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Four positive solutions for the semilinear elliptic equation: $-\Delta u + u = a(x)u^p + f(x)$ in $\mathbb{R}^N$

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Abstract. We consider the existence of positive solutions of the following semilinear elliptic problem in $\mathbb{R}^N$:

$$\begin{aligned} -\Delta u + u &= a(x)u^p + f(x) \qquad \text{in } \mathbb{R}^N, \\ u &> 0 \qquad \text{in } \mathbb{R}^N, \qquad (*) \\ u &\in H^1(\mathbb{R}^N), \end{aligned}$$

where $1 < p < \dfrac{N+2}{N-2}$ $(N \geq 3)$, $1 < p < \infty$ $(N = 1,\, 2)$, $a(x) \in C(\mathbb{R}^N)$, $f(x) \in H^{-1}(\mathbb{R}^N)$ and $f(x) \geq 0$. Under the conditions:

1° $a(x) \in (0, 1]$ for all $x \in \mathbb{R}^N$,
2° $a(x) \to 1$ as $|x| \to \infty$,
3° there exist $\delta > 0$ and $C > 0$ such that

$$a(x) - 1 \geq -Ce^{-(2+\delta)|x|} \qquad \text{for all } x \in \mathbb{R}^N,$$

4° $a(x) \not\equiv 1$,

we show that $(*)$ has at least four positive solutions for sufficiently small $\|f\|_{H^{-1}(\mathbb{R}^N)}$ but $f \not\equiv 0$.

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0. Introduction

In this paper, we study the existence and the multiplicity of positive solutions for the following semilinear elliptic problem:

$$-\Delta u + u = a(x)u^p + f(x) \qquad \text{in } \mathbb{R}^N, \tag{0.1}$$

$$u > 0 \qquad \text{in } \mathbb{R}^N, \tag{0.2}$$

$$u \in H^1(\mathbb{R}^N), \tag{0.3}$$

where $1 < p < \dfrac{N+2}{N-2}$ $(N \geq 3)$, $1 < p < \infty$ $(N = 1,\, 2)$. We assume that $a(x) \in C(\mathbb{R}^N)$ satisfies

$$a(x) > 0 \qquad \text{for all } x \in \mathbb{R}^N, \tag{0.4}$$

$$a(x) \to 1 \qquad \text{as } |x| \to \infty \tag{0.5}$$

and $f(x)$ satisfies

$$f(x) \in H^{-1}(\mathbb{R}^N), \tag{0.6}$$

$$f(x) \geq 0. \tag{0.7}$$

Under the assumptions (0.4)–(0.7), our problem (0.1)–(0.3) can be regarded as a perturbation problem of the following homogeneous problem:

$$-\Delta u + u = u^p \qquad \text{in } \mathbb{R}^N, \tag{0.8}$$

$$u > 0 \qquad \text{in } \mathbb{R}^N, \tag{0.9}$$

$$u \in H^1(\mathbb{R}^N). \tag{0.10}$$

It is known that (0.8)–(0.10) has a unique positive radial solution $\omega(x) = \omega(|x|)$ and any positive solution $u(x)$ of (0.8)–(0.10) can be written as

$$u(x) = \omega(x - x_0) \qquad \text{for some } x_0 \in \mathbb{R}^N.$$

(See Kwong [17], c.f. Kabeya-Tanaka [16]).

Our main question is whether positive solutions can survive after a perturbation of type (0.1)–(0.3) or not. Such a question was studied by Zhu [25], Cao-Zhou [11], Jeanjean [15], Hirano [14] and Adachi-Tanaka [1]. See also Ambrosetti and Badiale [3] for a perturbation result via Poincaré-Melnikov type arguments. Zhu [25] (c.f. Hirano [14]) were mainly concerned with the case $a(x) \equiv 1$ and $f(x) \geq 0$, $f(x) \not\equiv 0$ and succeeded to find the existence of at least two positive solutions under the situation

$$\|f\|_{H^{-1}(\mathbb{R}^N)} \leq M, \tag{0.11}$$

where the constant $M > 0$ was chosen so that the corresponding functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}\,dx - \int_{\mathbb{R}^N} fu\,dx$$

possesses the mountain pass environment. That is, there exist $\delta_0 > 0$, $\rho_0 > 0$ and $e \in H^1(\mathbb{R}^N)$ such that

$$I(u) \geq \delta_0 \qquad \text{for all } \|u\|_{H^1(\mathbb{R}^N)} = \rho_0$$

and

$$\|e\|_{H^1(\mathbb{R}^N)} > \rho_0,\ I(e) < 0.$$

Generalizations of the result of [25] were done by Cao-Zhou [11], Jeanjean [15] and Adachi-Tanaka [1]. They studied more general nonlinearities

$$\begin{aligned} -\Delta u + u &= g(x,u) + f(x) && \text{in } \mathbb{R}^N, \\ u &> 0 && \text{in } \mathbb{R}^N, \\ u &\in H^1(\mathbb{R}^N), \end{aligned} \tag{0.12}$$

under suitable conditions. [11] and [15] showed the existence of at least two positive solutions especially under the assumption:

$$a(x) \geq 1 \qquad \text{for all } x \in \mathbb{R}^N \tag{0.13}$$

or

$$g(x,u) \geq \overline{g}(u)\left(= \lim_{|x|\to\infty} g(x,u)\right) \qquad \text{for all } x \in \mathbb{R}^N \text{ and } u > 0. \tag{0.14}$$

Recently [1] has succeeded to show the existence without assuming (0.13), (0.14).

To find positive solutions, in [1], [11], [14], [15], [25], they argued in the following way: first they considered minimization problem:

$$\text{minimize } I(u) \text{ in } B(\rho_0) = \{u \in H^1(\mathbb{R}^N)\,;\ \|u\|_{H^1(\mathbb{R}^N)} < \rho_0\}$$

and found the first positive solution $u_0(x)$ as a minimizer of $I(u)$ in $B(\rho_0)$. We remark that if $f \not\equiv 0$, 0 is not a solution of our problem and the first positive solution is obtained as a perturbation of 0. The second positive solution $u_1(x)$ was obtained through the Mountain Pass Theorem. The key of their arguments is that the minimax value given by Mountain Pass Theorem — we call it *the MP level* in short — is lower than the first level of breaking down of the Palais-Smale condition.

From now on, we restrict ourselves to the problem (0.1)–(0.3) and we pay attention to the conditions (0.13), (0.14). This condition makes it easy to study (0.1)–(0.3) via variational methods. For the case $f(x) \equiv 0$, we can see the MP level is lower than the first level of breaking down of the Palais-Smale condition. Thus we can obtain a positive solution of (0.1)–(0.3) with $f \equiv 0$ via the Mountain Pass Theorem. On the other hand, if $a(x)$ satisfies

$$a(x) \in (0\,,\,1] \qquad \text{for all } x \in \mathbb{R}^N \tag{0.15}$$

and

$$a(x) \not\equiv 1,$$

the situation is completely different. For the case $f(x) \equiv 0$, we can see that the MP level is exactly equal to the first level of breaking down of the Palais-Smale condition and we can not get a positive solution through the Mountain Pass Theorem. Here we give two remarks:

(i) Bahri-Li [5] showed the existence of at least one positive solution of (0.1)–(0.3) with $f(x) \equiv 0$ under the conditions (0.4)–(0.5) and

$$a(x) - 1 \geq -C \exp(-(2+\delta)|x|) \qquad \text{for all } x \in \mathbb{R}^N \tag{0.16}$$

for some $\delta > 0$, $C > 0$. In Sect. 1, we observe that under (0.15), the critical value of their solution is strictly greater than the first break down of the Palais-Smale condition. See also Bahri-Lions [6], in which they showed the existence of at least one positive solution under condition $N \geq 2$ and

$$a(x) - 1 \geq -C \exp(-\delta|x|) \qquad \text{for all } x \in \mathbb{R}^N.$$

See also Bianchi [8] and Bianchi and Egnell [9], [10].

(ii) Adachi-Tanaka [1] showed that if $f(x) \geq 0$ and $f(x) \not\equiv 0$, the MP level is lower than the first level of breaking down of Palais-Smale condition even under the condition (0.15).

From the above remarks, under the condition (0.15), it is observed that the positive solution obtained in [1] is essentially different from one obtained in [5]. More precisely, even if $||f||_{H^{-1}(\mathbb{R}^N)} \to 0$, the solution of [1] does not approach to the solution of [5], since their critical values are different. Thus the existence of more than two positive solutions is expected.

In this paper, we study the multiplicity of positive solutions under (0.15) and our main result is the following

Theorem 0.1. *Assume (0.4), (0.5) and (0.15), (0.16). Then there exists an* $\delta_0 > 0$ *such that for non-negative function* $f(x)$ *satisfying* $0 < \|f\|_{H^{-1}(\mathbb{R}^N)} \leq \delta_0$, *the problem (0.1)–(0.3) possesses at least four positive solutions.*

As to an asymptotic behavior of solutions obtained in Theorem 0.1 as $\|f\|_{H^{-1}(\mathbb{R}^N)} \to 0$, we have

Theorem 0.2. *Assume that a sequence of non-negative functions* $(f_j(x))_{j=1}^{\infty} \subset H^{-1}(\mathbb{R}^N)$ *satisfies* $f_j(x) \not\equiv 0$ *and*

$$\|f_j\|_{H^{-1}(\mathbb{R}^N)} \to 0 \qquad \text{as } j \to \infty.$$

Then there exist a subsequence of $(f_j(x))_{j=1}^{\infty}$ *— still denoted by* $(f_j(x))_{j=1}^{\infty}$ *— and four sequences* $(u_j^{(k)}(x))_{j\in\mathbb{N}}$ $(k = 1, 2, 3, 4)$ *of positive solutions of (0.1)–(0.3) with* $f(x) = f_j(x)$ *such that*

(i) $\|u_j^{(1)}\|_{H^1(\mathbb{R}^N)} \to 0$ *as* $j \to \infty$.

(ii) *There exist sequences* $(y_j^{(2)})_{j=1}^{\infty}$, $(y_j^{(3)})_{j=1}^{\infty} \subset \mathbb{R}^N$ *such that*

$$|y_j^{(k)}| \to \infty,$$

$$\|u_j^{(k)}(x) - \omega(x - y_j^{(k)})\|_{H^1(\mathbb{R}^N)} \to 0$$

as $j \to \infty$ *for* $k = 2, 3$. *Here* $\omega(x)$ *is the unique positive radial solution of (0.8)–(0.10).*

(iii) *There exists a positive solution* $v_0(x)$ *of (0.1)–(0.3) with* $f \equiv 0$ *such that*

$$\|u_j^{(4)}(x) - v_0(x)\|_{H^1(\mathbb{R}^N)} \to 0 \qquad \text{as } j \to \infty.$$

We remark that the solutions $u_j^{(2)}(x)$, $u_j^{(3)}(x)$ do not converge strongly to solutions of (0.1)–(0.3) with $f \equiv 0$. As an immediate corollary to Theorem 0.2, we have the following result on symmetry-breaking of positive solutions for (0.1)–(0.3).

