the asymptotic behavior.

Let (U, ψ, V) be an isothermal coordinate system on Σ. Namely, $U \subset \Sigma$, $V \subset \mathbb{C}$ and $\psi : U \to V$ is biholomorphic. We define the homothetic transformation with the center $x \in U$ and factor $r > 0$ by

$$h_{x,r} : V_{x,r} \ni \xi \mapsto \psi^{-1}(\psi(x) + r\xi) \in U$$

where $V_{x,r} := \{\xi \in \mathbb{C}; \psi(x) + r\xi \in V\}$. This definition of $h_{x,r}$ depends on the local coordinate. But it does not matter for our purpose. Actually, we only consider h_{x_i,r_i} for the sequence with $x_i \to x$ and $r_i \to 0$. In this case, h_{x_i,r_i} is to be understood as the homothetical transformation defined for a *fixed* isothermal coordinate system which contains x. Note that if u satisfies equation (3.1) w.r.t. an isothermal coordinate, $u \circ h_{x,r}$ satisfies the same equation.

We shall start with the fundamental properties of solutions of (3.1) on $\mathbb{R}^2$.

Lemma 3.1. *Let $u \in C^{2,\alpha}(\mathbb{R}^2, \mathbb{R}^l)$ be a solution of (3.1) with finite Dirichlet integral. Then, by the stereographic projection, u is identified with a map $\overline{u} \in C^{2,\alpha}(S^2, \mathbb{R}^l)$ which satisfies (1.2).*

Proof. By stereographic projection, u is identified with the map $\overline{u}$ of $S^2 = \mathbb{R}^2 \cup \{\infty\}$ which is smooth except for the point $x = \infty$. And $\overline{u}$ satisfies equation (1.2) except for $x = \infty$. Then the removability of isolated singularities (cf. [SaU] or [J];Theorem 2.4.1) implies $\overline{u} \in C^2(S^2; \mathbb{R}^l)$ and $\overline{u}$ satisfies equation (1.2). Q.E.D.

We also give some notational conventions.

(1) c (small letter) denotes an absolute constant or a constant which depends on the choice of an isothermal coordinate of Σ and C (capital letter) denotes the constant depends on N and the two form ω on N. We specify what C depends on, if necessary. (e.g. $C(\Gamma)$)

(2) We denote balls in a suitable isothermal coordinate by $B_r(p)$. Namely, $B_r(p) = \psi^{-1}(\{\xi; |\xi - \psi(p)| < r\})$ for an isothermal coordinate (U, ψ, V). We also use the same notation $B_r(p)$ for geodesic balls in N. To denote the geodesic ball in Σ, we use the notation $B(x, r)$. When we specify where the ball is contained, we use the notation $B_r^{\Sigma}(p), B_r^N(p)$. In any case, we always assume that r is sufficiently small so that the coordinate is defined.

Lemma 3.2. *Let Σ be a closed Riemann surface. Suppose $\{u_i\} \subset H^{1,2}(\Sigma; \mathbb{R}^l)$ be a sequence with $\sup_i D(u_i; \Sigma) \leq M$. Then for any $\delta > 0$, we can choose a subsequence $\{u_{i_\mu}\}$ such that there is a finite set $\Lambda = \{x_1, \ldots, x_N\} \subset \Sigma$ with the following*

Property $(\sharp)_\delta$:

(i) There holds

$$\liminf_{\mu\to\infty} D(u_{i_\mu}; B(x_m, r)) > \delta$$

for any $r > 0, 1 \leq m \leq N$,

(ii) For any $x \in \Sigma \setminus \Lambda$, there exists $r > 0$ with $\limsup_{\mu\to\infty} D(u_{i_\mu}; B(x, r)) \leq \delta$.

Proof. For $\rho_\nu \downarrow 0$, we can choose a family of balls $\{B(x_k^\nu, \rho_\nu)\}_{k=1,\ldots,p_\nu}$ with $\Sigma \subset \overset{p_\nu}{\underset{k=1}{\to}} \bigcup$. Then by diagonal argument, we can choose a subsequence, denoted also by $\{u_i\}$, such that for any k, ν, there exists $\lim_{i\to\infty} D(u_i; B(x_k^\nu, \frac{\rho_\nu}{2}))$.

We put

$$\Lambda := \left\{x \in \Sigma; \lim_{i\to\infty} D(u_i; B(x_k^\nu, \frac{\rho_\nu}{2}) > \delta \text{ for any } \nu, k \text{ with } x \in B(x_k^\nu, \tfrac{\rho_\nu}{2})\right\}$$

Let $\{z_1, \ldots, z_N\}$ be a finite subset of Λ. Choosing ρ_ν sufficiently small, $B(x_k^\nu, \rho_\nu) \cap B(x_l^\nu, \rho_\nu) = \phi$ for $m \neq n$. Thus we have

$$\delta N \leq \sum_{m=1}^{N} D(u_i; B(z_m, \rho) \leq D(u; \Sigma) \leq M.$$

for sufficiently large i. Hence

$$N \leq \frac{M}{\delta},$$

i.e. Λ is a finite set. By our definition, it is easy to check that Λ satisfies the desired properties. Q.E.D.

Lemma 3.3. *For any $u \in H^{1,2}_{loc}(\mathbb{R}^2)$ and any $\varphi \in C_0^\infty(B_R(x))$ with $0 \le \varphi \le 1$ and $|\nabla\varphi| \le \frac{4}{R}$, there holds*

$$\int_{\mathbb{R}^2} |u|^4\varphi^2 dx \le c\left(\int_{B_R(x)} |u|^2 dx\right)\left(\int_{B_R(x)} |\nabla u|^2\varphi^2 dx + R^{-2}\int_{B_R(x)} |u|^2 dx\right)$$

Proof. See [Str3];Lemma 5.7. Q.E.D.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^2$. Suppose $u \in C^2(\Sigma; N)$ satisfies (3.1). Then, there exists $\varepsilon_0(|\Gamma|) > 0$ such that if $D(u; B_{2r}(x_0)) \le \varepsilon_0$ for some $0 < r$, then we have*

$$\int_{B_{2r}(x_0)} |\nabla^2 u|^2 \cdot \varphi^2 dx \le \frac{C}{r^2}\left\{\int_{B_{2r}(x_0)} |\nabla u|^2 dx\right\}^2$$

where $\varphi \in C_0^\infty(B_{2r}(x_0))$ satisfies $0 \le \varphi \le 1$ and $|\nabla\varphi| \le \frac{2}{r}$.

Proof. Since u satisfies (3.1), there holds

$$|\Delta u| \le |\Gamma| \cdot |\nabla u|^2.$$

Hence we have

$$\int_{B_{2r}(x_0)} |\Delta u|^2 \cdot \varphi^2 dx \le |\Gamma|^2 \int_{B_{2r}(x_0)} |\nabla u|^4 \varphi^2 dx. \tag{3.2}$$

By Lemma 3.3 and (3.2), we have

$$\int_{B_{2r}(x_0)} |\Delta u|^2 \cdot \varphi^2 dx \le C|\Gamma|^2 \left\{\int_{B_{2r}(x_0)} |\nabla u|^2 dx\right\} \times \left\{\int_{B_{2r}(x_0)} |\nabla^2 u|^2 \varphi^2 dx + \frac{1}{r^2}\int_{B_{2r}(x_0)} |\nabla u|^2 dx\right\}. \tag{3.3}$$

On the other hand, by integrating by parts twice and using binomial inequality, we have

$$\int_{B_{2r}(x_0)} |\Delta u|^2 \varphi^2 dx \ge \frac{1}{2}\int_{B_{2r}(x_0)} |\nabla^2 u|^2 \varphi^2 dx - \frac{C}{r^2}\int_{B_{2r}(x_0)} |\nabla u|^2 dx. \tag{3.4}$$

Combining (3.3) and (3.4) and putting $\varepsilon_0 = \frac{1}{8C|\Gamma|^2}$, we have the desired estimate by absorbing the right hand side to the left hand side. Q.E.D.

