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Calculus of Variations

Phase transition of an anisotropic Ginzburg–Landau equation

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Abstract

We study the effective geometric motions of an anisotropic Ginzburg–Landau equation with a small parameter $\varepsilon > 0$ which characterizes the width of the transition layer. For well-prepared initial datum, we show that as ε tends to zero the solutions will develop a sharp interface limit which evolves under mean curvature flow. The bulk limits of the solutions correspond to a vector field $\mathbf{u}(x, t)$ which is of unit length on one side of the interface, and is zero on the other side. The proof combines the modulated energy method and weak convergence methods. In particular, by a (boundary) blow-up argument we show that $\mathbf{u}$ must be tangent to the sharp interface. Moreover, it solves a geometric evolution equation for the Oseen–Frank model in liquid crystals.

Mathematics Subject Classification 53E10 · 35R35 · 35K58 · 35K57

Contents

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1 Introduction

In the study of liquid crystals one often encounters elastic energies with anisotropy, i.e. energies with distinct coefficients multiplying the square of the divergence and the curl of the order parameters. Typical examples involve the Oseen–Frank model [30], Ericksen's model [43, 44] and the Landau–De Gennes model [5]. From a microscopic point of view, the anisotropy of these models can be interpreted as excluded volume potential of molecular interaction, cf. [29]. Anisotropic models also arise in the theory of superconductivity, cf. [10]. The anisotropy brings various new challenges to the studies of both variational problems and their gradient flows of the aforementioned models. In contrast to the convergence analysis of isotropic models, i.e. the (scalar) Allen–Cahn equations (cf. [6, 19, 34, 47, 48, 50, 53, 54]), the powerful analytic tools such as maximum principle and monotonicity formula are not readily established for anisotropic ones.

The attempt of this work is to study an anisotropic system modeling the isotropic-nematic phase transition of a liquid crystal droplet. Let $d \in \{2, 3\}$ be the dimension of the physical domain Ω with C^3 boundary $\partial\Omega$. We consider the anisotropic Ginzburg–Landau type energy

$$A_\varepsilon(\mathbf{u}) = \int_\Omega \left(\frac{\varepsilon}{2}\mu |\operatorname{div} \mathbf{u}|^2 + \frac{\varepsilon}{2}|\nabla \mathbf{u}|^2 + \frac{1}{\varepsilon}F(\mathbf{u}) \right) dx. \tag{1.1}$$

Here $\mathbf{u} = (u_1, u_2, u_3) : \Omega \subset \mathbb{R}^d \mapsto \mathbb{R}^3$ is the order parameter describing the state of the system. The function $F(\mathbf{u})$ is a double equal-well potential which permits the isotropic-nematic phase transition. More precisely, it attains its global minimum value 0 at $\{0\} \cup \mathbb{S}^2$. An example of F is the Chern–Simons–Higgs model $F(\mathbf{u}) = |\mathbf{u}|^2(1 - |\mathbf{u}|^2)^2$. See for instance [31, 36] for the physics and [9, 27, 28] for the mathematical analysis of related variational problems. The parameter $\varepsilon > 0$ denotes the relative intensity of elastic and bulk energy, which is usually quite small. The parameter $\mu > 0$ is material dependent which measures the degree of anisotropy.

The energy (1.1) is a simplified case of the full Landau–De Gennes energy (cf. [35, 45]). The variational investigations of the isotropic-nematic phase transition involving (1.1) were first done by Golovaty, Novack, Sternberg and Venkatraman [27, 28] in the static case in 2D. The present paper is concerned with the L^2-gradient flow of (1.1), i.e. the following system.

$$\partial_t \mathbf{u}_\varepsilon - \mu \nabla(\operatorname{div} \mathbf{u}_\varepsilon) = \Delta \mathbf{u}_\varepsilon - \frac{1}{\varepsilon^2} DF(\mathbf{u}_\varepsilon) \quad \text{in } \Omega \times (0, T), \tag{1.2a}$$

$$\mathbf{u}_\varepsilon(x, 0) = \mathbf{u}_\varepsilon^{in}(x) \quad \text{in } \Omega, \tag{1.2b}$$

$$\mathbf{u}_\varepsilon(x, t) = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.2c}$$

where $DF(\mathbf{u})$ is the gradient of $F(\mathbf{u})$ with respect to $\mathbf{u}$. We shall study the small ε-asymptotics of this system with well-prepared initial datum $\mathbf{u}_\varepsilon^{in}$ that undergoes a sharp transition across a co-dimensional one interface $I_0 \subset \mathbb{R}^d$. We shall show that the energy density $\frac{\varepsilon}{2}|\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon}F(\mathbf{u}_\varepsilon)$ will be concentrated on a mean curvature flow $I := \bigcup_{t \geqslant 0} I_t \times \{t\}$ starting from I_0, namely

$$\lim_{\varepsilon \to 0} \int_\Omega \left(\frac{\varepsilon}{2}|\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon}F(\mathbf{u}_\varepsilon) \right) dx = \sigma \mathcal{H}^{d-1}(I_t), \tag{1.3}$$

where $\mathcal{H}^{d-1}$ is the $(d - 1)$ dimensional Hausdorff measure, and σ is a positive constant depending on F. Moreover, we shall derive bulk limit $\mathbf{u} := \lim_{\varepsilon \to 0} \mathbf{u}_\varepsilon$ away from I_t and its boundary condition on I_t.

