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Motion of a droplet by surface tension along the boundary

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Abstract. We give a description of the ultimate dynamics for the simplest evolution equation compatible with the Van der Waals Free Energy. We establish existence of stable sets of solutions corresponding to the physical motion of a small, almost semicircular interface (droplet) intersecting the boundary of the domain and moving towards a point where the curvature has a local maximum. Our results represent a particular extension of the Equilibrium theory of Modica and Sternberg to the next dynamic level in the Morse decomposition of the flow.

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1 Introduction

In this paper we study the functional

$$J_{\hat\varepsilon}(u) = \int_\Omega \left(\frac{\hat\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx,$$
$$u \in \left\{ v \in H^1(\Omega) \ : \ \frac{1}{|\Omega|} \int_\Omega v dx = m \right\} \tag{1.1}$$

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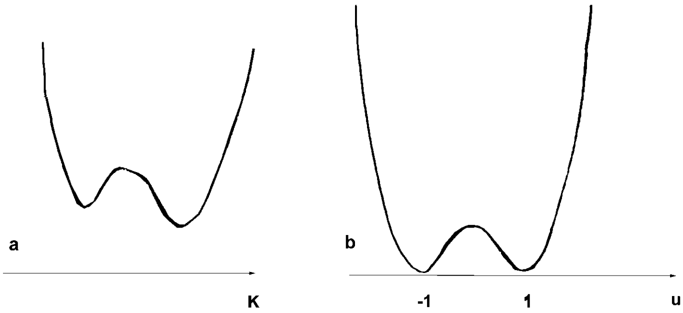

Fig. 1. (**a**). General $W(u)$, (**b**). Normalized $W(u)$

and its associated gradient flow in $L^2(\Omega)$. Here m is a constant and W is a double-well potential. By adding a linear function of u to W if necessary, we can normalize W without affecting the dynamics so as to have wells of equal depth. We take the global minimum of W to be 0 and attained only at $u = \pm 1$ (Fig. 1(a), (b)).

In the 70's, DeGiorgi et al. [66,67] introduced the family of functionals $\{\frac{1}{\hat{\varepsilon}} J_{\hat{\varepsilon}}(u)\}_{\hat{\varepsilon}>0}$ as a means of approximating the perimeter functional

$$P_\Omega(E) = |\partial E|, \quad E \subset \Omega. \tag{1.2}$$

Independently Cahn and his collaborators [5,19,20] introduced (1.1) for describing the evolution of the concentration u for a binary alloy. Some of these ideas had been introduced before by Van Der Waals [63].

The W term favors functions that take values close to its minima. We call such functions layered. We call **interfaces** the zero level sets of such a function, and we call **states**, the values close to ± 1 that u takes almost uniformly away from the interface. Note that the zero level set could be replaced by any other level set strictly between -1 and 1. Notice also that the mass constraint $\frac{1}{|\Omega|}\int_\Omega u dx = m, m \in (-1, 1)$, forces separation, that is both states have to be taken. In contrast, the gradient term favors the uniform unlayered state and penalizes interfaces by registering their perimeter. The result of this competition is the formation of layered functions with interfaces moving so as to reduce the total perimeter [41,42,64]. Surface tension energy is proportional to the perimeter and is a second order effect in relation to the bulk energy, and so $\hat{\varepsilon}$ is naturally small.

In the present paper we study the motion and stability properties of such interfaces for small $\hat{\varepsilon}$. We restrict ourselves to a single connected interface intersecting the boundary. We also limit ourselves to two space dimensions and therefore to interfaces that are curves. The discussion above suggests that most of the energy of a layered state is concentrated on and near the interface. This in turn suggests that perhaps for small $\hat{\varepsilon}$ the study of (1.1) can be reduced to a purely geometric problem associated to the perimeter functional. This intuition is generally false because the remaining part of the energy which is diffused through out can make a difference. This fact distinguishes diffuse from sharp interface models and makes the former much more interesting from the dynamic point of view. In the present paper size is an important parameter. After stimulating work of Carr, Gurtin and Slemrod [21] in one space dimension, Modica [55], improving on Modica and Mortola [67], and independently Sternberg [61] (see also Owen and Sternberg [57]) established a general relationship between (1.1) and (1.2), as $\hat{\varepsilon}$ tends to zero, for global minimizers. In two space dimensions[1] Chen and Kowalczyk [31] described the structure of local minimizers of (1.1) of small mass by showing that (in the limit) the interface is a circular arc intersecting the boundary orthogonally, and enclosing a point on the boundary where the curvature has a local maximum.[2] The constraint in (1.1) at the geometric level (1.2) is translated into fixed enclosed area . It is clear at the level of (1.2) that one can construct a circular arc intersecting the boundary orthogonally and enclosing a fixed area only at very special locations which are related to the critical points of the curvature of the boundary. It is also intuitive that the interface will be minimal when the curvature of the boundary is maximal. This intuition is behind the Chen-Kowalczyk result.

By the heuristic reasoning above one expects circular interfaces enclosing a point of local minimum of the curvature to correspond to unstable critical points of the functional (1.1). Moreover one expects this unstable equilibria to have unstable manifolds of dimension equal to that of the boundary $\partial\Omega$. A rigorous statement to this effect is stated in Theorem 1.1 below.

In this paper we study dynamics. We consider the simplest dynamical system associated to the constrained functional (1.1), which results after taking the gradient in L^2 of the functional on the Hilbert manifold made up of L^2 functions with fixed average [40, 65]. This produces the so called mass conserving Allen-Cahn equation studied by Rubinstein and Sternberg[59]

$$\begin{cases} \phi_t^{\hat{\varepsilon}}(y,t) = \hat{\varepsilon}^2 \Delta_y \phi^{\hat{\varepsilon}}(y,t) - f(\phi^{\hat{\varepsilon}}(y,t)) + ⨏_\Omega f(\phi^{\hat{\varepsilon}}(\cdot,t)), \\ \qquad\qquad y \in \Omega, t > 0 \\ \partial_n \phi^{\hat{\varepsilon}}(y,t) = 0, \quad y \in \partial\Omega, t > 0, \\ \phi^{\hat{\varepsilon}}(y,0) = \phi_0^{\hat{\varepsilon}}(y), \quad y \in \Omega \end{cases} \tag{1.3}$$

[1] We have not attempted higher space dimensions because of the complexity of the asymptotic expansion, in particular the difficulty of the geometric problem (see Sect. 2.6). Naturally several ingredients of our analysis extend effortlessly to higher dimensions.

[2] Ni and his collaborators (see [69] where also further references can be found) have for some time now identified the critical points of the (mean) curvature of $\partial\Omega$ as possible locations for the peaks of certain equilibrium solutions which they call spikes. The nonlinearities as well as the equilibrium they study are fundamentally different from these in the present paper. Nevertheless there are relationships (see Remark 4.5, and also the Appendix).

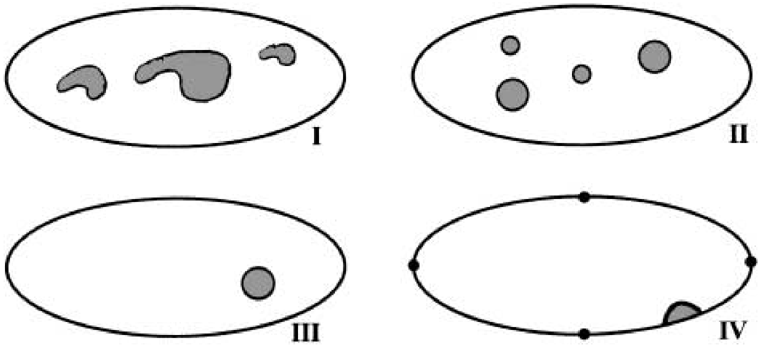

Fig. 2. Four key stages in the evolution for an elliptical domain Ω. Stage IV is the object of study in the present paper. We allow general two-dimensional domains Ω

where Ω is a fixed bounded domain with smooth boundary $\partial\Omega$, ∂_n is the exterior normal derivative to $\partial\Omega$, Δ_y represents the Laplacian with respect to y, and $⨏⨏_\Omega = \frac{1}{|\Omega|}\iint_\Omega$ represents the average over Ω. Here f is the derivative of W. We assume the following conditions for $f \in C^\infty(I\!R^1)$:

$$f(\pm 1) = 0, \quad f'(\pm 1) > 0, \quad \int_{-1}^{s} f = \int_{1}^{s} f > 0 \text{ for all } s \in (-1, 1). \tag{1.4}$$

For this dynamical system we construct a set in $L^2(\Omega)$ which captures all the unstable equilibria alluded to above, together with their unstable manifolds. We do this in appropriate coordinates so that the reduced flow on the one dimensional unstable manifolds corresponds in an unambiguous way to the motion of a (roughly) semicircular interface on the boundary moving towards the increasingly curved region, see Fig. 2.

If the interface happens to be close to a small semicircular shape (that we call **droplet**) one expects (on the basis of isoperimetric reasoning for example) that it will stay semicircular for economizing the perimeter and therefore that its evolution could be described in terms of the motion of one point on the boundary of the domain, which can be thought as the barycenter of the droplet.

Expressing our work in this paper in the language of dynamical systems we would say that we are describing a piece of the attractor of (1.3) for small $\hat{\varepsilon}$. This piece is lying in a sublevel set of energy very close to that of the global minimizer. It is also a very stable and attracting set, and therefore our result renders precise information on the ultimate dynamics of a typical solution to (1.3) see Fig. 3.

We consider the sublevel sets $H_c = \{\phi / J_\varepsilon(\phi) \leq c\}$ and look for the maximal compact invariant sets K_c of (1.3) contained in H_c. It is known that for gradient systems these sets are made up of unstable equilibria and their

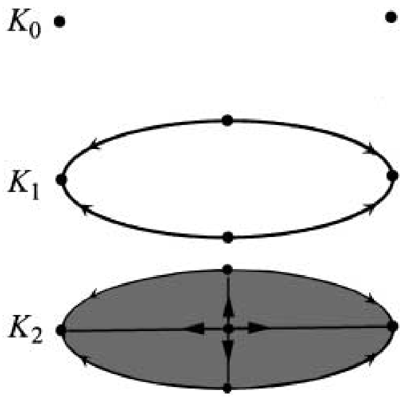

Fig. 3.

unstable manifolds [49]. We expect from bifurcation theory that there must be critical values $c_1^\varepsilon < c_2^\varepsilon < \ldots < c_n^\varepsilon < \ldots$ at which the maximal invariant set will change dimension, $c_0^\varepsilon < c_1^\varepsilon < \ldots < c_n^\varepsilon < \ldots$, $\dim K_c = i$, $c_i^\varepsilon < c < c_{i+1}^\varepsilon$. Figure 3 describes pictorially the first few invariant sets for an elliptical domain Ω. In the picture, which lives in the infinite dimensional phase space, we have indicated the equilibria, their unstable manifolds, and the sense of the flow. In this paper we study K_1.

This procedure of slicing the attractor in terms of the energy and identifying the maximal compact invariant sets contained therein, is known as the Morse Decomposition of the flow (Hale [49]). This approach was described for one-dimensional bistable gradient systems in Mischaikow [75] and implemented for the viscous Cahn-Hilliard equation in Grinfeld and Novick-Cohen [76]. We note that for the problem at hand, the limit of the whole attractor, as $\hat{\varepsilon}$ goes to zero, does not exist in any sense while the limits of these invariant sets are meaningful. Such geometrical ideas and methods were introduced in Fusco and Hale [45], Fusco [44], and in Carr and Pego [22,23] for the 1-dimensional Allen-Cahn equation. For related more recent work we refer to [1,13,7,8] and to the references therein.

We now state (informally) two of the main results in this paper. We use $z = z(\hat{\xi})$ to parametrize $\partial\Omega$ where $\hat{\xi}$ is the arc-length parameter. We denote by $\mathcal{K}_\Omega(\hat{\xi})$ the curvature of $\partial\Omega$ at $z(\hat{\xi})$.

Theorem 1.1 (Equilibria/Stability) [3] *Assume that*

$$m = 1 - \frac{\pi\delta^2}{|\Omega|}, \quad \delta \ll 1, \quad 0 < \hat{\varepsilon} \ll \delta^3. \tag{1.5}$$

[3] See Theorems 4.3, 4.4 for precise statements.

Let $z(\hat{\xi}_0)$ be a point on $\partial\Omega$ such that the curvature of $\partial\Omega$ experiences a strict extreme; namely,

$$\mathcal{K}'_{\Omega}(\hat{\xi}_0) = 0, \quad \mathcal{K}''_{\Omega}(\hat{\xi}_0) \neq 0.$$

Then there exists a unique equilibrium $\phi(y)$ of (1.3) such that the zero level set of $\phi(y)$ is close to the circle centered at $z(\hat{\xi}_0)$ with radius δ. In addition, if $\mathcal{K}''_{\Omega}(\hat{\xi}_0) > 0$, i.e., the curvature $\mathcal{K}_{\Omega}(\cdot)$ experiences a local minimum at $\hat{\xi}_0$, then the equilibrium is unstable with an one dimensional unstable manifold. If $\mathcal{K}''_{\Omega}(\hat{\xi}_0) < 0$, i.e, the curvature function $\mathcal{K}_{\Omega}(\cdot)$ experiences a local maximum at $\hat{\xi}_0$, then the equilibrium is exponentially stable.

Theorem 1.2 (Motion) *Assume (1.5) and that $\phi_0^{\hat{\varepsilon}}(y)$ is a "layered" initial data whose interface is close to a semicircle centered at $z(\hat{\xi}_0)$ with radius δ. Then the solution of (1.3) is also layered with interface close to a semicircle with radius δ centered at $z(\hat{\xi}(t))$. In addition, $\hat{\xi}(t)$ is determined by the following O.D.E.*

$$\begin{cases} \dfrac{d}{dt}\hat{\xi}(t) = \dfrac{4\hat{\varepsilon}^2\delta}{3\pi}\mathcal{K}'_{\Omega}(\hat{\xi}(t)) + O(\hat{\varepsilon}^2\delta^2), \qquad t \in (0,\infty), \\ \hat{\xi}(0) = \hat{\xi}_0 \end{cases} \tag{1.6}$$

where $O(\hat{\varepsilon}\delta^2)$ is bounded by $C\hat{\varepsilon}^2\delta^2$ with some positive C independent of $t \in (0,\infty)$.

We now proceed to explain some of the ideas. The success of the method employed depends on our ability to construct a good approximation to the invariant set. Our approach is based on perturbation theory. Our reference problem is (1.3) on the upper half plane or better yet, on a large circular disc.

This problem clearly possesses a one-dimensional manifold of equilibria (whose interfaces are semicircles centered on the real line with radius δ) and provides the first approximation to the manifold. The key idea is that shrinking the droplet is equivalent to flattening the boundary so size is the extra parameter.

A main obstruction however comes later after realizing that we can not shrink the droplet arbitrarily and therefore improve the approximation at will; There is a critical size ($\hat{\varepsilon}^{\frac{1}{3}}$) below which the droplet shape itself becomes unstable; it "melts down", and the uniform state becomes energetically more efficient (see Appendix). We are therefore forced to refine the approximation provided by the reference problem above by some other means that does not involve further shrinking of the droplet. We do this by the method of matched asymptotic expansions applied to **the equation of the manifold** (see (1.10)). This involves inner/outer expansions, boundary layer expansions, and the solution of a geometric problem for interfaces intersecting the boundary and enclosing a fixed area. We need several terms in the expansion: Seven terms for Theorem 1.1 and five terms for Theorem 1.2. In this way by truncating the expansion we construct a certain manifold $\mathcal{M}$, which in general is not invariant. Next we linearize the operator about a generic point on $\mathcal{M}$ and show that the spectrum splits into two parts: an $O(\hat{\varepsilon}^2)$ order eigenvalue (corresponding to the motion on the manifold), and the rest, which is bounded away from zero by a gap of the order $(\hat{\varepsilon})^2/\delta^2$, where δ can be thought as the "radius" of the droplet. The restriction on the radius not being too small (and also not too large) enters in establishing the gap. The conclusion we draw out of this spectral information concerns the stability of the set $\mathcal{M}$, in other words, the stability of the droplet shape. We also construct a thin invariant tube in L^2, about the set $\mathcal{M}$. By introducing coordinates about $\mathcal{M}$ we describe the solutions in the tube in terms of their projection on $\mathcal{M}$. The manifold $\mathcal{M}$ is a very good approximation to the maximal compact invariant set (manifold) $\tilde{\mathcal{M}}$ contained in the tube.

What we have presented above is a synthesis of certain results for describing the motion of interfaces intersecting the boundary.

We now give a more detailed description of this work. We begin by describing the contents of Sect. 2. We find it convenient throughout to introduce a change of variables that fixes the size of the droplet. Let

$$
\begin{aligned}
&y = \delta x, \quad \hat{\varepsilon} = \varepsilon\delta, \qquad u^{\varepsilon}(x,t) = \phi^{\hat{\varepsilon}}(y,t), \\
&\Omega_\delta = \delta^{-1}\Omega := \{x; \delta x \in \Omega\}\,.
\end{aligned}
\tag{1.7}
$$

We can write (1.3) as

$$
\begin{cases}
u_t^{\varepsilon}(x,t) = \varepsilon^2 \Delta u^{\varepsilon}(x,t) - f(u^{\varepsilon}(x,t)) + \,⨏\!\!\!\!\!\!\int_{\Omega_\delta} f(u^{\varepsilon}(\cdot,t)), \\
\qquad\qquad x \in \Omega_\delta, t > 0 \\
\partial_n u^{\varepsilon}(x,t) = 0, \qquad x \in \partial\Omega_\delta, t > 0, \\
u^{\varepsilon}(x,0) = u_0^{\varepsilon}(x), \qquad x \in \Omega_\delta
\end{cases}
\tag{1.8}
$$

where Δ is the Laplacian with respect to x, ∂_n is the normal derivative to $\partial\Omega_\delta$ with respect to x.

Similarly, we parameterize $\partial\Omega_\delta$ by $z^\delta(\xi)$ where ξ is the arc–length parameter of $\partial\Omega_\delta$; that is, we use the transformation

$$
\hat{\xi} = \xi\delta, \quad z^\delta(\xi) = \frac{1}{\delta} z\Big(\hat{\xi}\Big) = \frac{1}{\delta} z\Big(\delta\xi\Big)
\tag{1.9}
$$

where $z(\hat{\xi})$ is the arc–length parameterization of $\partial\Omega$.

We are seeking an invariant manifold $\tilde{\mathcal{M}}$ consisting of functions $u(\cdot,\xi,\varepsilon)$, parametrized in terms of the scalar ξ. The invariance of this manifold under

(1.8) is a purely geometric condition stating the tangency of the vector field to the manifold, can be written analytically in the form:

$$\begin{cases} -\varepsilon^2 \Delta u + f(u) + \varepsilon^2 c u_\xi + \varepsilon\sigma = 0, \\ \qquad\qquad x \in \Omega_\delta, t > 0, \xi \in I\!R^1, \\ \partial_n u(x,\xi,\varepsilon) = 0, \quad x \in \partial\Omega_\delta, \xi \in I\!R^1, \\ \iint_{\Omega_\delta} u(\cdot,\xi,\varepsilon) = |\Omega_\delta| - \pi. \end{cases} \tag{1.10}$$

Here $\sigma = \sigma(\xi,\varepsilon)$ and $c = c(\xi,\varepsilon)$ are constants in x. The scalings $\varepsilon^2 c$, $\epsilon\sigma$ can be guessed. Observe that if we define $u^\varepsilon(x,t) = u(x,\Xi,\varepsilon)$ where $\Xi = \Xi(t,\varepsilon)$ solves the ODE

$$\frac{d}{dt}\Xi(t,\varepsilon) = \varepsilon^2 c(\Xi,\varepsilon), \qquad t \in I\!R^1 \tag{1.11}$$

then $u^\varepsilon(x,t)$ is a solution to (1.8). Equation (1.11) represents the reduced flow on the manifold. We call c the **speed** of the droplet. If $c(\xi_0,\epsilon) = 0$ then $u(x,\xi_0,\epsilon)$ is an equilibrium solution.

We shall find (**approximate**) solutions to the **Manifold Equation** (1.10) (cf. [8ii]) We decompose u as $u = u^{\mathrm{I}} + u^{\mathrm{B}}$ and solve for $(u^{\mathrm{I}}, u^{\mathrm{B}}, \sigma, c)$ in the following four steps.

Step 1. First, we consider the differential equation (1.10a), neglecting the boundary condition (1.10b) and the area constraint (1.10c). Namely, for given parameters (σ, c), we find a solution u^{I} solving the following *interior* problem:

$$(\mathrm{P_I}) \qquad -\varepsilon^2 \Delta u^{\mathrm{I}} + f(u^{\mathrm{I}}) + \varepsilon^2 c {u^{\mathrm{I}}}_\xi + \varepsilon\sigma = 0, \qquad x \in \Omega_\delta, \xi \in I\!R^1. \tag{1.12}$$

Though there are infinitely many solutions, we are only interested in solutions having a certain special profile and whose interface $\Gamma(\xi,\varepsilon)$ defined by

$$\Gamma(\xi,\varepsilon) := \{x \in \Omega; u^I(x,\xi,\varepsilon) = 0\} \tag{1.13}$$

is a smooth (in space, in ξ, and in ε), simple curve intersecting $\partial\Omega_\delta$ at exactly two points. Let r (distance) and s (arc length) be the canonical coordinates of x with respect to the interface and $R = \frac{r}{\varepsilon}$ be the stretched variable. We seek u^I in the form $u^I(x,\xi,\varepsilon) = U(R) + \varepsilon \sum_{j\geq 0} \varepsilon^j u^I_j(R,s,\xi)$ where U is the heteroclinic solution to

$$\ddot{U} - f(U) = 0, \quad U(\pm\infty) = \pm 1, \quad U(0) = 0, \quad \int_R R\dot{U}^2(R)dR = 0, \tag{1.14}$$

The basic linear problem underlying the construction of $u^I_j, j = 0, 1, 2, \cdots$, is

$$\begin{cases} \phi''(R) - f'(U(R))\phi(R) = q(R), \qquad R \in I\!R, \\ \phi(0) = 0, \qquad \phi \in L^\infty(I\!R). \end{cases} \tag{1.15}$$

We remark here that (1.15) has a unique solution if and only if q satisfies the following compatibility (or solvability) condition

$$\int_{\mathbb{R}} q(R)\dot{U}(R)dR = 0 \tag{1.16}$$

Consider the following question: Given a family of curves $\{\Gamma(\xi,\varepsilon)\}_{\xi\in\mathbb{R}^1}$ and constants $\sigma(\xi,\varepsilon)$ and $c(\xi,\varepsilon)$, what is the necessary and sufficient condition for the existence of a unique solution $u^{\rm I}$ of $(\mathrm{P_I})$ having Γ as its interface? In step 1, we shall use asymptotic expansions to derive such a necessary and sufficient condition which can be expressed in terms of a set of differential equations governing Γ. We shall refer to these governing equations as the *interface* equation. They are derived from the compatibility condition (1.16).

Step 2. With $u^{\rm I}$ obtained in Step 1, we seek a function $u^{\rm B}$ such that if we define $u = u^{\rm I}+u^{\rm B}$, then u satisfies both the differential equation (1.10a) and the boundary condition (1.10b). Namely, we seek $u^{\rm B}$ to solve the following *boundary layer* problem

$$(\mathrm{P_B})\quad \begin{cases} \Big\{\varepsilon^2\Delta - f'(u^{\rm I})\Big\}u^{\rm B} = \varepsilon^2 c{u^{\rm B}}_\xi + \mathrm{N}(u^{\rm I},u^{\rm B}), \\ \qquad\qquad x\in\Omega_\delta, \xi\in\mathbb{R}^1, \\ \partial_n u^{\rm B} = -\partial_n u^I, \qquad x\in\partial\Omega_\delta, \xi\in\mathbb{R}^1, \\ u^{\rm B} = O\Big(\exp(-\frac{\nu}{\varepsilon}\mathrm{dist}(x,\partial\Omega_\delta))\Big). \end{cases} \tag{1.17}$$

Here and in the sequel, $\mathrm{N}(a,b) := f(a+b) - f(a) - f'(a)b$.

Denoting by $h(x)$ the distance from x to $\partial\Omega_\delta$ and by H the stretched variable $\frac{h}{\varepsilon}$, we seek u^B in the form $u^B(R,H,\xi,\varepsilon) = \sum_{j\geq 1}\varepsilon^j u_j^B(R,H,\xi)$. The basic underlying linear problem here is,

$$\begin{cases} \phi_{RR} + \phi_{HH} - f'(U)\phi = G, \quad \text{on } D := \mathbb{R}\times\mathbb{R}^+ \\ \phi_H(R,0) = g(R) \text{ on } \mathbb{R}, \quad \phi(0,0) = 0. \end{cases} \tag{1.18}$$

We have the following fact (Lemma 2.1): Problem (1.18) has a unique bounded solution if and only if

$$\iint_{\mathbb{R}\times\mathbb{R}^+} G(R,H)\dot{U}(R)dRdH + \int_{\mathbb{R}} g(R)\dot{U}(R) = 0. \tag{1.19}$$

Since we ask for $u^{\rm B}$ to decay exponentially fast for x away from the boundary $\partial\Omega_\delta$, it turns out that for such $u^{\rm B}$ to exist, it is necessary and sufficient that the angles at the intersections of $\partial\Omega_\delta$ with Γ have to satisfy certain relations which we call *contact angle* conditions. They are derived from the compatibility condition (1.19).

Step 3. We find conditions on σ and c such that $u = u^{\mathrm{I}} + u^{\mathrm{B}}$ satisfies the area constraint condition (1.10c); namely,

$$(\mathrm{P_A}) \qquad \iint_{\Omega_\delta} (u^{\mathrm{I}} + u^{\mathrm{B}})(\cdot, \xi, \varepsilon) = |\Omega_\delta| - \pi. \tag{1.20}$$

Step 4. We solve the following *geometric* problem: Find (Γ, σ, c) such that the interface equation from Step 1, the contact angle conditions from Step 2, and area constraint condition from Step 3 are all satisfied.

We remark that our geometric problem is different from a free boundary problem, which is frequently obtained after formal asymptotic expansions, and needs only to be solved for $t \in [0, \infty)$. In the current situation, if one considers ξ as time, then we are looking for a solution which is periodic with period equal to the arc length of the boundary $\partial\Omega_\delta$. Here, we are not going to establish an existence theorem (since what we want is more information about the solution) but instead, we shall again use formal asymptotic expansions to find an approximate solution.

In summary, we construct approximate solutions to (1.10) as follows.

Step 1: Assume Γ_ε is known, solve for u_ε^I. The solvability condition for u_ε yields the governing equation for Γ_ε.

Step 2: Solve for u_ε^B which satisfies $u_\varepsilon^B = O\left(e^{-\nu|R| - \nu H}\right)$. The solution for u_ε^B yields the contact angle of Γ_ε with $\partial\Omega$, $\frac{\pi}{2} + O(\varepsilon^2\delta)$.

Step 3: Solve (P_A). This yields the area constant on $\Omega_\varepsilon(\xi)$.

Step 4: Find $(\Gamma_\varepsilon, \sigma(\xi, \varepsilon), c(\xi, \varepsilon))$ such that the required condition in Steps 1-3 are fulfilled.

Finally, we have to emphasize that our solutions of (1.10) are only asymptotic solutions, in the sense that they can be accurate to $O(\varepsilon^K)$ for any a priori fixed integer K. This is a very brief description of the construction of the approximate manifold $\mathcal{M}$, which approximates the evolution stage we are describing in this paper.

The stability is done in Sect. 3 which we now describe. We remark that most of the spectral theory results are quite general and can be adapted to different situations by trivial modifications. Some, as in Lemma 3.9 for example, already read in generality. This section could have been written as a separate paper. For stability, one needs to study, for any $u \in \mathcal{M}$, the following eigenvalue problem:

$$\begin{cases} L\phi := -\varepsilon^2 \Delta\bar{\phi} + f'(u)\bar{\phi} = \bar{\lambda}\bar{\phi} - \hat{\lambda} \quad \text{in } \Omega_\delta \\ \frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega_\delta \\ \int_{\Omega_\delta} \phi dx = 0, \end{cases} \tag{1.21}$$

where $(\bar{\lambda}, \bar{\phi})$ is the unknown eigenvalue/eigenfunction and $\hat{\lambda} = -⨏⨏_{\Omega_\delta} f'(u)$ ϕdx.

A related more familiar eigenvalue problem is

$$\begin{cases} -\varepsilon^2 \Delta\phi + f'(u)\phi = \lambda\phi & \text{in } \Omega_\delta \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega_\delta \end{cases} \tag{1.22}$$

and corresponds to the standard Allen-Cahn equation. We study both eigenvalue problems although we need only the former for carrying out the proofs of our main results in the present paper.

We prove the following result.

Theorem 1.3 (Small Eigenvalues) *Let u be any point in $\mathcal{M}$ and let $\{\phi_j, \lambda_j\}_{j=1}^{\infty}$ and $\{(\bar\phi_j, \bar\lambda_j)\}_{j=1}^{\infty}$ be the complete solution of the eigenvalue problems (1.22) and (1.21) where the eigenvalues are ordered from small to large. Assume that for some large enough C^*, $\delta^2 \geq C^*\varepsilon$. Then*

$$\lambda_j = \frac{\varepsilon^2\pi^2}{|\Gamma|^2}\Big\{(j-1)^2 - 1 + O(\delta)\Big\}, \tag{1.23}$$

$$\phi_j = \varepsilon^{-1/2}\dot U\left(\tfrac{r}{\varepsilon}\right)\cos\Big((j-1)\pi\ell\Big) + O(\delta)$$

$$j = 1, 2, 3, ...$$

$$\bar\lambda_j = \frac{\varepsilon^2\pi^2}{|\Gamma|^2}\Big\{j^2 - 1 + O(\varepsilon^2\delta)\Big\}, \quad j = 2, 3, \ldots \tag{1.24}$$

$$\bar\phi_j = \varepsilon^{-1/2}\dot U\left(\tfrac{r}{\varepsilon}\right)\cos(j\pi\ell) + O(\delta), \quad j = 1, 2, 3, \ldots$$

$$\bar\lambda_1 = -\frac{4\varepsilon^2}{3\pi\hat\sigma_0}\frac{d^2}{d\xi^2}\mathcal{K}_{\Omega_\delta}(\xi) + O(\varepsilon^2\delta^4) \tag{1.25}$$

$$= -\frac{4\varepsilon^2\delta^3}{3\pi\hat\sigma_0}\frac{d^2}{d\hat\xi^2}\mathcal{K}_{\Omega_\delta}(\hat\xi)\Big|_{\hat\xi=\delta\xi} + O(\varepsilon^2\delta^4).$$

Here $|\Gamma|$ is the length of the interface and ℓ is a scaled arclength parameter of Γ, scaled so that it varies in $[0,1]$; $\hat\sigma_0$ is defined in Remark 2.4.