Lemma 3.5. *Let $\Omega \subset \Sigma$ and $u \in C^2(\Omega)$ a solution of (3.1). If there exists $r > 0$ such that*

$$\sup_{x\in\Sigma} D(u;B_r(x)) < \frac{\varepsilon_0}{2}$$

then there exists a constant C depending on r and $\Omega' \Subset \Omega$ such that there holds

$$\|\nabla^2 u\|_{C^\alpha(\Omega')} + \|\nabla u\|_{C^\alpha(\Omega')} < C.$$

Proof. Lemma 3.4 implies the $H^{2,2}$-bound;

$$\int_{\Omega'} |\nabla^2 u|^2 dx < C(r,\Omega').$$

Hence, by Sobolev's embedding theorem, we have

$$\int_{\Omega'} |\nabla u|^p dx < C(r,\Omega')$$

for any $1 \le p < \infty$. Since u satisfies equation (1.2), usual linear elliptic theory implies

$$\|\nabla^2 u\|_{L^p(\Omega')} + \|\nabla u\|_{L^p(\Omega')} < C(r,\Omega').$$

(Note that, we do not have the bound for $\|u\|_{L^p}$ in general. But, of course, since N is compact, $\|u\|_{L^p}$ is bounded by terms of N.) Then, again by Sobolev's embedding theorem, we have the bound for $\|\nabla u\|_{C^\alpha(\Omega')}$. Finally, using the interior Schauder's estimate, we obtain the desired result. Q.E.D.

Theorem 3.6. *Suppose $\{u_i\} \subset C^2(\Sigma;N)$ satisfies (1.2) and $\sup D(u_i;\Sigma) \le M$. Then, there exist a finite set (possibly an empty set) $\Lambda := \{x_1,\dots,x_N\}$ of points in Σ and $u_0 \in C^{2,\alpha}(\Sigma;N)$, $v_1,\dots,v_N \in C^{2,\alpha}(\mathbb{R}^2;N)$ satisfying the following conditions: (taking a suitable subsequence if necessary,)*

(a) $D(u_0;\Sigma) + \sum_{m=1}^{N} D(v_m;\mathbb{R}^2) < \infty$. *$u_0$ satisfies equation (1.2). $v_1,\cdot\cdot,v_N$ are non-trivial solutions of equation (1.2). $v_1,\cdot\cdot,v_N$ can be identified with maps $\overline{v_1},\dots,\overline{v_N}$ of S^2 by stereographic projection and $\overline{v_1},\dots,\overline{v_N}$ is a smooth solution of equation (1.2) in S^2,*

(b) $u_i \to u_0$ *in* $C^{2,\alpha}(\Sigma\setminus\Lambda;N)$,

(c) There exists a sequence $x_m^i \in \Sigma$, $r_m^i > 0$ (m=1,...,N) with $x_m^i \to x_m$, $r_m^i \to 0$ such that

$$v_m^i \longrightarrow v_m \quad \textit{locally in } C^{2,\alpha}(\mathbb{R}^2;N)$$

where

$$v_m^i(\xi) := u_i(h_{x_i,r_i}(\xi)),$$

(d) $D(u_0) + \sum_{m=1}^{N} D(v_m) \le \liminf_{i\to\infty} D(u_i)$.

Proof. Applying Lemma 3.2, we can choose a subsequence, which is again denoted by $\{u_i\}$*, and find a finite set* $\Lambda = \{x_1, \cdot\cdot, x_N\}$ *with property* $(\sharp)_{\frac{\varepsilon_0}{2}}$.

1° *Convergence at regular points*

Choosing an fixed isothermal coordinate system, we can apply Lemma 3.5 to u_i. Hence, u_i is uniformly bounded in $C^{2,\alpha}(K)$ for any $K \Subset \Sigma \setminus \Lambda$. Thus, there exists $u_0 \in C^{2,\alpha}(\Sigma \setminus \Lambda)$

$$u_i \longrightarrow u_0 \quad \text{in } C^{2,\alpha}(K)$$

for any $K \Subset \Sigma \setminus \Lambda$. Since the Dirichlet integral of u_i is uniformly bounded, Dirichlet integral of u_i is also bounded. Hence, the removability of isolated singularity (See [SaU] or [J]; Theorem 2.4.1) implies $u_0 \in C^{2,\alpha}(\Sigma)$ and u_0 satisfies equation (1.2) in Σ.

2°*Singularity*

We choose $\rho > 0$ so that $B(x_m, \rho) \cap B(x_{m'}, \rho) = \phi$ if $m \neq m'$ for $1 \leq m, m' \leq N$. Fixing an isothermal coordinate neighbourhood of x_m, we set

$$\begin{aligned} r_m^i \;:=\; & \inf\left\{ r > 0; \text{there exists x} \in B\left(x_m, \tfrac{\rho}{2}\right) \right. \\ & \left. \text{with } \tfrac{\varepsilon_0}{4} \leq D(u_i; B_r(x)) \right\}. \end{aligned}$$

Let x_m^i be a point which attains the infimum above. The definition of Λ and the convergence property on $\Sigma \setminus \Lambda$ proved above implies

$$\begin{aligned} & D(u_i; B_{r_m^i}(x_m^i)) = \frac{\varepsilon_0}{4}, \\ & x_m^i \to x_m \quad r_m^i \to 0. \end{aligned}$$

We define the rescaled map by $v_m^i := u_i(h_{x_m^i, r_m^i}(\xi))$ which satisfies equation (3.1). And there holds

$$D(v_m^i; B_1(z)) \leq \frac{\varepsilon_0}{2}$$

for a ball $B_1(z)$ in the isothermal coordinate which contained in the rescaled domain. Note that the rescaled domain exhausts $\mathbb{R}^2$ as $i \to \infty$. Then, applying Lemma 3.5, we have

$$v_m^i \longrightarrow v_m \quad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}^2).$$

Consequently v_m is a non-trivial solution of equation (3.1) in $\mathbb{R}^2$.

To obtain the bounds for the Dirichlet integral of $u_0, v_1.., v_N$, we take cut-off functions $\varphi_r^m \in C_0^\infty(B_{2r}(x_m))$ with $0 \leq \varphi_r \leq 1$ and $\varphi_r \equiv 1$ in $B_r(x_m)$ for $1 \leq m \leq N$. By the invariance of Dirichlet integral with respect to the scaling, we have

$$\lim_{i\to\infty} \frac{1}{2} \int |\nabla u_i|^2 \varphi_r^m dx \geq \lim_{i\to\infty} D(u_i; B_{Rr_m^i}(x_m^i)) \geq D(v_m; B_R(0)).$$

On the other hand, since u_i converges in $C^{2,\alpha}(\Sigma \setminus \Lambda)$, we have

$$\lim_{i\to\infty} \frac{1}{2} \int |\nabla u_i|^2 (1 - \varphi_r^1) \cdots (1 - \varphi_r^N) dx \geq D(u_0; \Sigma \setminus \bigcup_{m=1}^N B_r(x_m)).$$

Thus, by tending $r \to 0, R \to \infty$, we obtain

$$D(u_0) + \sum_{m=1}^{N} D(v_m) \leq \lim_{i \to \infty} D(u_i).$$

Since the Dirichlet integral of v_m is bounded, we can apply Lemma 3.1. Hence $v_1, .., v_N$ can be identified with maps $\overline{v}_1, .., \overline{v}_N$ of S^2 by stereographic projection and $\overline{v}_1, ..\overline{v}_N$ satisfies the equation (1,2). This proves the Theorem. Q.E.D.

4. Evolution of H-surfaces

We consider the evolution problem of equation (1.2). Namely, we consider the following equation;

$$\partial_t u = \Delta u - \Gamma(u)(\nabla u, \nabla u) \tag{4.1}$$

$$u(\cdot, 0) = u_0. \tag{4.2}$$

for given $u_0 \in H^{1,2}(\Sigma; N)$. As in the previous section, we treat the equation locally. Namely we consider the equation in isothermal coordinates. Equation (4.1) is written with respect to an isothermal coordinate (x^1, x^2) as follows.

$$\partial_t u = \sigma^{-2}(\Delta_0 u + \Gamma(u)(\nabla u, \nabla u)) \tag{4.3}$$

where σ is the conformal factor of the isothermal coordinate. Namely the metric tensor g of Σ is expressed with respect to the isothermal coordinate as $g = \sigma^2((dx^1)^2 + (dx^2)^2)$.

For $\Omega \subset \Sigma$, set

$$X(\Omega \times [0, T]) : = \left\{u \in L^{2,2}(\Omega \times [0, T]); [0, T] \ni t \mapsto u(\cdot, t) \in H^{1,2} \quad \text{is continuous.}\right\}.$$

We need a parabolic version of Lemma 3.2.