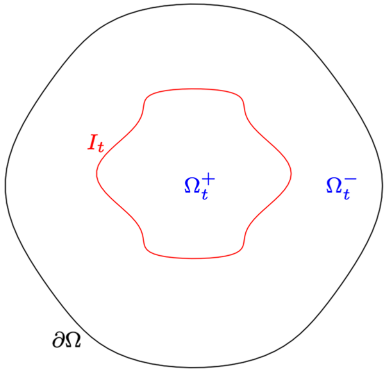


Fig. 1 I_t is the interface, Ω_t^+ is the nematic phase and Ω_t^- is the isotropic phase

System (1.2a) is a vectorial and anisotropic generalization of the scalar parabolic Allen–Cahn equation. In the scalar case, there have been many developments on its co-dimensional one limit to the (two-phase) mean curvature flow during the last two decades. Here we mention two classes of results and postpone the discussions of some others in the sequel. One is the convergence to a Brakke's flow by Ilmanen [34] using a version of Huisken's monotonicity formula [32] and tools from geometric measure theory. See also [11, 33, 47, 48, 50, 54] and the references therein for further renovations. Despite of its energetic nature, a major difficulty of such an approach is the control of the so-called *discrepancy measure*, and almost all existing literatures using this approach rely crucially on a version of Modica's maximum principle [46]. However, it is not clear whether Modica's maximum principle holds for elliptic/parabolic systems. Another approach, which relies more on parabolic comparison principle, is the global in time convergence towards the viscosity solution built by Chen–Giga–Goto [13] and independently by Evans–Spruck [20]. Such an approach has been implemented by Evans–Soner–Souganidis [19]. One can also refer to [6, 53] and the references therein for further discussions. These two approaches both give global in time (weak) convergences to weakly defined solutions of the mean curvature flow (up to their extinction times). However, as their technics involve parabolic maximum principle in one way or another, it is not clear how to use them to attack vectorial models in general. It is worth mentioning that for radially symmetric initial datum, Bronsard–Stoth [8] have obtained global in time convergence to the mean curvature flow of planar circles.

To the best of our knowledge, there are mainly two approaches to rigorously justify the convergence of the vectorial Allen–Cahn equations, both assuming that the limiting interface motion has a (local in time) classical solution. Compared with the aforementioned methods, which lead to global in time (weak) convergence, they have quite different natures. The first approach is the asymptotic expansion technics developed by De Mottoni–Schatzman [15] and by Alikakos–Bates–Chen [1]. It has been used recently by Fei–Wang–Zhang–Zhang [22] to study the isotropic-nematic phase transition in liquid crystals, and by Fei–Lin–Wang–Zhang [21] to study matrix-valued Allen–Cahn equations.

The second approach, which also assumes a classical solution of the limiting interface motion (but not the limiting flows in the bulk regions), is the modulated energy method developed by Fischer–Laux–Simon [24]. Such a method is motivated by Jerrard–Smets [37] and Fischer–Hensel [23], and has been generalized to a matrix–valued model by Laux–Liu [40].

In the present work, we shall use the methods employed in [24, 40] to derive the energy convergence (1.3) and the bulk limit $\mathbf{u} = \lim_{\varepsilon_k \to 0} \mathbf{u}_{\varepsilon_k}$ by establishing two modulated energy inequalities. Moreover, the derivation of the anchoring boundary condition of $\mathbf{u}$ (see (1.18c) below) uses a blow-up argument, which is inspired by a recent work of Lin–Wang [43]. There the authors have studied isotropic-nematic phase transitions in the static case based on an anisotropic Ericksen's model.

To state the main result, we assume that

$$I = \bigcup_{t \in [0,T]} I_t \times \{t\} \text{ is a smoothly evolving}\\ (d-1)\text{-dimensional submanifold in } \Omega, \tag{1.4}$$

starting from a $(d-1)$-dimensional submanifold $I_0 \subset \Omega$. Here a $(d-1)$-submanifold refers to an embedded closed smooth surface when $d = 3$ and curve when $d = 2$.