Corollary 0.3. *Suppose that* $a(x) = a(|x|)$, $f(x) = f(|x|)$ *are radially symmetric in addition to (0.4), (0.5), (0.15), (0.16). Then there exists a* $\delta_1 > 0$ *such that if* $f(x) \geq 0$, $f(x) \not\equiv 0$, $\|f\|_{H^{-1}(\mathbb{R}^N)} \leq \delta_1$, *then (0.1)–(0.3) possesses at least one positive solution which is not radially symmetric.*

Proof of Corollary 0.3. Suppose that the conclusion of Corollary 0.3 does not hold. Then there exists a sequence of non-negative radially symmetric functions $(f_j(x))_{j=1}^{\infty}$ such that $\|f_j\|_{H^{-1}(\mathbb{R}^N)} \to 0$ as $j \to \infty$ and all positive solutions of (0.1)–(0.3) with $f(x) = f_j(x)$ are radially symmetric. However, by Theorem 0.2, there exist a subsequence of $(f_j(x))_{j=1}^{\infty}$ — still denoted by $(f_j(x))_{j=1}^{\infty}$ — and sequences of solutions $(u_j^{(1)}(x))_{j=1}^{\infty}$, $(u_j^{(2)}(x))_{j=1}^{\infty}$, $(u_j^{(3)}(x))_{j=1}^{\infty}$, $(u_j^{(4)}(x))_{j=1}^{\infty}$ of (0.1)–(0.3) with $f(x) = f_j(x)$ satisfying the conclusion (i)–(iii) of Theorem 0.2. Clearly, $(u_j^{(2)}(x))_{j=1}^{\infty}$, $(u_j^{(3)}(x))_{j=1}^{\infty}$ are not radially symmetric for large j. This contradicts the assumption on $(f_j(x))_{j=1}^{\infty}$. Thus Corollary 0.3 holds. □

We remark that under the condition (0.13) such a symmetry breaking does not occur in general. In fact, we have

Theorem 0.4. *Suppose that* $a(x) = a(|x|)$, $f(x) = f(|x|)$ *are radially symmetric positive function and satisfy*

$$a(r) \geq 1 \qquad \textit{for all } r > 0,$$

$$a_r(r) \leq 0 \qquad \textit{for all } r > 0,$$

$$f(r) \geq 0 \qquad \textit{for all } r > 0,$$

$$f_r(r) \leq 0 \qquad \textit{for all } r > 0.$$

Then all positive solutions of (0.1)–(0.3) are radially symmetric.

Proof. We can derive Theorem 0.4 from the results of C. Li [18], [19] and Gidas-Ni-Nirenberg [12], [13]. □

In the following sections, we give proofs of Theorem 0.1 and Theorem 0.2. We use variational methods to find positive solutions of (0.1)–(0.3). We write $u_+(x) = \max\{u(x)\,, 0\}$, $\|u\|^2_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\, dx$ and we define for given $a(x)$ and $f(x)$,

$$I_{a,f}(u) = \frac{1}{2}||u||^2_{H^1(\mathbb{R}^N)} - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x) u_+^{p+1}\, dx - \int_{\mathbb{R}^N} f(x) u(x)\, dx$$
$$: H^1(\mathbb{R}^N) \to \mathbb{R}$$

and

$$J_{a,f}(v) = \max_{t>0} I_{a,f}(tv) : \Sigma_+ \to \mathbb{R},$$

where

$$\Sigma = \{v \in H^1(\mathbb{R}^N)\,;\ \|v\|_{H^1(\mathbb{R}^N)} = 1\},$$
$$\Sigma_+ = \{v \in \Sigma\,;\ v_+ \not\equiv 0\}.$$

We will see that critical points of $I_{a,f}(u) : H^1(\mathbb{R}^N) \to \mathbb{R}$ or $J_{a,f}(v) : \Sigma_+ \to \mathbb{R}$ are corresponding to positive solutions of (0.1)–(0.3).

We will find critical points of $I_{a,f}(u)$, $J_{a,f}(u)$ in the following way. First we find one positive solution $u^{(1)}(a, f; x) = u_{loc\,min}(a, f; x)$ as a local minimum of $I_{a,f}(u)$ near 0. Next we see the Palais-Smale compactness condition for $I_{a,f}(u)$ and $J_{a,f}(u)$ breaks down only at levels

$$I_{a,f}(u_0(x)) + kI_{1,0}(\omega) \qquad k = 1, 2, ...$$

where $I_{1,0}(u)$ is a functional corresponding to (0.8)–(0.10), $\omega(x)$ is a unique positive radial solution of (0.8)–(0.10) and $u_0(x)$ is a critical point of $I_{a,f}(u)$. In particular, we will see that the Palais-Smale condition holds under the level $I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega)$.

To find the second and third positive solutions, we use notation:

$$[J_{a,f} \leq c] = \{u \in \Sigma_+ \,;\, J_{a,f}(u) \leq c\}$$

for $c \in \mathbb{R}$. We will observe that for sufficiently small $\varepsilon > 0$

$$[J_{a,f} \leq I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega) - \varepsilon]$$

is not empty and

$$\mathrm{cat}\,([J_{a,f} \leq I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega) - \varepsilon]) \geq 2 \qquad (0.17)$$

provided $f(x) \geq 0$, $f(x) \not\equiv 0$ and $\|f\|_{H^{-1}(\mathbb{R}^N)}$ is sufficiently small. Here cat stands for the Lusternik-Schnirelman category. We find two positive solutions $u^{(2)}(a,f;x)$ and $u^{(3)}(a,f;x)$ satisfying

$$I_{a,f}(u^{(k)}(a,f;x)) < I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega) \qquad \text{for } k = 2, 3. \qquad (0.18)$$

We remark that for $f \equiv 0$, we see that

$$u_{loc\,min}(a,0;x) \equiv 0$$

and

$$[J_{a,0} \leq I_{a,0}(u_{loc\,min}(a,0;x)) + I_{1,0}(\omega)] = \emptyset \qquad (0.19)$$

and (0.17) is the key of our proof. To get (0.17), we use the following interaction phenomenon as in [1] (c.f. Bahri-Li [5], Bahri-Lions [6]):

$$I_{a,f}(u_{loc\,min}(a,f;x) + \omega(x-y)) < I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega)$$

for sufficiently large $|y| \geq 1$.

To find the fourth positive solution, we adapt the minimax method of Bahri-Li [5] to our functional $J_{a,f}(v)$ and we will find positive solution $u^{(4)}(a,f;x)$ with

$$I_{a,f}(u^{(4)}(a,f;x)) > I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega)$$

for sufficiently small $\|f\|_{H^{-1}(\mathbb{R}^N)}$. To show Theorem 0.2, we also use (0.18) and (0.19) in an essential way.

Thus our paper is organized as follows: In Sect. 1, we will give a functional frame work, some preliminaries and we will find a local minimizer of $I_{a,f}(u)$ near 0. Section 2 will be devoted to the proof of (0.17). Finally in Sects. 3 and 4, we complete proofs of our Theorems 0.1 and 0.2.

1. Preliminaries

1.1. Functional frame work

In what follows, we denote the usual Sobolev space by $E = H^1(\mathbb{R}^N)$ and we use notation:

$$\langle u, v\rangle_E = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} uv \, dx,$$

$$\|u\|_E = \langle u\,,\,u\rangle_E^{\frac{1}{2}}$$

for $u, v \in E$. We also denote the duality product between $E^* = H^{-1}(\mathbb{R}^N)$ and E by $\langle\cdot\,,\,\cdot\rangle_{E^*,E}$ and

$$\|f\|_{E^*} = \sup_{\|u\|_E=1} \langle f\,,\,u\rangle_{E^*,E}.$$

For all functions $a(x)$, $f(x) : \mathbb{R}^N \to \mathbb{R}$, we define a functional $I_{a,f}(u) : E \to \mathbb{R}$ by

$$I_{a,f}(u) = \frac{1}{2}\|u\|_E^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_+^{p+1}\, dx - \int_{\mathbb{R}^N} f(x)u(x)\, dx.$$

In what follows, we assume (0.4)–(0.7) and we have the following characterization of non-negative solutions of (0.1)–(0.3).

Lemma 1.1. *Assume (0.4)–(0.7). Then*

(i) $I_{a,f}(u) \in C^2(E\ ,\mathbb{R})$ *and*

$$I'_{a,f}(u)h = \langle u, h\rangle_E - \int_{\mathbb{R}^N} a(x)u_+^p h\, dx - \int_{\mathbb{R}^N} fh\, dx, \qquad (1.1)$$

$$I''_{a,f}(u)(h\,,\,h) = ||h||_E^2 - p\int_{\mathbb{R}^N} a(x)u_+^{p-1}h^2\, dx \qquad (1.2)$$

for $h \in E$.

(ii) *If* $u \in E$ *is a critical point of* $I_{a,f}(u)$, *then* $u(x)$ *is a non-negative solution of (0.1)–(0.3). Moreover if* $u(x) \not\equiv 0$ *or* $f \not\equiv 0$, *then* $u(x)$ *is a positive solution of (0.1)–(0.3).*

Proof. (i) can be proved in a standard way. We prove only (ii). Suppose that $u \in E$ satisfies $I'_{a,f}(u) = 0$. By (1.1),

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla h + uh - a(x)u_+^p h - fh)\, dx = 0$$

for all $h \in E$. Thus $u(x)$ is a weak solution of

$$-\Delta u + u = a(x)u_+^p + f \qquad \text{in } \mathbb{R}^N. \tag{1.3}$$

By the assumption (0.7), the right hand side of (1.3) is non-negative and we can see that $u(x)$ is non-negative by the maximal principle. If $u(x) \not\equiv 0$ or $f(x) \not\equiv 0$, we can see the right hand side of (1.3) is non-negative and not equivalently equal to 0. Thus $u(x)$ is positive in $\mathbb{R}^N$. □

Hereafter, we try to find critical points of $I_{a,f}(u)$. We will use the following estimate frequently:

$$\|u\|_{p+1} \le C_{p+1}\|u\|_E \qquad \text{for } u \in E, \tag{1.4}$$

where

$$\|u\|_{p+1} = \left(\int_{\mathbb{R}^N} |u|^{p+1}\, dx\right)^{\frac{1}{p+1}}.$$

The best constant $\bar{C}_{p+1}$ for (1.4) plays an important role. $\bar{C}_{p+1}$ can be characterized as

$$\bar{C}_{p+1}^{-1} = \inf_{u\neq 0} \frac{\|u\|_E}{\|u\|_{p+1}}. \tag{1.5}$$

It is known that the infimum in (1.5) is attained and the set of the minimizers can be written as

$$\{\lambda\omega(x-y)\,;\ \lambda \in \mathbb{R}\setminus\{0\},\ y \in \mathbb{R}^N\},$$

where $\omega(x)$ is the unique positive radial solution of (0.8)–(0.10), i.e., a critical point of $I_{1,0}(u)$. In particular, we have

$$\bar{C}_{p+1}^{-1} = \inf_{u\neq 0} \frac{\|u\|_E}{\|u\|_{p+1}} = \frac{\|\omega\|_E}{\|\omega\|_{p+1}}. \tag{1.6}$$

We can also see that $\omega(x)$ is a critical point of $I_{1,0}(u)$ corresponding to the mountain pass theorem, that is,

$$I_{1,0}(\omega) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{1,0}(\gamma(t)).$$

Here,

$$\Gamma = \{\gamma \in C([0,1],E)\,;\ \gamma(0) = 0,\ I_{1,0}(\gamma(1)) < 0\}.$$

In particular,

$$I_{1,0}(\omega) = \max_{t\ge 0} I_{1,0}(t\omega). \tag{1.7}$$

We can also see the set of critical points of $I_{1,0}(u)$ is

$$\{0\} \cup \{\omega(x-y)\,;\ y \in \mathbb{R}^N\}.$$

One of the virtue of the unperturbed functional $I_{1,0}(u)$ is the following fact — all critical points except 0 can be obtained through constraint problem:

$$\left(\int_{\mathbb{R}^N} v_+^{p+1}\,dx\right)^{-1} :\ \{v\in E;\ ||v||_E=1\}\to\mathbb{R}.$$

1.2. Properties of $I_{a,f}(u)$ and existence of a local minimizer

To find critical points of $I_{a,f}(u)$, first we observe that if $\|f\|_{E^*}$ is sufficiently small, then $I_{a,f}(u)$ has a similar feature to $I_{1,0}(u)$. That is,

($I_{a,f}$-1) $I_{a,f}(u)$ has a unique critical point $u_{loc\,min}(a,f;x)$ in a neighbourhood of 0.