Lemma 4.1. *Suppose $\zeta \in C_0^\infty(B_r(w_0))$ depend only on the distance $d(w, w_0)$ from w_0 and suppose ζ is non-increasing w.r.t.$d(w, w_0)$. Then, there exists constants C and $\overline{r}$ such that for any $T < \infty$, $r < \overline{r}$ and any $f \in L^{1,2}(B(w_0, r) \times [0, T]) \cap C^0([0, T], L^2(B(w_0, r))$, there holds*

$$\iint_{B(w_0,r)\times[0,T]} |\nabla f|^4 \zeta^2 dxdt \leq C \cdot \sup_{0<t<T} D(f_t; B_r(w_0)) \times \left\{ \iint_{B(w_0,r)\times[0,T]} |\nabla^2 f|^2 \zeta^2 dxdt + r^{-2} \iint_{B_r(w_0)\times[0,T]} |\nabla f|^2 \zeta^2 dxdt \right\}.$$

Proof. See [Str1];Lemma 3.2 for the proof.

To obtain the L^2 bound for $\nabla^2 u$, we begin with some computations.

Let $\zeta_r \in C_0^\infty(B(w, r))$ satisfy the conditions of Lemma 4.1 and

$$0 \le \zeta_{r,w} \le 1, \quad \zeta_{r,w} \equiv 1 \quad in B(w, \tfrac{1}{2}r),$$
$$|\nabla \zeta_{r,w}| < \frac{4}{r}.$$

In the sequel, we always assume $r < \bar{r}$ and we simply write ζ for $\zeta_{r,w}$, if there is no danger of confusion.

Lemma 4.2. *Suppose $u \in X(B(w,r) \times [0,T])$ satisfies (4.1)-(4.2). Then there holds*

$$\iint_{B(w,r)\times[0,T]} |\Delta u|^2 \zeta^2 dxdt + C\left\{\int |\nabla u_T|^2 \zeta^2 dx - \int |\nabla u_0|^2 \zeta^2 dx\right\}$$
$$\le C(\Gamma)\Big\{\iint_{B(w,r)\times[0,T]} |\nabla u|^4 \zeta^2 dxdt + \frac{T}{r^2} \sup_{0\le t\le T} \int_{B(w,r)} |\nabla u|^2 dx\Big\}$$

where $u_t(x) := u(x,t)$.

Proof. Integrating by parts, we have,

$$\text{(4.4)} \quad \frac{1}{2}\frac{\partial}{\partial t}\int |\nabla u|^2 \zeta_{r,w}^2 dx = \int_\Sigma \langle \nabla u, \nabla \partial_t u\rangle \zeta^2 dx$$
$$= -\int \langle \partial_t u, \Delta u\rangle \zeta^2 dx - 2\int \langle \partial_t u, \nabla u\rangle \nabla\zeta\zeta dx$$
$$= -\int |\partial_t u|^2 \zeta^2 dx + \int \langle \partial_t u, \Gamma(u)(\nabla u, \nabla u)\rangle \zeta^2 dx - 2\int \langle \partial_t u, \nabla u\rangle \nabla\zeta\zeta dx$$
$$\le -\frac{1}{2}\int |\partial_t u|^2 \zeta^2 dx + C(\Gamma)\int |\nabla u|^4 \zeta^2 dx + c\int |\nabla u|^2 |\nabla\zeta|^2 dx$$

where we used the binomial inequality to obtain the last inequality.
Integrating the inequality above w.r.t. t, we have

$$\text{(4.5)} \quad \frac{1}{2}\left\{\int |\nabla u_T|^2 \zeta^2 dx - \int |\nabla u_0|^2 \zeta^2 dx\right\}$$
$$+\frac{1}{2}\iint_{B(w_0,r)\times[0,T]} |\partial_t u|^2 \zeta^2 dxdt \le C(\Gamma)\Big\{\iint_{B(w_0,r)\times[0,T]} |\nabla u|^4 \zeta^2 dxdt$$
$$+\frac{T}{r^2}\sup_{0\le t\le T}\int_{B_r(w_0)} |\nabla u|^2 dx\Big\}.$$

Since u satisfies equation (4.1),

$$\text{(4.6)} \quad \iint |\Delta u|^2 \zeta_r^2 dxdt$$
$$\le C(\Gamma)\iint_{B(w,r)\times[0,T]} |\nabla u|^4 \zeta^2 dxdt + c\iint_{B(w,r)\times[0,T]} |\partial_t u|^2 \zeta^2 dxdt.$$

Hence, by (4.5) and (4.6), we obtain,

$$\iint_{B(w,r)\times[0,T]} |\Delta u|^2\zeta^2 dxdt + C\left\{\int |\nabla u_T|^2\zeta^2 dx - \int |\nabla u_0|^2\zeta^2 dx\right\}$$
$$\leq C(\Gamma)\Big\{\iint_{B(w,r)\times[0,T]} |\nabla u|^4\zeta^2 dxdt + \frac{T}{r^2}\sup_{0\leq t\leq T}\int_{B_r(w)} |\nabla u|^2 dx\Big\}$$

Q.E.D.

Next, for $u \in X(\Omega \times [0,T])$, we set

$$\varepsilon(u,r,T;\Omega) := \sup_{0\leq t\leq T, B(w,r)\subset\Omega}\int_{B(w,r)} |\nabla u|^2 dx$$

for $\Omega \subset \Sigma$. If there is no danger of confusion, we simply write $\varepsilon(r,T)$ for $\varepsilon(u,r,T;\Omega)$.

Lemma 4.3. *Suppose $u \in X(B(w_0,R)\times[0,T])$ satisfies (4.1)-(4.2). Then, there exists $\varepsilon(\Gamma) > 0$ such that if $\varepsilon(r,T;B(w_0,R)) < \varepsilon$, there holds,*

$$\iint_{B(w,r)\times[0,T]} |\nabla^2 u|^2\zeta^2 dxdt + \frac{1}{2}\left\{\int |\nabla u_T|^2\zeta^2 dx - \int |\nabla u_0|^2\zeta^2 dx\right\}$$
$$\leq \frac{CT}{r^2}\varepsilon(u,r,T;B_r(w)).$$

where $\zeta = \zeta_{r,w}$.

Proof. Lemma 4.1 and Lemma 4.2 imply,

$$\iint_{B(w,r)\times[0,T]} |\Delta u|^2\zeta^2 dxdt + C\left\{\int |\nabla u_T|^2\zeta^2 dx - \int |\nabla u_0|^2\zeta^2 dx\right\}$$
$$\leq C(\Gamma)\varepsilon(r,T)\Big\{\iint_{B(w,r)\times[0,T]} |\nabla^2 u|^2\zeta^2 dxdt + \frac{T}{r^2}\Big\} \tag{4.7}$$

On the other hand, integrating by parts twice, we have,

$$\begin{aligned}&\iint_{B(w,r)\times[0,T]} |\nabla^2 u|^2\zeta^2 dxdt \\ &\leq 2\iint_{B(w,r)\times[0,T]} |\Delta u|^2\zeta^2 dxdt + \iint_{B(w,r)\times[0,T]} |\nabla u|^2|\nabla\zeta|^2 dxdt \\ &\quad + C\iint_{B(w,r)\times[0,T]} |\nabla u|^2\zeta^2 dxdt \leq 2\iint_{B(w,r)\times[0,T]} |\Delta u|^2\zeta^2 dxdt \\ &\quad + \frac{cT}{r^2}\varepsilon(r,T).\end{aligned} \tag{4.8}$$

where the last term in the middle comes from the curvature term. From (4.7) and (4.8), we obtain,

$$\iint_{B(w,r)\times[0,T]} |\nabla^2 u|^2\zeta^2 dxdt + C\left\{\int |\nabla u_T|^2\zeta^2 dx - \int |\nabla u_0|^2\zeta^2 dx\right\}$$
$$\leq C(\Gamma)\varepsilon(r,T)\Big\{\iint_{B(w,r)\times[0,T]} |\nabla^2 u|^2\zeta^2 dxdt + \frac{T}{r^2}\Big\}$$

Choosing $\varepsilon := \frac{1}{2C}$ and absorbing the first term on the right hand side to the left, we obtain the desired result. Q.E.D.