Let Ω_t^+ be the domain enclosed by I_t, and $d_I(x,t)$ be the signed-distance from x to I_t which takes negative values in Ω_t^-, and positive values in $\Omega_t^+ = \Omega \backslash \overline{\Omega_t^-}$. Equivalently,

$$\Omega_t^\pm := \{x \in \Omega \mid d_I(x,t) \gtrless 0\}. \tag{1.5}$$

For $\delta > 0$, the (open) δ-neighborhood of I_t is denoted by

$$B_\delta(I_t) := \{x \in \Omega \mid |d_I(x,t)| < \delta\}. \tag{1.6}$$

Let $\delta_0 \in (0,1)$ be a sufficiently small number so that the nearest point projection

$$P_I(\cdot,t) : B_{4\delta_0}(I_t) \to I_t$$

is smooth for any $t \in [0,T]$, and that the interface (1.4) stays at least $4\delta_0$ distant away from the physical boundary $\partial\Omega$. A further description of the geometry can be found in Sect. 2.2 or in [12].

The first step to study the singular limit of (1.2) is to construct a modulated energy which encodes a distance between the energy in (1.1) and an energy corresponding to the moving interface I_t in (1.4). Following [23, 24, 37], we define an extension of the inward normal vector $\mathbf{n}(\cdot,t)$ of I_t by

$$\boldsymbol{\xi}(x,t) := \phi\left(\frac{d_I(x,t)}{\delta_0}\right) \nabla d_I(x,t) \quad \text{for } x \in \Omega,$$

where $\phi \in C_c^2(\mathbb{R};[0,1])$ is an appropriate cut-off function (see (2.11) below for its precise definition). Now we introduce

$$E_\varepsilon[\mathbf{u}_\varepsilon|I](t) := \int_\Omega \frac{\varepsilon}{2}\mu|\operatorname{div}\mathbf{u}_\varepsilon(\cdot,t)|^2\,dx\\ + \int_\Omega \left(\frac{\varepsilon}{2}|\nabla \mathbf{u}_\varepsilon(\cdot,t)|^2 + \frac{1}{\varepsilon}F(\mathbf{u}_\varepsilon(\cdot,t)) - \boldsymbol{\xi}\cdot\nabla\psi_\varepsilon(\cdot,t)\right)dx, \tag{1.7}$$

where ψ_ε is defined by

$$\psi_\varepsilon(x,t) := \int_0^{|\mathbf{u}_\varepsilon(x,t)|} g(s)\,ds. \tag{1.8}$$

We shall work with a class of potentials $F(\mathbf{u})$ under standard assumptions (see e.g. [11, 34]). That is,

$$F(\mathbf{u}) = f(|\mathbf{u}|) = g^2(|\mathbf{u}|)/2, \tag{1.9}$$

where f is a double equal-well potential, namely,

$$f \in C^\infty(\mathbb{R}_{\geqslant 0}), \quad f(s) > 0 \text{ for } s \in \mathbb{R}_{\geqslant 0}\backslash\{0,1\}, \tag{1.10a}$$
$$g \geqslant 0 \text{ and is locally Lipschitz continuous}, \; g(0) = g(1) = 0. \tag{1.10b}$$

Moreover, the following structural assumptions on f are made:

$$\exists s_0 \in (0,1) \text{ s.t. } f'(s) > 0 \text{ on } (0, s_0) \text{ and } f'(s) < 0 \text{ on } (s_0, 1); \tag{1.11a}$$
$$f'(0) = f'(1) = 0, \qquad f''(0), f''(1) > 0; \tag{1.11b}$$
$$\exists c_0 \in (0,1) \text{ s.t. } 2c_0^2 s^2 \leqslant f(s) \leqslant 2c_0^{-2} s^2 \text{ for any } s \geqslant 100. \tag{1.11c}$$

After an appropriate modification for large $|s|$, the function $g(s) = |s||s^2 - 1|$, which corresponds to the Chern–Simons–Higgs potential, satisfies (1.11).

To control the bulk errors, we need another modulated energy:

$$B[\mathbf{u}_\varepsilon|I](t) := \int_\Omega \Big(\sigma\chi - \sigma + 2(\psi_\varepsilon - \sigma)^-\Big)\eta \circ d_I \; dx + \int_\Omega (\psi_\varepsilon - \sigma)^+ \, |\eta \circ d_I| \, dx \tag{1.12}$$

Here $\chi(\cdot, t) := \mathbf{1}_{\Omega_t^+} - \mathbf{1}_{\Omega_t^-}$, $h^\pm$ denote the positive/negative parts of a function h respectively, and η is a truncation of the identity function defined by

$$\eta(z) := \begin{cases} z & \text{when } z \in [-\delta_0, \delta_0], \\ \delta_0 & \text{when } z \geqslant \delta_0, \\ -\delta_0 & \text{when } z \leqslant -\delta_0. \end{cases} \tag{1.13}$$

Note that $(\eta \circ d_I)\, \chi \geqslant 0$ in Ω due to our convention on the signed-distance function, and thus the two integrands in (1.12) are both non-negative. We refer the readers to the proof of Theorem 4.1 below for more details on the positivity of (1.12).