($I_{a,f}$-2) All critical points except $u_{loc\,min}(a,f;x)$ can be obtained through the following constraint problem:

$$J_{a,f}(v)=\max_{t>0} I_{a,f}(tv) :\ \Sigma_+\to\mathbb{R},$$

where

$$\Sigma=\{v\in E\,;\ \|v\|_E=1\},$$
$$\Sigma_+=\{v\in\Sigma\,;\ v_+\not\equiv 0\}.$$

We remark that if $f\equiv 0$,

$$\begin{aligned} J_{a,0}(v) &= I_{a,0}\left(\left(\int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx\right)^{-\frac{1}{p-1}} v\right)\\ &= \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx\right)^{-\frac{2}{p-1}}. \end{aligned} \tag{1.8}$$

In particular,

$$J_{1,0}(v)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\int_{\mathbb{R}^N} v_+^{p+1}\,dx\right)^{-\frac{2}{p-1}}$$

and

$$\begin{aligned} \inf_{v\in\Sigma_+} J_{1,0}(v) &= J_{1,0}\left(\frac{\omega}{\|\omega\|_E}\right)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{\|\omega\|_E}{\|\omega\|_{p+1}}\right)^{\frac{2(p+1)}{p-1}}\\ &= \left(\frac{1}{2}-\frac{1}{p+1}\right)\bar{C}_{p+1}^{-\frac{2(p+1)}{p-1}}. \end{aligned} \tag{1.9}$$

We remark that $J_{1,0}(\frac{\omega}{\|\omega\|_E})=I_{1,0}(\omega)$. Thus

$$\inf_{v\in\Sigma_+} J_{1,0}(v)=J_{1,0}\left(\frac{\omega}{\|\omega\|_E}\right)=I_{1,0}(\omega). \tag{1.10}$$

Recalling (0.4)–(0.5) and setting

$$\underline{a} = \inf_{x\in\mathbb{R}^N} a(x) > 0,$$
$$\overline{a} = \sup_{x\in\mathbb{R}^N} a(x) \geq 1,$$

we have from (1.8)

$$\overline{a}^{-\frac{2}{p-1}} J_{1,0}(v) = J_{\overline{a},0}(v) \leq J_{a,0}(v) \leq J_{\underline{a},0}(v) = \underline{a}^{-\frac{2}{p-1}} J_{1,0}(v) \quad \text{for all } v \in \Sigma_+.$$

Thus

$$\overline{a}^{-\frac{2}{p-1}} I_{1,0}(\omega) \leq \inf_{v\in\Sigma_+} J_{a,0}(v) \leq \underline{a}^{-\frac{2}{p-1}} I_{1,0}(\omega). \tag{1.11}$$

To see ($I_{a,f}$-1) and ($I_{a,f}$-2), first we observe

Lemma 1.2. (i) *For* $u \in E$ *and* $\varepsilon \in (0,1)$,

$$(1-\varepsilon) I_{\frac{a}{1-\varepsilon},0}(u) - \frac{1}{2\varepsilon}\|f\|_{E^*}^2 \leq I_{a,f}(u) \leq (1+\varepsilon) I_{\frac{a}{1+\varepsilon},0}(u) + \frac{1}{2\varepsilon}\|f\|_{E^*}^2. \tag{1.12}$$

(ii) *For* $v \in \Sigma_+$ *and* $\varepsilon \in (0,1)$,

$$(1-\varepsilon)^{\frac{p+1}{p-1}} J_{a,0}(v) - \frac{1}{2\varepsilon}\|f\|_{E^*}^2 \leq J_{a,f}(v) \leq (1+\varepsilon)^{\frac{p+1}{p-1}} J_{a,0}(v) + \frac{1}{2\varepsilon}\|f\|_{E^*}^2. \tag{1.13}$$

(iii) *In particular, there exists a* $d_0 > 0$ *such that if* $\|f\|_{E^*} \leq d_0$, *then*

$$\inf_{v\in\Sigma_+} J_{a,f}(v) > 0.$$

Proof. (i) Since for $\varepsilon \in (0,1)$

$$\left|\int_{\mathbb{R}^N} fu\,dx\right| \leq \|f\|_{E^*}\|u\|_E \leq \frac{\varepsilon}{2}\|u\|_E^2 + \frac{1}{2\varepsilon}\|f\|_{E^*}^2,$$

we have

$$\frac{1-\varepsilon}{2}\,\|u\|_E^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_+^{p+1}\,dx - \frac{1}{2\varepsilon}\|f\|_{E^*}^2 \leq I_{a,f}(u) \leq \frac{1+\varepsilon}{2}\|u\|_E^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_+^{p+1}\,dx + \frac{1}{2\varepsilon}\|f\|_{E^*}^2.$$

Thus we get (1.12).
(ii) From (i), we deduce for $v \in \Sigma_+$

$$(1-\varepsilon) J_{\frac{a}{1-\varepsilon},0}(v) - \frac{1}{2\varepsilon}\|f\|_{E^*}^2 \leq J_{a,f}(v) \leq (1+\varepsilon) J_{\frac{a}{1+\varepsilon},0}(v) + \frac{1}{2\varepsilon}\|f\|_{E^*}^2.$$

We also have from (1.8)

$$J_{\frac{a}{1\pm\varepsilon},0}(v) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\int_{\mathbb{R}^N} \frac{a(x)}{1\pm\varepsilon} v_+^{p+1}\, dx\right)^{-\frac{2}{p-1}}$$
$$= (1\pm\varepsilon)^{\frac{2}{p-1}} J_{a,0}(v).$$

Thus we get (1.13).
(iii) By (1.11) and (1.13), we have

$$\inf_{v\in\Sigma_+} J_{a,f}(v) \geq (1-\varepsilon)^{\frac{p+1}{p-1}} \inf_{v\in\Sigma_+} J_{a,0}(v) - \frac{1}{2\varepsilon}\|f\|_{E^*}^2$$
$$\geq (1-\varepsilon)^{\frac{p+1}{p-1}} \overline{a}^{-\frac{2}{p-1}} I_{1,0}(\omega) - \frac{1}{2\varepsilon}\|f\|_{E^*}^2.$$

Therefore $\inf_{v\in\Sigma_+} J_{a,f}(v) > 0$ for sufficiently small $\|f\|_{E^*}$. □

Next we study properties of a function

$$[0\,,\,\infty) \to \mathbb{R};\ t \mapsto I_{a,f}(tv)$$

for $v \in \Sigma_+$.

Lemma 1.3.

(i) *For every $v \in \Sigma_+$, the function $t \mapsto I_{a,f}(tv)$ has at most two critical points in $[0\,,\,\infty)$.*
(ii) *If $\|f\|_{E^*} \leq d_0$ (d_0 is given in Lemma 1.2), then for any $v \in \Sigma_+$ there exists a unique $t_{a,f}(v) > 0$ such that*

$$I_{a,f}(t_{a,f}(v)v) = J_{a,f}(v).$$

Moreover, $t_{a,f}(v) > 0$ satisfies

$$t_{a,f}(v) > \left(p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\, dx\right)^{-\frac{1}{p-1}} \geq \left(p\overline{a}\bar{C}_{p+1}^{p+1}\right)^{-\frac{1}{p-1}}, \quad (1.14)$$

$$I''_{a,f}(t_{a,f}(v)v)(v\,,\,v) < 0. \quad (1.15)$$

(iii) *If $t \mapsto I_{a,f}(tv)$ has a critical point different from $t_{a,f}(v)$, then it lies in $[0\,,\,(1-\frac{1}{p})^{-1}\|f\|_{E^*}]$.*

Proof. We set

$$g(t) = I_{a,f}(tv) = \frac{1}{2}t^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)v_+^{p+1}\, dx \cdot t^{p+1} - \int_{\mathbb{R}^N} fv\, dx \cdot t.$$

(i) We can see that

$$g''(t) = 1 - p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\, dx \cdot t^{p-1}.$$

Thus

$$g''(t) > 0 \quad \text{for } t < \left(p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx\right)^{-\frac{1}{p-1}},$$
$$g''(t) < 0 \quad \text{for } t > \left(p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx\right)^{-\frac{1}{p-1}}. \tag{1.16}$$

Therefore $g'(t)$ has at most two zeros t_1, t_2 and they satisfy

$$0 \le t_1 \le \left(p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx\right)^{-\frac{1}{p-1}} \le t_2.$$

(ii) Remark that $g(0) = 0$, $g(t) \to -\infty$ as $t \to \infty$ and $\sup_{t>0} g(t) > 0$ by (iii) of Lemma 1.2. Thus there exists a $t_{a,f}(v) \ge \left(p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx\right)^{-\frac{1}{p-1}}$ such that $I_{a,f}(t_{a,f}(v)v) = J_{a,f}(v)$. We remark that

$$t_{a,f}(v) = \left(p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx\right)^{-\frac{1}{p-1}}$$

cannot take a place. If it does, it follows from (1.16) that

$$g'(t) \le 0 \qquad \text{for all } t > 0.$$

It contradicts $\sup_{t>0} g(t) > 0$. Thus we get (1.14). Since $g''(t) = I''_{a,f}(tv)(v, v)$, (1.15) follows from (1.14).
(iii) Suppose that $g(t)$ has a critical point $\underline{t}$ different from $t_{a,f}(v)$. By (1.14) and (1.16), $\underline{t}$ satisfies

$$\underline{t} \le \left(p\int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx\right)^{-\frac{1}{p-1}}. \tag{1.17}$$

It follows from $g'(\underline{t}) = 0$ that

$$\underline{t} - \int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx \cdot \underline{t}^p - \int_{\mathbb{R}^N} fv\,dx = 0.$$