By examining the proof of Theorem 1.3 one sees that the result is meaningful for the more general class of u's characterized by the following structure:

$$u(x) = U\left(\frac{r}{\varepsilon}\right) + \varepsilon U_1^I\left(\frac{r}{\varepsilon}\right) + \varepsilon U_1^B\left(\frac{r}{\varepsilon}, \frac{h}{\varepsilon}\right) + O(\varepsilon^2)$$

where U is as in (1.14), and U_1^I, U_1^B satisfy:

$$\int_R f''(U(R))\dot U^2(R)U_1^I(R)dR = 0, \tag{1.26}$$

$$\iint_D f''(U(R))\dot{U}^2(R)U_1^B(R,H)dRdH = \frac{K_{\partial\Omega_\delta}(p^\pm)}{2}\int_R \dot{U}^2(R)dR \tag{1.27}$$

where p^+ and p^- are the intersections of Γ with $\partial\Omega_\delta$. Notice that U_1^B is the first term in the boundary layer expansion of u, and is significant near the points $p^\pm$. Conditions (1.26) and (1.27) are consequences of the solvability conditions (1.16), (1.19) for the interior and boundary layer expansions.

The reader should recall that the spectrum of $\hat{L}\phi = \phi'' - f'(U)\phi$ in $L^2(R)$ lies in $[0,\infty)$, with zero as a simple eigenvalue and with the continuous spectrum filling the entire interval $[\min(f'(\pm 1)),\infty)$. A finite number of eigenvalues below the bottom of the continuous spectrum is a possibility [53,68].

A general perturbation can be split into two parts. The one part is geometrical and is relevant to interface instabilities. The other part comes from the profile of the solution across level sets and relates to the tendency of the solution to stay layered (cf. (1.27) below).

The geometric perturbations are of special interest. Their corresponding eigenvalues $\lambda_n(\varepsilon)$ are called **critical**[4] and are characterized by the fact that $\lambda_n(\varepsilon) \to 0$ as $\varepsilon \to 0$ for fixed n, see [9]. In contrast the eigenvalues corresponding to the perturbations of the profile are associated to eigenvalues bounded away from zero uniformly in ε. We remark that the critical eigenvalues of L are coming from the zero eigenvalue of $\hat{L}$ above, and that they are of magnitude $O(\varepsilon^2)$. At first sight this may look peculiar since u is an ε-perturbation (and not an ε^2-perturbation) of U. The explanation lies with the conditions (1.26, 1.27) which have a cancellation effect on the ε-order term in the expansion of u (c.f.[9]).

Critical eigenfunctions capture motion relative to the moving interface. It is useful to think of these eigenfunctions in terms of a moving frame and in terms of relative speeds. It turns out that a perturbation of the interface away from $\partial\Omega$ evolves at $O(\varepsilon^2)$ speed, while angle adjustment near the boundary is faster and occurs at an $O(\varepsilon)$ speed. This fact allows us to disregard the motion of the interface in the determination of the boundary conditions that the eigenfunctions satisfy.

The critical eigenfunctions are studied via the **decomposition** (c.f. [9], [26],[10])

$$\phi = \phi^0\Theta(\ell) + \psi, \quad \|\phi\|_{L^2(\Omega)} = 1 \tag{1.28}$$

[4] Nishiura and Fujii [72], and Angenent, Mallet-Paret and Peletier [73] were among the first to identify critical eigenvalues in this sense, for related problems in one space dimension. There the interface is a point, there is no change of perimeter, and the relevant perturbations are translations.

where $\phi^0 = \dot{U}(\frac{r}{\varepsilon}) + O(\varepsilon)$ with the $O(\varepsilon)$ order term so that ϕ is an ε^2-approximate eigenfunction.

First, we ignore the ψ term, obtaining, for the unconstrained case the following **geometric eigenvalue problem** for $\Theta = \Theta(\ell)$:

$$\begin{cases} -(b_1\Theta')' + b_2\Theta = \mu\omega_2\Theta, \ell \in (0,1), \\ -b_1(0)\Theta'(0) + b^+\Theta(0) = 0, \quad b_2(1)\Theta'(1) + b^-\Theta(1) = 0 \end{cases} \tag{1.29}$$

where b_1, b_2, b^+, b^- are independent of Θ_1 and Θ_2 and satisfy the estimates

$$b^{\pm} = -|\Gamma|K_{\Omega_\delta}(p^{\pm}) + O(\varepsilon), \quad b_1(\ell) = 1 + O(\varepsilon), \quad \omega_2 = 1 + O(\varepsilon),$$
$$b_2(\ell) = \tfrac{3}{4}(|\Gamma|\mathcal{K})^2 + \tfrac{1}{4}(|\Gamma|\hat{\sigma})^2 + O(\varepsilon)$$

where $\mathcal{K}$ is the curvature of the interface Γ (a function of ℓ) and $\hat{\sigma}$ is a constant depending on σ.

We comment that for the standard Allen-Cahn equation (for which the corresponding geometric problem is evolution by mean curvature) we need to choose $U_1^I = 0$, $\delta = 0$. The Θ-equation (1.29) in the limit as, $\varepsilon \to 0$, takes the form ([10])

$$-\Theta'' - \frac{3}{4}\mathcal{K}^2\Theta = \mu\Theta.$$

Notice that the coefficient is $\frac{3}{4}\mathcal{K}^2$, as opposed to $\mathcal{K}^2$ that is obtained by linearizing directly the mean curvature operator ([68, 36, 35]). We refer to [10] for an explanation.

Theorem 1.3, and more generally Theorem 3.10 in Sect. 3, state that in the limit as $\varepsilon \to 0$, the critical eigenfunctions separate into $\dot{U}\left(\frac{r}{\varepsilon}\right)\Theta(\ell)$, and provide a one–dimensional eigenvalue problem determining Θ. The proof is based on the decomposition of the operator $-\varepsilon^2\Delta + f'(u)I$ into

$$L_r := -\frac{\varepsilon^2}{1+\mathcal{K}(s)}\partial_r\Big(\frac{\partial_r}{1+\mathcal{K}(s)}\Big) + f'(u)I,$$
$$L_s := -\frac{\varepsilon^2}{1+\mathcal{K}(s)}\,\partial_s\Big(\frac{\partial_s}{1+\mathcal{K}(s)}\Big). \tag{1.30}$$

We remark that the Θ-equation (1.29a) is an $O(\varepsilon^2)$ fact and so requires knowledge of the ε^2 terms in the expansion of u. On the other hand the boundary conditions (1.29b) result from an ε-order matching and so do not require any knowledge beyond the ε-level. The two derivations can in principle be decoupled.

A major stability issue for the droplet is persistence of its nearly circular shape. It is intuitive that unless the droplet is sufficiently small in relation to the curvature of the boundary, its circular shape should not be preserved. This intuition is confirmed by the following facts. First we note that the critical

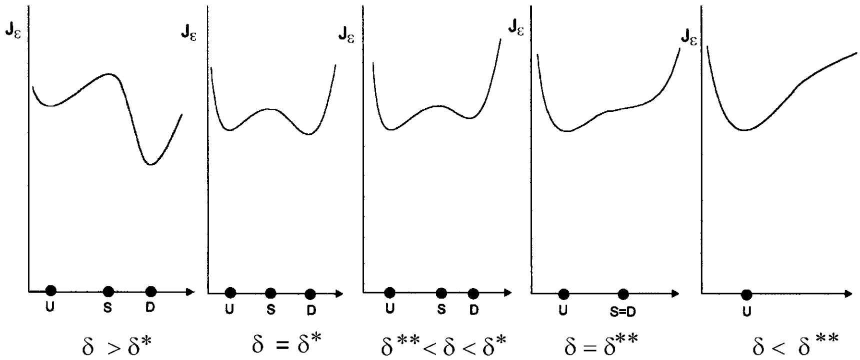

Fig. 4. The energy of the *uniform state*, the *spike*, and the *droplet*. The horizontal axis represents the phase space. The parameter is the mass and is proportional to δ^2. $\hat{\varepsilon}$ is fixed and small. For δ large (in relation always to $\hat{\varepsilon}$) the droplet is the global minimizer. Notice that the uniform state is always a local minimizer. The spike is unstable (see [12] and [69] in related context). At $\delta = \delta^*$ the uniform state and the droplet have equal energy. At $\delta = \delta^{**}$ droplet and spike coalesce. Finally for $\delta < \delta^{**}$ there is no droplet or spike, and the uniform state is the only critical point. δ is roughly the radius of the droplet

eigenvalues $\bar{\lambda}_j$, $j = 1, 2, \cdots$ scale like $C_j(j^2 - 1)\varepsilon^2\delta^{-2}$, and therefore perturbations away from the circular shape decay faster as $\delta \to 0$. The behavior of the principal eigenvalue becomes more subtle because of the strong dependence, of C_1 on δ. To argue this we first note that the principle eigenfunction is a perturbation related to the "shrinking" of the drop. Next we observe that, due to conservation, shrinking is possible only if the curvature of the boundary increases. On the other hand reducing δ is equivalent to flattening the boundary and therefore it antagonizes shrinking. Therefore on the basis of this understanding we expect C_1 to diminish as $\delta \to 0$, while we expect the rest of the C_j, $j = 2, 3, \ldots$ to remain largely unaffected.[5] As a result a **gap** appears between the principal eigenvalue, and the rest of the critical spectrum, when $\delta \to 0$, expressing the increasing stability of the droplet. However this argument focuses on the interface and disregards the part of the energy not due to the interface, which is related to the tendency of the profile to stay layered and which happens to become critical when $\delta^2 \sim \varepsilon$; this causes an extra complication in this work and forces us to establish several extra terms in the asymptotic expansion.

A first indication that the layered shape may be destabilized if $\delta^2 < C^*\varepsilon$ can be seen by comparing the energy J_ε of the droplet with the energy of the uniform state. In the appendix and in Remark 2.4 we analyse the energy

[5] This makes the calculation of $\bar{\lambda}_1$ especially delicate. Its different nature is suggested by the formula (1.24), which vanishes to principal order for $j = 1$. The calculation of $\bar{\lambda}_1$ is based on detailed knowledge of the speed c. Notice that (1.25) holds everywhere, including the equilibrium points.

near the critical δ where a bifurcation occurs. We compare the energies of the three state: The "drop", the "spiky" solution, and the uniform state, see Fig. 4. All this can be reconfirmed with a careful analysis of the spectrum.

We note that u_ξ can be shown to be close to the principle eigenfunction and so as a consequence of Theorem 1.3, we have the following result for the set $\mathcal{M}$:

Theorem 1.4 (Spectral Gap) *Let u be any point in $\mathcal{M}$, and assume that*

$$0 < \varepsilon < \varepsilon_0, \qquad \delta_0^2 > \delta^2 > C^*\varepsilon, \qquad \delta_0, C^*, \varepsilon_0 \quad \textit{constants.}$$

Then for any $v \in H^1(\Omega_\delta)$ satisfying

$$\int_{\Omega_\delta} v dx = 0, \quad \int_{\Omega_\delta} v\, u_\xi dx = 0 \quad \left(u_\xi := \frac{\partial u}{\partial \xi}\right)$$

we have

$$\int_{\Omega_\delta} \left(\frac{\varepsilon^2}{2}|\nabla v|^2 + f'(u)v^2\right) dx \geq \frac{2\varepsilon^2\pi^2}{|\Gamma|^2} \int_{\Omega_\delta} v^2 dx, \quad \textit{for } \varepsilon < \varepsilon_0. \tag{1.31}$$

Finally, we come to the dynamics of (1.8), which is done in detail in Sect. 4. For any solution $u^\varepsilon(x,t)$ of (1.8) whose initial data are close to $\mathcal{M}$, we decompose

$$u^\varepsilon(\cdot, t) = u(\cdot, \xi(t), \varepsilon) + v(\cdot, t)$$

where $u(\cdot, \xi(t), \varepsilon)$ is a certain projection of u^ε on $\mathcal{M}$. The proof of Theorem 1.2 is based on this decomposition. $v(\cdot, t)$ is controlled by the estimate (1.31) which allows the construction of an invariant tube around $\mathcal{M}$. The main term in (1.6) is obtained from the construction of u (cf. equation (1.10), (1.11)), and supplemented near the equilibria by linear analysis, (1.25).

For previous work related to the main theme of this paper, see Alikakos and Fusco [4]. For work on the stage of evolution described in this paper (see Fig. 1) for the related sharp interface models see Alikakos, Bates, Chen, and Fusco [3], and Bellettini and Fusco [14], and the references therein.

2 Approximate solutions to the manifold equation

2.1 Preliminaries

In this section, we carry out the four steps mentioned in Sect. 1 for the approximate solution of (1.10). First we introduce the coordinate systems we are going to use in the interior and boundary layer expansions (see Fig. 5 for a summary).

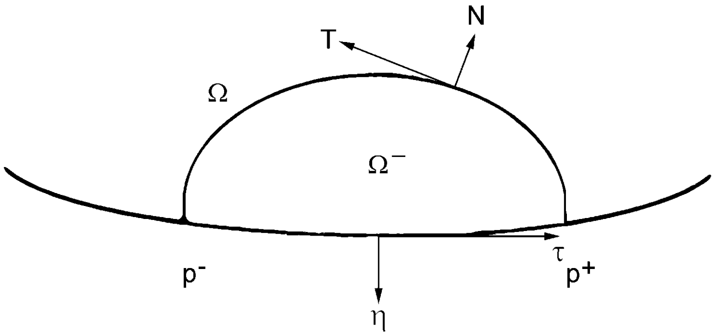

Fig. 5.

Representation of $\partial\Omega_\delta$

In the sequel, we shall identify points in $\mathbb{R}^2$, or vectors, with complex numbers.

We use a complex valued function $z = z(\hat{\xi})$ to parametrize $\partial\Omega$ where $\hat{\xi}$ is an arc length parameter oriented counter clockwise. We denote the curvature of $\partial\Omega$ at $z(\hat{\xi})$ by $\mathcal{K}_\Omega(\hat{\xi})$. Notice that if we write $z'(\hat{\xi}) = e^{\mathrm{i}\varphi(\hat{\xi})}$ where $\varphi(\hat{\xi})$ is a real valued function representing the angle between $\partial\Omega$ and the x–axis, then $\mathcal{K}(\hat{\xi}) = \varphi'(\hat{\xi})$.

We use

$$z^\delta = z^\delta(\xi) := \tfrac{1}{\delta} z(\hat{\xi})\Big|_{\hat{\xi}=\delta\xi} \tag{2.1}$$

to parametrize $\Omega_\delta = \frac{1}{\delta}\Omega$. Clearly, ξ is an arc length parameter for Ω_δ. We use $\tau(\xi), n(\xi)$ and $\mathcal{K}_{\Omega_\delta}(\xi)$ to denote the unit tangent, normal, and the curvature of $\partial\Omega_\delta$ at $z^\delta(\xi)$. Then, identifying vectors with complex numbers, one can derive

$$\tau(\xi) = z^\delta_\xi(\xi) := e^{\mathrm{i}\varphi^\delta(\xi)} \qquad n(\xi) = -\mathrm{i}e^{\mathrm{i}\varphi^\delta(\xi)}, \qquad \mathcal{K}_{\Omega_\delta}(\xi) = \varphi^\delta_\xi(\xi)$$

where $\varphi^\delta(\xi) = \varphi(\delta\xi)$. Assume that Ω is smooth. Then for any integer K,

$$\varphi^\delta(\xi + \varsigma) = \varphi^\delta(\xi) + \Sigma_{j=1}^K \varphi^\delta_j(\xi)\frac{\delta^j\varsigma^j}{j!} + O(\delta^{K+1}\varsigma^{K+1}), \tag{2.2}$$

where

$$\begin{aligned}\varphi^\delta_j(\xi) &= \delta^{-j}\frac{d^j}{d\xi^j}\varphi^\delta(\xi) = \delta^{-j}\frac{d^{j-1}}{d\xi^{j-1}}\mathcal{K}_{\Omega_\delta}(\xi) = \frac{d^{j-1}}{d\hat{\xi}^{j-1}}\mathcal{K}(\hat{\xi})\Big|_{\hat{\xi}=\delta\xi} \\ &= \frac{d^j}{d\hat{\xi}^j}\varphi(\hat{\xi})\Big|_{\hat{\xi}=\delta\xi}, \qquad j = 1, 2, \cdots.\end{aligned} \tag{2.3}$$

We use $h(x)$ to denote the distance of $x \in \bar{\Omega}_\delta$ from $\partial\Omega_\delta$. Then the change of coordinates from (h, ς) to x is given by

$$x = X^B(h, \varsigma) := z^\delta(\varsigma) - hn(\varsigma), \tag{2.4}$$

and is a differomorphism from $[0, h_0] \times (\mathbb{R}^1/|\partial\Omega_\delta|)$ to $\{x \in \bar{\Omega}_\delta; h(x) \leq h_0\}$. Here h_0 is a fixed positive constant and $\partial\Omega_\delta$ is the length of $\partial\Omega_\delta$. We shall use $h = \mathrm{h}(x)$ and $\varsigma = \varsigma(x)$ to denote the inverse of the change of variables $x = X^B(h, \varsigma)$ given by (2.4). One can verify

$$\nabla_x \mathrm{h} = n(\varsigma(x)), \qquad \nabla\varsigma(x) = (1 - h\mathcal{K}_{\Omega(\varsigma)})^{-1}\tau(\varsigma)\Big|_{\varsigma=\varsigma(x)}. \tag{2.5}$$

Representation of interface Γ.
We use a function $w(\cdot, \xi, \varepsilon) = w^1(\cdot, \xi, \varepsilon) + \mathrm{i}w^2(\cdot, \xi, \varepsilon))$ to describe $\Gamma(\xi, \varepsilon)$:

$$\Gamma(\xi, \varepsilon) = \{w(s, \xi, \varepsilon) \ : \ 0 \leq s \leq |\Gamma|(\xi, \varepsilon)\} \tag{2.6}$$

where s is the counterclockwise arc length parameter, $|\Gamma| = |\Gamma|(\xi, \varepsilon)$ is the total length of $\Gamma(\xi, \varepsilon)$ in Ω_δ, and $w(0, \xi, \varepsilon)$ and $w(L(\xi, \varepsilon), \xi, \varepsilon)$ are the intersection of Γ with $\partial\Omega_\delta$. We assume that w is well–defined and smooth in $(-h_0, L(\xi, \varepsilon) + h_0)$ for some positive constant h_0. We denote by $\mathbf{T} = \mathbf{T}(s, \xi, \varepsilon)$, $N = N(s, \xi, \varepsilon)$ and $\mathcal{K} = \mathcal{K}(s, \xi, \varepsilon)$ the unit tangent vector, unit normal vector, and curvature of Γ at $w(s, \xi, \varepsilon)$. Then there is a real valued function $\psi(s, \xi, \varepsilon)$ such that

$$\mathbf{T} = w_s := e^{\mathrm{i}[\psi(s,\xi,\varepsilon) + \varphi^\delta(\xi) + \pi/2]}, \quad N = -\mathrm{i}\mathbf{T}, \quad \mathcal{K} = \psi_s(s, \xi, \varepsilon). \tag{2.7}$$

We assume that Γ is smooth, so that there exists a fixed constant $m_0 > 0$ such that the transformation from (r, s) to x defined by

$$x = X^I(r, s) := w(s, \xi, \varepsilon) + rN(s, \xi, \varepsilon) \tag{2.8}$$

is a differomorphism from $D(m_0) := \{(r, s) \ : \ |r| < m_0, -h_0 < s < |\Gamma| + h_0\}$ to its image. We use $r = \mathrm{r}(x, \xi, \varepsilon)$ and $s = \mathrm{s}(x, \xi, \varepsilon)$ to denote the inverse of the transformation $x \to (r, s)$. Direct calculation shows that

$$\begin{aligned} &\nabla_x \mathrm{r} = N, \quad \nabla_x s = (1 + r\mathcal{K})^{-1}\mathbf{T}, \quad \Delta r = \mathcal{K}(1 + r\mathcal{K})^{-1}, \\ &\Delta s = -r\mathcal{K}_s(1 + r\mathcal{K})^{-3}. \end{aligned}$$

In addition, differentiating both sides of (2.8) with respect to ξ (considered x as independent of ξ), we obtain

$$0 = (1 + r\mathcal{K})\mathbf{T}\mathrm{s}_\xi + \mathrm{r}_\xi N + w_\xi + rN_\xi(s, \xi, \varepsilon).$$

It then follows that

$$\mathrm{r}_\xi(x,\xi,\varepsilon) = -w_\xi \cdot N = w_\xi^2 w_s^1 - w_\xi^1 w_s^2 \tag{2.9}$$

$$\begin{aligned}\mathrm{s}_\xi(x,\xi,\varepsilon) &= -(1+r\mathcal{K})^{-1}(w_\xi + N_\xi)\cdot \mathbf{T}\\ &= -(1+r\mathcal{K})^{-1}(w_\xi^1 w_s^1 + w_\xi^2 w_s^2 + w_{s\xi}^2 w_s^1 - w_{s\xi}^1 w_s^2).\end{aligned} \tag{2.10}$$

Here we observe that in (r,s) coordinates, r_ξ and $(1+r\mathcal{K})\mathrm{s}_\xi$ are independent of r.

For the interior expansion, we shall use the stretched variable $R=\varepsilon^{-1}r$; more precisely, we use the change of variables $x \to (R,s)$ defined by

$$x = w(s,\xi,\varepsilon) + \varepsilon R N(s,\xi,\varepsilon). \tag{2.11}$$

Under this change of variables, we can calculate

$$\begin{aligned}\varepsilon^2 \Delta_x = \partial_{RR} + \varepsilon\mathcal{K}(1+\varepsilon R\mathcal{K})^{-1}\partial_R + \varepsilon^2(1+\varepsilon R\mathcal{K})^{-2}\partial_{ss}\\ -\varepsilon^3 R\mathcal{K}(1+\varepsilon R\mathcal{K})^{-3}\partial_s.\end{aligned}$$

Corners – intersections of boundary and Interface
We denote by $p^\pm = p^\pm(\xi,\varepsilon)$ the intersections of Γ with $\partial\Omega_\delta$. To relate $\Gamma(\xi,\varepsilon)$ with $Z^\delta(\xi)$, we assume that, for some function $g=g(\xi,\varepsilon)$,

$$\begin{aligned}&p^\pm = w(L^\pm,\xi,\varepsilon) = z^\delta(\xi \pm g(\xi,\varepsilon)),\\ &\Big(L^+ := 0, L^- := |\Gamma|(\xi,\varepsilon)\Big) \quad \forall \xi \in I\!R^1.\end{aligned} \tag{2.12}$$

Also we use $\Omega_\delta^- = \Omega_\delta^-(\xi,\varepsilon)$ to represent the region bounded by Γ and $\{z^\delta(\varsigma); \xi - g(\xi,\varepsilon) < \varsigma < \xi + g(\xi,\varepsilon)\}$ and denote by $\Omega_\delta^+ = \Omega_\delta^+(\xi,\varepsilon)$ the compliment of $\Omega_\delta^- \cup \Gamma$ in Ω_δ.

In a small neighborhood of the “corner” $p^\pm$, we use the change coordinates

$$\begin{cases} r = \mathrm{r}(r,h,\xi,\varepsilon),\\ h = \mathrm{h}(x)\end{cases} \iff \quad x = X^C(r,h,\xi,\varepsilon). \tag{2.13}$$

Here $\mathrm{h}(x)$ and $\mathrm{x}(x,\xi,\varepsilon)$ are the signed distance from x to $\partial\Omega_\delta$ and to Γ respectively. It should be noticed that is is not trivial to write down the function $X^c(r,h)$ explicitly.

For the boundary expansion, we shall use the stretched variables (R,H) defined by

$$R = \varepsilon \mathrm{r}(x,\xi,\varepsilon),\ H = \varepsilon \mathrm{h}(x). \tag{2.14}$$

Under this change of variables, we have

$$\varepsilon^2 \Delta = \partial_{RR} + \partial_{HH} + 2N \cdot n \partial_{RH} + \varepsilon \mathcal{K}(1 + \varepsilon R\mathcal{K})^{-1} \partial_R \\ + \varepsilon \mathcal{K}_{\Omega_\delta}(1 + \varepsilon H \mathcal{K}_{\Omega_\delta}) \partial_H.$$

Here $N = N(\mathrm{s}(x, \xi, \varepsilon))|_{x = X^C(\varepsilon R, \varepsilon H, \xi, \varepsilon)}$, $n = n(\varsigma(x))|_{x = X^C(\varepsilon R, \varepsilon H, \xi, \varepsilon)}$.

2.2 The interior expansion

For the interior expansion, we seek solutions of the form

$$u^{\mathrm{I}}(x, \xi, \varepsilon) = U(R) + \varepsilon \Sigma_{j \geq 0} \varepsilon^j u^{\mathrm{I}}{}_j(R, s, \xi) \tag{2.15}$$

where $R = \frac{r}{\varepsilon}$, $x \to (r, s)$ is defined in (2.8), and $U(\cdot)$ is unique solution of (1.14) introduced in Sect. 1.

Under the new variables (R, s), the differential equation for u^{I} becomes

$$-u^{\mathrm{I}}{}_{RR} + f(u^{\mathrm{I}}) + \varepsilon \Big[\sigma + c \mathrm{r}_\xi u^{\mathrm{I}}{}_R - \mathcal{K}(1 + \varepsilon R\mathcal{K})^{-1} u^{\mathrm{I}}{}_R \Big] \\ + \varepsilon^2 \Big[c \mathrm{s}_\xi u^{\mathrm{I}}{}_s + c u^{\mathrm{I}}{}_\xi - (1 + \varepsilon R\mathcal{K})^{-2} u^{\mathrm{I}}{}_{ss} \Big] \\ + \varepsilon^3 \mathcal{K}_s (1 + \varepsilon R \mathcal{K} K)^{-3} u^{\mathrm{I}}{}_s = 0. \tag{2.16}$$

Here $u^{\mathrm{I}}{}_\xi$ represents the partial derivative when we consider u^{I} as a function of the variables R, s, ξ, ε, so that

$$\frac{d}{d_\xi} = \varepsilon^{-1} \mathrm{r}_\xi \partial_R + \mathrm{s}_\xi \partial_s + \partial_\xi.$$

To expand (2.16) as asymptotic power series of ε^j, we assume that $w(s, \xi, \varepsilon), \sigma(\xi, \varepsilon)$ and $c(\varepsilon)$ has the following expansions

$$w(s, \xi, \varepsilon) = \Sigma_{j \geq 0} \varepsilon^j w_j(s, \xi), \quad \sigma(\xi, \varepsilon) = \Sigma_{j \geq 0} \varepsilon^j \sigma_j(\xi), \\ c(\xi, \varepsilon) = \Sigma_{j \geq 0} \varepsilon^j c_j(\xi). \tag{2.17}$$

Clearly, we can use (2.7), (2.9), (2.10), and (2.17) to expand $\mathcal{K}$, r_ξ, and s_ξ as

$$\mathcal{K}(s, \xi, \varepsilon) = \Sigma_{j \geq 0} \varepsilon^j \mathcal{K}_j(s, \xi), \tag{2.18}$$

$$\mathrm{r}_\xi(x, \xi, \varepsilon) = \Sigma_{j \geq 0} \varepsilon^j \mathrm{r}^\xi_j(s, \xi), \tag{2.19}$$

$$\mathrm{s}_\xi(x, \xi, \varepsilon) = \Sigma_{j \geq 0} \varepsilon^j \mathrm{s}^\xi_j(R, s, \xi),$$

where $\mathcal{K}_j, \mathrm{r}^\xi_j$, and s^ξ_j depend only on $w_0, \cdots, w_j$.

Expressing equation (2.16) in terms of a power series in ε, we then obtain, for each coefficient of ε^{j+1} , $j = 0, 1, \cdots$, the following equation, for ${u^{\rm I}}_j$:

$$\begin{cases} \Big(\frac{d^2}{d^2R} - f'(U)\Big){u^{\rm I}}_j = \Big\{U'(R)[\mathcal{K}_j - (cr_\xi)_j] - \sigma_j\Big\} \\ \qquad\qquad +q_{j-1}(R, r, \xi), \\ u^{\rm I}(0, s, \xi) = 0, \quad \sup_{R\in I\!R} |{u^{\rm I}}_j(R, s, \xi)| < \infty \end{cases} \tag{2.20}$$

where $q_{-1} = 0$ and q_{j-1} depends only on expansions of order no bigger than $j-1$. Here we used the notation $(ab)_j = \Sigma_{i=0}^{j} a_i b_{j-i}$ if $a = \Sigma_{i\geq 0} a_i \varepsilon^i$ and $b = \Sigma_{i\geq 0} b_i \varepsilon^i$. We write ∂_{RR} as $\frac{d^2}{dR^2}$ since we consider s and ξ as parameters when we solve for ${u^{\rm I}}_j$. Notice that the condition ${u^{\rm I}}_j(0, s, \xi) = 0$ reflects the definition of Γ, being the $U(0)$ level set of $u^{\rm I}$.

Recall that for given bounded $q(R)$, the equation $\phi''(R) - f'(U(R))\phi(R) = q(R)$, $\phi(0) = 0$ has a unique bounded solution if and only if $\int_{-\infty}^{\infty} U'(R)q(R)dR = 0$. Hence to solve (2.20) uniquely, it is necessary and sufficient to have the following solvability condition:

$$(cr_\xi)_j(s, \xi) - \mathcal{K}_j(s, \xi) + a_0\sigma_j(s, \xi) = A_{j-1}(s, \xi) \tag{2.21}$$

where

$$a_0 := 2\Big/ \int_{I\!R} (U'(R))^2 dR \tag{2.22}$$

and $A_{-1} = 0$ and A_{j-1} depends only on the terms of order less than $j-1$.

One can easily verify (see Appendix in [10]) that if for all $i = 0, \cdots, j-1$, $(w_i, \sigma_i, c_i, {u^{\rm I}}_i)$ are known and smooth and satisfy

$$|D_R^m D_s^n D_\xi^l {u^{\rm I}}_j(z)| \leq O(1)e^{-\nu|R|} \quad \text{as} \ \ |R| \to \infty \tag{2.23}$$

for all non–negative integers m, n, l satisfying $m + n + l \geq 1$, then (2.20) has a unique solution ${u^{\rm I}}_j$ if and only if w_j satisfies (2.21). In addition, if ${u^{\rm I}}_j$ exists, it satisfies (2.23) also. Here ν is any fixed positive number $< \min\{f'(1), f'(-1)\}$.

In conclusion, in order to have a unique solution $u^{\rm I}$ of the form (2.15), it is necessary and sufficient for (w_j, c_j, σ_j) to satisfy (2.21) for $j = 0, 1, 2, \cdots$.