Now we state the main result of this work:

Theorem 1.1 *Let $d \in \{2, 3\}$, and the assumptions (1.10) and (1.11) be in place. Assume that the moving interface I in (1.4) evolves under mean curvature flow, and the initial datum of (1.2) satisfies the following conditions:*

$$\mathbf{u}_\varepsilon^{in} \in W_0^{1,2}(\Omega), \tag{1.14a}$$
$$A_\varepsilon(\mathbf{u}_\varepsilon^{in}) \leqslant c_1, \tag{1.14b}$$
$$E_\varepsilon[\mathbf{u}_\varepsilon^{in}|I_0] + B[\mathbf{u}_\varepsilon^{in}|I_0] \leqslant c_1 \varepsilon, \tag{1.14c}$$

where $c_1 > 0$ is independent of ε. Then there exists $C_1 > 0$ independent of ε such that

$$\sup_{t\in[0,T]} E_\varepsilon[\mathbf{u}_\varepsilon|I](t) + \sup_{t\in[0,T]} B[\mathbf{u}_\varepsilon|I](t) \leqslant C_1 \varepsilon, \tag{1.15}$$

$$\sup_{t\in[0,T]} \int_\Omega |\psi_\varepsilon - \sigma \mathbf{1}_{\Omega_t^+}| \, dx \leqslant C_1 \varepsilon^{1/4}. \tag{1.16}$$

Moreover, up to extraction of a subsequence $\varepsilon_k \downarrow 0$,

$$\mathbf{u}_{\varepsilon_k} \xrightarrow{k\to\infty} \mathbf{1}_{\Omega_t^+}\mathbf{u} \;\; \textit{in} \;\; C([0,T]; L^2_{loc}(\Omega\backslash I_t)), \tag{1.17}$$

where $\mathbf{u}$ satisfies the following properties:

$$\mathbf{u} \in L^\infty(0,T; W^{1,6/5}(\Omega_t^+; \mathbb{S}^2)), \; \partial_t \mathbf{u} \in L^2(0,T; L^{6/5}(\Omega_t^+)), \tag{1.18a}$$
$$\mathbf{u}(x,t) = 0 \;\; \textit{for every} \;\; t \in [0,T] \; \textit{and for a.e.} \;\; x \in \Omega_t^-, \tag{1.18b}$$
$$(\mathbf{u}\cdot\mathbf{n})(x,t) = 0 \; \textit{for a.e.} \; t \in [0,T] \; \textit{and for} \; \mathcal{H}^{d-1}\textit{-a.e.} \; x \in I_t. \tag{1.18c}$$

Among the conditions in (1.14), the crucial one is (1.14c), which is used to obtain the inequalities in Theorem 3.1 and in Theorem 4.1 below. To construct an initial datum satisfying (1.14), we need the following result.

Proposition 1.1 *Let $I_0 \subset \Omega$ be a $(d-1)$-dimensional submanifold. For any vector field*

$$\mathbf{u}^{in} \in W^{1,2}(\Omega; \mathbb{S}^2) \text{ with } \mathbf{u}^{in}|_{I_0} \cdot \mathbf{n}_{I_0} = 0 \; a.e. \text{ on } I_0, \tag{1.19}$$

there exists $\mathbf{u}_\varepsilon^{in} \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} \mathbf{u}_\varepsilon^{in} = \mathbf{u}^{in} & \text{in } \Omega_0^+ \backslash B_{2\delta_0}(I_0), \\ \mathbf{u}_\varepsilon^{in} = 0 & \text{in } \Omega_0^- \backslash B_{2\delta_0}(I_0), \end{cases} \tag{1.20}$$

and (1.14) holds for a constant $c_1 > 0$ which only depends on I_0 and $\|\mathbf{u}^{in}\|_{W^{1,2}(\Omega)}$.