That is,

$$\underline{t}\left(1 - \int_{\mathbb{R}^N} a(x)v_+^{p+1}\,dx \cdot \underline{t}^{p-1}\right) = \int_{\mathbb{R}^N} fv\,dx.$$

By (1.17), we have

$$\underline{t} \le \left(1 - \frac{1}{p}\right)^{-1} \|f\|_{E^*}$$ □

As to the behavior of $I_{a,f}(u)$ near 0, we have

Lemma 1.4. *There exist* $r_1 > 0$ *and* $d_1 \in (0\,,\,d_0]$ *such that*

(i) $I_{a,f}(u)$ *is strictly convex in* $B(r_1) = \{u \in E;\ ||u||_E < r_1\}$.
(ii) *If* $\|f\|_{E^*} \le d_1$, *then*

$$\inf_{\|u\|_E = r_1} I_{a,f}(u) > 0.$$

Moreover, $I_{a,f}(u)$ *has a unique critical point* $u_{loc\,min}(a, f; x)$ *in* $B(r_1)$ *and it satisfies*

$$u_{loc\,min}(a, f; x) \in B(r_1),$$
$$I_{a,f}(u_{loc\,min}(a, f; x)) = \inf_{u \in B(r_1)} I_{a,f}(u).$$

Proof. (i) By (1.2),

$$\begin{aligned} I''_{a,f}(u)(h,h) &= \|h\|_E^2 - p\int_{\mathbb{R}^N} a(x)u_+^{p-1}h^2\,dx \\ &\ge (1 - p\overline{a}\bar{C}_{p+1}^{p+1}\|u\|_E^{p-1})\|h\|_E^2. \end{aligned}$$

Thus $I''_{a,f}(u)$ is positive definite for $u \in B(r_1)$, where $r_1 = (p\overline{a}\bar{C}_{p+1}^{p+1})^{-\frac{1}{p-1}}$, and $I_{a,f}(u)$ is strictly convex in $B(r_1)$.
(ii) For $\|u\|_E = r_1$, we have again by (1.4),

$$\begin{aligned} I_{a,f}(u) &= \frac{1}{2}\|u\|_E^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_+^{p+1}\,dx - \int_{\mathbb{R}^N} fu\,dx \\ &\ge \frac{1}{2}r_1^2 - \frac{\overline{a}}{p+1}\bar{C}_{p+1}^{p+1}r_1^{p+1} - \|f\|_{E^*}r_1 \\ &= \left(\frac{1}{2} - \frac{\overline{a}}{p+1}\bar{C}_{p+1}^{p+1}r_1^{p-1}\right)r_1^2 - \|f\|_{E^*}r_1 \\ &= \left(\frac{1}{2} - \frac{1}{p(p+1)}\right)r_1^2 - \|f\|_{E^*}r_1. \end{aligned}$$

Thus there exists a $d_1 \in (0\,,\,d_0]$ such that

$$\inf_{\|u\|_E = r_1} I_{a,f}(u) > 0 \qquad \text{for } \|f\|_{E^*} \le d_1.$$

Since $I_{a,f}(u)$ is strictly convex in $B(r_1)$ and $\inf_{\|u\|_E = r_1} I_{a,f}(u) > I_{a,f}(0)$, there exist a unique critical point $u_{loc\,min}(a, f; x)$ of $I_{a,f}(u)$ in $B(r_1)$ and it satisfies

$$I_{a,f}(u_{loc\,min}(a, f; x)) = \inf_{\|u\|_E \le r_1} I_{a,f}(u). \qquad \square$$

Remark 1.5. (i) From the uniqueness of critical points, it follows that

$$u_{loc\,min}(a, 0; x) = 0.$$

(ii) In [1], [11], [15], [25] , the existence of a local minimizer is proved just under the assumption of the mountain pass geometry.

The following property of $u_{loc\,min}(a, f; x)$ can be proved easily.

Lemma 1.6. $u_{loc\,min}(a, f; x) \to 0$ *strongly in* E *as* $\|f\|_{E^*} \to 0$. □

Now we can prove ($I_{a,f}$-1) and ($I_{a,f}$-2).

Proposition 1.7. *Let* $d_2 = \min\{d_1\,,\,(1-\dfrac{1}{p})r_1\} > 0$ *and suppose that* $\|f\|_{E^*} \le d_2$. *Then*

(i) $J_{a,f}(v) \in C^1(\Sigma_+\,,\,\mathbb{R})$ *and*

$$J'_{a,f}(v)h = t_{a,f}(v)I'_{a,f}(t_{a,f}(v)v)h \tag{1.18}$$

for all $h \in T_v\Sigma_+ = \{h \in E\,;\,\langle h\,,\,v\rangle_E = 0\}$.

(ii) $v \in \Sigma_+$ *is a critical point of* $J_{a,f}(v)$ *if and only if* $t_{a,f}(v)v \in E$ *is a critical point of* $I_{a,f}(u)$.

(iii) *Moreover the set of critical points of* $I_{a,f}(u)$ *can be written as*

$$\{t_{a,f}(v)v\,;\,v \in \Sigma_+,\ J'_{a,f}(v) = 0\} \cup \{u_{loc\,min}(a, f; x)\}. \tag{1.19}$$

Proof. (i) By (1.15), we have

$$\frac{d^2}{dt^2}\bigg|_{t=t_{a,f}(v)} I_{a,f}(tv) < 0.$$

Thus by the implicit function theorem, we can see that $t_{a,f}(v) \in C^1(\Sigma_+, (0,\infty))$. Therefore

$$J_{a,f}(v) = I_{a,f}(t_{a,f}(v)v) \in C^1(\Sigma_+\,,\,\mathbb{R}).$$

Since

$$I'_{a,f}(t_{a,f}(v)v)v = 0, \tag{1.20}$$

we have

$$\begin{aligned} J'_{a,f}(v)h &= I'_{a,f}(t_{a,f}(v)v)(t_{a,f}(v)h + (t'_{a,f}(v), h)v) \\ &= t_{a,f}(v)I'_{a,f}(t_{a,f}(v)v)h \end{aligned}$$

for $h \in T_v\Sigma_+ = \{h \in E\,,\,\langle h\,,\,v\rangle_E = 0\}$.

(ii) By (i), $J'_{a,f}(v) = 0$ if and only if

$$I'_{a,f}(t_{a,f}(v)v)h = 0 \qquad \text{for all } h \in T_v\Sigma_+.$$

By (1.20), it is equivalent to $I'_{a,f}(t_{a,f}(v)v) = 0$.
(iii) Suppose that $u \in E$ is a critical point of $I_{a,f}(u)$. We write $u = tv$ with $v \in \Sigma_+$ and $t \geq 0$. By (iii) of Lemma 1.3, we have either

$$t = t_{a,f}(v) \quad \text{or} \quad t \leq \left(1 - \frac{1}{p}\right)^{-1} \|f\|_{E^*}.$$

Thus $u \in E$ is corresponding to a critical point of $J_{a,f}(v)$ or

$$\|u\|_E = t \leq \left(1 - \frac{1}{p}\right)^{-1} d_2 \leq r_1.$$

By Lemma 1.4, $I_{a,f}(u)$ has a unique critical point in $B(r_1)$ and it is $u_{loc\,min}(a, f; x)$. □

1.3. The Palais-Smale condition for $I_{a,f}(u)$ and $J_{a,f}(v)$

Next we study the break down of the Palais-Smale condition for $I_{a,f}(u)$ and $J_{a,f}(v)$. The unique positive solution $\omega(x)$ of (0.8)–(0.10) plays an important role to describe an asymptotic behavior of Palais-Smale sequence for $I_{a,f}(u)$.

Proposition 1.8. *Assume (0.4)–(0.7) and suppose that a sequence $(u_j)_{j=1}^{\infty} \subset E$ satisfies*

$$I'_{a,f}(u_j) \to 0 \quad \text{in } E^*,$$
$$I_{a,f}(u_j) \to c \in \mathbb{R}$$

as $j \to \infty$. Then there exist a subsequence — still we denote by $(u_j)_{j=1}^{\infty}$ —, a critical point $u_0(x)$ of $I_{a,f}(u)$, an integer $\ell \in \mathbb{N} \cup \{0\}$, and ℓ sequences of points $(y_j^1)_{j=1}^{\infty}, \dots, (y_j^{\ell})_{j=1}^{\infty} \subset \mathbb{R}^N$ such that

1° $|y_j^k| \to \infty$ *as* $j \to \infty$ *for all* $k = 1, 2, \dots, \ell$,
2° $|y_j^k - y_j^{k'}| \to \infty$ *as* $j \to \infty$ *for* $k \neq k'$,
3° $\left\| u_j(x) - \left(u_0(x) + \sum_{k=1}^{\ell} \omega(x - y_j^k) \right) \right\|_E \to 0$ *as* $j \to \infty$,
4° $I_{a,f}(u_j) \to I_{a,f}(u_0) + \ell I_{1,0}(\omega)$ *as* $j \to \infty$.

Proof. This is rather standard result. See [7], [20], [21] for analogous arguments. □

As to $J_{a,f}(v)$, we have the following

Proposition 1.9. *Suppose that $\|f\|_{E^*} \leq d_2$, where $d_2 > 0$ is given in Proposition 1.7. Then we have*

(i) $J_{a,f}(v) \to \infty$ *as* $\mathrm{dist}_E(v, \partial\Sigma_+) \equiv \inf\{\|v-u\|_E\,;\, u \in \Sigma,\ u_+ \equiv 0\} \to 0$.

(ii) *Suppose that* $(v_j)_{j=1}^{\infty} \subset \Sigma_+$ *satisfies as* $j \to \infty$

$$J_{a,f}(v_j) \to c \qquad \text{for some } c > 0, \tag{1.21}$$

$$\|J'_{a,f}(v_j)\|_{T^*_{v_j}\Sigma_+} \equiv \sup\{J'_{a,f}(v_j)h\,;\, h \in T_{v_j}\Sigma_+,\ \|h\|_E = 1\} \to 0. \tag{1.22}$$

Then there exist a subsequence — still we denote by $(v_j)_{j=1}^{\infty}$ *—, a critical point* $u_0(x) \in E$ *of* $I_{a,f}(u)$*, an integer* $\ell \in \mathbb{N} \cup \{0\}$ *and* ℓ *sequences of points* $(y_j^1)_{j=1}^{\infty}, \dots, (y_j^{\ell})_{j=1}^{\infty} \subset \mathbb{R}^N$ *such that*

1° $|y_j^k| \to \infty$ *as* $j \to \infty$ *for all* $k = 1, 2, \dots, \ell$,

2° $|y_j^k - y_j^{k'}| \to \infty$ *as* $j \to \infty$ *for* $k \neq k'$,

3° $\displaystyle \left\| v_j(x) - \frac{u_0(x) + \sum_{k=1}^{\ell} \omega(x - y_j^k)}{\left\| u_0(x) + \sum_{k=1}^{\ell} \omega(x - y_j^k) \right\|_E} \right\|_E \to 0$ *as* $j \to \infty$,

4° $J_{a,f}(v_j) \to I_{a,f}(u_0) + \ell I_{1,0}(\omega)$ *as* $j \to \infty$.