Lemma 4.4. *Suppose $u \in X(\Sigma \times [0,T])$ satisfies (4.1)–(4.2). Then there exists a constant $C = C(\Gamma)$ and $\bar{\varepsilon} = \bar{\varepsilon}(\Gamma) < \frac{\varepsilon}{4}$ such that if*

$$\sup_{w\in\Sigma} \int_\Sigma |\nabla u_0|^2 \zeta_{r,\omega}^2 dx < \bar{\varepsilon}$$

for some $0 < r < \bar{r}$, then there holds

$$\sup_{w\in\Sigma, 0\le t\le\tau} \int |\nabla u_t|^2 \zeta_{r,w}^2 dx \le 2\bar{\varepsilon} \ ,$$

where $\tau = \min(T, Cr^2)$.

Proof. Let L be the minimal number such that for any $0 < r < \bar{r}$ and any $x \in \Sigma$, geodesic ball $B(x,r)$ can be covered by L balls with radius $\frac{r}{2}$. Set $\bar{\varepsilon} := \frac{\varepsilon}{2L}$.

We set

$$\tau := \max\left\{ t_0 \in [0,T]; \sup_{w\in\Sigma, 0\le t\le t_0} \int |\nabla u_t|^2 \zeta_{r,w}^2 dx \le 2\bar{\varepsilon} \right\} .$$

Since $t \mapsto u_t \in H^{1,2}$ is continuous, $\tau > 0$. If $\tau \ne T$, choose $w_0 \in \Sigma$ so that

$$\int |\nabla u_\tau|^2 \zeta_{r,w_0}^2 dx = 2\bar{\varepsilon} \ .$$

Since we can find $x_i \in \Sigma (i = 1, \ldots, L)$ with $B_r(w_0) \subset \bigcup_{i=1}^L B_{\frac{r}{2}}(x_i)$ by the definition of L, we have

$$\sup_{0\le t\le\tau} \int_{B(w_0,r)} |\nabla u_t|^2 dx \le \sup_{0\le t\le\tau} \sum_{i=1}^L \int |\nabla u_t|^2 \zeta_{r,x_i}^2 dx \le 2\bar{\varepsilon} L = \varepsilon \ .$$

Thus, we can apply Lemma 4.3 for $R = r$, $w = w_0$. Then we obtain

$$\bar{\varepsilon} \le \int |\nabla u_\tau|^2 \zeta_{r,w_0}^2 dx - \int |\nabla u_\tau|^2 \zeta_{r,w_0}^2 dx \le C \cdot \frac{\tau}{r^2} \cdot \varepsilon \ .$$

This implies $\tau \ge Cr^2$. Q.E.D.

Lemma 4.5. *Suppose $u \in X(B(w_0, R) \times [0,T])$ satisfies (4.1)-(4.2). Then there holds,*

$$\begin{aligned} &\frac{1}{2} \left| \int |\nabla u_T|^2 \zeta_{r,w}^2 dx - \int |\nabla u_0|^2 \zeta_{r,w}^2 dx \right| \\ &\quad \le \iint_{B_r(w)\times[0,T]} |\nabla^2 u|^2 \zeta^2 dx dt + \frac{CT}{r^2} \varepsilon(r, T; B_R(w_0)) \end{aligned}$$

Proof. It follows from (4.4),

$$\frac{1}{2}\left|\frac{\partial}{\partial t}\int|\nabla u|^2\zeta^2dx\right|$$
$$\leq c\int|\partial_t u|^2\zeta^2dx + C\int|\nabla u|^4\zeta^2dx + c\int|\nabla u|^2|\nabla\zeta|^2dx.$$

Since $|\partial_t u|^2 \leq |\Delta u|^2 + C|\nabla u|^4$,

$$\frac{1}{2}\left|\frac{\partial}{\partial t}\int|\nabla u|^2\zeta^2dx\right|$$
$$\leq c\int|\nabla^2 u|^2\zeta^2dx + C\int|\nabla u|^4\zeta^2dx + c\int|\nabla u|^2|\nabla\zeta|^2dx.$$

Integrating the equation above w.r.t.t, we have,

$$\frac{1}{2}\left|\int|\nabla u_T|^2\zeta^2_{r,w}dx - \int|\nabla u_0|^2\zeta^2_{r,w}dx\right|$$
$$\leq c\iint_{B_r(w)\times[0,T]}|\nabla^2 u|^2\zeta^2dxdt + C\iint_{B_r(w)\times[0,T]}|\nabla u|^4\zeta^2dxdt + \frac{CT}{r^2}\varepsilon(r,T;B_R).$$

Estimating 2nd term on the left hand side by Lemma 4.1, we obtain the desired result. Q.E.D.

In the following Lemma, we work in an isothermal coordinate and obtain the estimate depending on conformal factor σ associated to the coordinate.

Lemma 4.6. *Suppose $u \in X(B_R(w_0)\times[0,t])$ satisfies (4.3) for some $t \leq T$. Set $R_0 := \sup\left\{r > 0; \varepsilon(r,t;B_R(w_0)) < \frac{\bar{\varepsilon}}{2}\right\}$.*

Then, $\partial_t u, \nabla u, u \in L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,t])$ for any $1 < p < \infty, \tau > 0$ and there exists a constant C which depends on $\tau, T, R_0, \sup_{B_R(w_0)}|\sigma|, \inf_{B_R(w_0)}|\sigma|$ and $\sup_{B_R(w_0)}|\nabla\sigma|$ such that

$$\|u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,t])} + \|\partial_t u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,t])} + \|\nabla u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,t])} \leq C$$

where all the norms are taken with respect to the isothermal coordinate.

Proof. From the estimate similar to the one used in Lemma 3.12 in [Str 2], we obtain

$$\int_{B_{\frac{3R}{4}}(w_0)}|\nabla^2 u|^2dx < C$$

for any $t \in [\tau, T]$. This implies $\|\nabla u(\cdot,t)\|_{L^p(B_{\frac{3R}{4}}(w_0))} < C$ for any $1 < p < \infty$ by Sobolev's embedding theorem.

Differentiating equation (4.3) w.r.t. t and x, we obtain

$$(\partial_t - \sigma^{-2}\Delta)\nabla u = D_u\Gamma(\nabla u,\nabla u)\cdot\nabla u + \Gamma(\nabla^2 u,\nabla u) + \Gamma(\nabla u,\nabla^2 u)$$
$$+D_x\Gamma(\nabla u,\nabla u) + \nabla\sigma^{-2}\Delta u, \tag{4.9}$$

$$(\partial_t - \sigma^{-2}\Delta)\partial_t u = D_u\Gamma(\nabla u,\nabla u)\cdot\partial_t u + \Gamma(\nabla\partial_t u,\nabla u) + \Gamma(\nabla u,\nabla\partial_t u). \tag{4.10}$$

Applying linear parabolic theory to equation (4.9), L^p-bounds for $\nabla u, \nabla^2 u$ implies the $L^{2,p}$-bound for ∇u. Especially, this gives the L^p-bounds for $\nabla \partial_t u$. Then applying the linear parabolic theory for equation (4.10), we obtain $L^{2,p}$-bounds for $\partial_t u$. Thus we obtain the desired result. Q.E.D

Lemma 4.7. *Suppose $u \in X(\Sigma \times (0,T))$ satisfies (4.1)-(4.2) and $CR^2 > T$ where C is the constant in Lemma 4.4. If there exists $R_0 > 0$ with*

$$\sup_{x \in \Sigma} D(u_0; B_{R_0}(x)) < \bar{\varepsilon}, \tag{4.11}$$

u extends to solution $\bar{u} \in S^{2,\alpha}(\Sigma \times (0,T])$.

Proof. By (4.11), we can apply Lemma 4.4. Then, we have $\varepsilon(u, \frac{R_0}{2}, T; \Sigma) < 2\bar{\varepsilon}$. Then applying Lemma 4.6, we obtain the following estimate.

$$\|u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,t])} + \|\partial_t u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,t])} + \|\nabla u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,t])} \leq C(R_0, \tau, T, \sigma)$$

for any $0 < \tau < t < T$. Since the constant is independent of t, we obtain

$$\|u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,T])} + \|\partial_t u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,T])} + \|\nabla u\|_{L^{2,p}(B_{\frac{R}{2}}(w_0)\times[\tau,T])} \leq C.$$

This implies that C^α-norm of $\nabla u, \nabla^2 u$ and $\partial_t u$ is uniformly bounded on $\Sigma \times [\tau, T)$. Thus we obtain the desired result. Q.E.D.