For easy reference, we provide some lower order solutions to the inner expansion:

First denote by U_1, U_{21}, U_{22} the solutions to the following problems:

$$U_1'' - f'(U)U_1 = 1 - a_0 U', \ a_0 = \frac{(1, U')_{L^2}}{(U', U')_{L^2}}, \tag{2.24}$$

$$U_{21}'' - f'(U)U_{21} = \frac{f''(U)}{2} U_1^2 - a_0, \ U_1' - a_1 U',$$

$$a_1 = \frac{\left(\frac{-f''(U)}{2}U_1^2 + a_0 U_1^2, U'\right)_{L^2}}{(U', U')_{L^2}} \tag{2.25}$$
$$U_{22}'' - f'(U)U_{22} = RU'$$

So for $j = 0, 1$ we have

$$c_0 r_0 - \mathcal{K}_0 + a_0\sigma_0 = 0,\ u_0^I = \sigma_0(\xi)U_1(R),$$

$$c_0 r_0 + c_1 r_0 - \mathcal{K}_1 + a_0\sigma_1 = a_1\sigma_0^2,$$
$$u_1^I = \sigma(\xi)U_1(R) + \sigma_0^2(\xi)U_{21}(R) + \mathcal{K}_0^2 U_{22}(R).$$

Note that all the geometric equations (2.21) for $j = 0, 1, \cdots$ can be combined in to the following single equation:

$$c\mathrm{r}_\xi - \mathcal{K} + a_0\sigma = a_1\sigma_0^2\varepsilon + \Sigma_{j\geq 2}\varepsilon^j\{\cdots\}. \tag{2.26}$$

Here and in the sequel, all the terms depending only on expansions of order $\leq j-1$ will be denoted by "$\cdots$".

Since later on we need explicit expansions up to order ε^2, it is convenient to introduce a new constant $\hat{\sigma}$ defined by

$$\hat{\sigma}(\xi,\varepsilon) = a_0\sigma(\xi,\varepsilon) - a_1\varepsilon\sigma^2(\xi,\varepsilon). \tag{2.27}$$

Clearly, finding σ is equivalent to finding $\hat{\sigma}$. With this new constant, we can write (2.26) as

$$c\mathrm{r}_\xi - \mathcal{K} + \hat{\sigma} = \Sigma_{j\geq 2}\varepsilon^j\{\cdots\}. \tag{2.28}$$

2.3 The boundary layer expansion

For the boundary layer expansion, we shall use the stretched variable R and H defined in (2.14). In the new coordinates (R, H), the differential equation for u^{B} becomes

$$\begin{aligned}\Big(-\partial_{RR} - \partial_{HH} + f'(U(R))\Big)u^{\mathrm{B}} &= (f'(U) - f'(u^{\mathrm{I}}))u^{\mathrm{B}} - \mathrm{N}(u^{\mathrm{I}}, u^{\mathrm{B}})\\ &+2N\cdot n {u^{\mathrm{B}}}_{RH}\\ &\quad+\varepsilon\mathcal{K}(1+\varepsilon R\mathcal{K})^{-1}{u^{\mathrm{B}}}_R + \varepsilon\mathcal{K}_{\Omega_\delta}(1+\varepsilon H\mathcal{K}_{\Omega_\delta})^{-1}{u^{\mathrm{B}}}_H\\ &\qquad-\varepsilon c[{u^{\mathrm{B}}}_R\mathrm{r}_\xi + \varepsilon {u^{\mathrm{B}}}_\xi], \qquad \text{on } I\!R\times I\!R^+.\end{aligned} \tag{2.29}$$

Here ${u^{\mathrm{B}}}_\xi$ on the right–hand is the partial derivative with respect to ξ while keeping R and H fixed. Also, s in $u^{\mathrm{I}}(R, s, \xi, \varepsilon)$, $N(s, \xi, \varepsilon)$, and $\mathcal{K}(s, \xi, \varepsilon)$ is evaluated at $s = \mathrm{s}(x, \xi, \varepsilon)|_{x=X^C(\varepsilon R, \varepsilon H, \xi, \varepsilon)}$ whereas ς in $n(\varsigma)$ and $\mathcal{K}_{\Omega_\delta}(\varsigma)$ is evaluated at $\varsigma = \varsigma(x)|_{x=X^C(\varepsilon R, \varepsilon H, \xi, \varepsilon)}$.

The boundary condition becomes

$$u^{\mathrm{B}}{}_H(R,0,\xi,\varepsilon) = -N\cdot n u^{\mathrm{B}}{}_R - \Big\{N\cdot n u^{\mathrm{I}}{}_R + \varepsilon \mathbf{T}\cdot n u^{\mathrm{I}}{}_s\Big\}\Big|_{s=\mathrm{s}(X^C(\varepsilon R,0,\xi,\varepsilon))}, \quad \text{on } I\!R^1\times\{0\}. \tag{2.30}$$

In the sequel, we use the superscript $-$ or $+$ to denote the neighborhood near p^- and p^+ respectively. We assume the following expansions, near $p^\pm$:

$$u^{\mathrm{B}\pm}(R,H,\xi,\varepsilon) = \Sigma_{j\geq 1}\varepsilon^j u^{\mathrm{B}}{}^{\pm}_j(R,H,\xi),$$

$$u^{\mathrm{B}}{}^{\pm}_0(R,H,\xi) := 0, \tag{2.31}$$

$$g(\xi,\varepsilon) = \Sigma_{j\geq 0}\varepsilon^j g_j(\xi), \tag{2.32}$$

$$L^+(\xi,\varepsilon) := 0 = \Sigma_{\geq 0}\varepsilon^j L^+_j(\xi),, \tag{2.33}$$

$$L^-(\xi,\varepsilon) := |\varGamma|(\xi,\varepsilon) = \Sigma_{j\geq 0}\varepsilon^j L_j(\xi), \tag{2.34}$$

$$N\cdot n|_{x=p^\pm} := N(L^\pm)\cdot n(\xi\pm g(\xi)) = \Sigma_{j\geq 1}\varepsilon^j\alpha^\pm_j(\xi),$$

$$\alpha^\pm_0 := 0. \tag{2.35}$$

In what follows, we shall call $\{w_j, u^{\mathrm{I}}{}_j, u^{\mathrm{B}}{}^{\pm}_j, \alpha^\pm_j, g_j, S_j, \sigma_j, c_j\}$ the j^{th} order expansion and we use $\cdots$ to denote various functions and/or constants that depend only on expansions of order $\leq j-1$.

To express (2.29) and (2.30) in power of ε, we need to write the coefficients in the equations as power series of ε.

Since $\alpha^\pm_0 = 0$, $\mathbf{T}\cdot n|_{x=p^\pm} = \pm 1 + \Sigma_{j\geq 2}\varepsilon^j(\cdots)(\xi)$, it then follows that

$$\begin{aligned}
\mathrm{s}(x,\xi,\varepsilon)|_{x=X^C(\varepsilon R,\varepsilon H,\xi,\varepsilon)} &:= S^\pm(R,H,\xi,\varepsilon)\\
&= \Sigma_{j\geq 0}\varepsilon^j L^\pm_j \mp \varepsilon H + \Sigma_{j\geq 2}\varepsilon^j(\cdots)(R,H,\xi),\\
\varsigma(x)|_{x=X^C(\varepsilon R,\varepsilon H,\xi,\varepsilon)} &= \pm\Sigma_{j\geq 0}\varepsilon^j g_j(\xi) \pm \varepsilon R\\
&\quad + \Sigma_{j\geq 2}(\cdots)(R,H,\xi)\\
N\cdot n|_{x=X^C(\varepsilon R,\varepsilon H,\xi,\varepsilon)} &= \Sigma_{j\geq 1}\varepsilon^j\alpha^\pm_j - \varepsilon H\mathcal{K}_0(L^\pm_0)\\
&\quad + \varepsilon R\mathcal{K}_{\varOmega_\delta}(\xi\pm g_0) + \Sigma_{j\geq 2}\varepsilon^j(\cdots)(R,H,\xi)\\
\mathcal{K} &= \Sigma_{j\geq 0}\varepsilon^j\mathcal{K}_j(L^\pm_0(\xi)) + \Sigma_{j\geq 1}\varepsilon^j(\cdots)(H,\xi),\\
\mathcal{K}_{\varOmega_\delta} &= \mathcal{K}_{\varOmega_\delta}(\xi\pm g_0) + \Sigma_{j\geq 1}\varepsilon^j\{\pm\mathcal{K}'_{\varOmega_\delta}(\xi\pm g_0)g_j\\
&\quad + (\cdots)(R,\xi)\}.
\end{aligned}$$

Substituting these expansions into the (2.29) and (2.30), we then obtain for, each $j=1,2,\cdots$, the equations

$$\begin{cases} [-\partial_{RR} - \partial_{HH} + f'(U(R))]u^{\mathrm{B}\pm}_{j} = B^{\pm}_{j-1}(R, H, \xi), \\ \qquad \text{in } I\!R \times (0, \infty), \\ u^{\mathrm{B}\pm}_{1,H} = -U'(R)[\alpha_1^{\pm} + R\mathcal{K}_{\Omega_\delta}(g_0^{\pm})], \quad \text{if } j = 1, \\ u^{\mathrm{B}\pm}_{j,H} = -U'(R)\alpha_j^{\pm}(\xi) + C_{j-1}, \quad \text{if } j \geq 2. \end{cases} \tag{2.36}$$

where $B^{\pm}_{-1}(R, H, \xi) \equiv 0$ and $B^{\pm}_{j-1}$, C_{j-1} depend only on the expansion of order $\leq j - 1$.

To solve for $u^{\mathrm{B}\pm}$, we need the following lemma:

Lemma 2.1 *Let U be defined as in (1.14). Consider the following linear problem*

$$\begin{cases} \Big(\partial_{RR} + \partial_{HH} - f'(U(R))\Big)\phi = G, & R \in I\!R^1, H > 0, \\ \phi_H(R, 0) = g(R), & R \in I\!R^1. \end{cases} \tag{2.37}$$

Assume that as $|R| + H \to \infty$, $|G| = O(e^{-\nu(|R|+H)})$ and $|g| = O(e^{-\nu|R|})$. Then (2.37) has a bounded solution if and only

$$\int_0^\infty \int_{I\!R^1} G(R, H)U'(R)dRdH + \int_{I\!R^1} g(R)U'(R)dR = 0. \tag{2.38}$$

In addition, bounded solutions are unique and satisfy $|\phi| = O(e^{-\nu(|R|+H)})$.

Proof. Under the hypotheses $g \in L^2(I\!R), G \in L^2(D), \int_H^\infty G \in L^1(D)$, $D = \mathbf{R} \times \mathbf{R}^+$, we will show the following:
i) there exist at most one solution $\phi \in H^1(D)$,
ii) There exist a $\phi \in H^1(D)$ if and only if

$$\iint_D GU'dRdH + \int_R gU'dR = 0.$$

iii) Let ν_2 be the second eigenvalue of $-\frac{d^2}{dR^2} + f'(U)I$. Then for any $\nu \in (0, \nu_2)$, for any integer $k \geq 0$ if f, g and G satisfy (2.38) and

$$D^i_R g(R) = O(e^{-\nu|R|}),\ j = 0, ..., k,$$

$$D^{\alpha_1}_R D^{\alpha_2}_H G(R, H) = O(e^{-\nu|R|-\nu H}),\ \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 \leq k$$

we then have

$$D^{\alpha_1}_R D^{\alpha_2}_H \phi(R, H) = O(e^{-\nu|R|-\nu H}),\ \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 \leq k + 2.$$

We give a sketch of argument:

i) If $V = V^1 - V^2$, then $\|V_H\|^2_{L^2} = \int_0^\infty \left\{\int_{I\!R}(V_R^2 + f(U)V^2)\right\} \leq 0$, hence $V = 0$.

ii) Multiplying (1.13) by U' and integrating over D, utilizing $U''' - f'(U)U' = 0$, establishes the necessity of (1.14).

For the sufficiency consider the "equivalent problem":
Find $V \in H^1(D)$ such that

$$J(V) = \inf_{\tilde{V} \in H^1(D)} J(\tilde{V}),$$

where

$$J(\tilde{V}) = \iint_D (|\nabla \tilde{V}|^2 + f'(U)\tilde{V}^2 + 2G\tilde{V})dRdH \\ - \int_{I\!R} \tilde{V}(R,0)g(R)dR.$$

If (1.14) holds, then it can be shown that

$$\inf_{\tilde{V} \in H^1} J(\tilde{V}) > -\infty$$

and V exists.

iii) If $G = \tilde{G}(H)U'(R), g = \left(\int_0^\infty \tilde{G}(H)dH\right) U'(R)$ then

$$V = U'(R) \int_H^\infty \int_H^\infty \tilde{G}(\hat{H})d\hat{H}d\tilde{H}.$$

If $G(\cdot, H) \perp U', g \perp U'$ (in L^2) then consider $w(H) = \|V(\cdot, H)\|_{L^2(I\!R)}$.
A calculation shows

$$\begin{cases} -\frac{d^2w}{dH^2} + \lambda_2 w \le \|G(\cdot, H)\|_{L^2(I\!R)},\ H \in (0, \infty) \\ w(\infty) = 0,\ w(0) < \infty \end{cases}.$$

From this it follows that $w = o(e^{-\nu H})$.

Finally elliptic estimates yield $D^\alpha \phi = O(e^{-\nu H})$ for $|\alpha| \le k + 2$. Also utilizing that $f'(U(\pm\infty)) = f'(\pm 1) \ge \nu_2$ we have

$$D^\alpha V = O(e^{-\nu H})O(e^{-\nu R}).$$

Note that the condition $\phi \in H^1(D)$ excludes solutions of the form $(a + bH)U'(R)$ with $a^2 + b^2 > 0$. If one allows solutions of the form $HU'(R)$, then (1.14) can be removed.

Hence, (2.36) has a unique bounded solution that decays $O(e^{-\nu|R| - \nu H})$ if and only if the following compatibility conditions are satisfied:

$$\begin{cases} \alpha_1^\pm(\xi) = -\mathcal{K}_{\Omega_\delta}(g_0^\pm) \int_{I\!R} (R(U')^2 \Big/ \int_{I\!R} (U')^2 = 0, \\ \alpha_j^\pm = \cdots \end{cases} \tag{2.39}$$

Remark 2.2
1. The preceding compatibility conditions show that $\alpha_j^{\pm}$ are uniquely determined by lower order expansions, and so is $u^{\mathrm{B}\pm}_{j-1}$. That is to say, $(\alpha_j^{\pm}, u^{\mathrm{B}\pm}_j)$ are decoupled from other j^{th} order expansion unknowns. They are readily available as soon as all expansions up to $j-1^{\mathrm{th}}$ order are available.

2. Since $\alpha_1^{\pm}(\xi)=0$, we then know that at the intersection of Γ and $\partial\Omega_\delta$, the contact angle is $\frac{\pi}{2}+O(\varepsilon^2)$. We shall be utilizing this fact later on in the eigenvalue analysis.

3. Since $\mathcal{K}_{\Omega_\delta}=O(\delta)$, one sees that

$$u^{\mathrm{B}}{}_j = O(\delta e^{-\nu(|R|+H)}), \quad j=1,2,$$

2.4 Extension to the whole domain (exterior expansion)

One may notice that the coordinates (r,s) and (r,h) are local. Hence, we need extensions of u^{I} and u^{B} to the whole domain Ω_δ.

Easy mathematical induction gives that $u^{\mathrm{I}}{}_j(R,s,\xi)=u_j^{\pm}(\xi)+O(e^{-\nu|R|})$ as $R\to\pm\infty$, where $u_j^{\pm}(\xi)$ is independent of s and if we write $u^{\pm}(\xi,\varepsilon):=\pm1+\varepsilon\Sigma_{j\geq0}\varepsilon^j u_j^{\pm}$, then

$$f(u^{\pm})+\varepsilon\sigma+\varepsilon^2 c u_\xi^{\pm}=0. \tag{2.40}$$

Therefore,

$$u_\varepsilon^{\pm}(\xi)=\pm1-\varepsilon\frac{\sigma(\xi,\varepsilon)}{f'(\pm1)}+\varepsilon\Sigma_{j\geq1}\varepsilon^j(\cdots)(\xi). \tag{2.41}$$

We define

$$\begin{aligned} u^{\mathrm{I}}(x,\xi,\varepsilon) &= (1-\zeta^+-\zeta^-)(U(R)+\varepsilon\Sigma_{j\geq0}\varepsilon^j u^{\mathrm{I}}{}_j) \\ &\quad +\zeta^+u^+(\xi)+\zeta^-u^-(\xi) \end{aligned} \tag{2.42}$$

where

$$\zeta^{\pm}=\zeta\Big(\pm\frac{\mathrm{r}(x,\xi,\varepsilon)}{\varepsilon\ln^2\varepsilon}-1\Big)$$

and $\zeta\in C^\infty$ is a fixed function satisfying

$$\zeta(s)=1 \text{ if } s>1, \quad \zeta(s)=0 \text{ if } s<0, \quad s\zeta'(s)\geq0 \text{ on } \mathbb{R}.$$

Similarly, since for any j, $u^{\mathrm{B}}{}_j=O(e^{-\nu|R|-\nu|H|})$, as $|R|+|H|\to\infty$, hence we can extend $u^{\mathrm{B}\pm}_j$ to Ω by a smooth extension by zero in a way we have done for u^{I}.

Now define $u=u^{\mathrm{I}}+u^{\mathrm{B}}$, we see that (1.10a) and (1.10b) are satisfied asymptotically.

Remark 2.3 The expansion for $u^{\pm}(x,\xi,\varepsilon) := u^{\pm}_{\varepsilon}(\xi)$ is called the **outer expansion** and usually depends on the space variable x (cf. [2]). In the current situation, it is independent of the space variable x, so that we do not need the complicated interior/exterior matched asymptotics introduced in [33] or in [2].

2.5 Area constraint condition

Next we solve the problem ($\mathrm{P_A}$).

Denote by Ω_δ^- the part of Ω_δ enclosed by Γ; we can calculate

$$\iint_{\Omega_\delta}(u^{\mathrm{I}}+u^{\mathrm{B}}) = u^+(\xi,\varepsilon)(|\Omega_\delta|-|\Omega_\delta^-|)+u^-(\xi,\varepsilon)|\Omega_\delta^-| \tag{2.43}$$

$$+\iint_{\Omega_\delta^+}(u^{\mathrm{I}}-u^+)+\iint_{\Omega_\delta^-}(u^{\mathrm{I}}-u^-) + \iint_{\Omega_\delta} u^{\mathrm{B}}. \tag{2.44}$$

First of all, since $\frac{\partial x}{\partial(R,H)} = \varepsilon^2\Big|n\cdot\mathbf{T}\Big| = \varepsilon^2\sqrt{1-(N\cdot n)^2} = \varepsilon^2(1+\Sigma_{j\geq 2}(\cdots))$ and since ${u^{\mathrm{B}}}_j$ decays exponentially to zero as $|R|+H\to\infty$,

$$\iint_{\Omega} u^{\mathrm{B}} = \left(\int_{I\!R} dR\int_0^\infty dH\varepsilon^2\Sigma_{j\geq 1}\varepsilon^j({u^{\mathrm{B}}}_j+\cdots)\right) + e^{-\frac{c}{\varepsilon}} = \varepsilon\Sigma_{j\geq 2}\varepsilon^j(\cdots)+e^{-\frac{c}{\varepsilon}},$$

for some $c>0$. The exponential term can be safely discarded.

Similarly, since $\frac{\partial x}{\partial(R,s)} = \varepsilon(1+\varepsilon R\mathcal{K})$,

$$\begin{aligned}
&\iint_{\Omega_\delta^+}(u^{\mathrm{I}+}-u^+)+\iint_{\Omega_\delta^-}(u^{\mathrm{I}}-u^-)\\
&= \varepsilon\int_0^\infty dR\int_{S^-(R,\xi,\varepsilon)}^{S^+(R,\xi,\varepsilon)}(1+\varepsilon R\mathcal{K})(u^{\mathrm{I}}-u^+)ds\\
&\quad +\varepsilon\int_{-\infty}^0 dR\int_{S^-(R,\xi,\varepsilon)}^{S^+(R,\xi,\varepsilon)}(1+\varepsilon R\mathcal{K})(u^{\mathrm{I}}-u^-)ds.\\
&= \Sigma_{j\geq 1}\varepsilon^j\{\cdots\}.
\end{aligned}$$

Therefore, equation (1.20) becomes

$$|\Omega_\delta|-\pi = |\Omega_\delta|u^+ + |\Omega_\delta^-|(u^- - u^+)+\Sigma_{j\geq 1}\varepsilon^j(\cdots).$$

Using the expansion of $u^\pm$ in (2.41), we then obtain

$$|\Omega_\delta^-(\xi,\varepsilon)| = \frac{\pi}{2} - \frac{-(\varepsilon|\Omega_\delta|)}{2f'(1)}\Big\{\sigma(\xi,\varepsilon) + \Sigma_{j\geq 1}\varepsilon^j[\cdots]\Big\} + \Sigma_{j\geq 1}\varepsilon^j(\cdots) \qquad (2.45)$$

where $\cdots$ depends only on expansions of order $\leq j-1$ and is independent of $|\Omega_\delta|$. Since $\varepsilon|\Omega_\delta| = \varepsilon\delta^{-2}|\Omega|$ and δ is small, here and in the sequel, we need the explicit dependence on $\varepsilon|\Omega_\delta|$.

To consider the case where both ε and δ are small, we introduce

$$\varepsilon^* = \varepsilon\delta^{-2}. \qquad (2.46)$$

In the sequel, we shall always assume that $\varepsilon^* \in (0,1]$. Under such an assumption, and thinking of ε^* as a new parameter, we can write (2.45) as

$$|\Omega_\delta^-(\xi,\varepsilon)| = \Big\{\frac{\pi}{2} - a_2\varepsilon^*\hat{\sigma}\Big\} + \Sigma_{j\geq 1}\varepsilon^j\{\cdots\} \qquad (2.47)$$

where $\hat{\sigma} = \hat{\sigma}(\xi,\varepsilon)$ is as in (2.27) and

$$a_2 = \frac{|\Omega|}{2f'(1)a_0} = \frac{|\Omega|}{4f'(1)}\int_{I\!R}(\dot{U}(R))^2 dR. \qquad (2.48)$$

In summary, problem (P_A) is equivalent to solving (2.47).

2.6 The asymptotic expansion of the solution to the geometric problem

A. The geometric problem

We first summarize all the conditions imposed on $(\Gamma(\xi,\varepsilon), \sigma(\xi,\varepsilon), c(\xi,\varepsilon))$.

1. The intersection condition

The intersections of Γ with Ω_δ are $z(\xi - g(\xi,\varepsilon))$ and $z(\xi + g(\xi,\varepsilon))$. Hence, from the equation $w_s = e^{\mathrm{i}[\psi+\varphi^\delta+\pi/2]}$ (cf (2.7)), it is convenient to take $w(s,\xi,\varepsilon)$ in the form

$$w(s,\xi,\varepsilon) = z^\delta(\xi + g(\xi,\varepsilon)) + \int_0^s \exp\Big(\mathbf{i}[\psi(\tilde{s},\xi,\varepsilon) + \varphi^\delta(\xi) + \pi/2]\Big)\, d\tilde{s} \qquad (2.49)$$

where $\psi(s,\xi,\varepsilon)$ is a real valued function to be determined. With such a choice of w, the intersection condition $w(0,\xi,\varepsilon) = z^\delta(\xi - g(\xi,\varepsilon))$ is automatically satisfied. The other intersection condition $z_\delta(\xi - g(\xi,\varepsilon)) = w(|\Gamma|(\xi,\varepsilon),\xi,\varepsilon)$ can be written as

$$\int_0^{|\Gamma|(\xi,\varepsilon)} e^{\mathbf{i}\psi(\tilde{s},\xi,\varepsilon)}d\tilde{s} = \mathbf{i}\int_{-g(\xi,\varepsilon)}^{g(\xi,\varepsilon)} e^{\mathbf{i}[\varphi^\delta(\xi+\varsigma)-\varphi^\delta(\xi)]}\, d\varsigma. \qquad (2.50)$$

2. The contact angle condition

The representation of $w(s,\xi,\varepsilon)$ in (2.49) implies $\mathbf{T} = e^{\mathbf{i}[\psi(s,\xi,\varepsilon)+\varphi^\delta(\xi)+\pi/2]}$, $N = -\mathbf{i}\mathbf{T} = e^{\mathbf{i}[\psi(s,\xi,\varepsilon)+\varphi^\delta(\xi)]}$, so that

$$N\cdot n\Big|_{x=p^+} = \sin\Big(\psi(0,\xi,\varepsilon)+\varphi^\delta(\xi)-\varphi^\delta(\xi+g)\Big),$$
$$N\cdot n\Big|_{x=p^-} = \sin\Big(\psi(|\Gamma|,\xi,\varepsilon)+\varphi^\delta(\xi)-\varphi^\delta(\xi-g)\Big).$$

It then follows from (2.39) and the definition of $\alpha_j^\pm$ in (2.35) that the contact angle condition is equivalent to

$$\begin{cases} \psi(0,\xi,\varepsilon) = \varphi^\delta(\xi+g(\xi,\varepsilon)) - \varphi^\delta(\xi) + \Sigma_{j\geq 2}\varepsilon^j\{\cdots\}, \\ \psi(|\Gamma|(\xi,\varepsilon),\xi,\varepsilon) = \pi + [\varphi^\delta(\xi-g(\xi,\varepsilon)) - \varphi^\delta(\xi)] \\ \qquad\qquad +\Sigma_{j\geq 2}\varepsilon^j\{\cdots\}. \end{cases} \tag{2.51}$$

Here and in the sequel, "$\cdots$" denotes terms depending only on expansions of order no greater than $j-1$; namely, one can assume that they are known constants or functions.

3. The equation of motion

With ψ given as in (2.49),

$$\mathcal{K} = \psi_s,$$
$$\mathrm{r}_\xi = -w_\xi\cdot N = -(1+g_\xi)\cos\{\varphi^\delta(\xi+g)-\varphi^\delta(\xi)-\psi\} +$$
$$+\int_0^s \Big\{\psi_\xi(\tilde{s},\cdot)+\varphi_\xi^\delta(\xi)\Big\}\cos\Big\{\psi(\tilde{s},\cdot)-\psi(s,\cdot)\Big\}\,d\tilde{s}.$$

Hence, the equation of motion (2.28) can be written as

$$\begin{aligned}\psi_s(s,\xi,\varepsilon) = \hat{\sigma} - c(1+g_\xi)\cos\{\varphi^\delta(\xi+g)-\varphi^\delta(\xi)-\psi\} \\ +c\int_0^s \Big[\psi_\xi(\tilde{s},\cdot)) + \varphi_\xi^\delta(\xi)\Big]\cos\Big\{\psi(\tilde{s},\cdot)-\psi(s,\cdot)\Big\}\,d\tilde{s} \\ +\Sigma_{j\geq 2}\varepsilon^j\{\cdots\}.\end{aligned} \tag{2.52}$$

4. The area constraint condition

Using "Im" to denote the imaginary part of a complex variable, we can calculate

$$\begin{aligned}|\Omega^-(\xi,\varepsilon)| &= \frac{1}{2}\int_{\partial\Omega^-}(xdy-ydx) \\ &= -\frac{1}{2}\mathrm{Im}\int_{\partial\Omega^-}(\mathrm{x}+\mathbf{i}\mathrm{y}-\mathrm{z}(\xi+\mathrm{g}))\mathrm{d}(\mathrm{x}-\mathrm{iy}) \\ &= -\frac{1}{2}\mathrm{Im}\Big\{\int_{\xi-\mathrm{g}}^{\xi+\mathrm{g}}[\mathrm{z}(\tilde{\xi})-\mathrm{z}(\xi+\mathrm{g})]\overline{\mathrm{z}'(\tilde{\xi})}\mathrm{d}\tilde{\xi}\end{aligned}$$

$$+\int_0^{|\Gamma|}[w(\tilde{s},\cdot)-w(0,\cdot)]\overline{w_s(\tilde{s},\cdot)}d\tilde{s}\Big\}$$
$$=-\frac{1}{2}\Big\{\int_{\xi-g}^{\xi+g}\int_{\xi+g}^{\tilde{\xi}}\sin\Big(\varphi^\delta(\hat{\xi})-\varphi^\delta(\tilde{\xi})\Big)d\hat{\xi}d\tilde{\xi}$$
$$+\int_0^{|\Gamma|}\int_0^{s}\sin\Big(\psi(\hat{s},\cdot)-\psi(\tilde{s},\cdot)\Big)d\hat{s}d\tilde{s}\Big\}.$$

Therefore, the area constraint condition (2.47) can be written as

$$\int_{\xi-g}^{\xi+g}\int_{\xi+g}^{\tilde{\xi}}\sin\Big(\varphi^\delta(\tilde{\xi})-\varphi^\delta(\hat{\xi})\Big)d\hat{\xi}d\tilde{\xi}$$
$$+\int_0^{|\Gamma|}\int_0^{s}\sin\Big(\psi(\tilde{s},\cdot)-\psi(\hat{s},\cdot)\Big)d\hat{s}d\tilde{s}$$
$$=[\pi-a_2\varepsilon^*\hat{\sigma}]+\Sigma_{j\geq 1}\varepsilon^j\{\cdots\}. \tag{2.53}$$

Definition of the geometric problem: *Find* $(\psi(s,\xi,\varepsilon)$*,* $g(\xi,\varepsilon)$*,* $|\Gamma|(\xi,\varepsilon)$*,* $\sigma(\xi,\varepsilon)$*,* $c(\xi,\varepsilon))$*, where* $\xi\in I\!R/|\partial\Omega_\delta|$ *and* ε *is a small parameter, such that the equations (2.50), (2.51), (2.52), and (2.53) are satisfied.*

B. The formal expansion set up

Since we are only interested in small δ, we can expand every coefficient in the ε power expansions in δ power expansions. This will lead to double series expansions. Because here we consider the case $\varepsilon^m<\delta<\sqrt{\varepsilon}$ where $m>0$ is fixed, to save calculation, we shall expand functions in a single series expansion; namely, we expand all unknowns in a δ power series. To compensate for the ε expansion, we introduce the parameter ε^* defined in (2.46). By considering $\varepsilon^*\in(0,1]$ as a fixed parameter, we replace ε by $\varepsilon^*\delta^2$ and seek expansions of the form

$$\begin{aligned}\psi(s,\xi,\varepsilon)&=\Sigma_{j\geq 0}\quad\delta^j\psi_j(s,\xi),\\ g(\xi,\varepsilon)&=\Sigma_{j\geq 0}\quad\delta^j g_j(\xi),\\ |\Gamma|(\xi,\varepsilon)&=\Sigma_{j\geq 0}\delta^j|\Gamma_j|(\xi),\\ \sigma(\xi,\varepsilon)&=\Sigma_{j\geq 0}\quad\delta^j\sigma_j(\xi),\\ c(\xi,\varepsilon)&=\Sigma_{j\geq 0}\quad\delta^j c_j(\xi).\end{aligned}$$

The modification for the expansion for $u(x,\xi,\varepsilon)$ is as follows:

$$u(x,\xi,\varepsilon)=u^{\mathrm{I}}(R,s,\xi,\varepsilon)\Big|_{R=\frac{\mathrm{r}(x,\xi,\varepsilon)}{\varepsilon},s=\mathrm{s}(x,\xi,\varepsilon)}$$
$$+u^{\mathrm{B}}(R,H,\xi,\varepsilon)\Big|_{R=\frac{\mathrm{r}(x,\xi,\varepsilon)}{\varepsilon},H=\frac{\mathrm{h}(x)}{\varepsilon}},$$

$$u^{\mathrm{I}}(R,s,\xi,\varepsilon) = U(R) + \delta^2 \Sigma_{j\geq 0} \quad \delta^j u^{\mathrm{I}}{}_j(R,s,\xi),$$
$$u^{\mathrm{B}}(R,H,\xi,\varepsilon) = \Sigma_{j\geq 2}\delta^j u^{\mathrm{B}}{}_j(R,H,\xi).$$

where $\mathrm{r}(x,\xi,\varepsilon)$ and $\mathrm{s}(x,\xi,\varepsilon)$ can be expressed as power expansion of δ via the δ power expansion of $\psi(s,\xi,\varepsilon)$. To obtain the expansion for $u^{\mathrm{I}}{}_j$, one first makes the change of variable $x \to (R,s)$, obtaining the differential equation for $u^{\mathrm{I}}(R,s,\xi,\varepsilon)$. Then one replaces ε by $\varepsilon^*\delta^2$, and expresses every function in power series of δ with coefficients of functions of the variables R, s and ξ. Collecting terms of the same power of δ one obtains, for each order of δ^j, a set of equations, with sufficient and necessary solvability conditions contributing to the geometric problem. Similarly, one can perform this procedure for u^{B}. After redoing all these expansions, one obtains the same geometric problem (2.50), (2.51), (2.52), (2.53), with the only difference being that $\Sigma_{j\geq 2}\varepsilon^j(\cdots)$ is replaced by $\Sigma_{j\geq 4}\delta^j(\cdots)$.