Proof. (i) By (ii) of Lemma 1.2 and (1.8), we have

$$\begin{aligned} J_{a,f}(v) &\geq (1-\varepsilon)^{\frac{p+1}{p-1}} J_{a,0}(v) - \frac{1}{2\varepsilon}\|f\|_{E^*} \\ &\geq (1-\varepsilon)^{\frac{p+1}{p-1}} \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(\int_{\mathbb{R}^N} a(x) v_+^{p+1} dx\right)^{-\frac{2}{p-1}} k! - \frac{1}{2\varepsilon}\|f\|_{E^*}. \end{aligned}$$

Since $\mathrm{dist}\,(v\,,\,\partial\Sigma_+) \to 0$ implies $v_+ \to 0$ in E, in particular,

$$\int_{\mathbb{R}^N} a(x) v_+^{p+1}\, dx \to 0,$$

we get (i).

(ii) Recalling (1.14) and (1.18), we have

$$\begin{aligned} \|I'_{a,f}(t_{a,f}(v_j)v_j)\|_{E^*} &= \frac{1}{t_{a,f}(v_j)} \|J'_{a,f}(v_j)\|_{T^*_{v_j}\Sigma_+} \\ &\leq (p\bar{a}\bar{C}_{p+1}^{p+1})^{\frac{1}{p-1}} \|J'_{a,f}(v_j)\|_{T^*_{v_j}\Sigma_+} \\ &\to 0 \qquad \text{as } j \to \infty. \end{aligned}$$

We also have

$$I_{a,f}(t_{a,f}(v_j)v_j) = J_{a,f}(v_j) \to c.$$

Applying Proposition 1.8, we get the conclusion (ii). □

As to the corollary to Proposition 1.9, we have

Corollary 1.10. *Suppose that* $\|f\|_{E^*} \le d_2$*. Then* $J_{a,f}(v)$ *satisfies the condition* $(\mathrm{PS})_c$ *for* $c < I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega)$*.*

Here we say that $J_{a,f}(v)$ satisfies $(\mathrm{PS})_c$ if and only if any sequence $(v_j)_{j=1}^{\infty} \subset \Sigma_+$ satisfying (1.21) and (1.22) has a strongly convergent subsequence in E.

Proof. By Proposition 1.9, $(\mathrm{PS})_c$ breaks down only for

$$c = I_{a,f}(u_0) + \ell I_{1,0}(\omega),$$

where $u_0 \in E$ is a critical point of $I_{a,f}(u)$ and $\ell \in \mathbb{N}$. By (1.19), (iii) of Lemma 1.2 and

$$I_{a,f}(u_{loc\,min}(a,f;x)) = \inf_{u \in B(r_1)} I_{a,f}(u) \le I_{a,f}(0) = 0,$$

we have

$$\begin{aligned} &I_{a,f}(u_{loc\,min}(a,f;x)) \\ &= \inf\{I_{a,f}(u_0);\ u_0 \in E \text{ is a critical point of } I_{a,f}(u)\}. \end{aligned}$$

Thus the lowest level of breaking down of $(\mathrm{PS})_c$ is $I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega)$. □

Later in Sect. 2, we will find two critical points below the level

$$I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega).$$

1.4. Properties of $J_{a,0}(v)$

Here we give some properties of the functional $J_{a,0}(v) \in C^1(\Sigma_+, \mathbb{R})$ under the condition (0.15) and (0.16) in addition to (0.4) and (0.5).

Lemma 1.11. *Assume (0.4), (0.5) and (0.15). Then*
(i) $\inf_{v \in \Sigma_+} J_{a,0}(v) = I_{1,0}(\omega)$.
(ii) $\inf_{v \in \Sigma_+} J_{a,0}(v)$ *is not attained.*
(iii) $J_{a,0}(v)$ *satisfies* $(\mathrm{PS})_c$ *for* $c \in (-\infty, I_{1,0}(\omega)) \cup (I_{1,0}(\omega), 2I_{1,0}(\omega))$.

Proof. This is a rather standard result. See for example [20],[21]. □

The following property of $J_{a,0}(v)$ is important to obtain the multiplicity of solutions for (0.1)–(0.3).

Lemma 1.12. *Assume that (0.4), (0.5) and (0.15). Then there exists a constant $\delta_0 > 0$ such that if $J_{a,0}(v) \le I_{1,0}(\omega) + \delta_0$, then*

$$\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla v|^2 + |v|^2)\, dx \neq 0. \tag{1.23}$$

Proof. Since $\inf_{v \in \Sigma_+} J_{a,0}(v) = I_{1,0}(\omega)$ is not attained, it follows from (ii) of Proposition 1.9 that for any $R \ge 1$ there exists an $\varepsilon = \varepsilon(R) > 0$ with the following property: if $v \in \Sigma_+$ satisfies

$$\begin{aligned} &J_{a,0}(v) \le I_{1,0}(\omega) + \varepsilon, \\ &\|J'_{a,0}(v)\|_{T^*_v \Sigma_+} \le \varepsilon \end{aligned}$$

then

$$\left\| v - \frac{\omega(x-y)}{\|\omega\|_E} \right\|_E \le \frac{1}{R} \qquad \text{for some } |y| \ge R. \tag{1.24}$$

We choose $R \ge 1$ sufficiently large so that (1.24) implies (1.23). Suppose that $v \in \Sigma_+$ satisfies $J_{a,0}(v) \le I_{1,0}(\omega) + \delta_0$. Then by Ekeland's principle, there exists $\tilde{v} \in \Sigma_+$ such that

$$\begin{aligned} \|\tilde{v} - v\|_E &\le \sqrt{\delta}_0, \\ \|J'_{a,0}(\tilde{v})\|_{T^*_{\tilde{v}} \Sigma_+} &\le \sqrt{\delta}_0, \\ J_{a,0}(\tilde{v}) &\le I_{1,0}(\omega) + \delta_0. \end{aligned}$$

Choosing $\delta_0 \le \min\{\varepsilon(2R)^2, \varepsilon(2R), \frac{1}{4R^2}\}$, we have

$$\begin{aligned} \left\| v - \frac{\omega(\cdot - y)}{\|\omega\|_E} \right\|_E &\le \|v - \tilde{v}\|_E + \left\| \tilde{v} - \frac{\omega(\cdot - y)}{\|\omega\|_E} \right\|_E \\ &\le \sqrt{\delta}_0 + \frac{1}{2R} \\ &\le \frac{1}{R} \end{aligned}$$

for some $|y| \ge 2R$. Thus we have (1.23). □

As to the existence of a critical point of $J_{a,0}(v)$, we deduce the following result from [5].

Proposition 1.13. (c.f. [5].) *Assume that (0.4), (0.5), (0.15) and (0.16). Then $J_{a,0}(v)$ has at least one critical point $v_a(x) \in \Sigma_+$, which can be characterized as*

$$J_{a,0}(v_a) = \inf_{\gamma \in \Gamma} \sup_{y \in \mathbb{R}^N} J_{a,0}(\gamma(y)).$$

Here Γ is given by

$$\Gamma = \{\gamma \in C(\mathbb{R}^N\,,\,\Sigma_+)\,;\,\gamma(y) = \frac{\omega(\cdot - y)}{\|\omega\|_E} \text{ for large } |y|\}. \tag{1.25}$$

Moreover $v_a(x)$ satisfies

$$J_{a,0}(v_a) \in (I_{1,0}(\omega), 2I_{1,0}(\omega)).$$

Proof. This result is essentially due to Bahri and Li [5]. See also [23] for a similar argument. Here we use notation in [23]. We consider two values:

$$\underline{b}_a = \inf_{v\in\Sigma_+} J_{a,0}(v),$$
$$\overline{b}_a = \inf_{\gamma\in\Gamma}\sup_{y\in\mathbb{R}^N} J_{a,0}(\gamma(y)).$$

Here Γ is given in (1.25). Using an idea from [5], we can see $\overline{b}_a < 2I_{1,0}(\omega)$ under the condition (0.16). By (0.15), we can see $\underline{b}_a = I_{1,0}(\omega)$. By the argument of [5], we can see if $\overline{b}_a = I_{1,0}(\omega)$, then there exists a critical point $v \in \Sigma_+$ such that $J_{a,0}(v) = \overline{b}_a = I_{1,0}(\omega)$. However this contradicts Lemma 1.11. Thus $\overline{b}_a \in (I_{1,0}(\omega)\,,\,2I_{1,0}(\omega))$ and $\overline{b}_a$ is a critical value of $J_{a,0}(v)$. □

Finally in this section we state some refinement of Corollary 1.10.

Proposition 1.14. *Assume that (0.4), (0.5), (0.15) and (0.16). Then for any $\varepsilon > 0$ there exists $d(\varepsilon) \in (0\,,\,d_2]$ such that for $\|f\|_{E^*} \le d(\varepsilon)$*
(i) $\|u_{loc\,min}(a,f;x)\|_E \le \varepsilon$.
(ii) $\inf_{v\in\Sigma_+} J_{a,f}(v) \in [I_{1,0}(\omega) - \varepsilon\,,\,I_{1,0}(\omega) + \varepsilon]$.
(iii) $J_{a,f}(v)$ *satisfies* (PS)$_c$ *for*

$$c \in (-\infty\,,\,I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega))$$
$$\cup(I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega)\,,\,2I_{1,0}(\omega) - \varepsilon).$$

Proof. (i) follows from Lemma 1.6. Since $\inf_{v\in\Sigma_+} J_{a,0}(v) = I_{1,0}(\omega)$, (ii) follows from Lemma 1.2. Now we prove (iii). As in Corollary 1.10, (PS)$_c$ breaks down only at levels $c = I_{a,f}(u_0) + \ell I_{1,0}(\omega)$, where $u_0(x)$ is a critical point of $I_{a,f}(u)$ and $\ell \in \mathbb{N}$. It follows from (iii) of Proposition 1.7 that $u_0(x) = u_{loc\,min}(a,f;x)$ or

$$I_{a,f}(u_0) \ge \inf_{v\in\Sigma_+} J_{a,f}(v) \ge I_{1,0}(\omega) - \varepsilon.$$

Thus, (PS)$_c$ breaks down for

$$c = I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega) \quad \text{or} \quad c \ge 2I_{1,0}(\omega) - \varepsilon.$$

Therefore we get (iii). □

2. Category of $[J_{a,f} \leq I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega) - \varepsilon]$

As in the introduction, we use notation:

$$[J_{a,f} \leq c] = \{v \in \Sigma_+ \,;\, J_{a,f}(v) \leq c\}$$

for $c \in \mathbb{R}$. To find critical points of $J_{a,f}(v)$, we show for a sufficiently small $\varepsilon > 0$

$$\mathrm{cat}([J_{a,f} \leq I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega) - \varepsilon]) \geq 2. \tag{2.1}$$

To prove (2.1), we need some preliminaries.

2.1. Some energy estimates

The following estimate plays an essential role in the proof of (2.1).