So far, we do not need special structure of non-linear term. But in the following Lemma, we need the assumption;

$$\Gamma(u)(\nabla u, \nabla u) = D^2\pi^i(u)(\nabla u, \nabla u) + 2D\pi \cdot H(u)(\nabla u \wedge \nabla u). \tag{4.12}$$

Lemma 4.8. *Suppose $u \in X(\Sigma \times [0,T])$ satisfies (4,1)-(4,2) and Γ satisfies (4.12). Then, $D(u(\cdot,t))$ and $I_\omega(u(\cdot,t))$ are absolute continuous in $t \in [0,T]$ and there holds*

$$-\iint_\Sigma |\partial_t u|^2 dxdt = I_\omega(u_T) - I_\omega(u_0), \tag{4.13}$$

where $u_t(x) := u(x,t)$.

Proof. Differentiating and integrating by parts, (noting that $\partial_t u = 0$ on $\partial\Sigma$.)

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|_{t=\tau} I_\omega(u_t) \\
&= \int_\Sigma \langle \nabla u_\tau, \nabla \partial_t u_\tau \rangle dx + \frac{\partial}{\partial t}\Big|_{t=\tau} \int_\Sigma \frac{1}{2} b_{ij}(u_\tau)\det(\nabla u_\tau^i, \nabla u_\tau^j) dx \\
&= \int_\Sigma \{\langle \nabla u_\tau, \nabla \partial_t u_\tau \rangle - \langle \Gamma(u_\tau)(\nabla u_\tau, \nabla u_\tau), \partial_t u_\tau \rangle\} dx \\
&= -\int_\Sigma |\partial_t u_\tau|^2 dx \in L^1([0,T]),
\end{aligned} \tag{4.14}$$

$$\frac{\partial}{\partial t}\Big|_{t=\tau} D(u_t)$$

(4.15) $$= \int_\Sigma \langle \nabla u_\tau, \nabla \partial_t u_\tau \rangle dx = -\int_\Sigma \langle \Delta u_\tau, \partial_t u_\tau \rangle dx \in L^1([0,T]).$$

Integrating (4.14) w.r.t. t, we obtain (4.13). Q.E.D.

Theorem 4.9.
(I) For any $u_0 \in H^{1,2}(\Sigma; N)$, there exists a solution u of (4.1)-(4.2) with the following properties:

(a) $u \in S^{2,\alpha}(\Sigma \times (0,\tau]) \cap X(\Sigma \times [0,\tau])$ for some $\tau > 0$,

(b) Maximal existence time $T > 0$ of solution u with property (a) above is characterized by the following property, if it is finite.

There exists $\bar{x} \in \Sigma$ such that

(4.16) $$\limsup_{t \to T} D(u(\cdot,t); B_R(\bar{x})) > \bar{\varepsilon}$$

for any $R > 0$.

(II) Let u_t be a solution in (I). If u_t satisfies $\sup_{t\in[0,T)} D(u_t;\Sigma) := D_0 < \infty$ and $\int\int_\Sigma |\partial_t u|^2 dxdt < \infty$ and suppose (4.16) holds for $0 < T \le \infty$, then we have the following asymptotic behavior:

There exists $r_i > 0, x_i \in \Sigma, t_i \to T$ such that

(c) $r_i \longrightarrow 0, \quad x_i \longrightarrow \bar{x}$, where $\bar{x} \in \Sigma$ is a point which satisfies (4.16),

(d) The rescaled map $v^i(\xi) = u(h_{x_i,r_i}(\xi), t_i)$ has the following convergence property

$$v^i(\xi) \longrightarrow v(\xi) \quad \textit{in } C^{2,\alpha}_{loc}(\mathbb{R}^{\not\models})$$

v is a solution of (1.1) with $D(v;\mathbb{R}^{\not\models}) < \infty$.

(III) If $T = \infty$ and there exists no point which satisfies (4.16), then we have

(e) There exists a time sequence t_i with $t_i \to \infty$ such that

$$u(\cdot,t_i) \longrightarrow u_\infty \quad \textit{in } C^{2,\alpha}(\Sigma)$$

and u_∞ is an extremal of I_ω.

Proof of (I). For $C^{1,\alpha}$ initial data u_0, we can establish the short time existence of $S^{2,\alpha}$ solution of (4.1)-(4.2) (cp.[Str2];Lemma 3.16). For a general initial data $u_0 \in H^{1,2}(\Sigma; N)$, take a sequence $u_0^{(k)} \in C^\infty$ such that

$$u_0^{(k)} \longrightarrow u_0 \ \text{ in } H^{1,2}(\Sigma; N).$$

Let $u^{(k)}(x,t)$ be the solution with initial data $u_0^{(k)}$ and $T^{(k)}$ the maximal time of existence of $u^{(k)}(x,t)$. Since $u_0^{(k)}$ converges in $H^{1,2}(\Sigma;N)$, there exists $R>0$ such that

$$D(u_0^{(k)};B(x,R))<\bar{\varepsilon}\quad\text{for any } x\in\Sigma. \tag{4.17}$$

By Lemma 4.7 and the definition of $T^{(k)}$, we have $CR^2<T^{(k)}$. Applying Lemma 4.6, we obtain

$$\|u^{(k)}\|_{L^{2,p}(\Sigma\times[\tau,CR^2])}+\|\nabla u^{(k)}\|_{L^{2,p}(\Sigma\times[\tau,CR^2])}+\|\partial_t u^{(k)}\|_{L^{2,p}(\Sigma\times[\tau,CR^2])}\le C \tag{4.18}$$

for any $1\le p<\infty$ and $\tau>0$.
On the other hand, (4.17) and Lemma 4.3 imply

$$\|\nabla^2 u^{(k)}\|_{L^2(\Sigma\times[0,CR^2])}<C\,. \tag{4.19}$$

Hence, by Lemma 4.1, we have

$$\|\nabla u^{(k)}\|_{L^4(\Sigma\times[0,CR^2])}<C\,. \tag{4.20}$$

Thus, since $|\partial_t u^{(k)}|^2\le C(|\Delta u|^2+|\nabla u|^4)$, (4.19) and (4.20) imply

$$\|\partial_t u\|_{L^2(\Sigma\times[0,CR^2])}<C\,. \tag{4.21}$$

By (4.18)-(4.21), we have the following convergence property.

$$\begin{aligned} u^{(k)} &\longrightarrow u &&\text{weakly in } L^{2,p}(\Sigma\times[\tau,CR^2])\cap L^{2,2}(\Sigma\times[0,CR^2])\\ \nabla u^{(k)} &\longrightarrow \nabla u &&\text{weakly in } L^{2,p}(\Sigma\times[\tau,CR^2]),\\ \partial_t u^{(k)} &\longrightarrow \partial_t u &&\text{weakly in } L^{2,p}(\Sigma\times[\tau,CR^2]),\end{aligned}$$

for any $1\le p<\infty$ and $0<\tau<CR^2$.

We shall check $u\in C^0([0,CR^2];H^{1,2}(\Sigma;N))$. By the $L^{2,p}$ estimate above, $u^{(k)}$ converge uniformly in $u\in C^0((0,CR^2];H^{1,2}(\Sigma;N))$. So we only have to prove the continuity at $t=0$. Since $\partial_t u\in L^2(\Sigma\times[0,T])$, u attains its initial value u_0 continuously in L^2 and, by (4.17), Dirichlet integral of u_t is uniformly bounded. Thus, for any $\delta>0$, there exists t_0 with

$$D(u_t;\Sigma)\ge D(u_0;\Sigma)-\delta$$

for any $0<t\le t_0$. Applying Lemma 4.3 for $u_t^{(k)}$ and letting $k\to\infty$, we have

$$\iint_{\Sigma\times[0,t]}|\nabla^2 u|\zeta^2 dxdt\le\frac{C}{R^2}t+\delta$$

for $0<t\le t_0$. By Lemma 4.5, we obtain

$$\left|\int|\nabla u_t|^2\zeta^2dx-\int|\nabla u_0|^2\zeta^2dx\right|\le\frac{C}{R^2}t+\delta$$

for $0 < t \leq t_0$. This proves $u \in C^0([0, CR^2]; H^{1,2}(\Sigma; N))$. Thus, we construct a local solution $u \in X(\Sigma \times [0, CR^2]) \cap S^{2,\alpha}(\Sigma \times (0, CR^2])$. Assertion (b) is the direct consequence of Lemma 4.7.