C. The asymptotic expansions

Since, as it turns out,

$$c_0(\xi) = c_1(\xi) = \hat{\sigma}_{0,\xi}(\xi) \equiv 0, \quad \hat{\sigma}_0 g_0 = 1,$$
$$\sigma_\xi = O(\delta^2), \quad g_\xi = O(\delta^2), \tag{2.54}$$

for the sake of simplicity, we assume this form from the beginning.

As c is the velocity of the droplet (cf.(1.11), we need the first nontrivial term. It turns out that c_2 is non trivial. Also, since we are particularly interested the stability of in equilibria, we need c_3 also. Hence, we carry out most of our calculation explicitly all the way up to the third order. (This is the reason we introduce $\hat{\sigma}$ in (2.27)).

In what follows, we shall need the expansion (2.2) for φ^δ.

Solving the ordinary differential equation (2.52) with the initial condition (2.51a), we obtain

$$\begin{aligned}\psi(s,\xi,\varepsilon) &= \varphi^\delta(\xi+g) - \varphi^\delta(\xi) + \hat{\sigma}s + O(\delta^2)\\ &= \varphi^\delta(\xi+g) - \varphi^\delta(\xi) + \hat{\sigma}s\\ &\quad + c\Big\{ -\frac{\sin(\hat{\sigma}s)}{\hat{\sigma}} + \delta\varphi_1 \frac{1-\cos(\hat{\sigma}s)}{\hat{\sigma}^2}\Big\}\\ &\quad + \Sigma_{j\geq 4}\delta^j\{\cdots\}.\end{aligned} \tag{2.55}$$

where in the first equation, we have used the fact that $c = O(\delta^2)$ and in the second equation, we have used the fact that $g_\xi = O(\delta^2)$, and $\psi_\xi + \varphi^\delta_\xi = \varphi^\delta_\xi(\xi+g)(1+g_\xi) + O(\delta^2) = \delta\varphi_1 + O(\delta^2)$.

Thus, to fulfill equation (2.51b), we need only to have

$$\begin{aligned}\hat\sigma|\Gamma| &= \pi + [\varphi^\delta(\xi - g) - \varphi^\delta(\xi + g)] + c[-\hat\sigma^{-1}\sin(\hat\sigma|\Gamma|)\\ &\quad +\delta\varphi_1\hat\sigma^{-2}(1-\cos(\hat\sigma|\Gamma|))] + \Sigma_{j\ge 4}\delta^j(\cdots)\\ &= \pi + [\varphi^\delta(\xi - g) - \varphi^\delta(\xi + g)] + \Sigma_{j\ge 4}\delta^j(\cdots)\\ &= \pi - 2\varphi_1 g\delta - \tfrac{1}{3}\varphi_3 g^3\delta^3 + \Sigma_{j\ge 4}\delta^j(\cdots) \end{aligned}\tag{2.56}$$

where in the second equation, we used the fact that $\hat\sigma|\Gamma| = \pi - 2\varphi_1 g\delta + O(\delta^2)$ (from the first equation) so that $-\hat\sigma^{-1}\sin(\hat\sigma|\Gamma|) + \delta\varphi_1\hat\sigma^{-2}(1 - \cos(\hat\sigma|\Gamma|)) = -2\varphi_1 g\hat\sigma^{-1}\delta + 2\delta\varphi_1 + O(\delta^2) = 2\delta\varphi_1\hat\sigma^{-2}(-\hat\sigma g + 1) + O(\delta^2) = O(\delta^2)$ since $\hat\sigma g = 1 + O(\delta)$. Hence, we obtain the following set of equations, for each order of expansion:

$$\begin{cases}\hat\sigma_0|\Gamma|_0 = \pi,\\ (\hat\sigma|\Gamma|)_1 = -2g_0\varphi_1,\\ (\hat\sigma|\Gamma|)_j = \cdots, \quad j\ge 2\end{cases}\tag{2.57}$$

where $(\hat\sigma|\Gamma|)_j := \Sigma_{i=0}^{j}\hat\sigma_i|\Gamma|_j$, cf. (2.34). and "$\cdots$" represents known terms (i.e., lower order expansion terms).

We continue with the solution of (2.50). First, we calculate the integral on the right–hand side. Using Taylor expansion, we can calculate

$$\begin{aligned}&\int_{-g}^{g} e^{\mathbf{i}[\varphi^\delta(\xi+\varsigma)-\varphi^\delta(\xi)]}d\varsigma\\ &\quad= \int_{-g}^{g}\Big\{1 + \Sigma_{k=1}^{\infty}\frac{1}{k!}\Big(\mathbf{i}\Sigma_{l\ge 1}(\delta\varsigma)^l\varphi_l/l!\Big)^k\Big\}d\varsigma\\ &\quad= 2g + \tfrac{1}{3}[\mathbf{i}\varphi_2 - \varphi_1^2]g^3\delta^2 + \Sigma_{j\ge 4}\delta^j\{\cdots\}.\end{aligned}$$

Using the expression of ψ in (2.55), we can calculate

$$\begin{aligned}\int_0^{|\Gamma|} e^{\mathbf{i}\psi}ds &= e^{\mathbf{i}[\varphi^\delta(\xi+g)-\varphi^\delta(\xi)]}\int_0^{|\Gamma|} e^{\mathbf{i}\hat\sigma s}\\ &\quad\times\Big\{1 + \mathbf{i}c\Big[-\frac{\sin(\hat\sigma s)}{\hat\sigma} + \varphi_1\delta\frac{1-\cos(\hat\sigma s)}{\hat\sigma^2}\Big]\\ &\quad + \Sigma_{j\ge 4}\delta^j(\ldots)\Big\}\, ds\\ &= e^{\mathbf{i}[\varphi^\delta(\xi+g)-\varphi^\delta(\xi)]}\Big\{\frac{e^{\mathbf{i}\hat\sigma|\Gamma|}-1}{\mathbf{i}\hat\sigma} + \mathbf{i}c\Big[\Big(\frac{e^{2\mathbf{i}\hat\sigma|\Gamma|}-1}{4\hat\sigma^2} - \frac{\mathbf{i}|\Gamma|}{2\hat\sigma}\Big)\\ &\quad + \varphi_1\delta\Big(\frac{2\mathbf{i}}{\hat\sigma^3} - \frac{|\Gamma|}{2\hat\sigma^2}\Big)\Big] + \Sigma_{j\ge 4}\delta^j(\ldots)\Big\}\\ &= \frac{\mathbf{i}}{\hat\sigma}\Big\{2 + (\mathbf{i}\varphi_2 - \varphi_1^2)g^2\delta^2 + c\Big[-\frac{\mathbf{i}|\Gamma|}{2} + \frac{\mathbf{i}\varphi_1\delta}{\hat\sigma^2}\Big]\\ &\quad + \Sigma_{j\ge 4}\delta^j(\ldots)\Big\}\end{aligned}$$

by using (2.56) and the assumption $\hat{\sigma} g = 1 + O(\delta)$. Hence, equation (2.50) becomes

$$2\hat{\sigma} g = 2 + [\mathbf{i}\varphi_2 - \varphi_1^2][1 - \tfrac{1}{3}\hat{\sigma} g] g^2\delta^2 + \mathbf{i}c[-|\Gamma|/2 + \varphi_1\delta/\hat{\sigma}^2] \\ + \Sigma_{j\geq 4}\delta^j(\cdots).$$

Equating the real and imaginary parts, we then obtain

$$\hat{\sigma} g = 1 - \tfrac{1}{3}\varphi_1^2 g^2\delta^2 + \Sigma_{j\geq 4}\delta^j(\cdots), \tag{2.58}$$

$$c = \tfrac{2}{3}\frac{\varphi_2 g^2\delta^2}{|\Gamma|/2 - \varphi_1\delta/\hat{\sigma}^2} + \Sigma_{j\geq 4}\delta^j(\cdots) = -\frac{4}{3\pi}\varphi_2 g\delta^2 \\ + \frac{16}{3\pi^2}\varphi_1\varphi_2 g^2\delta^3 + \Sigma_{j\geq 4}\delta^j(\cdots) \tag{2.59}$$

where in calculating c, we have used the previous result: $\hat{\sigma} g = 1 + O(\delta^2)$, $\hat{\sigma}|\Gamma| = \pi - 2\varphi_1 g\delta + O(\delta^2)$. In terms of the series expansion, we can write (2.58) and (2.59) as

$$\begin{cases} \hat{\sigma}_0 g_0 = 1, & c_0 = 0, \\ (\hat{\sigma} g)_1 = 0, & c_1 = 0, \\ a_0(\sigma g)_j = \cdots, & c_j = \cdots, \quad j \geq 2 \end{cases} \tag{2.60}$$

We remark that here we obtain $c_0 = c_1 = 0$ and $\hat{\sigma} g = 1 + O(\delta)$, part of the assumptions we assumed at the beginning in (2.54).

Finally, we solve the area constraint condition (2.53). We can compute, using Taylor expansion,

$$\int_{\xi-g}^{\xi+g}\int_{\xi+g}^{\tilde{\xi}} \sin\Big(\varphi^\delta(\tilde{\xi}) - \varphi^\delta(\hat{\xi})\Big) d\hat{\xi} d\tilde{\xi} \\ = \int_{-g}^{g}\int_{g}^{\varsigma} \sin\Big(\varphi^\delta(\xi+\varsigma) - \varphi^\delta(\xi+\hat{\varsigma})\Big) d\hat{\varsigma} d\varsigma \\ = \int_{-g}^{g}\int_{g}^{\varsigma} \Big[\delta\varphi_1(\varsigma - \hat{\varsigma}) + \tfrac{1}{2}\delta^2\varphi_2(\varsigma^2 - \hat{\varsigma}^2) \\ + \tfrac{1}{6}\delta^3\varphi_3(\varsigma^3 - \hat{\varphi}^3) - \tfrac{1}{6}\delta^3\varphi_1^3(\varsigma - \hat{\varsigma})^3 + O(\delta^4)\Big] \\ = \tfrac{4}{3}\varphi_1 g^3\delta + [\tfrac{1}{20}\varphi_3 - \tfrac{4}{15}\varphi_1^3] g^5\delta^3 + \Sigma_{j\geq 4}\delta^j(\cdots).$$

Similarly, we have

$$\int_0^{|\Gamma|}\int_0^s \sin\Big(\psi(s,\cdot) - \psi(\hat{s},\cdot)\Big) d\hat{s} ds \tag{2.61}$$

$$= \int_0^{|\Gamma|}\int_0^s \sin\Big\{\hat{\sigma}(s-\hat{s}) - c\Big([\sin(\hat{\sigma}s) - \sin(\hat{\sigma}\hat{s})]\hat{\sigma}^{-1}$$

$$
\begin{aligned}
&+\varphi_1\delta[\cos(\hat\sigma s)-\cos(\hat\sigma\hat s)]\hat\sigma^{-2}\Big)\Big\}d\hat s ds+\Sigma_{j\geq 4}\delta^j(\cdots)\\
&=\int_0^{|\Gamma|}\int_0^s \sin(\hat\sigma(s-\hat s))d\hat s ds-\frac{c}{\hat\sigma}\int_0^{|\Gamma|}\int_0^s\\
&\quad\times\cos(\hat\sigma(s-\hat s))[\sin(\hat\sigma s)-\sin(\hat\sigma\hat s)]d\hat s ds\\
&\qquad-\frac{c\varphi_1\delta}{\hat\sigma^2}\int_0^T\int_0^s\cos(\hat\sigma(s-\hat s))[\cos(\hat\sigma s)\\
&\quad-\cos(\hat\sigma\hat s)]d\hat s ds+\Sigma_{j\geq 4}\delta^j(\cdots)\\
&=\frac{\hat\sigma|\Gamma|-\sin(\hat\sigma|\Gamma|)}{\sigma^2}+\Sigma_{j\geq 4}\delta^j(\cdots) \qquad (2.62)\\
&=\frac{\pi-4\varphi_1 g\delta}{\hat\sigma^2}+\Sigma_{j\geq 3}\delta^j(\cdots).
\end{aligned}
$$

Here, in the last step, we have substituted from (2.56) and in obtaining (2.62), we used the fact that

$$
\begin{aligned}
&\int_0^{|\Gamma|}\int_0^s\cos(\hat\sigma(s-\hat s))[\sin(\hat\sigma s)-\sin(\hat\sigma\hat s)]d\hat s ds\\
&=\int_0^{|\Gamma|}ds\int_0^s\cos(\hat\sigma(s-\hat s))\sin(\hat\sigma s)d\hat s\\
&\quad-\int_0^{|\Gamma|}d\hat s\int_{\hat s}^{|\Gamma|}\cos(\hat\sigma(s-\hat s)\sin(\hat\sigma\hat s)ds\\
&=\frac{1}{\hat\sigma}\int_0^{|\Gamma|}\sin(\hat\sigma s)[\sin(\hat\sigma s)-\sin(\hat\sigma|\Gamma|-\hat\sigma s)]\,ds=O(\delta)
\end{aligned}
$$

by using $\hat\sigma|\Gamma|=\pi+O(\delta)$, and similarly, $\int_0^{|\Gamma|}\int_0^s\cos(\hat\sigma(s-\hat s))[\cos(\hat\sigma s)-\cos(\hat\sigma\hat s)]=O(\delta)$.

Now the equation (2.53) becomes

$$\hat\sigma^2(\pi-a_2\varepsilon^*\hat\sigma)=\pi-4\varphi_1 g\delta+\Sigma_{j\geq 2}\delta^j(\cdots). \qquad (2.63)$$

Namely,

$$
\begin{cases}
\hat\sigma_0^2[\pi-a_2\hat\sigma_0\varepsilon^*]=\pi,\\
\Big(\hat\sigma^2[\pi-a_2\hat\sigma\varepsilon^*]\Big)_1=\frac{1}{3}\phi_1 g\delta,\\
\Big(\hat\sigma^2[\pi-a_2\hat\sigma\varepsilon^*]\Big)_j=\cdots,\quad j\geq 2.
\end{cases} \qquad (2.64)
$$

We can now solve our geometric problem as follows: First, solve $\hat\sigma_0$ from (2.64) (See Remark to follow). Then, we solve for g_0, c_0, L_0 from (2.60) and (2.57) respectively. One observes that we have $c_0=0, \sigma_{0,\xi}=g_{0,\xi}=0, \hat\sigma_0 g_0=1$, $\hat\sigma_0 L_0=\pi$. Then consecutively for each integer $j\geq 1$, we can repeat the same process and solve first for $\hat\sigma_j$ from (2.64), and then for g_j, c_j from (2.60) and for finally L_j from (2.57). Here we remark that when $j = 1$, one obtains $\hat{\sigma}_{1,\xi} = O(\delta)$, $c_1 = 0$, and $g_{1,\xi} = O(\delta)$ so that $\hat{\sigma}_\xi = O(\delta^2)$ and $g_\xi = O(\delta^2)$. This is what we have assumed at the very beginning. If we do not make such an assumption, we can still carry out our calculation (but with much longer expressions) and at the end, obtain all the equalities in (2.54).

Remark 2.4
1. Since $\sigma_0|\Gamma|_0 = \pi$, we have $\sigma_0 > 0$. Hence, to ensure that the first equation in (2.64) has (at least) one solution we have to assume that $\hat{\varepsilon}^* \le \frac{2\pi}{3\sqrt{3}a_2}$. As $\varepsilon^* = \varepsilon\delta^{-2}$, this translates to

$$\hat{\varepsilon}\delta^{-3} = \varepsilon\delta^{-2} < C_1^* = \frac{2\pi}{3\sqrt{3}a_2} = \frac{4\pi a_0 f'(1)}{3\sqrt{3}|\Omega|}$$
$$= \frac{8\pi f'(1)}{3\sqrt{6}|\Omega| \int_{-1}^{1} \sqrt{W(s)}ds}. \tag{2.65}$$

Also, to make sure that there are solutions for $\hat{\sigma}_j$ for $j \ge 0$, we need that the derivative of $x^2(\pi - a_2\varepsilon^* x)$ at $x = \hat{\sigma}_0$ does not vanish. Hence, to make sure the expansions are bounded we need $\varepsilon < (C_1^* - \eta)\delta^2$ for any fixed η independent of δ and ε.

2. Assume (2.65) holds. Then (2.64) has exactly two positive solutions. Consequently, if $\varepsilon^* := \varepsilon\delta^{-2}$ is not too small (say, $\varepsilon^* > \sqrt{\delta}$) we have two legitimate asymptotic expansions. As $\hat{\sigma}|\Gamma|_0 = \pi$, the expansion with smaller $\hat{\sigma}_0$ has larger radius of interface than that of the expansion with larger $\hat{\sigma}_0$. In particular, if $\varepsilon^* = o(1)$ then the expansion produced by the smaller solution of (2.64) satisfies $|\Gamma|_0 = \pi + o(1)$ and the expansion produced by the larger solution of (2.64) satisfies $|\Gamma|_0 = \frac{2a_0\varepsilon^* + o(\varepsilon^*)}{\pi}$. We call the former a "droplet" solution and the latter a "spike" solution.

3. As we shall see from the eigenvalue analysis (cf. Remark 3.9), the spike solution is unstable. We include in the appendix a calculation (for the case of radial symetry) of the energy of three states: the "droplet", the "spike", and the "constant". We shall show that if (2.65) holds, there is a droplet and a spike solution. The spike solution always has higher energy than that of the constant state. The droplet solution has lower energy than that of constant state if $\varepsilon^* \in (C^{**}, C_1^*)$, where $C^{**} < C_1^*$ is a number which can be calculated. Our eigenvalue analysis shows that the droplet is stable if $\varepsilon^* < C^{***}$ for some $C^{***} < C^{**}$. That we did not obtain the best value of C^{***} is purely due to technical reasons.

4. For spike solutions we refer to Bates and Fife [12], Bates and Fusco [71], and Ward [70] for bistable nonlinearities, and to Ni and Takagi [69], Ward [70] and Kowalczyk [74] for one sided nonlinearities.

In the rest of this paper, we shall be interested only in the droplet solution; and we take $\hat{\sigma}_0$ to be the smaller positive solution of (2.64).

2.7 Conclusion

1. Existence of the approximate manifold

Assume that for some $\eta \in (0, C_1^*)$ independent of δ and ε, we have $\varepsilon \le (C_1^* - \eta)\delta^2$. Then taking $\hat{\sigma}_0$ as the smaller solution of (2.65), we can obtain an asymptotic solution to the equation of manifold (1.10). In the rest of this paper, we will always refer to this solution.

By truncating the expansion at finite order, say $O(\delta^K)$, we then obtain that (u, σ, c) satisfies all the equations (1.10) up to order $O(\delta^K)$. In addition, we can make the following refinement:

1) adding a constant term of order $O(\delta^K)$ so that the area constraint is exactly satisfied;

2) adding a function of order δ^K such that the boundary condition is satisfied exactly;

3) adding an $O(\delta^K)$ term to σ so that $\varepsilon\sigma = \fint\!\!\fint_{\Omega_\delta} f(u)$ since by integrating over Ω the differential equation for u, one finds that $\varepsilon\sigma(\xi, \varepsilon) = \fint\!\!\fint f(u) + O(\varepsilon^K)$. In summary, we have proved the following:

Theorem 2.5 *Assume that δ and ε are small parameters satisfying, for some $m \ge 2$,*

$$\delta^m \le \varepsilon \le \tfrac{1}{2} C_1^* \delta^2 \tag{2.66}$$

where C_1^* is defined in (2.65). Then for any integer K, if ε is sufficiently small, there exist $u = u(x, \xi, \varepsilon)$, $\sigma = \sigma(\xi, \varepsilon)$, $c = c(\xi, \varepsilon)$ such that

$$\begin{cases} \mathcal{L}^\varepsilon(u) := \varepsilon^2 \Delta u - f(u) & \\ \qquad +\varepsilon\sigma = \varepsilon^2 c u_\xi + O(\varepsilon^K) & \text{in } \Omega_\delta, \\ \partial_n u = 0 & \text{on } \partial\Omega_\delta, \\ \int_{\Omega_\delta} u = |\Omega_\delta| - \pi, & \\ \varepsilon\sigma = \fint\!\!\fint_{\Omega_\delta} f(u). & \end{cases} \tag{2.67}$$

In addition, (u, σ, c) has the asymptotic expansion (up to to $O(\varepsilon^K)$) detailed in the previous subsections. In particular, the following expansion holds:

$$\mathcal{K}^2(s, \xi, \varepsilon) = \psi_s^2 = \hat{\sigma}^2 + 2c\delta^2[-\cos(\hat{\sigma}s) + \delta\varphi_1 g \sin(\hat{\sigma}s)] + O(\delta^4), \tag{2.68}$$

$$c = -\frac{4\delta^2}{3\pi\hat{\sigma}_0} \mathcal{K}'(\hat{\xi})\Big|_{\hat{\xi}=\delta\xi} + O(\delta^3). \tag{2.69}$$

where $\hat{\sigma} = a_0\sigma - a_1\varepsilon\sigma^2 = \hat{\sigma}_0 + O(\delta)$, $g = g_0 + O(\delta)$, $g_0 = 1/\hat{\sigma}_0$, and $\hat{\sigma}_0$ is the smaller solution of (2.65).

2. Stability

This subsection, strictly speaking, belongs to the next section on the eigenvalue analysis. However, the principal eigenvalue is special since it can be obtained accurately only through the speed c. It is therefore appriopriate to present it here as a Corollary to Theorem 2.5. We define $\ell = s/|\Gamma|$. Then by utilizing Theorem 2.5 and (2.31), (2.33) we obtain

$$\begin{aligned} q(\ell) &:= |\Gamma|^2\Big(\tfrac{3}{4}\mathcal{K}^2 + \tfrac{1}{4}\hat{\sigma}^2\Big) && (2.70)\\ &= (\hat{\sigma}|\Gamma|)^2 + \frac{3(\hat{\sigma}|\Gamma|)^2 cg}{2\hat{\sigma}g}\Big[-\cos(\hat{\sigma}|\Gamma|\ell) \\ &\quad +\varphi_1 g\delta\sin(\hat{\sigma}|\Gamma|\ell)\Big] + O(\delta^4) && (2.71)\\ &= q_1 + \delta^2 q_2(\ell), && (2.72)\\ q_1 &:= \Big[\pi^2 - 4\pi\varphi_1 g\delta + 4\varphi_1^2 g^2\delta^2 - \tfrac{2\pi}{3}\varphi_3 g^3\delta^3\Big] && (2.73)\\ q_2(\ell) &:= 2\pi\varphi_2 g^2\Big[-\cos(\pi\ell) + \varphi_1 g(1-2\ell)\delta\sin(\pi\ell)\Big] \\ &\quad +O(\delta^2), \quad \ell\in(0,1). && (2.74) \end{aligned}$$

We remark that q_1 is a constant, and if $\varphi_2 = 0$, then $q_2 = O(\delta^4)$. Also, one notices that $2g$ is the arc length of the segment of Ω_δ between the two intersections with Γ.

Consider the following eigenvalue problem: Find $(\mu, \Theta(\ell))$ such that for some $\hat{\mu} = \hat{\mu}(\mu)$,

$$\begin{cases} -\Theta''(\ell) - q(\ell)\Theta(\ell) = \mu\Theta(\ell) + \hat{\mu}, \quad \ell\in(0,1),\\ \Theta'(0) = -K_1\Theta(0),\\ \Theta'(1) = K_2\Theta(1),\\ \int_0^1 \Theta(\ell)d\ell = 0, \end{cases} \tag{2.75}$$

where $K_1 = |\Gamma|\delta\varphi_1(\xi+g)$ and $K_2 = |\Gamma|\delta\varphi_1(\xi-g)$ are ($|\Gamma|$ multiples) the curvature of $\partial\Omega_\delta$ at the intersections with Γ. Using Taylor expansion, we have

$$\begin{aligned} K_i &= \frac{\sigma|\Gamma|}{\sigma g} g\Big[\delta\varphi_1 + (-1)^{i+1}|\Gamma|\delta^2 g\varphi_2 + \tfrac{1}{2}L\varphi_3 g^2\delta^3 + O(\delta^4)\Big]\\ &= \pi\varphi_1 g\delta + (-1)^{i+1}\pi\varphi_2 g^2\delta^2 - 2\varphi_1^2 g^2\delta^2 \\ &\quad +(-1)^i 2\varphi_1\varphi_2 g^3\delta^3 + \tfrac{1}{3}\pi\varphi_1^3 g^3\delta^3 + \tfrac{1}{2}\pi\varphi_3 g^3\delta^3. \end{aligned}$$

Corollary 2.6 *Let (μ_1, Θ_1) be the principal eigenvalue and eigenfunction of problem (2.75). Then,*

$$\mu_1 = -\frac{4\pi}{3}\varphi_3 g^3\delta^3 + O(\delta^4) = -\frac{4\pi}{3\hat{\sigma}^3}\delta^3 K(\hat{\xi})\Big|_{\hat{\xi}=\delta\xi}$$

$$+O(\delta^4), \tag{2.76}$$

$$\Theta_1(\ell) = \cos(\pi\ell) + O(\delta) \quad \ell \in (0,1) \tag{2.77}$$

Observe that we have used the relation $\hat{\sigma}g = 1+O(\delta^2)$ and the definition of φ_3 in the second equation of (2.76).

Proof. First we consider the case when $q = q_1$ is a constant. In this case, if we denote by (μ^*, Θ^*) the principal eigenvalue and eigenfunction, then they are given by

$$\Theta^* = b_1 + \sin\left\{b_1 + [\pi + 2b_3](\ell - \tfrac{1}{2})\right\}, \quad \mu^* = [\pi + 2b_3]^2 - q_1$$

where b_1, b_2, b_3 are given by

$$\begin{aligned} b_1 &= 2\sin b_2 \cos b_3 (\pi + 2b_3)^{-1}, \\ b_2 &= \varphi_2 g^2 \delta^2 [1 + 2\varphi_1 g\delta/\pi] + O(\delta^4), \\ b_3 &= -\varphi_1 g\delta - \tfrac{1}{2}\varphi_3 g^3\delta^3 + O(\delta^4). \end{aligned}$$

That is,

$$\mu^* = [\pi + 2b_3]^2 - q_1 = -\frac{4\pi}{3}\varphi_3 g^3 \delta^3 + O(\delta^4). \tag{2.78}$$

Next we consider when $q = q_1 + \delta^2 q_2$. A standard perturbation argument shows that

$$\mu = \mu^* - \delta^2 \frac{\int_0^1 q_2 \Theta^{*2}}{\int_0^1 \Theta^{*2}} + O(\delta^4) = \mu^* + O(\delta^4) = -\frac{4\pi}{3}\varphi_3 g^3\delta^3 + O(\delta^4).$$

This completes the proof. □

Remark 2.7 Corollary 2.6 is valid everywhere including the equilibrium point. In particular, if ξ_0 is a point of strict maximum curvature, then $\varphi_3(\xi_0)$, the second derivative of the curvature function, is negative. It then follows that the principal eigenvalue is positive, and this establishes stability.

3 Eigenvalue analysis

Let $u = u(x, \xi, \varepsilon)$ be the approximation we obtained in the previous section; namely, the solution to

$$\begin{cases} \mathcal{L}^\varepsilon(u) := \varepsilon^2 \Delta u - f(u) + \varepsilon\sigma \\ \qquad\qquad = \varepsilon^2 c u_\xi + O(\varepsilon^K) & \text{in } \Omega_\delta, \\ \partial_n u = 0 & \text{on } \partial\Omega_\delta, \\ \int_{\Omega_\delta} u = |\Omega_\delta| - \pi, \end{cases} \tag{3.1}$$

where K is an integer as large as it is needed. In this section, we shall study the eigenvalue problems (1.21) and (1.22).