We comment on the conditions in (1.19). When $d = 3$, I_0 is a smooth closed surface in Ω. Due to topological obstructions, a vector field satisfying (1.19) is usually not smooth. For instance, when I_0 is diffeomorphic to a 2-sphere, due to the hairy ball theorem, $\mathbf{u}^{in}|_{I_0}$ must have (at least) one pole. One example of such a pole, which is often encountered in the theory of liquid crystal, is given by the hedgehog profile. Locally the tangent vector field near such a pole is C^1-equivalent to the mapping $\mathbf{h}(x) = x/|x| : B_1 \cap \mathbb{R}^2 \to \mathbb{S}^1$. Note that $\mathbf{h} \in W^{\frac{1}{2},2}(B_1 \cap \mathbb{R}^2)$ but $\mathbf{h} \notin W^{1,2}(B_1 \cap \mathbb{R}^2)$. When $d = 2$, there are fewer constraints to arrange a vector field $\mathbf{f} : I_0 \mapsto \mathbb{S}^2 \subset \mathbb{R}^3$ that is orthogonal to the planar curve $I_0 \subset \mathbb{R}^2 \times \{0\}$. In general, using the extension lemma of Hardt–Lin (cf. [42, Lemma 2.2.10]), any tangent vector field $\mathbf{f} \in W^{\frac{1}{2},2}(I_0; \mathbb{S}^2)$ has an extension $\mathbf{u}^{in}$ satisfying (1.19).

An immediate consequence of Theorem 1.1 is the convergence in (1.3). Indeed, it follows from (1.15) and (2.26b) below that $\int_\Omega \frac{\varepsilon}{2}\mu|\operatorname{div}\mathbf{u}|^2\,dx \xrightarrow{\varepsilon\to 0} 0$, and thus such an energy does not contribute to the surface energy in the limit. However, it forces $\mathbf{u}$ to satisfy the boundary condition (1.18c). Now applying integration by parts to the last term of (1.7), and then using (1.16) and $\boldsymbol{\xi}|_{\partial\Omega} = 0$, we find

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_\Omega \left(\frac{\varepsilon}{2}|\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon}F(\mathbf{u}_\varepsilon)\right)dx \\ &= \lim_{\varepsilon\to 0} -\int_\Omega (\operatorname{div}\boldsymbol{\xi})\psi_\varepsilon\,dx = -\sigma\int_{\Omega_t^+}(\operatorname{div}\boldsymbol{\xi})\,dx = \sigma\mathcal{H}^{d-1}(I_t). \end{aligned} \tag{1.21}$$

Note that the last step is due to the Green's formula.

Under additional assumptions, we can show that the limit $\mathbf{u}$ in (1.17) solves a geometric evolution equation in the bulk region $\Omega^+ := \bigcup_{t\in[0,T]}\Omega_t^+ \times \{t\}$.

Theorem 1.2 *Let $d = 2$ and the assumptions of Theorem 1.1 be in place. Assume further that*

$$\begin{aligned} &f(s) = s^2 \text{ for } s \leqslant 1/4;\; f(s) = (s-1)^2 \text{ for } s \geqslant 3/4; \\ &f(s) \geqslant 1/16 \text{ for } s \in [1/4, 3/4]; \\ &\sup_{s\in[1/4,3/4]} |f'(s)| \leqslant 4. \end{aligned} \tag{1.22}$$

Then there exists a sufficiently small $\mu > 0$ *(independent of* ε*) such that the vector field* $\mathbf{u}$ *in* (1.17) *satisfies*

$$\int_\Omega \partial_t \mathbf{u} \wedge \mathbf{u} \cdot \mathbf{\Psi}\, dx + \int_\Omega (\nabla \mathbf{u} \wedge \mathbf{u}) \cdot \nabla \mathbf{\Psi}\, dx \\ = \mu \int_\Omega (\operatorname{div} \mathbf{u}) \Big((\operatorname{rot} \mathbf{\Psi}) \cdot \mathbf{u} - (\operatorname{rot} \mathbf{u}) \cdot \mathbf{\Psi} \Big)\, dx \tag{1.23}$$

for almost every $t \in (0, T)$ *and for every* $\mathbf{\Psi} \in C_c^1(\Omega_t^+; \mathbb{R}^3)$.

In the above equation $\wedge$ is the wedge product in $\mathbb{R}^3$ and rot is the curl operator. The equation (1.23) is the weak formulation of an Oseen–Frank flow, written as

$$\partial_t \mathbf{u} = \Delta \mathbf{u} + \mu (\mathbb{I}_3 - \mathbf{u} \otimes \mathbf{u}) \nabla (\operatorname{div} \mathbf{u}) + |\nabla \mathbf{u}|^2 \mathbf{u}, \qquad \text{for } t \in (0, T], x \in \Omega_t^+. \tag{1.24}$$

It can be verified that when $\mathbf{u}$ is sufficiently regular, then (1.23) implies (1.24). It is worth mentioning that equation of the form (1.24) is the L^2-gradient flow of the variational problem

$$\inf \int_U \left(\mu |\operatorname{div} \mathbf{u}|^2 + |\nabla \mathbf{u}|^2 \right)\, dx, \tag{1.25}$$

where the infimum is taken among mappings $\mathbf{u} \in W^{1,2}(U; \mathbb{S}^2)$ fulfilling certain boundary conditions on ∂U. Note that (1.25) is a special case of the full Oseen–Frank model (cf. [30]).