Proof of (II). Suppose (4.16) holds at $t = T$. For a sequence r_i with $r_i \downarrow 0$, we set

$$t_i := \sup\{t \in (0,T); D(u_\tau; B(x, r_i)) < \bar{\varepsilon} \quad \text{for any } x \in \Sigma \text{ and } 0 < \tau < t\}.$$

If $T = \infty$, let t_i satisfy

$$\iint_{\Sigma \times [t_i - 1, t_i]} |\partial_t u|^2 dxdt \to 0 \tag{4.22}$$

by choosing a suitable subsequence. Let $x_i \in \Sigma$ be a point which attains the supremum in the definition of t_i. From the definition, it follows immediately that

$$t_i \longrightarrow T, \quad x_i \longrightarrow \bar{x}.$$

for some $\bar{x} \in \Sigma$ which satisfies (4.16). Then fix an isothermal coordinate which contains $\bar{x}$ and $\sigma = 1$ at $\bar{x}$ and define the rescaled map v_i by $v^i(\xi, s) = u(h_{x_i, r_i}(\xi), t_i + r_i^2 s)$. v_i also satisfies equation

$$\partial_t v_i = \sigma^{-2}(\Delta_0 v_i + \Gamma(v_i)(\nabla v_i, \nabla v_i)). \tag{4.23}$$

Note that the conformal factor σ satisfies

$$\sigma \to 1, \quad \nabla\sigma \to 0 \quad \text{uniformly on any compact subset}$$

as the rescaling factor r_i tends to 0. By the definition of t_i, for any $K \Subset \mathbb{R}^2$ and sufficiently large i, there exists $\delta > 0$

$$\sup\{D(v_i(\cdot, s); B_{1-\delta}(\xi)); \xi \in K \quad \text{and } s \in [-1, 0]\} \leq \bar{\varepsilon}, \tag{4.24}$$

$$D(v_i(\cdot, 0); B_{1+\delta}(\xi)) \geq \bar{\varepsilon}. \tag{4.25}$$

Thus by (4.23),(4.24) and Lemma 4.6, we have uniform $L^{2,p}(K \times [-\frac{1}{2}, 0])$ bound for $\partial_t v_i, \nabla v_i$. Since

$$\iint_{K \times [-1,0]} |\partial_t v_i|^2 dxdt \leq \iint_{\Sigma \times [t_i - r_i^2, t_i]} |\partial_t u|^2 dxdt \longrightarrow 0,$$

$v_i(\cdot, 0)$ converges to a non-trivial extremal v in $C^{2,\alpha}_{loc}(\mathbb{R}^2)$. Finiteness of the Dirichlet integral of v follows from the condition; $D(u_t; \Sigma) := D_0 < \infty$. Thus, we can apply Lemma 3.1 to v, v can be identified with an extremal of S^2. This proves (c) and (d).

Proof of (III). Choose $t_i \to \infty$. If $T = \infty$ and there exists no point which satisfies (4.16), there exists $r > 0$ with

$$\sup\left\{D(u(\cdot, t); B_r(x)); x \in \Sigma, \quad t \in [t_i, t_i + Cr^2]\right\} < 2\bar{\varepsilon}$$

by Lemma 4.4. Hence, by Lemma 4.6, we have the uniform bound

$$\|u\|_{S^{2,\alpha}(\Sigma\times[t_i+\frac{1}{2}Cr^2,t_i+Cr^2])} < C .$$

This implies

$$u(\cdot, t_i + Cr^2) \longrightarrow u_\infty \quad \text{in } C^{2,\alpha}(\Sigma; N)$$

choosing a suitable subsequence. Since

$$\iint_{\Sigma\times[0,\infty)} |\partial_t u|^2 dxdt < \infty,$$

we may assume

$$\iint_{\Sigma\times[t_i,t_i+Cr^2]} |\partial_t u|^2 dxdt \to 0.$$

This implies that u_∞ satisfies equation (1.1). Q.E.D.

5. Results

Theorem 5.1. *Let N be a compact Riemannian manifold with $\pi_2(N) = 0$ and Σ a closed Riemann surface. If there exists a map $u_0 \in H^{1,2}(\Sigma; N)$ which satisfies*

$$I_\omega(u_0) \cdot |d\omega|^2 < \frac{1}{27\gamma(2)^2}, \tag{5.1}$$

$$D(u_0; \Sigma) \cdot |d\omega|^2 < \frac{1}{9\gamma(2)^2}, \tag{5.2}$$

where $\gamma(2)$ is the isoperimetric constant defined in Theorem 2.1, then there exists a local minimizer $\underline{u} \in C^{2,\alpha}(\Sigma, N)$ of I_ω which is homotopic to u_0. In fact, $\underline{u}$ satisfies

$$I_\omega(\underline{u}) = \inf\Big\{I_\omega(u); u \in H^{1,2}(\Sigma; N), \quad D(u; \Sigma) < \frac{1}{9\gamma(2)^2|d\omega|^2}, \quad u \in [u_0]\Big\},$$

where $[u_0] \in [\Sigma : N]$ denotes the free homotopy class induced by u_0.

Proof. By Proposition 2.4, for $u \in H^{1,2}(\Sigma; N)$, we have

$$I_\omega(u) = D(u; \Sigma) + 2V_\omega(u) \geq D(u; \Sigma) - 2K \cdot D(u; \Sigma)^{\frac{3}{2}}, \tag{5.3}$$

where $K := \gamma(2)|d\omega|$. Set $f(t) = t - 2Kt^{\frac{3}{2}}$. Observe that $f(t)$ is monotone increasing in the interval $[0, \frac{1}{9K^2}]$. Let $g(s)$ denotes the inverse function of $f(t)$ $(0 \leq t \leq \frac{1}{9K^2})$ defined in the interval $[0, \frac{1}{27K^2}]$. Inequality (5.3) implies:

$$\text{If} \quad D(u; \Sigma) \leq \frac{1}{9K^2}, \quad I_\omega \leq s < \frac{1}{27K^2}, \text{then, } D(u; \Sigma) \leq g(s) < \frac{1}{9K^2}. \tag{5.4}$$

Set

$$m := \inf\left\{ I_\omega(u); u \in H^{1,2}(\Sigma; N), \quad D(u;\Sigma) < \frac{1}{9K^2}, \quad u \text{ is homotopic to } u_0 \right\}.$$

Choose $u_i \in H^{1,2}(\Sigma; N)$, which is homotopic to u_0, such that

$$D(u_i;\Sigma) < \frac{1}{9K^2}, \tag{5.5}$$

$$I_\omega(u_i;\Sigma) \longrightarrow m. \tag{5.6}$$

Let w_i be the solution of the evolution problem:

$$\partial_t w_i = \Delta w_i - D^2\pi(w_i)(\nabla w_i, \nabla w_i) - 2D\pi(w_i^* d\omega),$$
$$w_i(\cdot, 0) = u_i$$

constructed in Theorem 4.9. Let T_i be the maximal existence time of smooth solution w_i. We shall prove for sufficiently large i:

(1) There exists $D_0 < \frac{1}{9K^2}$ such that $D(w_i(\cdot,t)) \le D_0$ for $0 \le t < T_i$,
(2) $T_i = \infty$. Moreover, w_i produces no singularity as $t \to \infty$ in the sense of Theorem 4.9.

Proof of (1). Suppose there exists $0 \le t_0 < T_i$ with $D(w_i(\cdot,t_0)) > \frac{1}{9K^2}$. Since the map $[0,T_i) \ni t \mapsto w_i(\cdot,t) \in H^{1,2}(\Sigma;N)$ is continuous by Theorem 4.9, there exists $0 \le t_1 < t_0$ with $D(w_i(\cdot,t_1);\Sigma) = \frac{1}{9K^2}$. Then, by (5.4), $I_\omega(w_i(\cdot,t_1);\Sigma) \ge \frac{1}{27K^2}$. But this is a contradiction, since by Lemma 4.8,

$$I_\omega(w_i(\cdot,t_1)) \le I_\omega(u_i) < \frac{1}{27K^2}.$$

Hence, by (5.4), we obtain the desired result.