One sees that both eigenvalue problems are related to the bilinear form $\langle L\phi,\,\psi\rangle$ defined by

$$\langle L\phi,\,\psi\rangle = \iint_{\Omega_\delta}\Big[\varepsilon^2\nabla\phi\nabla\psi + f'(u)\phi\psi\Big]dx,$$
$$\phi,\psi\in H^1(\Omega_\delta). \tag{3.2}$$

In fact, (λ,ϕ) is an eigenvalue/eigenfunction to (1.22) if and only if $\phi\in H^1(\Omega_\delta)$ and

$$\langle L\phi,\,\psi\rangle = \lambda(\phi,\psi)\quad \forall\psi\in H^1(\Omega) \tag{3.3}$$

and $(\bar\lambda,\bar\phi)$ is an eigenvalue/eigenfunction of (1.21) if and only if $\bar\phi\in\bar H^1(\Omega)$ and

$$\langle L\bar\phi,\,\psi\rangle = \bar\lambda(\bar\phi,\psi)\quad \forall\psi\in \bar H^1(\Omega). \tag{3.4}$$

Here and in the sequel, $(\cdot,\cdot)$ stands for the $L^2(\Omega_\delta)$ inner product, $\|\cdot\|$ the $L^2(\Omega_\delta)$ norm, and

$$\bar H^1(\Omega_\delta) := \{\phi\in H^1(\Omega_\delta)\ ;\ \textstyle\iint_{\Omega_\delta}\phi = 0\}.$$

The idea of our analysis is a separation of variables technique, [8,9, 26,10]. Observe that the principal eigenvalue of (1.21) (or (1.22)) is the infinimum of $\langle L\phi,\,\phi\rangle$ in $\bar H^1(\Omega_\delta)$ (or $H^1(\Omega_\delta)$) subject to $\|\phi\| = 1$. Since away from the interface Γ, $f'(u)$ is uniformly positive, it is reasonable to believe that the mass of the corresponding eigenfunction is concentrated near Γ. That is, one needs only to study the behavior of L near Γ where the local coordinate (r,s) is well–defined. As mentioned in the introduction, in the thin neighborhood of Γ, L can be decomposed as $\mathrm{L} = L_r + L_s$ where

$$L_r := \frac{\varepsilon^2}{1+r\mathcal{K}(s)}\frac{\partial}{\partial r}\Big(\frac{1}{1+r\mathcal{K}(s)}\frac{\partial}{\partial r}\Big) + f'(u)\mathbf{I}\ ,$$
$$L_s := -\frac{\varepsilon^2}{1+r\mathcal{K}(s)}\frac{\partial}{\partial s}\Big(\frac{1}{1+r\mathcal{K}(s)}\frac{\partial}{\partial s}\Big). \tag{3.5}$$

To leading order, one can ignore the $r\mathcal{K}(s)$ term and replace u by $U(\frac{r}{\varepsilon})$. In such a case, L_r becomes $L_r^0 := -\varepsilon^2\partial^2_{rr} + f'(U(\frac{r}{\varepsilon}))$ and L_s becomes $L_s^0 := -\varepsilon^2\partial^2_{ss}$. If one regards the domain for (r,s) as a rectangle, then all the eigenfunctions of $\mathrm{L}^0 := L_r^0 + L_s^0$ have the form $A(r)B(s)$ where $A(r)$ is the eigenfunction of L_r^0 with eigenvalue λ^r and B is the eigenfunction of L_s^0 with eigenvalue λ^s. The corresponding eigenvalue for L^0 is $\lambda^r+\lambda^s$. If the boundary condition for B is Neumann and the interval for s is $[0,L]$, then the eigenfunctions and eigenvalues of L_s^0 are given by

$$B_j = \cos(\tfrac{(j-1)\pi}{L}s),\quad \lambda_j^s = \varepsilon^2(j-1)^2,\quad j=1,2,\cdots.$$

On the other hand, the principal eigenvalue of $L_r^0 := -\varepsilon^2\partial_{rr}^2 + f'(U(\frac{r}{\varepsilon}))$ is zero with (unnormalized) eigenfunction $A_0(r) = \dot{U}(\frac{r}{\varepsilon})$. This eigenvalue relates to the shrinking or expansion of the interface. The next eigenvalue of L^0 is $\geq \nu_0$ for some positive constant ν_0 independent of ε. This corresponds to the fact that in the cross section of the interface, the solution u must stay close to the profile $U(\frac{r}{\varepsilon})$. All this indicates that for small eigenvalues of L the eigenfunction has the form, to leading order, $A_0(r)B_j(s)$, and the corresponding eigenvalue should be $\lambda_j = \lambda_r^0 + \lambda_s^j$, for any finite integer j. From the graph of $A_0(r)B(s)$, one sees that these eigenfunctions corresponds to the change of shape of the interface.

One may notice that the eigenvalue of L_s is of order ε^2 whereas the preceeding argument is based on a leading order approximation. Nevertheless, decomposing the bilinear form $\langle L\phi,\, \phi\rangle$ into $(L_r\phi, \phi) + (L_s\phi, \phi)$, one still sees that the eigenfunction has the decomposition $A(r)B(s)$ up to $O(\varepsilon^2)$. It is well known that, the principal eigenvalue of L_r vanishes not only to the leading order, but also vanishes to $O(\varepsilon)$ order (assuming that the depths of the double well potential $F(u) = \int f(u)du$ is equal). This phenomenon relates to the following identity for the $O(\varepsilon)$ order expansion of the solution u (cf. (1.26):

$$\int_R f''(U(R))\dot{U}^2(R)u_1^I(R,s)dR = 0 \quad \forall s \in [0, L]. \tag{3.6}$$

Therefore, at least formally, one can calculate the eigenvalue up to order $O(\varepsilon^2)$ by assuming that the eigenfunction has the form $\phi = A(r)B(s)$. To make the calculation rigorous, one can use the decomposition

$$\phi = U'(\tfrac{r}{\varepsilon})\Theta(s) + \psi(r,s), \quad \Big(\int \psi(r,s)\dot{\Theta}(\tfrac{r}{\varepsilon})dr = 0 \quad \forall s\Big). \tag{3.7}$$

We remark that $\dot{U}(\frac{r}{\varepsilon})$ is only the leading order approximation to the principal eigenfunction of L_r. This is sufficient for capturing the leading order eigenvalues (of order $O(\varepsilon^2)$) of L since the next eigenvalue of L_r is uniformly (in ε) positive and $\psi(\cdot, s)$ has a large portion on the subspace orthogonal to the principal eigenfunction of L_r.

One may find that to evaluate $\langle L\phi,\, \phi\rangle$ with $\phi = \dot{U}(\frac{r}{\varepsilon})B(s)$ up to $O(\varepsilon^2)$ order, one needs the explicit expansion of the solution up to $O(\varepsilon^2)$, and the calculation may not be straightforward since there are many places contributing to the $O(\varepsilon^2)$ order expansion. Such a calculation was performed in [68] (see [10] and also [26]). The conclusion is that the eigenvalue is proportional (with known proportional constant) to the eigenvalue of the eigenvalue problem, for (Θ, μ), $-\Theta''(s) + K(s)\Theta(s) = \frac{\mu}{\varepsilon^2}\Theta(s)$, $s \in S^1$, where S^1 is the unit circle and $K(s)$ is a known function depending the geometry of the interface Γ. Here the calculation is based on the Allen–Cahn equation: $u_t - \varepsilon^2 \Delta u + f(u) = 0$ with the interface strictly contained in the thin domain.

In the current situation, as we shall see later, the first several eigenvalues of (1.21) have the expansion $(\frac{\varepsilon\pi}{L})^2(j^2-1) + o(\varepsilon^2)$, $j = 1, 2, \cdots$. Thus, the principal eigenvalue (corresponding to $j = 1$) is close to zero, and the $O(\varepsilon^2)$ order expansion gives no information for the sign of the principal eigenvalue. Hence, to find the sign of the principal eigenvalue, which is important to the stability issue, expansions of order higher than $O(\varepsilon^2)$ may be needed. Indeed, the principle eigenvalue is of order $O(\varepsilon^2\delta^3)$. This suggests (for $\varepsilon \sim \delta^2$), an $O(\varepsilon^4)$ order expansion or an $O(\delta^7)$ expansion. Unlike the expansion that we displayed in the previous section, where we only needed the existence of the solution, here we need a more explicit form the principal eigenvalue and determine its sign. One may also notice that the first order boundary layer expansion u_1^B has no closed form. This poses an extra difficulty in performing the calculation.

Another point that one needs to handle is the boundary condition for $\Theta(s)$ since in the current situation Γ intersects the boundary and therefore, the domain for s is an interval, instead of a circle with no boundary.

Due to all these considerations, we seek a better approximation of $A(r)$, the principle eigenfunction of L_r, which, hopefully, will automatically take care of all the lower order contributions, as well as the boundary conditions.[6] We find that the following function meets the requirement (see also (11.3) in [9] for a related point):

$$\phi^0(x) = \sqrt{\varepsilon} e^{-crr_\xi/2} u_r = \sqrt{\varepsilon} e^{-crr_\xi} \nabla r \cdot \nabla u. \tag{3.8}$$

As we shall see later, such chosen ϕ^0 will automatically take care of the cancellation of lower order expansions, due to the fact that u satisfies the differential equation (3.1) and the boundary condition.

The structure of this section is as follows:

In Sect. 3.1, we study a few properties of ϕ^0 defined in (3.8), by utilizing the equation (3.1).

In Sect. 3.2, we introduce a global coordinate system on a band enclosing the interface, and utilizing it we decompose the space $H^1(\Omega_\delta)$ into $\mathcal{X}^0 \oplus_{L^2} \mathcal{X}^{0\perp}$ by setting $\phi = \Theta(s)\phi^0 + \psi(r,s)$.

In Sect. 3.3 we study the quadratic form $\langle L\phi, \phi\rangle$ on $\mathcal{X}^0$ and the corresponding eigenvalue problem: Find $\phi \in \mathcal{X}^0$ and $\lambda \in \mathbb{R}$ such that $\langle L\phi, \psi\rangle = \lambda(\phi, \psi)$ for all $\psi \in \mathcal{X}^0$.

In Sect. 3.4, we study $\langle L\phi, \phi\rangle$ on $\mathcal{X}^{0\perp}$ and then on $H^1(\Omega_\delta) = \mathcal{X}^0 \oplus \mathcal{X}^{0\perp}$ in Sect. 3.5, and finally on $\bar{H}^1(\Omega_\delta)$ in Sect. 3.6.

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[6] In particular the boundary analog of identity (3.6) is satisfied (cf. (1.27)).

In Sect. 3.7, we give an abstract perturbation result that quantifies and refines the following idea: $\langle L\phi,\ \phi\rangle \ll 1 \Rightarrow \phi$ has a significant χ^0 component and so ψ is small (cf. [9,26]).

Finally, in Sect. 3.8, we provide our main result concerning the principal and the second eigenvalue/eigenfunction of the eigenvalue problems (1.21) and (1.22).

3.1 Approximation of the principal eigenfunction for L_r

In the rest of this section, we shall suppress the variable (ξ, ε) for functions such as $\mathrm{r}(x,\xi,\varepsilon)$, $\mathrm{s}(x,\xi,\varepsilon)$, $w(s,\xi,\varepsilon)$, and $\mathrm{x}(r,s,\xi,\varepsilon)$. Also, we introduce

$$\begin{aligned}
\mathrm{I}_\varepsilon &:= (-m_\varepsilon, m_\varepsilon) \quad \text{where} \quad m_\varepsilon := \varepsilon|\ln\varepsilon|^2,\\
\Omega_\delta^1 &:= \{x \in \Omega_\delta : \mathrm{dist}(x, \Gamma) \le \mathrm{I}_\varepsilon\}\\
&= \{\mathrm{x}(r,s)\ :\ |r| < m_\varepsilon, S^+(r) < s < S^-(r)\},\\
\Omega_\delta^2 &:= \Omega_\delta \setminus \overline{\Omega}_\delta^1.
\end{aligned}$$

We denote by $w = w(s), s \in [0, L]$ the representation of the interface where L is the length of the interface. We denote by $p^\pm$ the intersections of Γ with $\partial\Omega_\delta$. Since $\partial\Omega_\delta$ intersects Γ almost orthogonally (up to an error of $O(\varepsilon^2)$), near $p^\pm$, we can represent $\partial\Omega_\delta$ by $s = S^\pm(r)$ where $s = \mathrm{s}(x)$ and $r = \mathrm{r}(x)$ is the local coordinates of Ω_δ^1. We set

$$\begin{aligned}
&\Sigma := \partial\Omega_\delta^1 \cap \partial\Omega_\delta = \Sigma^+ \cup \Sigma^-\\
&\qquad \text{where}\ \ \Sigma^\pm := \{\mathrm{x}(r,s)\ :\ r \in \mathrm{I}_\varepsilon, s = S^\pm(r)\}.
\end{aligned}$$

Note that

$$\begin{aligned}
&L = S^-(0) - S^+(0), \quad S^+(r) = O(\varepsilon^2 + r^2),\\
&S^-(r) = L + O(r^2 + \varepsilon^2), \quad S_r^\pm(r) = O(\varepsilon + |r|).
\end{aligned} \tag{3.9}$$

Certain properties of ϕ^0 are shown in the following lemma.

Lemma 3.1 *Let ϕ^0 be defined as in (3.8). Then*

$$\begin{aligned}
\varepsilon^2\Delta\phi^0 - f'(u)\phi^0 &= \varepsilon^2 \frac{\frac{3}{4}\mathcal{K}^2 + \frac{1}{4}\hat{\sigma}^2}{(1 + r\mathcal{K})^2}\phi^0\\
&\quad + \frac{2\varepsilon^{5/2}\mathcal{K}}{(1 + r\mathcal{K})^3} u^{\mathrm{B}}{}_{ss} + O(\varepsilon^{5/2}\delta^2),
\end{aligned} \tag{3.10}$$

$$\partial_n\phi^0\Big|_{\Sigma^i} = \mathcal{K}_{\Omega_\delta}(\zeta)\phi^0 + O(\varepsilon^{\frac{1}{2}}\delta^2), \tag{3.11}$$

$$\int_{\Sigma^\pm} \phi^0\partial_n\phi^0 = \mathcal{K}_{\Omega_\delta}(p^\pm)\int_{\Sigma^\pm} {\phi^0}^2 + O(\varepsilon\delta^2). \tag{3.12}$$

Proof. In the (r,s) coordinates, $\frac{d}{d\xi} = \mathrm{r}_\xi\partial_r + \mathrm{s}_\xi\partial_s + \partial_\xi$. Differentiating equation (3.1) with respect to r yields (recalling that r_ξ is independent of r)

$$\begin{aligned}\varepsilon^2\Delta u_r - f'(u)u_r &= \frac{\varepsilon^2\mathcal{K}^2}{(1+r\mathcal{K})^2}u_r + \frac{2\varepsilon^2\mathcal{K}}{(1+r\mathcal{K})^3}u_{ss}\\ &\quad+\frac{\varepsilon^2\mathcal{K}_s(1-2r\mathcal{K})}{(1+r\mathcal{K})^4}u_s\\ &\quad\;+\varepsilon^2\mathrm{cr}_\xi u_{rr} + \varepsilon^2 c(\mathrm{s}_\xi u_{sr} + \mathrm{s}_{\xi r}u_s + u_{r\xi})\\ &\quad+O(\varepsilon^{K-1})\\ &= \frac{\varepsilon^2\mathcal{K}^2}{(1+r\mathcal{K})^2}u_r + \frac{2\varepsilon^2\mathcal{K}}{(1+r\mathcal{K})^3}u^B_{ss} + \varepsilon^2\mathrm{cr}_\xi u_{rr}\\ &\quad+O(\varepsilon^2\delta^2),\end{aligned}$$

where in the second equality, the following estimates have been used: $u_{ss} = {u^{\mathrm{I}}}_{ss} + {u^{\mathrm{B}}}_{ss} = O(\varepsilon) + {u^{\mathrm{B}}}_{ss}$, $\mathcal{K}_s = O(\delta^2)$, $u_s = O(1)$, $c = O(\delta^2)$, $u_{sr} = O(1) + {u^{\mathrm{B}}}_{sr} = \frac{1}{\varepsilon}O(e^{-(h+|r|)/\varepsilon}) + O(1)$, $\mathrm{s}_\xi = O(\varepsilon + h + |r|)$, and $u_{r\xi} = O(1)$. By introducing the multiplier $e^{-cr r_\xi/2}$ we can absorb the term $\varepsilon^2\mathrm{cr}_\xi u_{rr}$ and write the equation in the form

$$\begin{aligned}\varepsilon^2\Delta(e^{-crr_\xi/2}u_r) - f'(u)e^{-crr_\xi/2}u_r &= \varepsilon^2 e^{-crr_\xi}u_r\frac{\mathcal{K}^2 - \frac{1}{2}cr_\xi\mathcal{K} + \frac{1}{4}(cr_\xi)^2}{(1+r\mathcal{K})^2}\\ &\quad+\frac{2\varepsilon^2\mathcal{K}}{(1+r\mathcal{K})^3}{u^{\mathrm{B}}}_{ss} + O(\varepsilon^2\delta^2)\end{aligned}$$

Here r has been rescaled by ε everywhere it occurs since $ru_r = O(1)$ and $ru_{sr} = O(1)$. Recall that $cr_\xi - \mathcal{K} = \hat{\sigma} + O(\varepsilon^2)$. It follows that $\mathcal{K}^2 - \frac{1}{2}cr_\xi\mathcal{K} + \frac{1}{4}(cr_\xi)^2 = \frac{3}{4}\mathcal{K}^2 + \frac{1}{4}\hat{\sigma}^2 + O(\varepsilon^2)$. The equation (3.10) thus follows from the definition of ϕ^0.

We proceed to prove (3.11) and (3.12). For this purpose, we use the local coordinates (h,ς). Differentiating the relation

$$x = z(\varsigma) - hn(\varsigma) = w(s) + r\mathbf{N}(s)$$

gives

$$dx = (1 - h\mathcal{K}_{\Omega(\varsigma)})\tau(\varsigma)d\varsigma - n(\varsigma)dh = (1 + r\mathcal{K}(s))\mathbf{T}(s)ds + \mathbf{N}(s)dr.$$

That is,

$$\begin{aligned}&\begin{pmatrix}\frac{\partial\varsigma}{\partial s} & \frac{\partial\varsigma}{\partial r}\\ \frac{\partial h}{\partial s} & \frac{\partial h}{\partial r}\end{pmatrix}\\ &= \begin{pmatrix}\frac{1+r\mathcal{K}(s)}{1-h\mathcal{K}_{\Omega(\varsigma)}}\tau(\varsigma)\cdot\mathbf{T}(s) & \frac{1}{1-h\mathcal{K}_{\Omega(\varsigma)}}\tau(\varsigma)\cdot\mathbf{N}(s)\\ -(1+r\mathcal{K}(s))n(\varsigma)\cdot\mathbf{T}(s) & -n(\varsigma)\cdot\mathbf{N}(s)\end{pmatrix},\end{aligned} \tag{3.13}$$

$$\begin{pmatrix} \frac{\partial s}{\partial \varsigma} & \frac{\partial s}{\partial h} \\ \frac{\partial r}{\partial \varsigma} & \frac{\partial r}{\partial h} \end{pmatrix} = \begin{pmatrix} \frac{1-h\mathcal{K}_{\Omega(\varsigma)}}{1+r\mathcal{K}(s)}\tau(\varsigma)\cdot\mathbf{T}(s) & -\frac{1}{1+r\mathcal{K}(s)}n(\varsigma)\cdot\mathbf{T}(s) \\ (1-h\mathcal{K}_{\Omega(\varsigma)})\tau(\varsigma)\cdot\mathbf{N}(s) & -n(\varsigma)\cdot\mathbf{N}(s) \end{pmatrix}. \quad (3.14)$$

In the (h,ς) coordinates, $\partial_n = -\frac{\partial}{\partial h}$, and $u_h = u_{h\varsigma} = u_{\varsigma h} = 0$ on $\partial\Omega_\delta$. Hence we obtain

$$\begin{aligned}
\partial_n u_r\Big|_{\Sigma^i} &= \frac{\partial}{\partial h}\Big(\frac{\partial u}{\partial h}\frac{\partial h}{\partial r} + \frac{\partial u}{\partial \varsigma}\frac{\partial \varsigma}{\partial r}\Big) = \frac{\partial^2 u}{\partial h^2}\frac{\partial h}{\partial r} + \frac{\partial u}{\partial \varsigma}\frac{\partial}{\partial h}\Big(\frac{\partial \varsigma}{\partial r}\Big) \\
&= \frac{\partial h}{\partial r}\Big\{u_{rr}\Big(\frac{\partial r}{\partial h}\Big)^2 + 2u_{rs}\frac{\partial r}{\partial h}\frac{\partial s}{\partial h} + u_{ss}\Big(\frac{\partial s}{\partial h}\Big)^2 \\
&\quad + u_s\frac{\partial^2 s}{\partial h^2} + u_r\frac{\partial^2 r}{\partial h^2}\Big\} \\
&\qquad + \frac{\partial}{\partial h}\Big(\frac{\partial \varsigma}{\partial r}\Big)\Big\{u_r\frac{\partial r}{\partial \varsigma} + u_s\frac{\partial s}{\partial \varsigma}\Big\}. \\
&= u_r\Big\{\frac{\partial h}{\partial r}\frac{\partial^2 r}{\partial h^2} + \frac{\partial r}{\partial \varsigma}\frac{\partial}{\partial h}\Big(\frac{\partial \varsigma}{\partial r}\Big)\Big\} \\
&\quad + \frac{1}{\varepsilon}u^{\mathrm{B}}{}_{1,HH}\frac{\partial h}{\partial r} + O(\varepsilon).
\end{aligned}$$

Here, in the second equality, the change of coordinates from (h,ς) back to (r,s) was used and in the third equality, the following estimates were used: on Σ, $\frac{\partial h}{\partial r} = \frac{\partial r}{\partial h} = O(\varepsilon + |r|)$, $u_{rr} = O(\frac{1}{\varepsilon^2}e^{-|r|/\varepsilon})$, $u_{rs} = O(\frac{1}{\varepsilon}e^{-|r|/\varepsilon})$, $u_s = O(e^{-|r|/\varepsilon})$, $u_{ss} = O(1) + \frac{1}{\varepsilon}u^{\mathrm{B}}{}_{1,HH}$, $\frac{\partial s}{\partial \varsigma} = O(\varepsilon + |r|)$, and $(\frac{\partial s}{\partial h})^2 = 1 + O(\varepsilon + |r|)$.

From $u^{\mathrm{B}}{}_{1,HH} = O(\delta)$, and on $\Sigma^\pm$ (denoting $\varsigma^\pm = \xi \pm g(\xi)$),

$$\begin{aligned}
\frac{\partial h}{\partial r} &= -n(\varsigma)\cdot\mathbf{N}(s) = [-n(\varsigma^\pm) + O(\mathcal{K}_\Omega(\varsigma^\pm)r)]\cdot[N(S^\pm) + O(r^2)] \\
&= O(\varepsilon^2 + \delta|r| + r^2),
\end{aligned}$$

we obtain $\frac{1}{\varepsilon}u^{\mathrm{B}}{}_{1,HH}\frac{\partial h}{\partial r} = O(\delta^2)$. Also, direct calculation shows that

$$\begin{aligned}
\frac{\partial h}{\partial r}\frac{\partial^2 r}{\partial h^2} + \frac{\partial r}{\partial \varsigma}\frac{\partial}{\partial h}\Big(\frac{\partial \varsigma}{\partial r}\Big) &= \mathcal{K}_{\Omega(\varsigma)}\Big(\tau(\varsigma)\cdot\mathbf{N}(s)\Big)^2 \\
&= \mathcal{K}_{\Omega(\varsigma)}\Big(1 + O(\varepsilon^2 + |r|^2)\Big).
\end{aligned}$$

Hence,

$$\partial_n u_r\Big|_{\Sigma^i} = \mathcal{K}_{\Omega(\varsigma)}u_r + O(\delta^2). \quad (3.15)$$

Observe that $crr_\xi = O(\delta^2|r|)$ and $\partial_n(crr_\xi) = O(\delta^2(\varepsilon+|r|))$. Equation (3.11) thus follows from (3.15) and the definition of ϕ^0.

Finally, utilizing the expansion

$$\mathcal{K}_{\Omega(\varsigma)} = \mathcal{K}_\Omega(\varsigma^\pm) + \frac{\partial \mathcal{K}_\Omega}{\partial \varsigma}(\varsigma^\pm)\frac{\partial \varsigma}{\partial r}(p^\pm)r + O(|r|^2 + \varepsilon^2)$$

and the fact that $\int_{\Sigma^\pm} r{\phi^0}^2 = \varepsilon \int_{I\!R} R(\dot{U}(R))^2 dR + O(\varepsilon^2) = O(\varepsilon^2)$, we then obtain from (3.11) equation (3.12). This completes the proof of the lemma.
□

3.2 The decomposition

Now we define precisely the separation of variables in the limit as $\varepsilon \to 0$ for the eigenfunctions. We would like to decompose every H^1 function into the form $\phi^0\Theta(s) + \psi$ where (r, s) is the coordinate system near the interface Γ, Θ is an H^1 function on the interval $(0, L)$, and $\psi \perp_{L^2} \phi^0\Theta(s)$. Later on, we shall show that for small eigenvalues, the ψ part of the eigenfunction is small and can be neglected.

Since u_r decays exponentially fast away from the interface Γ, in the rest of this section, without loss of generality, we shall assume that $\phi^0 = 0$ in Ω^2_δ.

As the coordinate system (r, s) does not work very well near the boundary $\partial\Omega_\delta$, we introduce a new coordinate system (r, ℓ) where $r = r(x, \xi, \varepsilon)$ is as before and $\ell = \ell(x, \xi, \varepsilon)$ is defined by

$$\begin{aligned} \ell &= \frac{s - S^+(r)}{S^-(r) - S^+(r)}, \quad \text{or} \\ s &= \mathrm{s}(r,\ell) := S^+(r) + \ell(S^-(r) - S^+(r)). \end{aligned} \tag{3.16}$$

Note that under the coordinates (r, ℓ), the domain Ω^1_δ becomes $\mathrm{I}_\varepsilon \times (0, 1)$.

Using (3.9) and $\mathcal{K}(s) = \hat{\sigma} + O(\delta^2)$, we calculate:

$$\begin{aligned} J(r,\ell) := \frac{\partial x}{\partial(r,\ell)} &= [1 + r\mathcal{K}(s(r,\ell))][S^-(r) - S^+(r)] \\ &= [1 + r\hat{\sigma}]|\Gamma| + O(\delta^2|r| + |r^2|), \end{aligned} \tag{3.17}$$

$$\begin{aligned} |\nabla_x \ell|^2 &= |\ell_s|^2|\nabla_x s|^2 + |\ell_r|^2|\nabla_x r|^2 \\ &= \frac{1}{|\Gamma|^2(1 + r\hat{\sigma})^2} + O(\delta^2|r| + r^2 + \varepsilon^2). \end{aligned} \tag{3.18}$$

In what follows we denote $\phi^0(x(r, \ell))$ by $\phi^0(r, \ell)$.

We define function spaces $\mathcal{X}^0$ and $\mathcal{X}^{0\perp}$ by

$$\mathcal{X}^0 := \{\phi \in H^1(\Omega_\delta) \ : \ \phi = 0 \text{ in } \Omega_\delta^2, \phi = \Theta(\ell)\phi^0(r,\ell) \text{ in } \ \Omega_\delta^1\}, \tag{3.19}$$

$$\mathcal{X}^{0\perp} := \{\psi \in H^1(\Omega_\delta) \ : \ (\phi,\psi) = 0 \text{ for all } \ \phi \in \mathcal{X}^0\}. \tag{3.20}$$

Similarly, we define

$$\begin{aligned} \bar{\mathcal{X}}^0 &:= \mathcal{X}^0 \cap \bar{H}(\Omega_\delta), \\ \bar{\mathcal{X}}^{0\perp} &:= \{\psi \in \bar{H}^1(\Omega_\delta) \ ; \ (\psi,\phi) = 0 \ \forall \phi \in \bar{\mathcal{X}}^0\}. \end{aligned} \tag{3.21}$$

We remark that in the (r,ℓ) coordinates, for every $\phi = \Theta(\ell)\phi^0 \in \mathcal{X}^0$ and $\tilde{\phi} = \tilde{\Theta}(\ell)\phi^0 \in \mathcal{X}^0$,

$$\iint_{\Omega_\delta} \phi \, dx = \int_0^1 \Theta(\ell)\omega_1(\ell)\, d\ell, \tag{3.22}$$

$$\iint_{\Omega_\delta} \phi\tilde{\phi} = \int_0^1 \omega_2(\ell)\Theta(\ell)\tilde{\Theta}(\ell)\, d\ell, \tag{3.23}$$

where

$$\omega_i(\ell) := \int_{\mathrm{I}_\varepsilon} J(r,\ell){\phi^0}^i(r,\ell)\, dr, \quad i = 1,2. \tag{3.24}$$

Therefore, $\psi \in \mathcal{X}^{0\perp}$ (or $\bar{\mathcal{X}}^0$) if and only if $\psi \in H^1(\Omega_\delta)$ (or $\bar{H}^1(\Omega_\delta)$) and

$$\int_{\mathrm{I}_\varepsilon} J(r,\ell)\phi^0(r,\ell)\psi(r,\ell)dr = 0 \quad \forall\, \ell \in [0,1]. \tag{3.25}$$

In addition, $H^1(\Omega_\delta) = \mathcal{X}^0 \oplus_{L^2} \mathcal{X}^{0\perp}$ since for every $\phi \in H^1(\Omega_\delta)$,

$$\begin{aligned} &\phi = \Theta(\ell)\phi^0 + \psi \in \mathcal{X}^0 \oplus_{L^2(\Omega_\delta)} \mathcal{X}^{0\perp}, \\ &\Theta(\ell) := \frac{1}{\omega_2(\ell)} \int_{\mathrm{I}_\varepsilon} J(r,\ell)\phi^0(r,\ell)\phi(r,\ell)dr. \end{aligned}$$

Next, we characterize $\bar{\mathcal{X}}^0$ and show that $\bar{H}^1(\Omega_\delta) = \bar{\mathcal{X}}^0 \oplus_{L^2} \bar{\mathcal{X}}^{0\perp}$.

From (3.22), $\phi \in \bar{\mathcal{X}}^0 = \mathcal{X}^0 \cap \bar{H}^1(\Omega_\delta)$ if and only if $\phi = \Theta(\ell)\phi^0$ for some $\Theta \in H^1((0,1))$ satisfying $\int_0^1 \omega_1(\ell)\Theta(\ell)\, d\ell = 0$.