This work will be organized as follows: In Sect. 2, we shall adapt the modulated energy method of [24] to the vectorial and anisotropic system (1.2), and then derive a differential inequality, i.e. Proposition 2.1. Such an inequality was previously derived in [40] for a matrix-valued equation. When applied to (1.2), it includes a term which does not have an obvious sign due to the additional div term. This problem will be solved in Sect. 3 during the proof of the inequality in Theorem 3.1. This theorem, which leads to the first part of Theorem 1.1, is a major novelty of the present work, and will be employed in Sect. 4 (see Theorem 4.1) to derive the L^1-estimate of ψ_ε in (1.16). Such an estimate will be used in Lemma 4.3 to identify appropriate level sets of ψ_ε which converge to I_t in certain sense. With this key lemma, we derive in Sect. 5 the anchoring boundary condition (1.18c), and thus finish the proof of Theorem 1.1. Section 6 is devoted to the proof of Theorem 1.2. The proof of Proposition 1.1 is quite similar to the construction given in [40]. We present a proof in Appendix A for the convenience of the readers.

2 Preliminaries

2.1 Notation and conventions

We shall adopt the following conventions throughout the paper. Unless specified otherwise, $C > 0$ is a generic constant whose value might change from line to line, and will depend on the geometry of the interface (1.4) but not on ε or $t \in [0, T]$. For two square matrices A and B, their Frobenius inner product is defined by $A : B := \operatorname{tr} A^{\mathsf{T}} B$, which induces the norm $|A| := \sqrt{\operatorname{tr} A^{\mathsf{T}} A}$. We shall also use the following notation for a vector-valued function

$\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ where

$$x = \begin{cases} (x_1, x_2, x_3) & \text{when } d = 3, \\ (x_1, x_2, 0) & \text{when } d = 2. \end{cases} \tag{2.1}$$

$$\begin{aligned} &\partial_0 = \partial_t, \quad \partial_i = \partial_{x_i} \quad 1 \leqslant i \leqslant 3, \\ &\nabla u_1 = (\partial_1 u_1, \partial_2 u_1, \partial_3 u_1), \quad \operatorname{div} \mathbf{u} = \sum_{i=1}^{3} \partial_i u_i, \\ &\operatorname{rot} \mathbf{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1). \end{aligned} \tag{2.2}$$

To ease computations when $d = 2$, $\nabla \mathbf{u}$ will be understood as the matrix $\begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & 0 \\ \partial_1 u_2 & \partial_2 u_2 & 0 \\ \partial_1 u_3 & \partial_2 u_3 & 0 \end{pmatrix}$, and

$$\begin{aligned} &\text{any planar vector field is understood as a} \\ &\text{3D vector field with vanishing 3rd component.} \end{aligned} \tag{2.3}$$

In particular, the latter applies to the normal and the mean curvature vector fields (cf. (2.9) and (2.13) respectively below). For a function of $\mathbf{u}$, like $F(\mathbf{u})$, its gradient will be denoted by

$$DF = (\partial_{u_1} F, \partial_{u_2} F, \partial_{u_3} F).$$

We end this section by the following assumptions regarding various constants. Theorem 1.1 will be proved for any fixed constant $\mu > 0$, while Theorem 1.2 is valid for a sufficiently small (fixed) μ. To simplify the presentation, we shall assume without loss of generality that

$$\mu \in (0, 1) \text{ is a fixed constant.} \tag{2.4}$$

Finally we can normalize g (cf. (1.9)) to have

$$\sigma := \int_0^1 g(s)\, ds = 1. \tag{2.5}$$

As the L^2-gradient flow of (1.1), the system (1.2) enjoys the following energy dissipation law:

$$A_\varepsilon(\mathbf{u}_\varepsilon(\cdot, \hat{T})) + \int_0^{\hat{T}} \int_\Omega \varepsilon |\partial_t \mathbf{u}_\varepsilon|^2 \, dxdt = A_\varepsilon(\mathbf{u}_\varepsilon^{in}(\cdot)) \tag{2.6}$$

for arbitrarily large time $\hat{T}$. Combining this with the theory of gradient flow and the regularity theory for elliptic system (cf. [4, 45]), one can construct a unique solution to system (1.2) that satisfies

$$\mathbf{u}_\varepsilon \in L^2(0, \hat{T}; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \text{ and } \partial_t \mathbf{u}_\varepsilon \in L^2(\Omega \times (0, \hat{T})).$$

So for almost every $\hat{t} \in (0, \hat{T})$, we have

$$\mathbf{u}_\varepsilon(\cdot, \hat{t}) \in W^{2,2}(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow C^{0,1/2}(\overline{\Omega}).$$