Proposition 2.1. (c.f. Proposition 3.1 of [1]) *Assume (0.4), (0.5), (0.16) and suppose that* $\|f\|_{E^*} \leq d_2$, $f \geq 0$ *and* $f \not\equiv 0$. *Then there exists* $R_0 > 0$ *such that*

$$\begin{aligned} &I_{a,f}(u_{loc\,min}(a,f;x) + t\omega(x-y)) \\ &< I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega) \end{aligned} \tag{2.2}$$

for all $|y| \geq R_0$ *and* $t \geq 0$.

Proof. Straightforward computation gives us

$$\begin{aligned} I_{a,f}(u_{loc\,min}(a,f;x) + t\omega(x-y)) &= \frac{1}{2}\|u_{loc\,min}(x) + t\omega(x-y)\|_E^2 \\ &- \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)(u_{loc\,min}(x) + t\omega(x-y))^{p+1}\,dx \\ &- \int_{\mathbb{R}^N} f(u_{loc\,min}(x) + t\omega(x-y))\,dx \\ = \frac{1}{2}\|u_{loc\,min}(x)\|_E^2 + \frac{t^2}{2}\|\omega\|_E^2 &+ t\langle u_{loc\,min}(x)\,,\,\omega(x-y)\rangle_E \\ &- \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)u_{loc\,min}(x)^{p+1}\,dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N} a(x)\omega(x-y)^{p+1}\,dx \\ &- \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)\{(u_{loc\,min}(x) + t\omega(x-y))^{p+1} \\ &- u_{loc\,min}(x)^{p+1} - t^{p+1}\omega(x-y)^{p+1}\}\,dx \\ &- \int_{\mathbb{R}^N} f(x)(u_{loc\,min}(x) + t\omega(x-y))\,dx\,. \end{aligned} \tag{2.3}$$

Here we write $u_{loc\,min}(x) = u_{loc\,min}(a, f; x)$. Since $I'_{a,f}(u_{loc\,min}(x)) = 0$, we have

$$\langle u_{loc\,min}(x)\,,\, h\rangle_E = \int_{\mathbb{R}^N} a(x) u_{loc\,min}(x)^p h\,dx + \int_{\mathbb{R}^N} fh\,dx \quad \text{for all } h \in E.$$

Setting $h(x) = t\omega(x-y)$, we have

$$t\langle u_{loc\,min}\,,\, \omega(x-y)\rangle_E = t\int_{\mathbb{R}^N} a(x) u_{loc\,min}(x)^p \omega(x-y)\,dx + t\int_{\mathbb{R}^N} f(x)\omega(x-y)\,dx.$$

Thus

$$\begin{aligned}
I_{a,f}(\, & u_{loc\,min}(x) + t\omega(x-y)) \\
&= I_{a,f}(u_{loc\,min}(x)) + I_{1,0}(t\omega) \\
&\quad + \frac{1}{p+1}\int_{\mathbb{R}^N} (1-a(x)) t^{p+1}\omega(x-y)^{p+1}\,dx \\
&\quad - \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)\{(u_{loc\,min}(x) + t\omega(x-y))^{p+1} - u_{loc\,min}(x)^{p+1} \\
&\qquad\qquad - t^{p+1}\omega(x-y)^{p+1} - (p+1)u_{loc\,min}(x)^p t\omega(x-y)\}\,dx \\
&= I_{a,f}(u_{loc\,min}(x)) + I_{1,0}(t\omega) + (\mathrm{I}) - (\mathrm{II}).
\end{aligned}$$

Using (1.7), we have

$$I_{a,f}(u_{loc\,min}(x) + t\omega(x-y)) \le I_{a,f}(u_{loc\,min}(x)) + I_{1,0}(\omega) + (\mathrm{I}) - (\mathrm{II}).$$

Since

$$I_{a,f}(u_{loc\,min}(x) + t\omega(x-y)) \to I_{a,f}(u_{loc\,min}(x)) < 0 \qquad \text{as } t \to 0,$$
$$I_{a,f}(u_{loc\,min}(x) + t\omega(x-y)) \to -\infty \qquad \text{as } t \to \infty,$$

we can easily find $0 < m < M$ such that

$$I_{a,f}(u_{loc\,min}(x) + t\omega(x-y)) \le 0 \qquad \text{for } t \in [0\,, m] \cup [M\,, \infty).$$

Thus it suffices to prove (2.2) for $t \in [m\,,\, M]$.

We recall the fact that for some $c > 0$

$$\omega(|x|)\,|x|^{\frac{N-1}{2}} \exp(|x|) \to c \qquad \text{as } |x| \to \infty.$$

(See Bahri-Li [5], Bahri-Lions [6], Gidas-Ni-Nirenberg [12, 13] and Kwong [17]). In particular, we have

(i) there exists a constant $C_0 > 0$ such that

$$\omega(x) \le C_0 \exp(-|x|) \qquad \text{for all } x \in \mathbb{R}^N;$$

(ii) for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\omega(x) \ge C_\varepsilon \exp(-(1+\varepsilon)|x|) \qquad \text{for all } x \in \mathbb{R}^N.$$

We also remark that

(i) $(s+t)^{p+1} - s^{p+1} - t^{p+1} - (p+1)s^p t \ge 0$ for all $(s,t) \in [0,\infty)\times[0,\infty)$;
(ii) for any $r > 0$ we can find a constant $A(r) > 0$ such that

$$(s+t)^{p+1} - s^{p+1} - t^{p+1} - (p+1)s^p t \ge A(r)t^2 \quad \text{for all } (s,t) \in [r,\infty)\times[0,\infty).$$

Thus, setting $A = A(\min_{|x|\le 1} u_{loc\,min}(a,f;x)) > 0$, we have

$$\begin{aligned}
(\mathrm{II}) &\ge \frac{1}{p+1}\int_{|x|\le 1} a(x)\{(u_{loc\,min} + t\omega)^{p+1} - {u_{loc\,min}}^{p+1} - t^{p+1}\omega^{p+1} \\
&\qquad - (p+1){u_{loc\,min}}^{p} t\omega\}\,dx \\
&\ge \underline{a}A \int_{|x|\le 1} \omega(x-y)^2\,dx \\
&\ge \underline{a}AC'_\varepsilon \exp(-2(1+\varepsilon)|y|). \qquad (2.4)
\end{aligned}$$

We also have from (0.16)

$$\begin{aligned}
(\mathrm{I}) &\le \frac{1}{p+1}\int_{\mathbb{R}^N} C\exp(-(2+\delta)|x|)C_0^{p+1}\exp(-(p+1)|x-y|)\,dx \\
&\le C'\exp(-\min\{p+1\,,\,2+\delta\}|y|). \qquad (2.5)
\end{aligned}$$

Since (2.4) holds for any $\varepsilon > 0$, choosing $2\varepsilon < \delta$, we can find $R_0 > 0$ such that

$$(\mathrm{I}) < (\mathrm{II}) \qquad \text{for } |y| \ge R_0.$$

Thus we get (2.2). □

Remark 2.2. A similar argument is given in [1] to show the existence of at least two positive solutions for (0.12). This idea is originally used by Bahri-Li [5] to show $\overline{b}_a < 2I_{1,0}(\omega)$. See also Bahri-Lions [6], Bahri-Coron [4], Taubes [24].

Remark 2.3. (2.2) does not hold for $f(x) \equiv 0$. In fact, if $f(x) \equiv 0$, then $u_{loc\,min}(a,0;x) = 0$ and

$$I_{a,0}(u_{loc\,min}(a,0;x) + \omega(x-y)) = I_{a,0}(\omega(x-y)) > I_{1,0}(\omega).$$

2.2. Lusternik-Schnirelman category

Now we are going to show (2.1). First of all, we recall the definition of Lusternik-Schnirelman category.

Definition. (i) For a topological space X, we say a non-empty, closed subset $A \subset X$ is *contractible to a point in* X if and only if there exists a continuous mapping

$$\eta : [0, 1] \times A \to X$$

such that for some $x_0 \in X$

1° $\eta(0, x) = x$ for all $x \in A$,
2° $\eta(1, x) = x_0$ for all $x \in A$.

(ii) We define

$$\operatorname{cat}(X) = \min\{k \in \mathbb{N};\ \text{there exist closed subsets } A_1, \dots, A_k \subset X \text{ such that } 1^\circ\ A_j \text{ is contractible to a point in } X \text{ for all } j,\ 2^\circ \bigcup_{j=1}^{k} A_j = X\}.$$

When there do not exist finitely many closed subsets $A_1,\dots,A_k \subset X$ such that 1° and 2° hold, we say $\operatorname{cat}(X) = \infty$.

For fundamental properties of Lusternik-Schnirelman category, we refer to Ambrosetti [2] and Schwartz [22]. Here we use the following property:

Proposition 2.4. *Suppose that M is a Hilbert manifold and $\Psi \in C^1(M, \mathbb{R})$. Assume that for $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$*

1° *$\Psi(x)$ satisfies $(PS)_c$ for $c \le c_0$,*
2° $\operatorname{cat}(\{x \in M;\ \Psi(x) \le c_0\}) \ge k$.

Then $\Psi(x)$ has at least k critical points in $\{x \in M;\ \Psi(x) \le c_0\}$. □

To estimate Lusternik-Schnirelman category, the following lemma is useful.

Lemma 2.5. *Let $N \ge 1$ and X be a topological space. Suppose that there exist two continuous mapping*

$$F : S^{N-1} = \{y \in \mathbb{R}^N;\ |y| = 1\} \to X,\ G : X \to S^{N-1}$$

such that $G \circ F$ is homotopic to identity : $S^{N-1} \to S^{N-1}$; $x \mapsto x$, that is, there exists a continuous mapping $\zeta : [0, 1] \times S^{N-1} \to S^{N-1}$ such that

$$\zeta(0, x) = (G \circ F)(x) \quad \text{for all } x \in S^{N-1},$$
$$\zeta(1, x) = x \quad \text{for all } x \in S^{N-1}.$$

Then

$$\mathrm{cat}(X) \geq 2.$$

Proof. We argue indirectly and suppose that $\mathrm{cat}(X) = 1$. That is, X is contractible to a point in itself. Thus there exists

$$\eta : [0, 1] \times X \to X$$

such that for some $x_0 \in X$

$$\eta(0, x) = x \qquad \text{for all } x \in X,$$
$$\eta(1, x) = x_0 \qquad \text{for all } x \in X.$$

Now we consider a homotopy $\beta : [0, 1] \times S^{N-1} \to S^{N-1}$ defined by

$$\beta(s, x) = G(\eta(s, F(x))).$$

Clearly

$$\beta(0, x) = (G \circ F)(x) \qquad \text{for all } x \in X,$$
$$\beta(1, x) = G(x_0) \qquad \text{for all } x \in X.$$

Thus $G \circ F$ is homotopic to a constant mapping. However, by assumption, $G \circ F$ is homotopic to the identity. This is a contradiction and we have $\mathrm{cat}(X) \geq 2$. □

From now on, we construct two mappings

$$F : S^{N-1} \to [J_{a,f} \leq I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega) - \varepsilon],$$

$$G : [J_{a,f} \leq I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega) - \varepsilon] \to S^{N-1},$$

so that $G \circ F$ is homotopic to the identity.