To characterize $\bar{\mathcal{X}}^{0\perp}$, we introduce a function $\omega_3(\ell)$ defined by

$$\omega_3(\ell) := \frac{\omega_1(\ell)}{\omega_2(\ell)}. \tag{3.26}$$

We claim that

$$\bar{\mathcal{X}}^{0\perp} = \{\bar{\psi} \in \bar{H}^1(\Omega_\delta) \ : \ \bar{\psi} = m\omega_3\phi^0 + \psi, \ m \in I\!R^1, \psi \in \mathcal{X}^{0\perp}\}. \tag{3.27}$$

In fact, if $\bar{\psi} := m\omega_3\phi^0 + \psi \in \bar{H}^1(\Omega_\delta)$ where $\psi \in \mathcal{X}^{0\perp}$, then for any $\phi = \Theta\phi^0 \in \bar{\mathcal{X}}^0$, $(\bar{\psi}, \phi) = m(\omega_3\phi^0, \Theta\phi^0) = m\int_0^1 \omega_3\omega_2\Theta\, d\ell = \int_0^1 \omega_1\Theta = 0$. That is, $\bar{\psi} \in \bar{\mathcal{X}}^{0\perp}$. On the other hand, for every $\phi \in \bar{H}^1(\Omega_\delta)$, writing $\phi = \Theta\phi^0 + \psi \in \mathcal{X}^0 + \mathcal{X}^{0\perp}$ and defining $m = \int_0^1 \Theta\omega_1 \Big/ \int_0^1 \omega_1\omega_3$, then $\phi = [m\omega_3\phi^0 + \psi] + [(\Theta - m\omega_3)\phi^0]$, and $(\Theta - m\omega_3)\phi^0 \in \bar{\mathcal{X}}^0$ since $\int_0^1 \omega_1(\Theta - m\omega_3) = 0$. Thus (3.27) holds. In addition, $\bar{H}^1(\Omega_\delta) = \bar{\mathcal{X}}^0 \oplus_{L^2} \bar{\mathcal{X}}^{0\perp}$.

Finally, we establish a few properties needed later for the functions ω_i, $i = 1, 2, 3$.

Note that $crr_\xi = O(\delta^2|r|)$, $J(r,\ell) = |\Gamma| + |\Gamma|r\mathcal{K} + O(\varepsilon^2 + r^2)$, and

$$\varepsilon\int_{\mathrm{I}_\varepsilon} r u_r^2(r,\ell)\, dr = \varepsilon\int_{I\!R} R\dot{U}(R)^2 dr + O(\varepsilon^2) = O(\varepsilon^2). \tag{3.28}$$

Hence,

$$\begin{aligned}\omega_2(\ell) &= \varepsilon|\Gamma|\int_{\mathrm{I}_\varepsilon} u_r^2(r,\ell)\, dr + O(\varepsilon\delta^2)\\ &= \int_{\mathrm{I}_\varepsilon}\left\{\frac{1}{\varepsilon}\Big(\dot{U}(\tfrac{r}{\varepsilon})\Big)^2 + \dot{U}(\tfrac{r}{\varepsilon})({u^{\mathrm{I}}}_{0,R} + {u^{\mathrm{B}}}_{1,R})\right\} dr\\ &\quad + O(\varepsilon\delta^2).\end{aligned} \tag{3.29}$$

From the expansion of ${u^{\mathrm{I}}}_0$, one can show that ${u^{\mathrm{I}}}_0(R,s)$ is independent of s, up to $O(\delta^2)$ order, since σ and $\mathcal{K}$ are so. Also, from ${u^{\mathrm{B}}}_1 = O(\mathcal{K}_{\Omega_\delta}) = O(\delta)$, it follows from the last equation that

$$\omega_2(\ell) = \omega_2(\tfrac{1}{2})[1 + O(\varepsilon\delta^2) + O(\varepsilon\delta e^{-h/\varepsilon})] \quad \forall\, \ell \in [0,1]. \tag{3.30}$$

In a similar manner, we have,

$$\begin{aligned}\frac{1}{\sqrt{\varepsilon}}\omega_1(\ell) &= \int_{\mathrm{I}_\varepsilon} u_r(r,\ell)\, d\ell + O(\varepsilon\delta^2)\\ &= \int_{\mathrm{I}_\varepsilon} u_r(r,s)\, dr\Big|_{\text{ s fixed}} + O(\varepsilon\delta^2) = u^+ - u^- + O(\varepsilon\delta^2)\\ &= \frac{1}{\sqrt{\varepsilon}}\omega_1(\tfrac{1}{2})[1 + O(\varepsilon\delta^2)] \quad \forall\, \ell \in [0,1].\end{aligned}$$

The definition of ω_3 then implies that

$$\omega_3(\ell) = \omega_3(\tfrac{1}{2})[1 + O(\varepsilon\delta^2) + O(\varepsilon\delta e^{-h/\varepsilon})] \quad \forall\, \ell \in [0,1]. \tag{3.31}$$

Finally, from

$$\varepsilon\frac{\partial}{\partial\ell}\int_{\mathrm{I}_\varepsilon} u_r^2(r,\ell)dr = 2\varepsilon\int_{\mathrm{I}_\varepsilon} u_r u_{rs}\frac{\partial s(r,\ell)}{\partial\ell}\,dr$$
$$= 2\int_{\mathrm{I}_\varepsilon}\Big(\dot{U}(\tfrac{r}{\varepsilon})+O(\varepsilon)\Big)\Big(\frac{1}{\varepsilon}u^{\mathrm{B}}{}_{1,RH}+O(e^{-|r|/\varepsilon})\Big)dr$$
$$= O(\varepsilon)$$

it follows that $\omega_2'(\ell) = O(\varepsilon)$. Similarly, one can show that $\omega_1'(\ell) = \omega_1(\frac{1}{2})O(\varepsilon)$, so that

$$\omega_3'(\ell) = \omega_3(\tfrac{1}{2})O(\varepsilon) \quad \forall\,\ell\in[0,1]. \tag{3.32}$$

3.3 The restriction of $\langle L\cdot,\,\cdot\rangle$ on $\mathcal{X}^0$ and $\bar{\mathcal{X}}^0$

In this subsection, we study the restriction of $\langle L\phi,\,\psi\rangle$ on the spaces $\mathcal{X}^0$ and $\bar{\mathcal{X}}^0$ defined in (3.19) and (3.21) respectively. In fact, we study the following eigenvalue problems:

(a) Find (λ^0,ϕ^0) such that $\phi\in\mathcal{X}^0$ and

$$\langle L\phi^0,\,\phi\rangle = \lambda^0(\phi^0,\phi) \quad \forall\,\phi\in\mathcal{X}^0; \tag{3.33}$$

(b) Find $(\bar{\lambda},\bar{\phi})$ such that $\bar{\phi}\in\bar{\mathcal{X}}^0$ and

$$\langle L\bar{\phi}^0,\,\phi\rangle = \bar{\lambda}^0(\bar{\phi}^0,\phi) \quad \forall\,\phi\in\bar{\mathcal{X}}^0. \tag{3.34}$$

Later we shall show that the small eigenvalues to the original eigenvalue problems (1.21) and (1.22) are accurately approximated by (a) and (b), respectively.

First, we characterize the bilinear form $\langle L\cdot,\,\cdot\rangle$ restricted to the Hilbert space $\mathcal{X}^0$.

Lemma 3.2 *For every $\phi_1 = \Theta_1(\ell)\phi^0$ and $\phi_2 = \Theta_2(\ell)\phi^0$ in $\mathcal{X}^0$,*

$$\frac{|\Gamma|^2}{\varepsilon^2\omega_2(\frac{1}{2})}\langle L\phi_1,\,\phi_2\rangle = b^+\Theta_1(0)\Theta_2(0) + b^-\Theta_1(1)\Theta_2(1) + \int_0^1\Big\{b_1\Theta_1'\Theta_2' + b_2\Theta_1\Theta_2\Big\}d\ell \tag{3.35}$$

where b_1, b_2, b^+, b^- are independent of Θ_1 and Θ_2 and satisfy

$$\begin{cases} b^\pm = -|\Gamma|\mathcal{K}_{\Omega_\delta}(p^\pm) + O(\varepsilon\delta^2), \\ b_1(\ell) = 1 + O(\varepsilon\delta^2) + O(\varepsilon\delta e^{-h/\varepsilon}), \quad \ell\in[0,1], \\ b_2(\ell) = \frac{3}{4}(|\Gamma|\mathcal{K}(|\Gamma|\ell))^2 + \frac{1}{4}(|\Gamma|\hat{\sigma})^2 \\ \qquad +O(\varepsilon\delta^2) + O(\varepsilon\delta e^{-h/\varepsilon}), \quad \ell\in[0,1]. \end{cases} \tag{3.36}$$

Proof. Integration by parts for the integral in $\langle L\phi_1, \phi_2\rangle$ gives

$$\begin{aligned}\varepsilon^{-2}\langle L\phi_1, \phi_2\rangle &= \int_{\Sigma} \Theta_1\Theta_2\phi^0\partial_n\phi^0 + \iint_{\Omega_\delta^1}\Big\{\Theta_1'\Theta_2'|\nabla\ell|^2\phi^{0\,2} \\ &\quad +[-\Delta\phi^0 + \varepsilon^{-2}f'(u)\phi^0]\phi^0\Theta_1\Theta_2\Big\} \\ &= \bar{b}^+\Theta_1(0)\Theta_2(0) + \bar{b}^-\Theta(1)\Theta_1(1) \\ &\quad + \int_0^1 \Big\{\bar{b}_1(\ell)\Theta_1'(\ell)\Theta_2'(\ell) + \bar{b}_2(\ell)\Theta_1(\ell)\Theta_2(\ell)\Big\}d\ell\end{aligned}$$

where

$$\begin{aligned}\bar{b}^\pm &= \int_{\Sigma^\pm} \phi^0\partial_n\phi^0, \\ \bar{b}_1(\ell) &= \int_{\mathrm{I}_\varepsilon} J(r,\ell)|\nabla_x\ell|^2\phi^{0\,2}(r,\ell)\,d\ell, \\ \bar{b}_2(\ell) &= \int_{\mathrm{I}_\varepsilon} J(r,\ell)[-\Delta\phi^0 + \varepsilon^{-2}f'(u)\phi^0]\phi^0(r,\ell)\,d\ell.\end{aligned}$$

Because $\Sigma^\pm$ is parameterized by $x = w(s) + r\mathbf{N}(s)|_{s=S^\pm(r)}$, the arclength element of $\Sigma^\pm$ is

$$\sqrt{1 + [(1 + r\mathcal{K}(S^\pm(r))S_r^\pm(r)]^2}dr = (1 + O(r^2 + \varepsilon^2))dr.$$

It then follows from (3.12) that

$$\bar{b}^\pm = \mathcal{K}_{\Omega_\delta}(p^\pm)\int_{\mathrm{I}_\varepsilon}\phi^{0\,2}(r,\ell^\pm)\,dr + O(\varepsilon\delta^2) \quad (\ell^+ := 0,\ \ell^- := 1).$$

Consequently, it follows from (3.30), (3.28), and $\mathcal{K}_{\Omega_\delta} = O(\delta)$, that

$$\bar{b}^\pm = |\Gamma|^{-2}\omega_2(\tfrac{1}{2})\{|\Gamma|\mathcal{K}_{\Omega_\delta}(p^\pm) + O(\varepsilon\delta^2)\} =: |\Gamma|^{-2}\omega_2(\tfrac{1}{2})b^\pm.$$

Similarly, from (3.17), (3.18), (3.28), and (3.30), we have

$$\bar{b}_1(\ell) = |\Gamma|^{-2}\omega_2(\tfrac{1}{2})[1 + O(\varepsilon\delta^2) + O(\varepsilon\delta e^{-h/\varepsilon})] =: |\Gamma|^{-2}\omega_2(\tfrac{1}{2})b_1(\ell).$$

Finally, we estimate $\bar{b}_2(\ell)$. Using (3.10) and noting that $\int_{\mathrm{I}_\varepsilon}|\phi^0| = O(\sqrt{\varepsilon})$, we have

$$\begin{aligned}\bar{b}_2(\ell) &= |\Gamma|\int_{\mathrm{I}_\varepsilon}\frac{\frac{3}{4}\mathcal{K}^2 + \frac{1}{4}\hat{\sigma}^2}{1 + r\hat{\sigma}}\phi^{0\,2}(r,\ell)\,dr \\ &\quad + |\Gamma|\int_{\mathrm{I}_\varepsilon}\frac{2\varepsilon^{\frac{1}{2}}\mathcal{K}}{L(1 + r\hat{\sigma})^2}u_{ss}^B\phi^0 ds + O(\varepsilon\delta^2).\end{aligned}$$

The first term on the right–hand side can be written as, in view of (3.28) and (3.30), $\omega_2(\frac{1}{2})\{\frac{3}{4}\mathcal{K}^2 + \frac{1}{4}\hat{\sigma}^2\} + O(\varepsilon\delta^2) + O(\varepsilon\delta e^{-h/\varepsilon})$. Since $u_r = \varepsilon^{-1}\dot{U}(\frac{r}{\varepsilon}) + O(1)$ and $u^{\mathrm{B}}{}_{ss} = \varepsilon^{-1}u^{\mathrm{B}}{}_{1,HH} + O(\delta e^{-h/\varepsilon}))$, the second term can be estimated by

$$O(\varepsilon\delta e^{-|h|/\varepsilon}) + C\Big|\int_{I\!R}\dot{U}(R)u^{\mathrm{B}}{}_{1,HH}(R,H)\,dR\Big|.$$

Using the equation for $u^{\mathrm{B}}{}_1 : u^{\mathrm{B}}{}_{1,RR} + u^{\mathrm{B}}{}_{1,HH} - f'(U)u^{\mathrm{B}}{}_1 = 0$, we have that

$$\begin{aligned}\int_{I\!R} u^{\mathrm{B}}{}_{1,HH}(R,H)U'(R)dR &= \int_{I\!R}[f'(U)u^{\mathrm{B}}{}_1 - u^{\mathrm{B}}{}_{1,RR}]U' \\ &= \int_{I\!R}[U'' - f(U)]'u^{\mathrm{B}}{}_1 = 0\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{b}_2(\ell) &= |\Gamma|^{-2}\omega_2(\tfrac{1}{2})\{\frac{3}{4}(|\Gamma|\mathcal{K})^2 + \frac{1}{4}(|\Gamma|\hat{\sigma})^2 + O(\varepsilon\delta^2) + O(\varepsilon\delta e^{-h/\varepsilon})\} \\ &=: |\Gamma|^{-2}\omega_2(\tfrac{1}{2})b_2(\ell)\ .\end{aligned}$$

This completes the proof of the lemma. □

Now we study the eigenvalue problem (3.33) and (3.34). Observe that the bilinear form $\langle L\cdot,\,\cdot\rangle$ in the Hilbert space $Y := H^1(\Omega_\delta)$ is (i) bounded (i.e., $\langle L\phi,\,\psi\rangle \le C\|\phi\|_Y\|\psi\|_Y$), (ii) symmetric (i.e., $\langle L\phi,\,\psi\rangle = \langle L\psi,\,\phi\rangle$ for all $\phi,\psi \in Y$), and (iii) coercive on the Hilbert space $X := L^2(\Omega_\delta)$ (i.e., $\langle L\phi,\,\phi\rangle \ge \frac{1}{C}\|\phi\|_Y^2 - C\|\phi\|_X^2$ for any $\phi \in Y$). Also, we know that Y is a compact subspace of X (i.e., any sequence bounded in Y has a subsequence convergent in X). Hence, by the Lax–Milgram theorem, for any closed subspace $Z \subset Y$, the restriction of $\langle L\cdot,\,\cdot\rangle$ on Z has a complete eigen–family $\{\lambda_j^Z, \phi_j^Z\}_{i=1}^\infty$ (assume that the dimension of Z is infinite) in the sense that $\{\phi_j^Z\}_{j=1}^\infty$ is an X-orthonormal basis for Z, and

$$\langle L\phi_j^Z,\,\phi\rangle = \lambda_j^Z(\phi_j^Z,\phi)_X \quad \forall\,\phi \in Z.$$

Furthermore, $\lambda_1^Z \le \lambda_2^Z \le \lambda_3^Z \le \cdots$ and $\lim_{j\to\infty}\lambda_j^Z = \infty$.

Clearly, the eigenvalue problems (3.33) and (3.34) correspond to the following two situations respectively:

(a) $Y = H^1(\Omega_\delta)$ and $Z = \mathcal{X}^0$. We shall denote (λ_j^Z, ϕ_j^Z) by (λ_j^0, ϕ_j^0) in this case.

(b) $Y = \bar{H}^1(\Omega_\delta)$ and $Z = \bar{\mathcal{X}}^0 := \mathcal{X}^0 \cap \bar{H}^1(\Omega_\delta)$. In this case, we denote (λ_j^Z, ϕ_j^Z) by $(\bar{\lambda}_j^0, \bar{\phi}_j^0)$.

The original eigenvalue problems (3.3) and (3.4) correspond to the following situations, respectively:

(c) $Y = Z = H^1(\Omega)$. In this case we denote (λ_j^Z, ϕ_j^Z) by (λ_j, ϕ_j).

(d) $Y = Z = \bar{H}^1(\Omega)$. We denote (λ_j^Z, ϕ_j^Z) by $(\bar{\lambda}_j, \bar{\phi}_j)$.

Now we characterize $(\{\lambda_j^0, \phi_j^0)\}_{j=1}^{\infty}$ and $\{(\bar{\lambda}_j^0, \bar{\phi}_j^0)\}_{j=1}^{\infty}$. Later in this section we shall use a perturbation argument to show that they approximate $\{(\lambda_j \phi_j)\}$ and $\{(\bar{\lambda}_j, \bar{\phi}_j)\}$ respectively.

Lemma 3.3 *(1) (λ, ϕ) solves the eigenvalue problem (3.3) if and only if (μ, Θ), where $\lambda = \varepsilon^2 |\Gamma|^{-2} \mu$ and $\phi = \Theta(\ell)\phi^0$, solves the following eigenvalue problem:*

$$\begin{cases} -(b_1\Theta')' + b_2\Theta = \mu \frac{\omega_2(\ell)}{\omega_2(\frac{1}{2})}\Theta, \quad \ell \in (0,1), \\ -b_1(0)\Theta'(0) + b^+\Theta(0) = 0, \quad b_1(1)\Theta'(1) + b^-\Theta(1) = 0, \end{cases} \tag{3.37}$$

where b_1, b_2, b_3, b_4 are as in Lemma 3.6 and $\omega_i(\cdot)$ $(i = 1, 2)$ as in (3.24);

(2) $(\bar{\lambda}, \bar{\phi})$ solves the eigenvalue problem (3.4) if and only if (μ, Θ), where $\bar{\lambda} = \varepsilon^2 |\Gamma|^{-2} \mu$ and $\bar{\phi} = \Theta(\ell)\phi^0$, solves the following eigenvalue problem:

$$\begin{cases} -(b_1\Theta')' + b_2\Theta = \mu \frac{\omega_2(\ell)}{\omega_2(\frac{1}{2})}\Theta - \hat{\mu}(\mu)\frac{\omega_1(\ell)}{\omega_1(\frac{1}{2})}, \quad \ell \in (0,1), \\ -b_1(0)\Theta'(0) + b^+\Theta(0) = 0, \quad b_1(1)\Theta'(1) + b^-\Theta(1) = 0, \\ \int_0^1 \Theta(\ell)\omega_1(\ell)\, d\ell = 0. \end{cases} \tag{3.38}$$

Proof. The assertion follows from the characterization of $\langle L\cdot, \cdot\rangle$ on $\mathcal{X}^0$ in (3.35) and a standard variational principle argument. We leave the details to the reader. □

Using the estimate of $b_1, b_2, b^{\pm}$, we can now prove the following:

Theorem 3.4 *Let $\{\lambda_j^0, \phi_j^0)\}_{j=1}^{\infty}$ and $\{(\bar{\lambda}_j^0, \bar{\phi}_j^0)\}_{j=1}^{\infty}$ be the complete solutions of (3.3) and (3.4) respectively. Then, the following hold:*

$$\lambda_j^0 = \varepsilon^2 |\Gamma|^{-2} \Big\{ [(j-1)^2 - 1]\pi^2 + O(j^2\delta) \Big\},$$

$$\phi_j^0 = \sqrt{\frac{2}{\omega_2(\frac{1}{2})}}\phi^0 \cos((j-1)\pi\ell) + O(j^2\delta), \;\; j = 1, 2, \cdots, \tag{3.39}$$

$$\bar{\lambda}_j^0 = \varepsilon^2 |\Gamma|^{-2} \Big\{ [j^2 - 1]\pi^2 + O(j^2\delta) \Big\},$$

$$\bar{\phi}_j^0 = \sqrt{\frac{2}{\omega_2(\frac{1}{2})}}\phi^0 \cos(j\pi\ell) + O(j^2\delta), \;\; j = 1, 2, \cdots, \tag{3.40}$$

$$\bar{\lambda}_1^0 = -\frac{4\varepsilon^2}{3\pi\hat{\sigma}_0}\frac{d^2}{d\varsigma^2}\mathcal{K}_{\Omega_\delta}(\varsigma)\Big|_{\varsigma=\xi} + O(\delta^4)$$

$$\phi_1^0 = \sqrt{\frac{2}{\omega_2(\frac{1}{2})}}\phi_0 + O(\delta). \tag{3.41}$$

Proof. Since from the last section, $|\Gamma|\mathcal{K} = L\hat{\sigma} + O(\delta^2) = \pi + O(\delta)$, we have $b_2 = \pi^2 + O(\delta)$. Hence, replacing $b^{\pm}$ by 0, b_1 by 1, b_2 by π^2, and $\frac{\omega_i(\ell)}{\omega_i(\frac{1}{2})}$ by 1 and using a perturbation argument, we obtain from the previous lemma the assertion (3.39) and (3.40). Note that when $j = 1$, the estimate (3.40) yields no information about the sign of the eigenvalue. Hence, we need to refine the estimate.

Recall the following estimates:

$$b_i = \mathcal{K}_{\Omega_\delta}(p^i) + O(\varepsilon\delta^2), \quad i = 1, 2, \quad \frac{\omega_1(\ell)}{\omega_1(\frac{1}{2})} = 1 + O(\varepsilon\delta^2),$$
$$\|b_3 - 1\|_{L^p((0,1))} + \Big\|b_4 - [\tfrac{3}{4}(L\mathcal{K})^2 + \tfrac{1}{4}(L\hat{\sigma})^2]\Big\|_{L^p((0,1))}$$
$$+\Big\|\frac{\omega_2(\cdot)}{\omega_2(\frac{1}{2})} - 1\Big\|_{L^p((0,1))} \leq C\varepsilon\delta[\delta + \varepsilon^{1/p}],$$

for all $p \in [1, \infty) \cup \{\infty\}$. Hence, comparing (3.38) to (2.75), and using the fact that the normalized (in L^2) eigenfunction for both problems are bounded in $C^2((0,1))$ with a bound independent of ε and δ, we conclude that the corresponding principal eigenvalues of both problems differ by $O(\varepsilon\delta^2)$ (here we can take any $p \in [1, 2]$). Hence,

$$\lambda = \varepsilon^2|\Gamma|^{-2}\Big(\mu_1 + O(\varepsilon\delta^2)\Big) = -\frac{4\varepsilon^2\pi}{3}\frac{g^3\hat{\sigma}_0^3}{|\Gamma|^2\hat{\sigma}_0^3}\frac{d^2\mathcal{K}_{\Omega_\delta}}{d\varsigma^2}(\xi) + O(\varepsilon^2\delta^4)$$
$$= -\frac{4\varepsilon^2}{3\pi\hat{\sigma}_0}\frac{d^2\mathcal{K}_{\Omega_\delta}}{d\varsigma^2}(\xi) + O(\delta^4). \tag{3.42}$$

This completes the proof of the theorem. □

The following corollary will be used later.

Corollary 3.5 *(a) There exists a positive constant C such that for every $\phi = \Theta(\ell)\phi^0 \in \mathcal{X}^0$,*

$$\|\Theta\|^2_{C^0([0,1])} + \|\Theta\|^2_{H^1((0,1))} \leq C\varepsilon^{-2}\langle L\phi,\, \phi\rangle + C^2\|\phi\|^2. \tag{3.43}$$

(b) There exists a positive constant C such that for every $\phi = \Theta\phi^0 \in \bar{\mathcal{X}}^0$,

$$|\Theta(0) + \Theta(1)|^2 \leq C\delta^2\|\phi\|^2 + C\varepsilon^{-2}\langle L\phi,\, \phi\rangle. \tag{3.44}$$

Proof. (a) The first assertion follows directly from the characterization of $\langle L\cdot,\, \cdot\rangle$ on $\mathcal{X}^0$ stated in Lemma 3.2, and the details are omitted.

(b) To prove the second assertion, we write ϕ as $\phi = \alpha\bar{\phi}_1^0 + \hat{\phi} = \alpha\bar{\Theta}_1^0\phi^0 + \hat{\Theta}\phi^0$ where $\alpha \in \mathbb{R}$, $\bar{\phi}_1^0 = \Theta_1^0(\ell)\phi^0$ is the principal eigenfunction of (3.4) and $\hat{\phi} = \hat{\Theta}\phi^0 \perp_{L^2(\Omega_\delta)} \bar{\phi}_1^0$. Then,

$$\langle L\hat{\phi},\, \hat{\phi}\rangle \geq \bar{\lambda}_2^0\|\hat{\phi}\|^2 \geq 2L^{-2}\pi^2\varepsilon^2\|\hat{\phi}\|^2.$$

It then follows from (a) that

$$\begin{aligned}\|\hat{\Theta}\|^2_{C^0([0,1])} &\le \tilde{C}\varepsilon^{-2}\langle L\hat{\phi},\, \hat{\phi}\rangle = \tilde{C}\varepsilon^{-2}\Big\{\langle L\phi,\, \phi\rangle - \bar{\lambda}^0_1\|\alpha\bar{\phi}^0_1\|^2\Big\}\\ &\le \tilde{C}\varepsilon^{-2}\langle L\phi,\, \phi\rangle + C\delta^3\|\phi\|^2.\end{aligned}$$

Here in the last inequality, we have used the fact that $\bar{\lambda}^0_1 = O(\varepsilon^2\delta^3)$ and $\|\alpha\bar{\phi}^0_1\|^2 \le \|\phi\|^2$.

Finally, observe that $\bar{\Theta}^0_1(\ell) = \sqrt{\frac{2}{\omega_2(\frac{1}{2})}}\cos(\pi\ell) + O(\delta)$, so that

$$|\bar{\Theta}(1) + \bar{\Theta}^0_1(1)| \le C\delta\|\bar{\Theta}^0_1\|_{L^2((0,1))}.$$

Therefore,

$$\begin{aligned}|\Theta(0) + \Theta(1)|^2 &\le 2\frac{|\bar{\Theta}^0_1(0) + \bar{\Theta}^0_1(1)|^2}{\|\Theta^0_1\|^2_{L^2((0,1))}}\|\alpha\bar{\Theta}^0_1\|^2 + 4\|\hat{\Theta}\|^2_{C^0([0,1])}\\ &\le C\delta^2\|\phi\|^2 + C\varepsilon^{-2}\langle L\phi,\, \phi\rangle + C\delta^3\|\phi\|^2.\end{aligned}$$

The second assertion of the corollary thus follows. □

In the rest of this section, we shall show that the projection of the eigenfunctions of (1.21) and (1.22) on $\mathcal{X}^{0\perp}$ is insignificant for small eigenvalues, so that for every fixed positive integer j, λ_j and $\bar{\lambda}_j$ are well approximated by λ^0_j and $\bar{\lambda}^0_j$ as given in Theorem 3.4.

3.4 The restriction of $\langle L\cdot,\, \cdot\rangle$ on $\mathcal{X}^{0\perp}$

Lemma 3.6 *There exists a constant ν_0 independent of ξ and ε such that for every $\psi \in \mathcal{X}^{0\perp}$,*

$$\langle L\psi,\, \psi\rangle \ge \nu_0 \iint_{\Omega^1_\delta} [\varepsilon^2|\nabla\psi|^2 + \psi^2]. \tag{3.45}$$

Proof. We divide the integral in the definition of $\langle L\psi,\, \psi\rangle$ into integrals on Ω^2_δ and Ω^1_δ respectively.

In Ω^2_δ, $f'(u)$ is uniformly positive, so that for some positive ν_2 independent of ε and ψ,

$$\iint_{\Omega^2_\delta} [\varepsilon^2|\nabla\psi|^2 + f'(u)\psi^2] \ge \nu_2 \iint_{\Omega^2_\delta} [\varepsilon^2|\nabla\psi|^2 + \psi^2]. \tag{3.46}$$

In Ω_δ^1, we use the coordinates (r,ℓ), obtaining

$$\begin{aligned}\iint_{\Omega_\delta^1}&[\varepsilon^2|\nabla\psi|^2 - f'(u)\psi^2]\\ &\geq L^{-1}\int_0^1 d\ell\int_{\mathrm{I}_\varepsilon}[\varepsilon^2\psi_r^2(r,\ell) - f'(U(r/\varepsilon))\psi^2(r,\ell)]dr\\ &\quad -\iint_{\Omega_\delta^1}[C(\varepsilon+|r|)\varepsilon^2|\nabla\psi|^2 + C(|r|+\varepsilon)\psi^2].\end{aligned}\tag{3.47}$$

Since $\psi\in\mathcal{X}^{0\perp}$, (3.25) implies that for any ℓ, $\int_{\mathrm{I}_\varepsilon}U'(r/\varepsilon)\psi(r,\ell)dr = O(\varepsilon)\|\psi(\cdot,\ell)\|_{2,\mathrm{I}_\varepsilon}$. It then follows that $\int_{\mathrm{I}_\varepsilon}[\varepsilon^2\psi_r^2(r,\ell) - f'(U(r/\varepsilon))\psi^2(r,\ell)]dr \geq \nu_3\int_{\mathrm{I}_\varepsilon}\psi^2(r,\ell)dr$ for some positive ν_3 independent of ε (cf. [33,26]). Therefore, it follows from (3.47) that

$$\iint_{\Omega_\delta^1}[\varepsilon^2|\nabla\psi|^2 - f'(u)\psi^2] \geq \frac{\nu_3}{C}\iint_{\Omega_\delta^1}\psi^2 - Cm_\varepsilon\iint_{\Omega_\delta^1}[\varepsilon^2|\nabla\psi|^2+\psi^2].$$

Consequently, taking a small positive η, we have

$$\begin{aligned}\iint_{\Omega_\delta{}^1{}_\mathrm{I}}&[\varepsilon^2|\nabla\psi|^2+\psi^2]\\ &\geq (1-\eta)\Big\{\frac{\nu_3}{C}\iint_{\Omega_\delta^1}\psi^2 - Cm_\varepsilon\iint_{\Omega_\delta^1}[\varepsilon^2|\nabla\psi|^2+\psi^2]\Big\}\\ &\quad +\eta\iint_{\Omega_\delta^1}I[\varepsilon^2|\nabla\psi|^2 - \|f'(u)\|_\infty\psi^2]\\ \geq \nu_4&\iint_{\Omega_\delta^1}[\varepsilon^2|\nabla\psi|^2+\psi^2].\end{aligned}$$

Combining this with (3.46), we obtain the assertion of the lemma. □

Lemma 3.6 shows that all the eigenvalues of the bilinear form $\langle L\cdot,\,\cdot\rangle$ restricted to the closed subspace $\mathcal{X}^{0\perp}$ of

$H^1(\Omega_\delta)$ have a positive lower bound ν_0. To show that small eigenvalues of (1.21) (or (1.22)) are characterized by the restriction of $\langle L\cdot,\,\cdot\rangle$ on $\mathcal{X}^0$ (or $\bar{\mathcal{X}}^0$), we need to show that $\mathcal{X}^0$ is almost invariant for the operator L. It suffices to study the behavior of $\langle L\phi,\,\psi\rangle$ for $\phi\in\mathcal{X}^0$ and $\psi\in\mathcal{X}^{0\perp}$, as we shall do in the next two subsections.