Proof of (2). Suppose $T_i < \infty$. By Theorem 4.9 (II) and (1), there exists a singular point $\overline{x} \in \Sigma$. Then there exists t_k, r_k, x_k (Omitting the index i, since we fix the index i for a while)with

$$t_k \to T_i, \quad r_k \to 0, \quad x_k \to \overline{x}$$

such that

$$v_k(\xi) \longrightarrow v \quad \text{in } C^{2,\alpha}_{loc}(\mathbb{R}^2)$$

where $v_k(\xi) = w_i(h_{x_k,r_k}(\xi), t_k)$. And v satisfies (1.1) and

$$D(v;\mathbb{R}^2) > \overline{\varepsilon}. \tag{5.7}$$

By the removability of isolated singularity, v can be identified with the extremal of S^2 by stereographic projection. Hence for sufficiently large $R > 0$, we may assume

$$v_k(\partial B_R(0)) \subset B_\rho(v(\infty))$$

for $0 < \rho < \min(\frac{\pi}{2\kappa_N}, i(N))$. Hence there exists a energy minimizing harmonic map h_k with

$$h_k(B_R(0)) \subset B_\rho(p),$$
$$v_k|_{\partial B_R(0)} = h_k|_{\partial B_R(0)}$$

(cf. [J]; Lemma 4.1.4, or [Mr]). Moreover, taking $R > 0$ sufficiently large, we may assume

$$D(h_k; B_R(0)) < \min(\frac{A}{12}, \frac{\bar{\varepsilon}}{2}). \tag{5.8}$$

where $A := \inf\left\{D(u;\Sigma); u \in H^{1,2}(\Sigma; N), \quad u \text{ is homotopic to } u_0\right\}$.(We may assume $A > 0$, since, by Theorem of Sacks-Uhlenbeck, $A > 0$ for non-trivial homotopy class.) Then we can construct two auxiliary maps $W_k \in Lip(\Sigma)$, $V_k \in Lip(\mathbb{R}^2)$(which is identified with $\overline{V}_k \in Lip(S^2)$ by stereographic projection) as follows;

$$W_k(x) := \begin{cases} w_k(x) & \text{if } x \in \Sigma \setminus B_{Rr_k}(x_k), \\ h_k(\frac{\phi_{x_k}(x)}{r_k}) & \text{if } x \in B_{Rr_k}(x_I). \end{cases}$$

$$V_k(\xi) := \begin{cases} v(\phi_{x_k}^{-1}(\xi)) & \text{if } \xi \in B_R(0), \\ h_k(\frac{R^2\cdot\xi}{|\xi|^2}) & \text{if } \xi \in \mathbb{R}^2 \setminus B_R(0). \end{cases}$$

Here, we set $w_k(x) := w(x, t_k)$ and (U, ϕ_{x_k}, V) is a isothermal coordinate system centered at x_k. By our definition,

$$\int_\Sigma w_k^*\omega = \int_\Sigma W_k^*\omega + \int_{S^2} \overline{V}_k^*\omega, \tag{5.9}$$

$$D(w_k;\Sigma) = D(W_k;\Sigma) + D(\overline{V}_k : S^2) - 2D(h_k; B_R(0)). \tag{5.10}$$

Hence

$$I_\omega(w_k) = I_\omega(W_k) + I_\omega(V_k) - 2D(h_k; B_R(0)). \tag{5.11}$$

By (5.7), (5.8) and (5.11), we have

$$D(W_k;\Sigma) = D(w_k;\Sigma) + 2D(h_k;\Sigma) - D(\overline{V}_k; S^2) < D(w_k;\Sigma) < \frac{1}{9K^2}. \tag{5.12}$$

Since $\pi_2(N) = 0$, w_k and W_k are homotopic. Hence, by the definition of m and (5.12),

$$I_\omega(W_k) \geq m.$$

By inequality (5.3), we have

$$\begin{aligned} I_\omega(\overline{V}_k) &\geq D(\overline{V}_k; S^2) - 2KD(\overline{V}_k; S^2)^{\frac{3}{2}} \\ &\geq D(\overline{V}_k; S^2)\left\{1 - 2KD(w_k;\Sigma)^{\frac{1}{2}}\right\} \\ &\geq \frac{1}{3}D(\overline{V}_k; S^2) > \frac{1}{3}A. \end{aligned}$$

Hence, by (5.11), we obtain

$$I_\omega(u_i) \geq I_\omega(w_k) > m + \frac{1}{3}A - 2D(h_k; B_R(0)) > m + \frac{1}{6}A.$$

This contradicts to our choice of u_i. This proves $T_i = \infty$. The same argument implies w_i produces no singularity as $t_k \to \infty$. Thus we have proved (2).

Now, we shall complete the proof of the theorem. By (1), (2) and Theorem 4.9 (e), there exists a time sequence, $t_k \to \infty$ such that

$$w_i(\cdot, t_k) \longrightarrow \tilde{u}_i \quad \text{in } C^{2,\alpha}$$

where $\tilde{u}_i$ is an extremal of I_ω. And $\tilde{u}_i$ satisfies

$$D(\tilde{u}_i; \Sigma) \leq D_0 < \frac{1}{9K^2},$$
$$I_\omega(\tilde{u}_i) \longrightarrow m.$$

Hence by Theorem 3.6, $\tilde{u}_i$ converges to an extremal $\underline{u}$ in $C^{2,\alpha}$ except for finitely many singular points. But by the argument similar to the one used in the proof of (2), there can be no singular point. Hence, $\tilde{u}_i$ converges to $\underline{u}$ in $C^{2,\alpha}(\Sigma, N)$. By the continuity of I_ω implies that

$$D(\underline{u}; \Sigma) \leq D_0 < \frac{1}{9K^2}, \quad I_\omega(\underline{u}) = m$$

and it is obvious $\underline{u} \in [u_i] = [u_0]$. Thus $\underline{u}$ satisfies desired properties. Q.E.D.

Proof of the Main Theorem. It follows from the following corollary of Theorem 5.1.

Corollary 5.2. *Let Σ and N be as in Theorem 5.1. Suppose $\alpha \in [\Sigma : N]$ is a given homotopy class. Set*

$$A_\alpha = \inf\left\{D(u; \Sigma); u \in H^{1,2}(\Sigma; N), \quad u \in \alpha\right\}.$$

If

$$A_\alpha \cdot |d\omega|^2 < \frac{\mu}{\gamma(2)^2} \tag{5.13}$$

where μ is a unique solution of equation $\mu + \mu^{\frac{3}{2}} = \frac{1}{27}$ then there exists a local minimizer $\underline{u} \in \alpha$ of I_ω with

$$I_\omega(\underline{u}) = \inf\left\{I_\omega(u); u \in H^{1,2}(\Sigma; N), \quad D(u; \Sigma) < \frac{1}{9\gamma(2)^2|d\omega|^2}, \quad u \in \alpha\right\}.$$

Proof. By Theorem of Sacks-Uhlenbeck, there exists an energy minimizing harmonic map u_0 in a given homotopy class α. Then, by the isoperimetric inequality, we have

$$\begin{aligned} I_\omega(u_0) &\leq D(u_0; \Sigma) + 2\gamma(2) \cdot |d\omega| \cdot D(u_0; \Sigma)^{\frac{3}{2}} \\ &= A_\alpha + 2\gamma(2) \cdot |d\omega| A_\alpha^{\frac{3}{2}} < \frac{4}{27\gamma(2)^2}. \end{aligned}$$

Hence u_0 satisfies (5.1). (5.2) is automatically satisfied by (5.13). Thus, by Theorem 5.1, we obtain the result. Q.E.D.

The proof of Theorem 5.1 and Corollary 5.2 imply the following theorem.