2.3. A mapping $F_R : S^{N-1} \to \Sigma_+$

We define a mapping $F_R : S^{N-1} \to \Sigma_+$ in the following way: in Proposition 2.1, we observed that for $|y| \geq R_0$

$$I_{a,f}(u_{loc\,min}(a, f; x) + t\omega(x - y)) < I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega) \quad \text{for all } t \geq 0.$$

For $|y| \geq R_0$, we find $s = s(f, y) > 0$ such that

$$u_{loc\,min}(a, f; x) + s\omega(x - y) = t_{a,f}\left(\frac{u_{loc\,min}(a, f; x) + s\omega(x - y)}{\|u_{loc\,min}(a, f; x) + s\omega(x - y)\|_E}\right) \times \frac{u_{loc\,min}(a, f; x) + s\omega(x - y)}{\|u_{loc\,min}(a, f; x) + s\omega(x - y)\|_E},$$

that is,

$$\|u_{loc\,min}(a,f;x)+s\omega(x-y)\|_E = t_{a,f}\left(\frac{u_{loc\,min}(a,f;x)+s\omega(x-y)}{\|u_{loc\,min}(a,f;x)+s\omega(x-y)\|_E}\right). \tag{2.6}$$

This implies

$$\begin{aligned} &J_{a,f}\left(\frac{u_{loc\,min}(a,f;x)+s\omega(x-y)}{\|u_{loc\,min}(a,f;x)+s\omega(x-y)\|_E}\right)\\ &\quad = I_{a,f}(u_{loc\,min}(a,f;x)+s\omega(x-y))\\ &\quad < I_{a,f}(u_{loc\,min}(a,f;x))+I_{1,0}(\omega). \end{aligned}$$

Proposition 2.6. *Assume (0.4), (0.5) and (0.16). Then there exist* $d_3 \in (0\,,\,d_2]$ *and* $R_1 > R_0$ *such that for any* $\|f\|_{E^*} \le d_3$ *and for any* $|y| \ge R_1$ *there exists a unique* $s = s(f,y) > 0$ *in a neighborhood of* 1 *satisfying (2.6). Moreover*

$$\{y \in \mathbb{R}^N\,;\,|y| \ge R_1\} \to (0\,,\,\infty)\,;\, y \mapsto s(f,y)$$

is continuous.

Proof. We use the implicit function theorem to prove Proposition 2.6. Set

$$\begin{aligned} \Phi(s,f,y) &= \langle I'_{a,f}(u_{loc\,min}(a,f;x)+s\omega(x-y)),\\ &\qquad u_{loc\,min}(a,f;x)+s\omega(x-y)\rangle_{E^*,E}\\ &= \|u_{loc\,min}(a,f;x)+s\omega(x-y)\|_E^2\\ &\qquad -\int_{\mathbb{R}^N} a(x)(u_{loc\,min}(a,f;x)+s\omega(x-y))^{p+1}\,dx\\ &\qquad -\int_{\mathbb{R}^N} f(u_{loc\,min}(a,f;x)+s\omega(x-y))\,dx. \end{aligned}$$

Then (2.6) holds if and only if $\Phi(s,f,y)=0$. Since $\omega(x)$ is a unique positive radial solution of (0.8)–(0.10), we have

$$\|\omega\|_E^2 - \int_{\mathbb{R}^N} \omega^{p+1}\,dx = 0.$$

Using this fact, we see that

$$\begin{aligned} \Phi(1,0,y) &= \langle I'_{a,0}(\omega(x-y))\,,\,\omega(x-y)\rangle_{E^*,E}\\ &= ||\omega||_E^2 - \int_{\mathbb{R}^N} a(x)\omega(x-y)^{p+1}\,dx\\ &\to 0 \qquad \text{as } |y|\to\infty,\\ \left.\frac{\partial}{\partial s}\right|_{s=1}\Phi(s,0,y) &= 2||\omega||_E^2 - (p+1)\int_{\mathbb{R}^N} a(x)\omega(x-y)^{p+1}\,dx\\ &\to -(p-1)||\omega||_E^2 < 0 \qquad \text{as } |y|\to\infty. \end{aligned}$$

Thus by the implicit function theorem, we can find a unique $s = s(f, y)$ in a neighborhood of 1 such that $\Phi(s, f, y) = 0$. The continuity of $y \mapsto s(f, y)$ is also clear. $\square$

Now we define a function $F_R : S^{N-1} = \{y \in \mathbb{R}^N \,;\, |y| = 1\} \to \Sigma_+$ by

$$F_R(y) = \frac{u_{loc\,min}(a, f; x) + s(f, Ry)\omega(x - Ry)}{\|u_{loc\,min}(a, f; x) + s(f, Ry)\omega(x - Ry)\|_E}$$

for $\|f\|_{E^*} \le d_3$ and $R \ge R_1$. Then we have

Proposition 2.7. *For $0 < \|f\|_{E^*} \le d_3$ and $R \ge R_1$ there exists $\varepsilon_0(R) > 0$ such that*

$$F_R(S^{N-1}) \subset [J_{a,f} \le I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega) - \varepsilon_0(R)].$$

Proof. By construction, we have

$$F_R(S^{N-1}) \subset [J_{a,f} < I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega)].$$

Since $F(S^{N-1})$ is compact, the conclusion holds. $\square$

Thus we construct a mapping

$$F_R \,:\, S^{N-1} \to [J_{a,f} \le I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega) - \varepsilon_0(R)].$$

2.4. A mapping $G \,:\, [J_{a,f} < I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega)] \to S^{N-1}$

First we remark that

Lemma 2.8. *There exists $d_4 \in (0\,,\, d_3]$ such that if $\|f\|_{E^*} \le d_4$, then*

$$[J_{a,f} < I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega)] \subset [J_{a,0} < I_{1,0}(\omega) + \delta_0], \quad (2.7)$$

where $\delta_0 > 0$ is given in Lemma 1.12.

Proof. By (1.13), we have for any $\varepsilon \in (0\,,\,1)$

$$J_{a,0}(v) \le (1-\varepsilon)^{-\frac{p+1}{p-1}} \left(J_{a,f}(v) + \frac{1}{2\varepsilon}\|f\|^2_{E^*} \right) \qquad \text{for all } v \in \Sigma_+. \quad (2.8)$$

Recalling $I_{a,f}(u_{loc\,min}(a, f; x)) \le 0$,

$$v \in [J_{a,f} < I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega)]$$

implies $J_{a,f}(v) < I_{1,0}(\omega)$. Thus by (2.8), we have

$$J_{a,0}(v) \le (1-\varepsilon)^{-\frac{p+1}{p-1}} \left(I_{1,0}(\omega) + \frac{1}{2\varepsilon}\|f\|^2_{E^*} \right)$$
$$\text{for all} \qquad v \in [J_{a,f} \le I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega)].$$

Since $\varepsilon \in (0\,,\,1)$ is arbitrary, we have (2.7) for sufficiently small $\|f\|_{E^*}$. □

Now we can define

$$G\,:\,[J_{a,f} < I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega)] \to S^{N-1}$$

by

$$G(v) = \int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla v|^2 + |v(x)|^2)\,dx \Bigg/ \left|\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla v|^2 + |v(x)|^2)\,dx\right|.$$

By Lemma 2.8 and Lemma 1.12,

$$\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla v|^2 + |v(x)|^2)\,dx \neq 0$$

for all $v \in [J_{a,f} < I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega)]$ and $G(v)$ is well-defined. Moreover we have

Proposition 2.9. *For a sufficiently large $R \geq R_1$ and for sufficiently small $\|f\|_{E^*} > 0$*

$$G \circ F_R\,:\,S^{N-1} \to S^{N-1}\,;\; y \mapsto G(F_R(y))$$

is homotopic to the identity.

Proof. We define

$$\zeta(\theta, y)\,:\,[0\,,\,1] \times S^{N-1} \to S^{N-1}$$

by

$$\zeta(\theta,y) = \begin{cases} G\left(\dfrac{(1-2\theta)F_R(y) + 2\theta\omega(x-Ry)}{\|(1-2\theta)F_R(y) + 2\theta\omega(x-Ry)\|_E}\right), & \text{for } \theta \in [0\,,\,1/2), \\ G\left(\dfrac{\omega(x - \frac{R}{2(1-\theta)}y)}{\|\omega(x - \frac{R}{2(1-\theta)}y)\|_E}\right), & \text{for } \theta \in [1/2\,,\,1), \\ y, & \text{for } \theta = 1. \end{cases}$$

We can easily see that $\zeta(\theta, y) \in C([0\,,\,1] \times S^{N-1}\,,\,S^{N-1})$ and

$$\zeta(0,y) = G(F_R(y)) \qquad \text{for all } y \in S^{N-1},$$

$$\zeta(1,y) = y \qquad \text{for all } y \in S^{N-1},$$

provided $R > 0$ is sufficiently large and $\|f\|_{E^*} > 0$ is sufficiently small. □

Therefore, applying Lemma 2.5, we have

Proposition 2.10. *For sufficiently large $R \geq R_1$,*

$$\mathrm{cat}([J_{a,f} < I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega) - \varepsilon_0(R)]) \geq 2. \qquad \square$$

Thus we have

Theorem 2.11. *Assume (0.4), (0.5), (0.15) and (0.16). Then there exists $d_5 > 0$ such that if $\|f\|_{E^*} \leq d_5$, $f \geq 0$, $f \not\equiv 0$, then $J_{a,f}(v)$ has at least two critical points in*

$$[J_{a,f} < I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega)].$$

Proof. Since $(\mathrm{PS})_c$ holds for $J_{a,f}(v)$ for $c \in (-\infty\,,\, I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega))$, Theorem 2.11 follows from Proposition 2.4 and Proposition 2.10. $\square$

3. A positive solution related to Bahri-Li's solution and proof of Theorem 0.1

Next we observe that if $\|f\|_{E^*}$ is sufficiently small, Bahri-Li's minimax argument also works for $J_{a,f}(v)$. We define

$$b_{a,f} = \inf_{\gamma \in \Gamma} \sup_{y \in \mathbb{R}^N} J_{a,f}(\gamma(y))$$

where Γ is defined in (1.25). By (1.13), we have for any $\varepsilon \in (0\,,\,1)$

$$(1-\varepsilon)^{\frac{p+1}{p-1}} \overline{b}_a - \frac{1}{2\varepsilon}\|f\|_{E^*}^2 \leq b_{a,f} \leq (1+\varepsilon)^{\frac{p+1}{p-1}} \overline{b}_a + \frac{1}{2\varepsilon}\|f\|_{E^*}^2, \qquad (3.1)$$

where $\overline{b}_a = \inf_{\gamma \in \Gamma} \sup_{y \in \mathbb{R}^N} J_{a,0}(\gamma(y))$. Thus we have

Lemma 3.1. *For any $\delta > 0$ there exists $d_6 > 0$ such that if $\|f\|_{E^*} \leq d_6$, then*

$$b_{a,f} \in (\overline{b}_a - \delta\,,\, \overline{b}_a + \delta). \qquad \square$$

In particular, since $\overline{b}_a \in (I_{1,0}(\omega)\,,\, 2I_{1,0}(\omega))$ by Proposition 1.13, we have

Theorem 3.2. *There exists a $d_7 > 0$ such that if $\|f\|_{E^*} \leq d_7$, then $J_{a,f}(v)$ has a critical point $\overline{v}_{a,f}(x)$ such that*

$$J_{a,f}(\overline{v}_{a,f}) = b_{a,f} \geq I_{a,f}(u_{loc\,min}(a,f;x)) + I_{1,0}(\omega).$$

Proof. Since $\bar{b}_a \in (I_{1,0}(\omega)\,,\,2I_{1,0}(\omega))$, choosing $\delta > 0$ small and applying Lemma 3.1, we can find for $\|f\|_{E^*} \le \delta$

$$b_{a,f} \in (I_{1,0}(\omega) + \varepsilon\,,\,2I_{1,0}(\omega) - \varepsilon)$$

for some $\varepsilon > 0$. By (iii) of Proposition 1.14, $J_{a,f}(v)$ satisfies (PS)$_c$ for $c = b_{a,f}$ and there exists a critical point corresponding to $b_{a,f}$. □

Now we can complete the proof of Theorem 0.1.