3.5 Properties of $\langle L\cdot, \cdot\rangle$ on $H^1(\Omega_\delta)$

Lemma 3.7 *There exist positive constants C_1 and C_2 such that for every $\phi \in \mathcal{X}^0$ and $\psi \in \mathcal{X}^{0\perp}$,*

$$|\langle L\phi, \psi\rangle| \le C_1\varepsilon^2\langle L\phi, \phi\rangle + \frac{1}{4}\langle L\psi, \psi\rangle + C_2\varepsilon^4\|\phi\|^2_{2,\Omega}, \quad (3.48)$$

$$\langle L(\phi+\psi), (\phi+\psi)\rangle \ge (1 - C_1\varepsilon^2)\langle L\phi, \phi\rangle + \tfrac{\nu_0}{2}\|\psi\|^2 - C_2\varepsilon^4\|\phi\|^2_{2,\Omega}. \quad (3.49)$$

Proof. We need only prove the first inequality since the second is a direct consequence of the first, (3.45), and the linearity and symmetry of $\langle L\cdot, \cdot\rangle$.

Writing $\phi = \Theta\phi^0$, and integrating by parts twice for the integral in $\langle L\phi, \psi\rangle$, we obtain

$$\langle L\phi, \psi\rangle = \iint_{\Omega^1_\delta}(-\varepsilon^2\Delta\phi^0 + f'(u)\phi^0)\Theta\psi + \varepsilon^2\int_\Sigma \Theta\psi\partial_n\phi^0 + \varepsilon^2\iint_{\Omega^1_\delta}\nabla\Theta(\phi^0\nabla\psi - \psi\nabla\phi^0) =: I + II + III.$$

Estimation of I: From (3.10), we see that

$$\sup_{\ell\in(0,1)}\int_{\mathrm{I}_\varepsilon}\Big|\Delta\phi^0 - \varepsilon^{-2}f'(u)\phi^0\Big|^2(r,\ell)dr \le C.$$

It then follows that

$$|I| \le C\varepsilon^2\|\Theta\|_{2,(0,1)}\|\psi\|_{2,\Omega} \le C\varepsilon^4\|\phi\|^2 + C^{-1}\|\psi\|^2.$$

Here we used the fact that $\|\phi\|$ and $\|\Theta\|_{2,(0,1)}$ are equivalent.

Estimation of II: We need only consider the integral on Σ^+. Since $\Theta = \Theta(0)$ on Σ^+, we need only consider the integral of $\int_{\Sigma^+}\psi\partial_n\phi^0 dS_{\Sigma^+}$. Recall that on Σ^+, $\partial_n\phi^0 = [\mathcal{K}_{\Omega_\delta}(p^+) + O(|r|\delta^2)]\phi^0(r, 0) + O(\sqrt{\varepsilon}\delta^2)$, and the arc length element is $dS_{\Sigma^+} = (1 + O(\varepsilon^2 + r^2))dr$. It then follows that

$$\int_{\Sigma^+}\psi\partial\phi^0 = L^{-1}\mathcal{K}_{\Omega_\delta}\int_{\mathrm{I}_\varepsilon}J(r,0)\psi(r,0))\phi^0(r,0)dr\Big|_{\ell=0} + O(\varepsilon)\|\psi\|_{2,\Sigma^+} = O(\varepsilon)\|\psi\|_{2,\Sigma^+}$$

since by the orthogonality criterion (3.25), the integral on the right–hand side vanishes. Thus,

$$\begin{aligned}|II| &\le C\varepsilon^3\|\psi\|_{2,\Sigma}\Big[|\Theta(0)| + |\Theta(1)|\Big]\\ &\le C\varepsilon^5[\Theta^2(0) + \Theta^2(1)] + C^{-2}\varepsilon\|\psi\|^2_{2,\Sigma}\\ &\le C\varepsilon^5[\Theta^2(0) + \Theta^2(1)] + C\iint_{\Omega_\delta}[\varepsilon^2|\nabla\psi|^2 + \psi^2]\end{aligned}$$

by the Sobolev imbedding $\int_{\Sigma^\pm}\psi^2 \le 2\iint_{B(p,2m_\varepsilon)\cap\Omega}[\varepsilon|\nabla\psi|^2+\varepsilon^{-1}\psi^2]$.

Estimation of III:. In the (r,ℓ) coordinates, we can write

$$\begin{aligned} III &= \varepsilon^2\int_0^1 \Theta'(\ell)\int_{\mathrm{I}_\varepsilon} J(r,\ell)|\nabla_x\ell|^2(\phi^0\psi_\ell-\phi^0{}_\ell\psi)dr d\ell \\ &=: \varepsilon^2\int_0^1 \Theta'(\ell)W(\ell)d\ell. \end{aligned}$$

Differentiating (3.25) with respect to ℓ we have

$$\int_{\mathrm{I}_\varepsilon} J(r,\ell)\phi^0\psi_\ell = -\int_{\mathrm{I}_\varepsilon} J\phi^0{}_\ell\psi - \int_{\mathrm{I}_\varepsilon} J_\ell\phi^0\psi.$$

It then follows that

$$\begin{aligned} W(\ell) = &\int_{\mathrm{I}_\varepsilon} J\Big[|\nabla_x\ell|^2-|\Gamma|^{-2}\Big]\phi^0\psi_\ell dr \\ &-\int_{\mathrm{I}_\varepsilon}\Big[|\Gamma|^{-2}J\phi^0{}_\ell+|\Gamma|^{-2}J_\ell\phi^0+J|\nabla_x\ell|^2\phi^0{}_\ell\Big]\psi dr. \end{aligned}$$

Observe that $L^2|\nabla_x\ell|^2-1=O(|r|+|S_r^\pm(r)|)=O(\varepsilon+|r|)$. Then we can invoke the property that

$$\sup_{\ell\in(0,1)}\int_{\mathrm{I}_\varepsilon}\Big\{(1+|r/\varepsilon|)^2\phi^{0\,2}(r,\ell)+\phi^{0\,2}_{\ell}(r,\ell)\Big\}dr\le C$$

to conclude that

$$|W(\ell)|\le C\Big[\varepsilon\|\nabla\psi(\cdot,\ell)\|_{2,\mathrm{I}_\varepsilon}+\|\psi(\cdot,\ell)\|_{2,\mathrm{I}_\varepsilon}\Big].$$

Therefore,

$$\begin{aligned} |III| &\le \varepsilon^2\int_0^1|\Theta'|[\varepsilon\|\nabla_x\psi(\cdot,\ell)\|_{2,\mathrm{I}_\varepsilon}+\|\psi(\cdot,\ell)\|_{2,\mathrm{I}_\varepsilon}]d\ell \\ &\le C\varepsilon^4\int_0^1\Theta'^2 d\ell+\frac{1}{C}\iint_\Omega(\varepsilon^2|\nabla\psi|^2+\psi^2). \end{aligned}$$

In summary, we have

$$\begin{aligned} |\langle L\phi,\,\psi\rangle| &\le C\varepsilon^4\int_0^1[\Theta'^2+\Theta^2]d\ell+\frac{1}{C}\iint_\Omega[\varepsilon^2|\nabla\psi|^2+\psi^2] \\ &\le C\varepsilon^4\int_0^1[\Theta'^2+\Theta^2]+\tfrac{1}{4}\langle L\psi,\,\psi\rangle \qquad (3.50) \end{aligned}$$

by taking C large and utilizing (3.45). Finally, via Corollary 3.5 (a), we obtain the assertion of the lemma. □

As we shall see later, estimate (3.49) is sufficient for concluding that all "small" eigenvalues of (1.22) are close to the "small" eigenvalues of (3.33).

3.6 Properties of $\langle L\cdot,\,\cdot\rangle$ on $\bar{H}^1(\Omega_\delta)$

Lemma 3.8 *There exist constants C_3 and C_4 which are independent of ε and δ such that for any $\phi\in\bar{\mathcal{X}}^0$ and $\phi^\perp\in\bar{\mathcal{X}}^{0\perp}$,*

$$\langle L(\phi+\phi^\perp),\,(\phi+\phi^\perp)\rangle \ge \Big\{1-C_3\varepsilon\Big\}\langle L\phi,\,\phi\rangle - C_4\varepsilon^3\delta^2\|\phi\|^2 \quad (3.51)$$
$$+\Big\{\tfrac{a_0\nu_0\varepsilon}{2|\Omega_\delta|}-(C_4\varepsilon^4+\pi^2\varepsilon^2|\Gamma|^{-2})\Big\}\|\phi^\perp\|^2.$$

where a_0 is as in (2.22).

Remark Observe that if δ is too small, then $\frac{a_0\nu_0\varepsilon}{2|\Omega_\delta|}$ will not be larger than $C_4\varepsilon^4+\pi^2\varepsilon^2|\Gamma|^{-2}$, so the coefficient in front of $\|\phi^\perp\|^2$ in (3.51) is negative, and we cannot show the positivity of the principal eigenvalue and the stability of the droplet. Similarly, for the spiky solution obtained in Sect. 2 (see Remark 2.4).

Proof. By the characterization of $\bar{\mathcal{X}}^0$ and $\bar{\mathcal{X}}^{0\perp}$, we can write $\phi=\Theta\phi^0$ and $\phi^\perp=\hat{\phi}+\psi$ where $\psi\in\mathcal{X}^{0\perp}$ and $\hat{\phi}=m\omega_3\phi^0$, $m\in I\!R^1$. Set $\varphi=\phi+\hat{\phi}$. Then $\phi+\phi^\perp=\varphi+\psi\in\mathcal{X}^0+\mathcal{X}^{0\perp}$. Applying (3.49) we obtain

$$\begin{aligned}\langle L(\phi+\phi^\perp),\,(\phi+\phi^\perp)\rangle &= \langle L(\varphi+\psi),\,(\varphi+\psi)\rangle\\ &\ge (1-C_1\varepsilon^2)\langle L\varphi,\,\varphi\rangle - C_2\varepsilon^4\|\varphi\|^2+\tfrac{\nu_0}{2}\|\psi\|^2\\ &= (1-C_1\varepsilon^2)\Big\{\langle L\phi,\,\phi\rangle+2\langle L\phi,\,\hat{\phi}\rangle+\langle L\hat{\phi},\,\hat{\phi}\rangle\Big\}\\ &\quad -C_2\varepsilon^4\|\phi\|^2+\Big\{\tfrac{\nu_0}{2}\|\psi\|^2-C_2\varepsilon^4\|\hat{\phi}\|^2\Big\} \quad (3.52)\end{aligned}$$

where in the second equation, we have used identity $\|\varphi\|^2=\|\phi\|^2+\|\hat{\phi}\|^2$. We shall now estimate each term on the right–hand side.

First of all, since $\hat{\phi}\in\mathcal{X}^0$, we have, by Theorem 3.4

$$\langle L\hat{\phi},\,\hat{\phi}\rangle \ge \lambda_1^0\|\hat{\phi}\|^2 \ge -2\varepsilon^2\pi^2|\Gamma|^{-2}\|\hat{\phi}\|^2. \quad (3.53)$$

Next, we can use Lemma 3.2 to write

$$\frac{|\Gamma|^2}{\varepsilon^2\omega_2(\frac12)}\langle L\phi,\,\hat{\phi}\rangle = m\omega_3(\tfrac12)\Big\{b^+\Theta(0)\frac{\omega_3(0)}{\omega_3(\frac12)}+b^+\Theta(1)\frac{\omega_3(1)}{\omega_3(\frac12)}$$
$$+\int_0^1\Big[b_1\Theta'\frac{\omega_3'}{\omega_3(\frac12)}+b_2\Theta\frac{\omega_3}{\omega_3(\frac12)}\Big]d\ell\Big\}.$$

Since $b^\pm=O(\delta)$, $|b^+-b^-|=O(\delta^2)$, and $|\frac{\omega_3(\ell)}{\omega_3(\frac12)}-1|=O(\varepsilon\delta)$, we have that

$$\Big|b_1\Theta(0)\frac{\omega_3(0)}{\omega_3(\frac12)}+b_2\Theta(1)\frac{\omega_3(1)}{\omega_3(\frac12)}\Big|$$

$$\le \frac{|b_1\Theta(0)\omega_3(0) - b_2\Theta(1)\omega_3(1)|}{\omega_3(\frac{1}{2})}|\Theta(1)|\frac{|b_1\Theta(0)\omega_3(0)|}{\omega_3(\frac{1}{2})}|\Theta(0)+\Theta(1)|$$
$$\le C\delta^2|\Theta(1)| + C\delta|\Theta(0)+\Theta(1)|.$$

Recall that $\phi \in \bar{\mathcal{X}}^0$ implies that $\int_0^1 \Theta\omega_1 = 0$, so we can estimate

$$\Big|\int_0^1 \Theta b_3 \frac{\omega_3}{\omega_3(\frac{1}{2})}\Big| = \Big|\int_0^1 \Theta\Big[\frac{b_2(\frac{1}{2})\omega_1(\ell)}{\omega_1(\frac{1}{2})} - \frac{b_2(\ell)\omega_3(\ell)}{\omega_3(\frac{1}{2})}\Big]d\ell\Big|$$
$$\le C\delta^2\int_0^1 |\Theta|. \tag{3.54}$$

Here we used the fact that $\mathcal{K} = \hat{\sigma} + O(\delta^2)$ so that $b_2(\ell) = b_2(\frac{1}{2}) + O(\delta^2)$.

Using (3.32), we have

$$\Big|\int_0^1 b_1\Theta'\omega_3'\Big| \le C\varepsilon\int_0^1 |\Theta'|.$$

In summary, we have the estimate

$$\Big|\langle L\phi,\,\hat{\phi}\rangle\Big| \le C\varepsilon^2|m\omega_3(\tfrac{1}{2})|\Big\{\varepsilon\int_0^1 |\Theta'|$$
$$+\delta^2\int_0^1 |\Theta| + \delta^2|\Theta(1)| + \delta|\Theta(0)+\Theta(1)|\Big\}$$
$$\le \frac{\varepsilon}{C|\Omega_\delta|}\|\hat{\phi}\|^2 + C\varepsilon^3|\Omega_\delta|\Big\{\varepsilon^2\|\Theta\|^2_{H^1((0,1))}$$
$$+\delta^4\|\Theta\|^2_{C^0([0,1])} + \delta^2|\Theta(0)+\Theta(1)|^2\Big\}$$

where in the second inequality we use the fact that $\|\hat{\phi}\| = \|m\omega_3\phi^0\|$ is equivalent to the value $|m\omega_3(\frac{1}{2})|$. Using Corollary 3.5 we then obtain

$$\Big|\langle L\phi,\,\hat{\phi}\rangle\Big| \le \frac{\varepsilon}{C|\Omega_\delta|}\|\hat{\phi}\|^2$$
$$+C\varepsilon^3|\Omega_\delta|\Big\{(\varepsilon^2+\delta^4+\delta^2)\varepsilon^{-2}\langle L\phi,\,\phi\rangle + \delta^4\|\phi\|^2\Big\}$$
$$\le \frac{\varepsilon}{C_1|\Omega_\delta|}\|\hat{\phi}\|^2 + C\varepsilon|\Omega_\delta|\delta^2\langle L\phi,\,\phi\rangle$$
$$+C\varepsilon^3|\Omega_\delta|\delta^4\|\phi\|^2. \tag{3.55}$$

Substituting the estimates in (3.55) and (3.53) into (3.52) then yields

$$\langle L(\phi+\phi^\perp),\,\phi+\phi^\perp\rangle \ge \Big(1 - C\varepsilon^2 - C\varepsilon\delta^2|\Omega_\delta|\Big)\langle L\phi,\,\phi\rangle$$
$$-C(\varepsilon^4 + \varepsilon^3\delta^4|\Omega_\delta|)\|\phi\|^2 \tag{3.56}$$

$$+\Big\{\tfrac{\nu_0}{2}\|\psi\|^2-(C\varepsilon^4+\varepsilon^2\pi^2|\Gamma|^{-2}$$
$$+\frac{\varepsilon}{C_1|\Omega_\delta|})\|\hat{\phi}\|^2\Big\}. \tag{3.57}$$

Finally, we convert $\|\psi\|$ and $\|\hat{\phi}\|$ into $\|\phi^\perp\|$. Since $\phi^\perp=\hat{\phi}+\psi\in\bar{H}^1(\Omega_\delta)$,

$$0=\iint_{\Omega_\delta}\phi^\perp=\iint_{\Omega_\delta}\hat{\phi}+\iint_{\Omega_\delta}\psi.$$

It then follows that

$$\|\hat{\phi}\|^2=\frac{\|\hat{\phi}\|^2}{|\iint_{\Omega_\delta}\hat{\phi}|^2}\Big|\iint_{\Omega_\delta}\hat{\phi}\Big|^2=\frac{\int_0^1\omega_2\omega_3^2d\ell}{|\int_0^1\omega_1\omega_3|^2}\Big|\iint_{\Omega_\delta}\psi\Big|^2$$
$$\le\frac{\int_{I\!R}(\dot{U}(R))^2+o(1)}{4\varepsilon+o(\varepsilon)}|\Omega_\delta|\|\psi\|^2\le\frac{2|\Omega_\delta|}{3a_0\varepsilon}\|\psi^2\|^2.$$

where a_0 is defined as in (2.22). Therefore,

$$\|\psi\|^2=\|\psi\|^2\frac{\|\phi^\perp\|^2}{\|\hat{\phi}\|^2+\|\psi\|^2}\ge\frac{a_0\varepsilon}{|\Omega_\delta|^2}\|\phi^\perp\|^2.$$

It then follows from (3.57) (replacing $\|\hat{\phi}\|$ by $\|\phi^\perp\|$) that

$$\langle L(\phi+\phi^\perp),\,\phi+\phi^\perp\rangle\ge\Big(1-C\varepsilon^2-C\varepsilon\delta^2|\Omega_\delta|\Big)\langle L\phi,\,\phi\rangle$$
$$-C(\varepsilon^4+\varepsilon^3\delta^4|\Omega_\delta|)\|\phi\|^2$$
$$+\Big\{\frac{\nu_0a_0\varepsilon}{|\Omega_\delta|}-\Big(C\varepsilon^4+\varepsilon^2\pi^2|\Gamma|^{-2}$$
$$+\frac{\varepsilon}{C_1|\Omega_\delta|}\Big)\Big\}\|\phi^\perp\|^2.$$

Taking C_1 small enough we then obtain the assertion of the lemma. □

With the estimate for $\langle L\phi,\,\psi\rangle$, $\phi\in\bar{\mathcal{X}}^0$ and $\psi\in\bar{\mathcal{X}}^{0\perp}$ (this corresponds to the off diagonal entries of a matrix), we can now establish the relationship between the eigenvalue problems (3.3), (3.4) and (3.33), (3.34). To make the presentation clearer, we shall first establish a general perturbation result.

3.7 A perturbation result

Lemma 3.9 *Let X and Y be two Hilbert spaces and Y be compactly imbedded in X. Assume that $\langle Ly,\,z\rangle$ is a bounded bilinear form defined on Y such that it is symmetric and coercive (with respect to X). Let Z and $Z^\perp$ be closed subspaces of Y such that $Y=Z+Z^\perp$ and $Z\perp_X Z^\perp$. Assume that for* *some positive constants* $\eta_1 \in (0,1)$*,* $\eta_2 > 0$*, and* $\nu > 0$*, we have for all* $z \in Z, z^\perp \in Z^\perp$*,*

$$\langle L(z+z^\perp),\, (z+z^\perp)\rangle \geq (1-\eta_1)\langle Lz,\, z\rangle + \nu\|z^\perp\|^2 - \eta_2\|z\|^2 \quad (3.58)$$

where $\|\cdot\| = \|\cdot\|_X$*. Then, for any positive integer* j*, if* $\lambda_j^Y < \nu$*, we have*

$$\lambda_j^Z - (\eta_2 + \eta_1\lambda_j^Z) \leq \lambda_j^Y \leq \lambda_j^Z. \quad (3.59)$$

In addition, if $\lambda_j^Y > \lambda_{j-1}^Y$ *(*$\lambda_0^Y := -\infty$*) and* $\lambda_{j+1}^Z > \lambda_j^Z$*, then either* $\|y_j - z_j\|^2$ *or* $\|y_j + z_j\|^2$ *is bounded by*

$$\frac{3(\eta_2 + \eta_1\lambda_j^Z)}{\min\{\lambda_j^Y - \lambda_{j-1}^Y, (1-\eta_1)(\lambda_{j+1}^Z - \lambda_j^Z), \nu + \eta_2 - (1-\eta_1)\lambda_j^Z\}}. \quad (3.60)$$

Proof. Recall that for any positive integer i, λ_i^Y and λ_i^Z can be obtained by the Min–Max characterization,

$$\lambda_i^Y = \min_{M_i \subset Y} \max_{x \in M_i, \|x\|=1} \langle Lx,\, x\rangle, \quad \lambda_i^Z = \min_{M_i \subset Z} \max_{x \in M_i, \|x\|=1} \langle Lx,\, x\rangle,$$

where M_i denotes a generic i dimensional subspace of Y. Since $Z \subset Y$, we immediately conclude that $\lambda_i^Y \leq \lambda_i^Z$ for any positive integer i.

Now assume that $\lambda_j^Z < \nu$. We shall establish the lower bound for λ_j^Y.

For each $i \leq j$, we write $y_i = \hat{y}_i + \hat{y}_i^\perp \in Z + Z^\perp$. We claim that $\hat{y}_1 \cdots, \hat{y}_j$ are linearly independent. In fact, if it were not true, then $\Sigma_{i=1}^j c_i\hat{y}_i = 0$ for some non–zero vector $(c_1, \cdots, c_j)$. It then follows that $y := \Sigma_{i=1}^j c_i y_i \in Z^\perp$. But this is impossible since on the one hand we have $\langle Ly,\, y\rangle = \Sigma\lambda_i^Y c_i^2 \leq \lambda_j^Y\|y\|^2 < \nu\|y\|^2$ and on the other hand by taking $z = 0$ and $z^\perp = y$ in (3.58) we obtain $\langle Ly,\, y\rangle \geq \nu\|y\|$. Hence, the dimension of $M :=$ span$\{\hat{y}_1, \cdots, \hat{y}_j\}$ is j. Consequently, there exists a non-trivial $z = \Sigma_{i=1}^j \alpha_i\hat{y}_i \in M$ such that $z \perp z_i$ for all $i \leq j-1$.

Define $y = \Sigma_{i=1}^j \alpha_i y_i = z + \hat{y}^\perp$ where $\hat{y}^\perp = \Sigma_{i=1}^j \alpha_i\hat{y}_i^\perp \in Z^\perp$. We write $y = \alpha y_j + y_j^\perp$ and $z = \beta z_j + z_j^\perp$ where $\alpha = \alpha_j$, $y_j^\perp = \Sigma_{i=1}^{j-1}\alpha_i y_i$, $z_j^\perp \perp z_i$ for all $i = 1, \cdots, j$. Since $\hat{y}$ is non–trivial, we can assume that $1 = \|y\|^2 = \alpha^2 + \|y_j^\perp\|^2 = \beta^2 + \|z_j^\perp\|^2 + \|\hat{y}^\perp\|^2$.

Note that, by definition of (λ_i^Y, y_i),

$$\langle Ly,\, y\rangle \geq \lambda_j^Y\alpha^2 + \lambda_{j-1}^Y\|y_j^\perp\|^2 = \lambda_j^Y - (\lambda_j^Y - \lambda_{j-1}^Y)\|y_j^\perp\|^2. \quad (3.61)$$

Note also that, since $y = \hat{y} + \hat{y}^\perp \in Z + Z^\perp$, we have from (3.58),

$$\langle Ly,\, y\rangle \geq (1-\eta_1)\langle L\hat{y},\, \hat{y}\rangle + \nu\|\hat{y}^\perp\|^2 - \eta_2\|\hat{y}\|^2$$

$$\geq (1-\eta_1)\Big(\lambda_j^Z \beta^2 + \lambda_{j+1}^Z \|z_j^\perp\|^2\Big) + \nu \|\hat{y}^\perp\|^2 - \eta_2 \|\hat{y}\|^2$$
$$= (1-\eta_1)\lambda_j^Z - \eta_2 + (1-\eta_1)(\lambda_{j+1}^Z - \lambda_j^Z)\|z_j^\perp\|^2$$
$$+[\nu + \eta_2 - (1-\eta_1)\lambda_j^Z]\|\hat{y}^\perp\|^2$$

by using the identity $\beta^2 = 1 - \|z_j^\perp\|^2 - \|\hat{y}^\perp\|^2$ and $\|\hat{y}\|^2 = 1 - \|\hat{y}^\perp\|^2$. The last inequality, together with (3.61), implies that

$$\lambda_j^Y - \lambda_j^Z + \eta_1 \lambda_j^Z + \eta_2$$
$$\geq (\lambda_j^Y - \lambda_{j-1}^Y)\|y_j^\perp\|^2 + (1-\eta_1)(\lambda_{j+1}^Z - \lambda_j^Z)\|z_j^\perp\|^2$$
$$+(\nu + \eta_2 + (1-\eta_1)\lambda_Z^j)\|\hat{y}^\perp\|^2.$$

Since the right–hand side is non–negative ($\lambda_j^Z < \nu$ implies that $\nu + \eta_2 - (1-\eta_1)\lambda_j^Y > 0$), we immediately obtain (3.59).

Set $\triangle = \min\{\lambda_j^Y - \lambda_{j-1}^Y, (1-\eta_1)(\lambda_{j+1}^Z - \lambda_j^Z), \nu + \eta_2 - (1-\eta_1)\lambda_Z^j)\}$. Then from the last estimate, we obtain,

$$\|\hat{y}^\perp\|^2 + \|z_j^\perp\|^2 + \|y_j^\perp\|^2 \leq [\eta_2 + \eta_1 \lambda_j^Z]/\triangle. \tag{3.62}$$

Now if the right–hand side is $\geq 2/3$, there is nothing to prove since the right–hand side in (3.60) is $\geq 2 \geq \min\{\|y_j - z_j\|^2, \|y_j + z_j\|^2\}$. Otherwise, we have $\alpha^2 = 1 - \|y_j^\perp\|^2 \geq 1/3$ and $\beta^2 = 1 - \|\hat{y}^\perp\|^2 - \|z_j^\perp\|^2 \geq 1/3$. Now, assuming, without loss of generality that α and β are positive (otherwise, change z_j to $-z_j$ and/or y_j to $-y_j$), we then have $\beta+\alpha \geq 2/\sqrt{3} > 1$. Finally, multiplying the relation $0 = y - y = [\alpha y_j + y_j^\perp] - [\beta z_j + z_j^\perp + \hat{y}^\perp]$ by $y_j - z_j$ we obtain $(\alpha+\beta)\|y_j - z_j\|^2 + (y_j - z_j, y_j^\perp - z_j^\perp - \hat{y}^\perp) = 0$, which implies that $\|y_j - z_j\|^2 \leq (\|y_j^\perp\| + \|z_j^\perp\| + \|\hat{y}^\perp\|)^2 \leq 3(\|y_j^\perp\|^2 + \|z_j\|^2 + \|\hat{y}^\perp\|^2)$. Using (3.62), we then obtain (3.60), thereby completing the proof of the lemma. □

3.8 Conclusion

With the previous preparation, we can now establish the main results of this section.

First using estimate (3.49) and applying Lemma 3.9 with $Y = H^1(\Omega_\delta)$, $X = L^2(\Omega_\delta)$, $Z = \mathcal{X}^0$, $\eta_1 = C_1\varepsilon^2$, $\eta_2 = C_2\varepsilon^4$, and $\nu = \nu_0/2$, we immediately obtain the following estimates:

Theorem 3.10 *Let $\{(\mu_j, \Theta_j)\}_{j=1}^\infty$ be the solutions of (3.37) where Θ_j is normalized so that*
$\|\Theta_j \phi^0\| = 1$. *Let $\{(\lambda_j, \phi_j)\}_{j=1}^\infty$ be the solution of (1.22). Then for any integer j, if $\mu_j < \frac{|\Gamma|^2\nu_0}{2\varepsilon^2}$ we have*

$$\varepsilon^2|\Gamma|^{-2}\mu_j \geq \lambda_j \geq \varepsilon^2|\Gamma|^{-2}\mu_j - (C_2 + C_1|\Gamma|^{-2}\mu_j)\varepsilon^4 \tag{3.63}$$

where ν_0, C_1, C_2 are the positive constants in (3.49). If in addition, we assume that $\lambda_{j-1} < \lambda_j$ and $\mu_{j+1} > \mu_j$, then

$$\|\phi_j - \Theta_j \phi^0\|^2 \leq \frac{3\varepsilon^4(C_2 + C_1|\Gamma|^{-2}\mu_j)}{\min\{\lambda_j - \lambda_{j-1}, \varepsilon^2|\Gamma|^2(\mu_{j+1} - \mu_j), \frac{\nu_0}{2} + C_2\varepsilon^4 + (1 - C_1\varepsilon^2)\varepsilon^2|\Gamma|^{-2}\mu_j\}}. \tag{3.64}$$

In particular, for any fixed positive integer J independent of ε and δ, we have

$$\lambda_j = \frac{\varepsilon^2\pi^2}{|\Gamma|^2}\Big\{(j-1)^2 - 1 + O(\delta)\Big\}, \tag{3.65}$$

$$\phi_j = \sqrt{\frac{2}{\varepsilon\Omega_2(\frac{1}{2})}}\dot{U}(\tfrac{r}{\varepsilon})\cos\Big((j-1)\pi\ell\Big) + O(\delta) \quad \forall j = 1, \cdots, J$$

Proof. Utilizing estimate (3.26) and applying Lemma 3.9 with $Y = H^1(\Omega_\delta)$, $X = L^2(\Omega_\delta)$, $Z = \mathcal{X}^0$, $\eta_1 = C_1\varepsilon^2$, $\eta_2 = C_2\varepsilon^4$, and $v = v_0/2$, we immediately obtain the first assertion of the theorem.

Utilizing Theorem 3.4 and the fact that $\phi^0 = \sqrt{\varepsilon}\dot{U}(\frac{r}{\varepsilon}) + O(\delta)$, we also obtain the second assertion. □

Similarly, applying Lemma (3.9) with $Y = \bar{H}(\Omega_\delta)$ and $Z = \bar{\mathcal{X}}^0$, and utilizing Lemma 3.2 and Theorem 3.4, we obtain the following.