Theorem 5.3. *Let $u_{0\sharp} : \pi_1(\Sigma) \to \pi_1(\Sigma)$ be a homomorphism induced by a given map $u_0 \in H^{1,2}(\Sigma; N)$. If u_0 satisfies (5.1) and (5.2), there exists a local minimizer $\underline{u}$ of I_ω with*

$$\underline{u}_\sharp = u_{0\sharp}$$
$$I_\omega(\underline{u}) = m := \inf \left\{ I_\omega(u); |d\omega|^2 \cdot D(u;\Sigma) < \frac{1}{27\gamma(2)^2}, \quad u_{o\,\sharp} = u_\sharp \right\}.$$

Set

$$A_{u_{0\sharp}} = \inf \left\{ D(u;\Sigma); u \in H^{1,2}(\Sigma; N), \quad u_\sharp = u_{0\sharp} \right\}.$$

Especially, if

$$A_{u_{0\sharp}} \cdot |d\omega|^2 \le \frac{\mu}{\gamma(2)^2}$$

where μ is as in Corollary 5.2, there exists a minimizer $\underline{u}$ of I_ω with

$$u_{0\sharp} = \underline{u}_\sharp, \quad I_\omega(\underline{u}) = m.$$

Proof. The proof of Theorem 5.1 and Corollary 5.2 can be adapted to the proof of the theorem. In fact, the auxiliary map W_k constructed in the proof of (2) in the proof of Theorem 5.1 induces the same homomorphism as the one induced by w_k. Q.E.D.

Finally, we shall study extremals of S^2. Set

$$M := \Big\{ \alpha \in \pi_2(N); \text{there exists a map } u_0 : S^2 \to N \text{ with}$$
$$D(u_0; S^2) < \frac{1}{9|d\omega|^2 \cdot \gamma(2)^2}, \quad I_\omega(u_0) < \frac{1}{27|d\omega|^2 \cdot \gamma(2)^2}, \quad [u_0] = \alpha \Big\}.$$

Theorem 5.4. *Suppose $\alpha \in M$. Then, there exists a finite subset $\{\alpha_1, \ldots, \alpha_N\}$ in M such that*

(1) $\alpha = \alpha_1 + \cdots + \alpha_N$,
(2) Each α_i $(1 \le i \le N)$ is induced by extremal u_i of I_ω with

$$D(u_i; S^2) < \frac{1}{27|d\omega|^2 \cdot \gamma(2)^2},$$
$$I_\omega(u_i) < \frac{1}{9|d\omega|^2 \cdot \gamma(2)^2}.$$

Proof. Let $u_0 \in H^{1,2}(S^2; N)$ be a map with

$$D(u_0; S^2) < \frac{1}{9K^2}, \tag{5.14}$$

$$I_\omega(u_0) < \frac{1}{27K^2}, \tag{5.15}$$

$$[u_0] = \alpha. \tag{5.16}$$

By Theorem 4.9, there exists a solution of the evolution problem;

$$\begin{aligned} &\partial_t w = \Delta w - D^2\pi(w)(\nabla w, \nabla w) - 2D\pi(w^* d\omega) \\ &w(\cdot, 0) = u. \end{aligned}$$

Let $0 < T \leq \infty$ be the maximal time of existence of smooth solution w. As in the proof of (1) in the proof of Theorem 5.1, we can show

$$D(w(\cdot, t), S^2) < \frac{1}{27K^2} \quad \text{for } t \in [0, T).$$

We also have

$$I_\omega(w(\cdot, t)) < \frac{1}{9K^2} \quad \text{for } t \in [0, T).$$

Either of the following two cases can happen;

(1) $T = \infty$ and the solution produces no singularity at $t = \infty$ in the sense of Theorem 4.9 (II).
(2) $0 < T \leq \infty$ and the solution produces singularity at $t = T$ in the sense of Theorem 4.9 (II).

If case (1) happens, we obtain the desired result by Theorem 4.9 (III). Suppose case (2) happens. Then, there exist $t_i \to T$, $x_i \to \overline{x}$, $r_i \to 0$ such that the rescaled map $u_i(\xi) = w_i(exp_{x_i} r_i \xi, t_i)$ converges to extremal u of $\mathbb{R}^2$ in $C^{2,\alpha}_{loc}(\mathbb{R}^2, N)$. For sufficiently large R, we may assume that there exists an energy minimizing harmonic map $h_i \in C^{2,\alpha}(B_R(0); N)$ whose image is contained in some convex ball in N (cf.[J]; Lemma 4.1.4. or [Mr]) with

$$\begin{aligned} &h_i|_{\partial B_R(0)} = v_i|_{\partial B_R(0)} \\ &D(h_i; B_R(0)) < \frac{\overline{\varepsilon}}{2} \end{aligned}$$

We construct auxiliary maps as in the proof of Theorem 5.1.

$$W_i(x) := \begin{cases} w(x) & \text{if } x \in S^2 \setminus B_{Rr_i}(x_i), \\ h_i(\frac{\phi_{x_i}(x)}{r_i}) & \text{if } x \in B_{Rr_i}(x_i). \end{cases}$$

$$V_i(\xi) := \begin{cases} v(\phi_{x_i}^{-1}(\xi)) & \text{if } \xi \in B_R(0), \\ h_i(\frac{R^2 \cdot \xi}{|\xi|^2}) & \text{if } \xi \in \mathbb{R}^2 \setminus B_R(0). \end{cases}$$

V_i can be identified with the Lipschitz map $\bar{V}$ of S^2 by the stereographic projection. By our choice of h_i and our definition of W_i and V_i, there holds for sufficiently large i

$$[\bar{V}_i]=[u] \in \pi_2(N),$$
$$[w_i] = [W_i] + [V_i]=[W_i] + [u] \in \pi_2(N). \tag{5.17}$$

We also have,

$$\begin{aligned} D(W_i;S^2)&=D(w_i;S^2) + D(h_i;B_R(0)) - D(v_i;B_R(0)) \\ &\le \frac{1}{27K^2} - \bar{\varepsilon} + \frac{\bar{\varepsilon}}{2} \le \frac{1}{27K^2} - \frac{\bar{\varepsilon}}{2}, \end{aligned} \tag{5.18}$$

$$\begin{aligned} I_\omega(W_i)&=I_\omega(w_i) - I_\omega(V_i) \\ &\le \frac{1}{9K^2} - \frac{1}{3}D(v_i) < \frac{1}{9K^2}. \end{aligned} \tag{5.19}$$

Hence taking W_i as the initial value of evolution problem in stead of u_0, we take the same procedure. Since by (5.18), Dirichlet integral of the map decreases at least $\frac{\bar{\varepsilon}}{2}$ at each step of the procedure, finally the case (1) happens and the procedure ends in finitely many steps. Thus we obtain the desired result by (5.17). Q.E.D.

References

[Al] Almgren, F.J., Optimal isoperimetric inequalities. Ind. U. Math. J. **35**, 451–547 (1986)

[F] Federer, H., Geometric Measure Theory. Springer Verlag, New York, 1969

[F-F] Federer, H., Fleming, W.H., Normal and integral currents. Ann. Math. **72**, 458–520 (1960)

[Gr1] Grüter, M., Conformally invariant variational integrals and the removability of isolated singularities. Manuscripta Math. **47**, 85–104 (1984)

[Gr2] Grüter, M., Regularity of weak H-surfaces. J. Reine Angew. Math. **329**, 1–15 (1981)

[H-S] Hardt, R., Simon, L., Seminar on Geometric Measure Theory. Birkhäuser Verlag, Basel, 1986

[J] Jost, J., Two-dimensional geometric variational problems. Pure and Applied Mathematics. Chicester: John Wiley & Sons, 1991

[Mg] Morgan, F., Geometric Measure Theory – A Beginners Guide. Academic Press, 1988

[Mr] Morrey, C.B., The problem of Plateau on a Riemannian manifold. Ann. Math. **49**, 807–851 (1948)

[SaU] Sacks, J., Uhlenbeck, K., The existence of minimal immersions of 2-spheres. Ann. Math. **113**, 1–24 (1981)

[ScU] Schoen, R.M., Uhlenbeck, K., Boundary regularity and miscellaneous results on harmonic maps. J. Diff. Geom. **18**, 253–268 (1983)

[Si] Simon, L., Lectures on Geometric Measure Theory. Proc. of the Centre for Math. Anal., Australian National University, vol 3, 1983

[Ste] Steffen, K., On the existence of surfaces with prescribed mean curvature and boundary. Math. Z. **146**, 113–135 (1976)

[Str1] Struwe, M., On the evolution of harmonic mappings of Riemannian surfaces. Comm. Math. Helv. **60**, 558–581 (1985)

[Str2] Struwe, M., The existence of surfaces of constant mean curvature with free boundaries. Acta. Math. **160**, 19–64 (1988)

[Str3] Struwe, M., Variational methods. Springer Verlag, Berlin, 1990

[W] Wente, H.C., An existence theorem for surfaces of constant mean curvature. J. Math. Anal. Appl **26**, 318–344 (1969)