End of the proof of Theorem 0.1. First we set $u^{(1)}(x) = u_{loc\,min}(a, f; x)$. $u^{(1)}(x)$ satisfies

$$I_{a,f}(u^{(1)}(x)) < 0. \tag{3.2}$$

By Theorem 2.11, there exist two critical points $v^{(2)}(x)$, $v^{(3)}(x)$ in

$$[J_{a,f} < I_{a,f}(u_{loc\,min}(a, f; x)) + I_{1,0}(\omega)].$$

Let $u^{(2)}(x) = t_{a,f}(v^{(2)})v^{(2)}(x)$, $u^{(3)}(x) = t_{a,f}(v^{(3)})v^{(3)}(x)$ be corresponding solutions. They satisfy

$$\begin{aligned} 0 &< I_{a,f}(u^{(k)}(x)) = J_{a,f}(v^{(k)}(x)) \\ &< I_{a,f}(u^{(1)}(x)) + I_{1,0}(\omega) \quad \text{for } k = 2,\ 3. \end{aligned} \tag{3.3}$$

By Theorem 3.2, there exists another critical point $v^{(4)}(x)$. Let $u^{(4)}(x)$ be a corresponding positive solution. Then it satisfies

$$I_{a,f}(u^{(4)}(x)) = J_{a,f}(v^{(4)}(x)) \ge I_{a,f}(u^{(1)}(x)) + I_{1,0}(\omega). \tag{3.4}$$

By (3.2)–(3.4), $u^{(1)}(x)$, $u^{(2)}(x)$, $u^{(3)}(x)$, $u^{(4)}(x)$, are distinct and (0.1)–(0.3) possesses at least four positive solutions. □

4. Proof of Theorem 0.2

Finally in this section, we give a proof of Theorem 0.2.

Proof of Theorem 0.2. Suppose that $(f_j(x))_{j=1}^{\infty} \subset H^{-1}(\mathbb{R}^N)$ is a sequence of non-negative functions such that $f_j(x) \not\equiv 0$ and $\|f_j\|_{H^{-1}(\mathbb{R}^N)} \to 0$ as $j \to \infty$. Let

$$u_j^{(1)}(x) = u_{loc\,min}(a, f_j; x)$$

and $u_j^{(2)}(x)$, $u_j^{(3)}(x)$ be positive solutions corresponding to Theorem 2.11 and let $u_j^{(4)}(x)$ be a positive solution corresponding to Theorem 3.2. By Lemma 1.6,

$$u_j^{(1)}(x) = u_{loc\,min}(a, f_j; x) \to 0 \qquad \text{in } E \text{ as } j \to \infty.$$

On the other hand, by (1.13), $v_j^{(2)} = u_j^{(2)}(x)/\|u_j^{(2)}(x)\|_E$, $v_j^{(3)} = u_j^{(3)}(x)$ $/\|u_j^{(3)}(x)\|_E$ satisfy for any $\varepsilon \in (0,1)$

$$\begin{aligned} J_{a,0}(v_j^{(k)}(x)) &\leq (1-\varepsilon)^{-\frac{p+1}{p-1}}(J_{a,f_j}(v_j^{(k)}(x)) + \frac{1}{2\varepsilon}\|f_j\|_{E^*}) \\ &\leq (1-\varepsilon)^{-\frac{p+1}{p-1}}(I_{a,f_j}(u_{loc\,min}(a,f_j;x) \\ &\quad +I_{1,0}(\omega) + \frac{1}{2\varepsilon}\|f_j\|_{E^*}) \\ &\leq (1-\varepsilon)^{-\frac{p+1}{p-1}}(I_{1,0}(\omega) + \tfrac{1}{2\varepsilon}\|f_j\|_{E^*}) \end{aligned}$$

for $k = 2$, 3.

Letting $j \to \infty$, we have

$$\limsup_{j\to\infty} J_{a,0}(v_j^{(k)}(x)) \leq (1-\varepsilon)^{-\frac{p+1}{p-1}} I_{1,0}(\omega).$$

Since $\varepsilon \in (0,1)$ is arbitrary, we have

$$\limsup_{j\to\infty} J_{a,0}(v_j^{(k)}(x)) \leq I_{1,0}(\omega).$$

Thus by (i) of Lemma 1.11, $J_{a,0}(v_j^{(k)}(x)) \to I_{1,0}(\omega)$ for $k = 2$, 3.

Recalling that $\inf_{v\in\Sigma_+} J_{a,0}(v) = I_{1,0}(\omega)$ is not achieved and using (ii) of Proposition 1.9, we can get the second statement (ii) of Theorem 0.2.

(iii) For $u_j^{(4)}(x)$, we have from (3.1) that

$$\begin{aligned} (1-\varepsilon)^{-\frac{p+1}{p-1}}\overline{b}_a - \frac{1}{2\varepsilon}\|f_j\|_{E^*}^2 &\leq I_{a,f_j}(v_j^{(k)}(x)) = b_{a,f_j} \\ &\leq (1+\varepsilon)^{-\frac{p+1}{p-1}}\overline{b}_a + \frac{1}{2\varepsilon}\|f_j\|_{E^*}^2. \end{aligned}$$

Letting $j \to \infty$ and $\varepsilon \to 0$, we find

$$\lim_{j\to\infty} I_{a,f_j}(u_j^{(4)}(x)) = \overline{b}_a \in (I_{1,0}(\omega), 2I_{1,0}(\omega)). \tag{4.1}$$

We can also see that

$$\begin{aligned} I'_{a,0}(u_j^{(4)}(x))h &= I'_{a,f_j}(u_j^{(4)}(x))h + \int_{\mathbb{R}^N} f_j(x)h(x)\,dx \\ &= \int_{\mathbb{R}^N} f_j(x)h(x)\,dx \qquad \text{for all } h \in E. \end{aligned}$$

Thus,

$$\|I'_{a,0}(u_j^{(4)}(x))\|_{E^*} = \|f_j\|_{E^*} \to 0 \qquad \text{as } j \to \infty. \tag{4.2}$$

By (4.1) and (4.2), we can deduce that $u_j^{(4)}(x)$ is bounded in E as $j \to \infty$ and

$$I_{a,0}(u_j^{(4)}(x)) \to \bar{b}_a \in (I_{1,0}(\omega), 2I_{1,0}(\omega)). \tag{4.3}$$

Since $I_{a,0}(u)$ satisfies $(PS)_c$ for $c \in (I_{1,0}(\omega), 2I_{1,0}(\omega))$, we can extract a convergent subsequence — still we denote by $u_j^{(4)}(x)$ — such that

$$u_j^{(4)}(x) \to u_0(x) \qquad \text{in } E.$$

Clearly $u_0(x)$ is a critical point of $I_{a,0}(x)$, that is, a positive solution of (0.1)–(0.3) with $f \equiv 0$. □

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References

1. S. Adachi and K. Tanaka, Existence of positive solutions for a class of nonhomogeneous elliptic equations in $\mathbb{R}^N$, Preprint.
2. A. Ambrosetti, Critical points and nonlinear variational problems. Mém. Soc. Math. France (N.S.) **49** (1992), 139 pp.
3. A. Ambrosetti and M. Badiale, Homoclinics: Poincaré-Melnikov type results via a variational approach. Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), 233–252.
4. A. Bahri and J. M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, Comm. Pure Appl. Math. **41** (1988), 253–294.
5. A. Bahri and Y. Y. Li, On the min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$, Rev. Mat. Iberoamericana **6** (1990), 1–15.
6. A. Bahri and P. L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 365–413.
7. V. Benci and G. Cerami, Positive solutions of some nonlinear elliptic problems in exterior domains, Arch. Rational Mech. Anal. **99** (1987), 283–300.
8. G. Bianchi, Non-existence of positive solutions to semilinear elliptic equations on $\mathbb{R}^n$ or $\mathbb{R}^n_+$ through the method of moving planes, Commun. Partial Differential Equations **22** (1997), 1671–1690.
9. G. Bianchi and H. Egnell, An ODE approach to the equation $\Delta u + K u^{\frac{n+2}{n-2}} = 0$, in $\mathbb{R}^n$, Math. Z. **210** (1992), 137–166.
10. G. Bianchi and H. Egnell, A variational approach to the equation $\Delta u + K u^{\frac{n+2}{n-2}} = 0$, in $\mathbb{R}^n$, Arch. Rat. Mech. Anal. **122** (1993), 159–182.
11. D. M. Cao and H. S. Zhou, Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh **126A** (1996), 443–463.
12. B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. **68** (1979), 209–243.
13. B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, Math. Anal. Appli. Part A, Advances in Math. Suppl. Studies 7A, (Ed. L. Nachbin), Academic Press, (1981), 369–402.

14. N. Hirano, Existence of entire positive solutions for nonhomogeneous elliptic equations, Nonlinear Anal. **29** (1997), 889–901.
15. L. Jeanjean, Two positive solutions for a class of nonhomogeneous elliptic equations, Differential Integral Equations **10** (1997), 609–624.
16. Y. Kabeya and K. Tanaka, Uniqueness of positive radial solutions of semilinear elliptic equations in $\mathbb{R}^n$ and Séré's non-degeneracy condition, to appear in Comm. Partial Diff. Eq.
17. M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, Arch. Rational Mech. Anal. **105** (1989), 234–266.
18. C. Li, Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on bounded domains, Commun. Partial Differential Equations **16** (1991), 491–526.
19. C. Li, Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains, Commun. Partial Differential Equations **16** (1991), 585–615.
20. P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 1, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 109–145.
21. P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 223–283.
22. J. T. Schwartz, Nonlinear Functional Analysis, Gordon Breach, 1969.
23. K. Tanaka, Periodic solutions for singular Hamiltonian systems and closed geodesics on non-compact Riemannian manifolds, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire.
24. C. Taubes, Min-Max theory for the Yang-Mills-Higgs equations, Comm. Math. Phys. **97** (1985), 473–540.
25. X. P. Zhu, A perturbation result on positive entire solutions of a semilinear elliptic equation, J. Differential Equations **92** (1991), 163–178.