Under the assumption (1.11c), the nonlinearity of (1.2a) has a linear growth. So considering the system with initial datum $\mathbf{u}_\varepsilon(\cdot, \hat{t})$, and using the Hölder estimates for parabolic system (cf. [51]), we deduce that

$$\mathbf{u}_\varepsilon \text{ is a classical solution of (1.2a) in } \Omega \times (0, \hat{T}]. \tag{2.7}$$

For initial datum undergoing phase transitions near the initial interface I_0, formal asymptotic analysis suggests that $\nabla \mathbf{u}_\varepsilon$ will be singular near I_t. However, the global dissipation law (2.6) is not sufficient to yield the (strong) convergence of $\mathbf{u}_\varepsilon$, not even in the domain away from I_t. Following a recent work of Fisher et al. [24], we shall establish in this section a differential inequality which modulates the concentration and leads to the compactness of solutions in Sobolev spaces.

2.2 The modulated energy

We first set up the geometry of the moving interface I defined in (1.4). Under a local parametrization $\boldsymbol{\varphi}_t(s) : U \subset \mathbb{R}^{d-1} \to I_t$, the mean curvature flow reads

$$\partial_t \boldsymbol{\varphi}_t(s) = \kappa \mathbf{n} \tag{2.8}$$

where $\kappa = \kappa(\boldsymbol{\varphi}_t(s), t)$ is the mean curvature and $\mathbf{n} = \mathbf{n}(\cdot, t) : I_t \mapsto \mathbb{S}^{d-1}$ is the inward normal vector. For any $t \in [0, T]$ we assume that the nearest-point projection $P_I(\cdot, t) : B_{4\delta_0}(I_t) \mapsto I_t$ is smooth for some sufficiently small $\delta_0 \in (0, 1)$ which only depends on the geometry of I. Analytically we have $P_I(x, t) = x - \nabla d_I(x, t) d_I(x, t)$. So for each fixed $t \in [0, T]$, any point $x \in B_{4\delta_0}(I_t)$ corresponds to a unique pair (r, s) with $r = d_I(x, t)$ and $s \in U$, and the identity

$$d_I\Big(\boldsymbol{\varphi}_t(s) + r\mathbf{n}(\boldsymbol{\varphi}_t(s), t), t\Big) = r$$

holds with independent variables (r, s, t). Differentiating this identity with respect to r and t leads to the following identities:

$$\begin{aligned} \nabla d_I(x, t) &= \mathbf{n}(P_I(x, t), t), \\ -\partial_t d_I(x, t) &= \partial_t \boldsymbol{\varphi}_t(s) \cdot \mathbf{n}(\boldsymbol{\varphi}_t(s), t) =: V(s, t). \end{aligned} \tag{2.9}$$

The significance of these equations is that they extend the normal vector and the normal velocity from I_t to a neighborhood of it. So we shall also use $\mathbf{n}$ to denote ∇d_I when the latter is smooth. We shall extend $\mathbf{n}$ to the whole computational domain Ω by defining

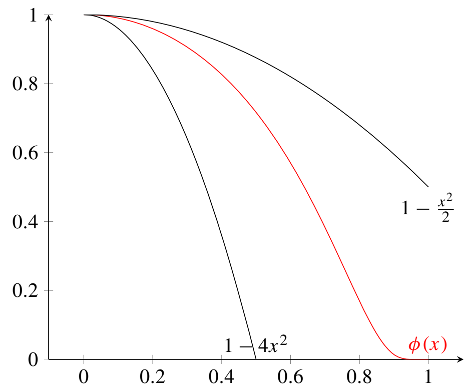
$$\boldsymbol{\xi}(x, t) := \phi\left(\frac{d_I(x, t)}{\delta_0}\right) \nabla d_I(x, t) \tag{2.10}$$

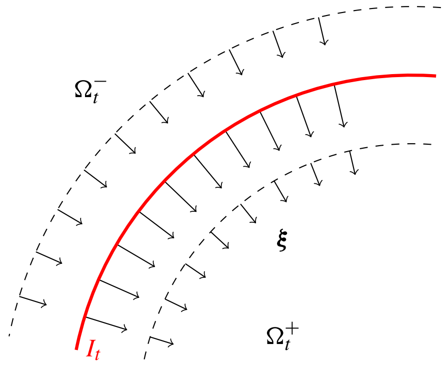
where $\phi : \mathbb{R} \mapsto \mathbb{R}_+$ is an even, smooth function that decreases on $[0, 1]$, and satisfies

$$\begin{cases} \phi(z) > 0 & \text{for } |z| < 1, \\ \phi(z) = 0 & \text{for } |z| \geqslant 1, \\ 1 - 4z^2 \leqslant \phi(z) \leqslant 1 - \frac{1}{2}z^2 & \text{for } |z| \leqslant 1/2. \end{cases} \tag{2.11}$$