Theorem 3.11 (The Principal Eigenvalue) *Let $u = u(x, \xi, \varepsilon)$ be constructed as in the previous section. Assume that for some large constant C^*,*

$$\delta^2 > C^*\varepsilon. \tag{3.66}$$

Let $\{(\bar{\mu}_j, \bar{\Theta}_j)\}_{j=1}^\infty$ be the solutions of (3.37) where $\bar{\Theta}_j$ is normalized so that $\|\bar{\Theta}_j\phi^0\| = 1$. Let $\{(\bar{\lambda}_j, \bar{\phi}_j)\}_{j=1}^\infty$ be the solution of (1.21). Then for any integer j, if $\bar{\mu}_j < \frac{\varepsilon\delta^2}{2C_4}$ we have

$$\varepsilon^2|\Gamma|^{-2}\bar{\mu}_j \geq \bar{\lambda}_j \geq \varepsilon^2|\Gamma|^{-2}\bar{\mu}_j - C_5(\varepsilon^2\delta^4 + \varepsilon^3\bar{\mu}_j) \tag{3.67}$$

where C_5 is a positive constant independent of δ and ε. If in addition, we assume that $\bar{\lambda}_{j-1} < \bar{\lambda}_j$ and $\bar{\mu}_{j+1} > \bar{\mu}_j$, then we have

$$\|\bar{\phi}_j - \bar{\Theta}_j\phi^0\|^2 \leq \frac{C_5[\varepsilon^2\delta^4 + \varepsilon^3\bar{\mu}_j]}{\min\{\bar{\lambda}_j - \bar{\lambda}_{j-1}, \varepsilon^2|\Gamma|^{-2}(\bar{\mu}_{j+1} - \bar{\mu}_j), \frac{\varepsilon}{C_5|\Omega_\delta|}\}}. \tag{3.68}$$

In particular the following hold:

$$\bar{\phi}_j = \sqrt{\frac{2}{\varepsilon\Omega_2(\frac{1}{2})}}\phi^0\dot{U}(\tfrac{r}{\varepsilon})\cos(j\pi\ell) + O(\delta), \quad j = 1, 2 \tag{3.69}$$

$$\bar{\lambda}_1 = -\frac{4\varepsilon^2}{3\pi\hat{\sigma}_0}\frac{d^2\mathcal{K}_{\Omega_\delta}}{d\xi^2}(\xi) + O(\varepsilon^2\delta^4). \tag{3.70}$$

$$\bar{\lambda}_2 = \frac{3\varepsilon^2\pi^2}{|\Gamma|^2} + O(\varepsilon^2\delta). \tag{3.71}$$

Recall from the expansion that $\mathrm{r}_\xi(s,\xi,\varepsilon) = -\cos(\pi s/|\Gamma|) + O(\delta)$ and that

$$\begin{aligned} u_\xi &= \mathrm{r}_\xi(s,\xi,\varepsilon)u_r(r,s,\xi,\varepsilon) + \mathrm{s}_\xi u_s + u_\xi(r,s,\xi,\varepsilon) \\ &= -\frac{1}{\varepsilon}\Big\{\cos(\pi s/|\Gamma|)U'(r/\varepsilon) + O(\varepsilon)\Big\} \end{aligned}$$

which is close to the principal eigenfunction. Hence we have the following:

Theorem 3.12 (Spectral Gap) *Let $u = u(x,\xi,\varepsilon)$ be constructed as in the previous section. Assume that (3.66) holds. Then, for any $v \in H^1(\Omega_\delta)$ satisfying*

$$\iint_{\Omega_\delta} v = 0, \quad \iint_{\Omega_\delta} v u_\xi = 0, \tag{3.72}$$

we have

$$\langle Lv,\, v\rangle := \iint_{\Omega_\delta}\Big\{\varepsilon^2|\nabla v|^2 + f'(u)v^2\Big\}dx \geq \frac{2\varepsilon^2\pi^2}{|\Gamma|^2}\iint_{\Omega_\delta} v^2. \tag{3.73}$$

4 Dynamics

In Sect. 2, we constructed a function $u(x,\xi,\varepsilon)$ and a scalar field $c(\xi,\varepsilon)$ such that

$$\begin{cases} \mathcal{L}^\varepsilon(u) := \varepsilon^2\Delta u - f(u) + ⨌ f(u) = \varepsilon^2 c u_\xi + O(\varepsilon^K) & \text{in } \Omega_\delta, \\ \partial_n u = 0 & \text{on } \partial\Omega_\delta, \\ \int_{\Omega_\delta} u = |\Omega_\delta| - \pi. \end{cases} \tag{4.1}$$

In this section, we shall study the dynamics of (1.8) in a small neighborhood of a manifold $\mathcal{M}$ defined by

$$\mathcal{M} := \{u(\cdot,\xi,\varepsilon); \xi \in \partial\Omega_\delta\}. \tag{4.2}$$

4.1 The tubular coordinate system

For any positive constant η, we define

$$\begin{aligned} \mathcal{N}^\eta_{L^2} &:= \{\varphi \in L^2(\Omega_\delta); d_{L^2}(\varphi,\mathcal{M}) \leq \eta\}, \\ \mathcal{N}_{H^1_\varepsilon} &:= \{\varphi \in H^1(\Omega_\delta); d_{H^1_\varepsilon}(\varphi,\mathcal{M}) \leq \eta\} \end{aligned} \tag{4.3}$$

where d_{L^2} and $d_{H^1_\varepsilon}$ represent, respectively, the distance in the $L^2(\Omega_\delta)$ norm and in H^1_ε norm defined by

$$\|\varphi\|^2_{H^1_\varepsilon} := \varepsilon^2\|\nabla\varphi\|_{L^2(\Omega_\delta)} + \|\varphi\|^2_{L^2(\Omega_\delta)}.$$

The following result concerning the $L^2(\Omega_\delta)$ projection of a neighborhood of $\mathcal{M}$ on $\mathcal{M}$ is proved in [6, Lemma 2.5]. We state it without proof.

Lemma 4.1 *Assume that a is a sufficiently small number. Then for each $w \in \mathcal{N}^{a\sqrt{\varepsilon}}_{L^2}$, there is a unique $\xi_w \in \partial\Omega_\delta$ such that*

$$\|w - u(\cdot, \xi_w, \varepsilon)\|_{L^2} = d_{L^2}(w, \mathcal{M}) := \inf_{\xi\in\partial\Omega_\delta} \|w - u(\cdot, \xi, \varepsilon)\|_{L^2(\Omega_\delta)}.$$

In addition, ξ is a smooth function of w and

$$\Big(w - u(\cdot, \xi_w, \varepsilon), u_\xi(\cdot, \xi_w, \varepsilon)\Big)_{L^2} = 0.$$

Furthermore, if $w \in \mathcal{N}^{a\sqrt{\varepsilon}}_{H^1_\varepsilon}$ and $u(\cdot, \xi_w, \varepsilon)$ is the L^2 projection of w on $\mathcal{M}$ as stated above, then

$$\|w - u(\cdot, \xi_w, \varepsilon)\|_{H^1_\varepsilon} \leq C d_{H^1_\varepsilon}(w, \mathcal{M})$$

where C is a constant independent of w, ε and ξ.

One observes that $w \to (\xi_w, v := w - u(\cdot, \xi_w, \varepsilon))$ is a smooth change of coordinates in $\mathcal{N}^{a\sqrt{\varepsilon}}_{L^2}$, as well as in $\mathcal{N}^{a\sqrt{\varepsilon}}_{H^1_\varepsilon}$.

4.2 Stability of the manifold

It is convenient to write (1.8) as an abstract evolution equation

$$w_t = \mathcal{L}^\varepsilon(w), \quad w(0) = w_0 \tag{4.4}$$

where

$$\mathcal{L}^\varepsilon(w) := \varepsilon^2 \Delta w - f(w) + ⨏\!\!⨏_{\Omega_\delta} f(w) \text{ in } \Omega_\delta, \quad \partial_n w = 0 \text{ on } \partial\Omega_\delta.$$

Theorem 4.2 *Assume that k in (4.1) is ≥ 5. Then there exists a small positive but ε–independent constant $\eta > 0$ such that the neighborhood $\mathcal{N}^{\eta\varepsilon^{k-2}}_{L^2}$ is positively invariant under the flow (4.4); namely,*

$$w(0) \in \mathcal{N}^{\eta\varepsilon^{k-2}}_{L^2} \implies w(t) \in \mathcal{N}^{\eta\varepsilon^{k-2}}_{L^2} \quad \forall\, t \geq 0. \tag{4.5}$$

In addition, the neighborhood $\mathcal{N}^{\eta\varepsilon^3}_{H^1_\varepsilon}$ is stable in the sense that

$$w(0) \in \mathcal{N}^{\eta\varepsilon^{k-2}}_{H^1_\varepsilon} \implies w(t) \in \mathcal{N}^{C\eta\varepsilon^{k-2}}_{H^1_\varepsilon} \quad \forall\, t \geq 0 \tag{4.6}$$

where C is a positive constant independent of $w(0)$ and ε.

Proof. By continuous dependence, $u(t)$ stays in $\mathcal{N}_{L^2}^{2\eta\varepsilon^{k-2}}$ for $t \in [0,\tau)$ for some $\tau > 0$. For any such t, we can write, by Lemma 4.1,

$$w(t) = u(\cdot, \xi(t), \varepsilon) + v(t), \quad v(t) \perp_{L^2} u_\xi(\cdot, \xi(t), \varepsilon).$$

Abbreviating $u(\cdot, \xi(t), \varepsilon)$ as u, (4.4) can be written as

$$u_\xi \dot{\xi}(t) + v_t = \mathcal{L}^\varepsilon(u) + Lv + N(u,v) \tag{4.7}$$

where L is the linearization of $\mathcal{L}^\varepsilon$ at u and $N(u,v)$ is the remaining part which is at least quadratic in v. Taking the L^2 inner product of (4.7) with v and using the condition that $v \perp_{L^2} u_\xi$ and that $\partial_n u = \partial_n v = 0$ on $\partial\Omega_\delta$, we then obtain

$$\tfrac{1}{2}(v,v)_t = (\mathcal{L}^\varepsilon(u), v) + \langle Lv,\, v\rangle + (N(u,v), v). \tag{4.8}$$

Using the equation for u, we have

$$(\mathcal{L}^\varepsilon(u), v) \le C\varepsilon^k \|v\|_{L^1(\Omega_\delta)} \le C\varepsilon^k \Big(\|v\|_{L^2(\Omega_\delta^2)} + C m_\varepsilon^{1/2} \|v\|_{L^2(\Omega_\delta^1)} \Big).$$

Since $v \perp_{L^2} u_\xi$, by the eigenvalue analysis in Sect. 4, we have

$$\tfrac{1}{2}\langle Lv,\, v\rangle \ge C^{-1}\varepsilon^2 \|v\|_{H_\varepsilon^1}^2 + C^{-1}\|v\|_{2,\Omega_\delta^1}^2 \ge C^{-1}\varepsilon \|v\|_{2,\Omega_\delta^2} \|v\|_{H_\varepsilon^1}.$$

Assume that $f''(s) > 0$ when $|s| \ge C_0$. Then $N(u,v)v \ge -C|v|^3$ (see [2, Lemma 2.2]), it then follows that

$$\begin{aligned} -(N(u,v), v) &\le C\|v\|_{3,\Omega_\delta}^3 \le CC\|v\|_{2,\Omega_\delta}^2 \|v\|_{H^1(\Omega_\delta)} \\ &\le C^2\varepsilon^{-1}\|v\|_{2,\Omega_\delta}^2 \|v\|_{H_\varepsilon^1} \le 2\eta\varepsilon^{k-3}C^2 \|v\|_{2,\Omega_\delta} \|v\|_{H_\varepsilon^1} \end{aligned}$$

where in the second inequality we have used the Nirenberg–Gagliardo inequality, and in the last inequality, we have used the assumption that $w(t) \in \mathcal{N}_{L^2}^{2\eta\varepsilon^{k-2}}$.

Inserting all these estimates into (4.8), we then obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|_{L^2(\Omega_\delta)}^2 \le{}& \|v\|_{2,\Omega_\delta^2}\Big[C\varepsilon^k - C^{-1}\varepsilon\|v\|_{H_\varepsilon^1}\Big] \\ &+ \|v\|_{2,\Omega_\delta}\Big[C\varepsilon^k m_\varepsilon^{1/2} + (2C^2\eta\varepsilon^{k-3} - C^{-1}\varepsilon^2)\|v\|_{H_\varepsilon^1}\Big]. \end{aligned}$$

Since $k \ge 5$, we can take η small (and independent of ε) such that $2C^2\eta\varepsilon^{k-3} \le \frac{1}{2}C^{-1}\varepsilon^2$. Hence, if $\|v\|_{2,\Omega_\delta} > \eta\varepsilon^{k-2}$, then so does $\|v\|_{H_\varepsilon^1}$, and therefore,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|_{2,\Omega_\delta}^2 \le{}& \|v\|_{2,\Omega_\delta^2}\Big[C\varepsilon^k - C^{-1}\eta\varepsilon^{k-1}\Big] \\ &+ \|v\|_{2,\Omega_\delta}\Big[C\varepsilon^{k+1/2}|\ln\varepsilon| - \tfrac{1}{2}C^{-1}\eta\varepsilon^k\Big] < 0. \end{aligned}$$

This implies that $\|v\|_{2,\Omega_\delta}$ can never exceed $\eta\varepsilon^{k-2}$, thereby proving (4.5).

Using local regularity theory for parabolic equations and for the function $W(y) := w(x)|_{x=\varepsilon y}$, one can easily establish (4.6). This completes the proof of the lemma. □

4.3 Motion on the manifold $\mathcal{M}$

Theorem 4.3 *Assume that k in (4.1) is* ≥ 5 *and* $w(0) \in \mathcal{N}_{H^1_\varepsilon}^{\eta\varepsilon^{k-2}}$. *Let* $w(t) = u(\cdot, \xi(t), \varepsilon) + v(t)$ *be the decomposition given by Lemma 4.1. Then*

$$\dot{\xi}(t) = \varepsilon^2 c(\xi(t), \varepsilon) + O(\varepsilon^{k-1/2}\delta^2) = \frac{4}{3\pi} g_0^2 \mathcal{K}'_{\Omega_\delta}(\xi)\varepsilon^2\delta[1+O(\delta)] + O(\delta^4\varepsilon^2) \tag{4.9}$$

where $g_0 = 1$ *if* $\varepsilon = O(\delta^3)$.

Proof. Taking the L^2 inner product of (4.7) with u_ξ we have

$$\dot{\xi}(t)\|u_\xi\|^2_{2,\Omega_\delta} + (v_t, u_\xi) = (\mathcal{L}^\varepsilon u, u_\xi) + \langle Lv,\ u_\xi\rangle + (N(u,v), u_\xi). \tag{4.10}$$

Differentiating the identity $(v, u_\xi) = 0$ with respect to t we have

$$\begin{aligned} \Big|(v_t, u_\xi)\Big| &= \Big|(v, u_{\xi\xi})\dot{\xi}\Big| \leq |\dot{\xi}|\ \|v\|_{2,\Omega_\delta}\|u_{\xi\xi}\|_{2,\Omega_\delta} \\ &\leq C|\dot{\xi}|C\eta\varepsilon^{k-2}\varepsilon^{-3/2} \leq C^2\dot{\xi}\varepsilon^{k-5/2}\|u_\xi\|^2_{2,\Omega_\delta} \end{aligned}$$

since $\|u_\xi\|_{2,\Omega_\delta} \geq C^{-1}\varepsilon^{-1/2}$ and $\|u_{\xi\xi}\|_{2,\Omega_\delta} \leq C\varepsilon^{-3/2}$.

Assuming that $f(s)$ grows at most with a power of three, then $|N(u,v)| \leq Cv^2(1+|v|)$ so that

$$\begin{aligned} \Big|(N(u,v), u_\xi)\Big| &\leq C\Big[\|v\|^2_{4,\Omega_\delta} + \|v\|^3_{6,\Omega_\delta}\Big]\|u_\xi\|_{2,\Omega_\delta} \\ &\leq C^2\Big[(\varepsilon^{-1}\|v\|_{H^1_\varepsilon}\|v\|_{2,\Omega_\delta} \\ &\quad +(\varepsilon^{-1}\|v\|_{H^1_\varepsilon})^2\|v\|_{2,\Omega_\delta}\Big]\|u_\xi\|_{2,\Omega_\delta} \\ &\leq C\varepsilon^{2k-5}\varepsilon^{1/2}\|u_\xi\|^2_{2,\Omega_\delta}. \end{aligned}$$

Also, from the equation for u, we have

$$(\mathcal{L}^\varepsilon(u), u_\xi) = \varepsilon^2 c\ (u_\xi, u_\xi) + (O(\varepsilon^k), u_\xi) = \Big(\varepsilon^2 c + O(\varepsilon^{k+1})\Big)\|u_\xi\|^2_{2,\Omega_\delta}$$

since $\|u_\xi\|_{L^1} = O(1)$.

Utilizing all these estimates in (4.10), we then obtain

$$\dot{\xi}\Big(1 + O(\varepsilon^{k-5/2})\Big)\|u_\xi\|^2 = \Big(\varepsilon^2 c + O(\varepsilon^{2k-9/2}) + O(\varepsilon^{k+1})\Big)\|u_\xi\|^2_{2,\Omega_\delta}.$$

That is

$$\begin{aligned} \dot{\xi} &= \Big[\varepsilon^2 c + O(\varepsilon^{2k-9/2}) + O(\varepsilon^{k+1})\Big]\Big[1 + O(\varepsilon^{k-5/2})\Big] \\ &= \varepsilon^2 c + O(\varepsilon^{k-1/2}\delta^2) \end{aligned}$$

since $k \geq 5$. This completes the proof of the Lemma. □

4.4 Equilibria and their stability

Theorem 4.4 *Let $z(\xi_0)$ be a point on $\partial\Omega_\delta$ such that the curvature of $\partial\Omega_\delta$ experiences a strict extreme; namely,*

$$\varphi_2 := \delta^{-2}\mathcal{K}'_{\Omega_\delta}(\xi_0) = 0, \quad \varphi_3 := \delta^{-3}\mathcal{K}''_{\Omega_\delta}(\xi_0) \neq 0.$$

Then in a small neighborhood of $u(\cdot,\xi_0,0)$, there exists a unique equilibrium of (1.8). In addition, if $\varphi_3 > 0$, i.e., the curvature experiences a local minimum, then the equilibrium is unstable with an one dimensional unstable manifold. If $\varphi_3 < 0$, then the equilibrium is exponentially stable.

Proof. 1. Existence. For any $\tilde{\xi} \in I\!R^1$, let $w(t,\tilde{\xi})$ be solution of the flow of (4.4) with initial data $w(0) = u(\cdot,\tilde{\xi},\varepsilon)$. If we denote by $\xi(\tilde{\xi},t)$ as the point such that the L^2 projection of $w(t,\tilde{\xi})$ is $u(\cdot,\xi(\tilde{\xi},t),\varepsilon)$, then from Theorem 4.3, we know that

$$\dot{\xi}(t,\tilde{\xi}) = \varepsilon^2\Big[c(\xi(t,\tilde{\xi}),\varepsilon) + O(\varepsilon^{7/2})\Big], \quad \xi(\tilde{\xi},0) = \tilde{\xi}. \tag{4.11}$$

Let ξ_1,ξ_2 be any two fixed points such that

$$\xi_1 < \xi_0 < \xi_2, \quad \delta^{3/2} \le |\xi_1-\xi_0|, |\xi_2-\xi_0| \le \delta^{-1/2}.$$

Then since $|\mathcal{K}''_{\Omega_\delta}(\xi_0)| = \delta^3|\varphi_3| > 0$ and φ_3 is independent of δ, we see that

$$[c(\xi_1,\varepsilon) + O(\varepsilon^{7/2})][c(\xi_2,\varepsilon) + O(\varepsilon^{7/2})] < 0. \tag{4.12}$$

Now define $\mathcal{A}_i$ as the set consisting of all those $\tilde{\xi} \in (\xi_1,\xi_2)$ such that there exists a time $T(\tilde{\xi}) > 0$ satisfying $\xi(T(\tilde{\xi}),\tilde{\xi}) = \xi_i$. By continuous dependence of initial data of the flow (4.4), both $\mathcal{A}_1$ and $\mathcal{A}_2$ are open. In addition, from (4.11) and (4.12), we see that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$. Hence, there exists $\xi^* \in (\xi_1,\xi_2)$ such that $\xi \notin \mathcal{A}_1 \cup \mathcal{A}_2$; namely, $\xi(t,\xi^*) \in (\xi_1,\xi_2)$ for all $t \ge 0$. Furthermore, since $|c(\xi)+O(\varepsilon^{7/2})| \neq 0$ as long as $\delta^{-1/2} \ge |\xi-\xi_0| \ge \delta^{3/2}$, we see that for t large enough,

$$\xi(t,\xi^*) \in (\xi_0 - \delta^{3/2}, \xi_0 + \delta^{3/2}) \quad \text{for all } \ t \ge T^*. \tag{4.13}$$

Recalling that (4.4) is a gradient flow, we know that ω–limit set of any trajectory is non–empty and consisting of equilibria. In particular, the ω–limit set of $w(t,\xi^*)$ is non–empty and consists of equilibria of (4.4). From (4.13), we then conclude that there exists at least one equilibrium of (4.4) near $u(\cdot,\xi_0,\varepsilon)$. This proves the existence.

2. Uniqueness and Stability Observe that the principal eigenvalue of L at $u = u(\cdot,\xi,\varepsilon)$ is $\varepsilon^2[\mathcal{K}''_{\Omega_\delta} + O(\delta^4)]$. Also if we replace u by $u+v$, then for any $\phi \in H^1(\Omega_\delta)$,

$$\int_{\Omega_\delta} |(f'(u) - f'(u+v))|\phi^2 \le C\varepsilon^{-1}\|v\|_{2,\Omega_\delta}\|\phi\|_{H^1_\varepsilon}\|\phi\|_{2,\Omega_\delta}.$$

We then conclude that if we replace u by $u+v$ with $\|v\|_{2,\Omega_\delta} \le \varepsilon^3\delta^4$, then the principal eigenvalue of L will change at most by $C\varepsilon^2\delta^4$.

Now from the eigenvalue analysis in Sect. 3, we see that for any $w \in \{w; w = u(\cdot,\xi,\varepsilon)+v, \xi \in [\xi_0-\delta^{-1/2}, \xi_0+\delta^{-1/2}], \|v\|_{2,\Omega_\delta} \le \varepsilon^3\delta^4\}$, the second eigenvalue of L at w is $\varepsilon^2[3L_0^{-2}+O(\delta)]$, uniformly for $\xi \in [\xi_0-\delta^{-1/2}, \xi_0+\delta^{-1/2}]$, while the principal eigenvalue is $\frac{-4\pi}{3L_0^2}\varepsilon^2\delta^3[\varphi_3(\xi_0)+O(\delta^{1/2})]$, which is uniformly bounded away from zero. From this, and standard linearization theory we obtain both the uniqueness of the equilibrium and its exponential stability/instability. □

Remark 4.5 As far as we know, the existence of unstable equilibria stated in Theorem 4.4 has not been rigorously verified before in the literature, in the case of bistable nonlinearities in higher space dimensions. In the case of one-sided nonlinearities, Ni and Takagi [[69]] have established existence of unstable equilibria near critical points of the curvature of $\partial\Omega$. The existence of stable equilibria near strict local maxima of the curvature has been proved in several places; see, for example, [31, 51].

5 Appendix: Energy comparison

In this appendix, we shall compare the energies of the constant solution (i.e., homogeneous state) with those of single interface layered solutions. More precisely, we calculate the energy

$$\mathcal{E}(u) := \int_0^1 r^{N-1}\Big(\frac{\hat{\varepsilon}^2}{2}u_r^2 + W(u)\Big)\,dr, \quad W(u) := \int_{-1}^u f(s)ds$$

for $u=u(r)$, which, together with a constant σ, solves

$$\begin{cases} \varepsilon^2 u_{rr} + \hat{\varepsilon}^2\frac{N-1}{r}u_r = f(u)+\sigma, \quad r\in(0,1), \\ u_r(0)=u_r(1)=0, \quad \int_0^1 r^{N-1}u(r)\,dr = \frac{1}{N}m, \\ \qquad m := 1-2\delta^N. \end{cases} \tag{A.1}$$

If (u,σ) is a constant solution, then it is uniquely given by $u=m$ and $\sigma = -f(m)$, and its corresponding energy is

$$\mathcal{E}(m) = \tfrac{1}{N}W(m).$$

Now let u be a singled layered (“bubble”) solution of (A.1) in the sense that $u(r)$ is monotonic. Let ρ be the “radius”; namely $u(\rho)=0$. We assume that $\rho \in [2\hat{\varepsilon}|\ln\hat{\varepsilon}|^2, 1/2]$ and want to calculate the energy associated with it.

From the differential equation (A.1), one can show that $\sigma + f(u(1)) = O(e^{-\nu(1-\rho)/\hat{\varepsilon}})$, where ν can be any positive constant $< f'(u(1))$. Also, define $w(z) = u(\rho + \hat{\varepsilon} z)$. Then

$$w'' - f(w) = \sigma - \Big\{ \tfrac{(N-1)\hat{\varepsilon}}{\rho} + O(\tfrac{\hat{\varepsilon}^2 |\ln\hat{\varepsilon}|^2}{\rho^2}) \Big\} w', \quad z \in [-|\ln\hat{\varepsilon}|^2, |\ln\hat{\varepsilon}|^2].$$

A phase plane analysis then yields (since u is single layered),

$$\begin{aligned} w(z) &= U(z) + \sigma U_1(z) + O(\tfrac{\hat{\varepsilon}^2 |\ln\hat{\varepsilon}|^2}{\rho^2})\Big), \quad \Big(U_1 \text{ is as in (2.24)}\Big) \\ \sigma &= \frac{(N-1)\hat{\varepsilon}}{a_0 \rho} + O\Big(\tfrac{\hat{\varepsilon}^2 |\ln\hat{\varepsilon}|^2}{\rho^2})\Big), \quad \Big(a_0 \text{ is as in (2.22)}\Big) \qquad \text{(A.2)} \\ \int_0^1 r^N u_r &= \rho^N (2 + o(1)), \\ \int_0^1 r^{N-1} u_r^2 &= \rho^{N-1} \Big(\tfrac{2}{a_0} + o(1) \Big). \end{aligned}$$

From the area constraint $m = N \int_0^1 r^{N-1} u = u(1) - \int_0^1 r^N u_r$, one derives

$$u(1) = m + \int_0^1 r^N u_r = 1 - 2\delta^N + \rho^N (2 + o(1)).$$

Therefore,

$$\begin{aligned} \sigma &= -f(u(1)) + O(e^{-\nu(1-\rho)/\hat{\varepsilon}}) = -f'(1)(u(1) - 1) + O(|u(1) - 1|^2) \\ &= f'(1)[2\delta^N - 2\rho^N + o(\rho^N + \delta^N)]. \end{aligned}$$

This relation, together with (A.2), then implies that ρ must satisfy the algebraic equation

$$\hat{\varepsilon} = G(\rho) + o(\delta^{N+1} + \rho^{N+1}), \quad G(\rho) := \frac{2f'(1)a_0}{N-1} \rho(\delta^N - \rho^N). \qquad \text{(A.3)}$$

Clearly, this equation has a solution if and only if

$$\begin{aligned} &\hat{\varepsilon} < c_N^* \delta^{N+1}, \\ &c_n^* := \frac{2f'(1)a_0 N}{(N^2-1)(1+N)^{1/N}} \\ &\quad = \frac{4Nf'(1)}{(N^2-1)(1+N)^{1/N}\sqrt{2}\int_{-1}^1 \sqrt{W(s)}ds}. \end{aligned} \qquad \text{(A.4)}$$

Assume that the above relation holds. We now calculate the energy of u:

$$
\begin{aligned}
N\mathcal{E}(u) &= N\int_0^1 r^{N-1}\Big(W(u)+\sigma u-\tfrac{\hat\varepsilon^2}{2}u_r^2\Big)\\
&\quad +N\hat\varepsilon^2\int_0^1 r^{N-1}u_r^2-\sigma N\int_0^1 r^{N-1}u\\
&= r^N[W(u)+\sigma u-\tfrac{\hat\varepsilon^2}{2}u_r^2]\Big|_{r=0}^{r=1}\\
&\quad +\int_0^1\Big\{r^N u_r[u_{rr}-f(u)-\sigma]+Nr^{N-1}\hat\varepsilon^2u_r^2\Big\}-\sigma m\\
&= W(u(1))+\sigma[u(1)-m]\\
&\quad +\hat\varepsilon^2\int_0^1 r^{N-1}u_r^2\quad\Big(\text{by (A.1)}\Big)\\
&= W(m)+(f(u(1)+\sigma)(u(1)-m)-\tfrac12 f'(\xi)[u(1)-m]^2\\
&\quad +\hat\varepsilon^2\int_0^1 r^{N-1}u_r^2\Big(\xi\in[m,u(1)]\Big)\\
&= W(m)-\tfrac12 f'(1)[2\delta^N-2\rho^N]^2[1+o(1)]+\frac{2\hat\varepsilon\rho^{N-1}}{a_0}\\
&= N\mathcal{E}(m)-\tfrac12 f'(1)[2\delta^N-2\rho^N]^2[1+o(1)]\\
&\quad +4f'(1)\rho^N[\delta^N-\rho^N](1+o(1))\\
&= N\mathcal{E}(m)-2f'(1)[\delta^N-\rho^N+o(\rho^N)][\delta^N-3\rho^N+o(\rho^N)]
\end{aligned}
$$

Hence,

$$\mathcal{E}(u)<\mathcal{E}(m)\quad\iff\rho>3^{-1/N}\delta.$$

From the equation for ρ in (A.3), to have $\rho>3^{-1/N}\delta$, it is necessary and sufficient to have $G(3^{-1/N}\delta)>\hat\varepsilon$; namely, it is sufficient and necessary to have

$$\hat\varepsilon\delta^{-(N+1)}<\frac{4f'(1)a_0}{33^{1/N}}=\frac{8f'(1)}{33^{1/N}\sqrt2\int_{-1}^1\sqrt{W(s)}ds}=:C_N^*.\tag{A.5}$$

One observes that when $N=2$, $C_N^*=c_N^*=C_1^*$ where C_1^* is defined in (2.65).

In conclusion, for the bubble to be the global minimum energy, one needs condition (A.5). In terms of $\varepsilon=\hat\varepsilon/\delta$, condition (A.5) is equivalent to $\varepsilon\le C_N^*\delta^N$, where N is the space dimension.

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