To fulfill these requirements, we can simply choose

$$\phi(z) = \begin{cases} e^{\frac{1}{z^2-1}+1} & \text{for } |z| < 1, \\ 0 & \text{for } |z| \geqslant 1. \end{cases}$$





We proceed with the extension of the mean curvature. Choosing a cut-off function $\eta_0(x,t)$ such that

$$\eta_0(\cdot,t) \in C_c^\infty(B_{2\delta_0}(I_t);[0,1]) \text{ and } \eta_0 \equiv 1 \text{ in } B_{\delta_0}(I_t), \tag{2.12}$$

we constantly extend the inward mean curvature vector by defining

$$\mathbf{H}(x,t) := \kappa \nabla d_I(x,t) \quad \text{with} \quad \kappa(x,t) = -\Delta d_I(P_I(x,t))\eta_0(x,t). \tag{2.13}$$

These combined with (2.10) imply that

$$(\mathbf{n}\cdot\nabla)\mathbf{H} = 0 \text{ in } B_{\delta_0}(I_t), \tag{2.14a}$$
$$(\boldsymbol{\xi}\cdot\nabla)\mathbf{H} = 0 \text{ in } \Omega, \tag{2.14b}$$
$$\boldsymbol{\xi} = 0 \text{ and } \mathbf{H} = 0 \text{ on } \partial\Omega. \tag{2.14c}$$

Lemma 2.1 *There exists a constant $C>0$ depending only on the geometry of the interface* (1.4) *such that the following properties hold for every $t\in[0,T]$:*

$$|\nabla\cdot\boldsymbol{\xi} + \mathbf{H}\cdot\boldsymbol{\xi}| \leqslant C\,|d_I| \quad \text{in } B_{\delta_0}(I_t), \tag{2.15a}$$
$$\partial_t d_I + (\mathbf{H}\cdot\nabla)d_I = 0 \quad \text{in } B_{\delta_0}(I_t), \tag{2.15b}$$
$$\partial_t\boldsymbol{\xi} + (\mathbf{H}\cdot\nabla)\,\boldsymbol{\xi} + (\nabla\mathbf{H})^{\mathsf{T}}\,\boldsymbol{\xi} = 0 \quad \text{in } B_{\delta_0}(I_t), \tag{2.15c}$$

where $\nabla\mathbf{H} := \{\partial_j H_i\}_{1\leqslant i,j\leqslant 3}$ is a matrix with i being the row index.

Proof By introducing $\phi_0(\tau) := \phi(\frac{\tau}{\delta_0})$, we can rewrite (2.10) as $\boldsymbol{\xi} = \phi_0(d_I)\nabla d_I$. Since ϕ is even, we have $\phi_0'(0) = 0$. This combined with Taylor's expansion in d_I implies that

$$\begin{aligned}\nabla\cdot\boldsymbol{\xi} &= |\nabla d_I|^2\phi_0'(d_I) + \phi_0(d_I)\Delta d_I(x,t)\\ &= \qquad O(d_I) + \phi_0(d_I)\Delta d_I(P_I(x,t),t).\end{aligned}$$

This and (2.13) lead to (2.15a). Using (2.9) and (2.13), we can write (2.8) as the transport equation (2.15b), which leads to the following identities in $B_{\delta_0}(I_t)$:

$$\begin{aligned}\partial_t\nabla d_I + (\mathbf{H}\cdot\nabla)\nabla d_I + (\nabla\mathbf{H})^{\mathsf{T}}\nabla d_I &= 0,\\ \partial_t\phi_0(d_I) + (\mathbf{H}\cdot\nabla)\phi_0(d_I) &= 0.\end{aligned}$$

These two equations together imply (2.15c). □

It will be convenient to introduce

$$\psi_\varepsilon = \mathrm{d}^F\circ\mathbf{u}_\varepsilon \quad \text{where } \mathrm{d}^F(\mathbf{v}) := \int_0^{|\mathbf{v}|} g(s)\,ds. \tag{2.16}$$

It can be verified using (1.10b) that

$$\mathrm{d}^F(\mathbf{v})\in C^1(\mathbb{R}^3), \quad \text{and} \quad D\mathrm{d}^F(\mathbf{v}) = 0 \text{ iff } \mathbf{v}\in\{0,\mathbb{S}^2\}. \tag{2.17}$$

By (1.9) we have

$$|D\mathrm{d}^F(\mathbf{v})| = \sqrt{2F(\mathbf{v})}, \qquad \forall\mathbf{v}\in\mathbb{R}^3. \tag{2.18}$$

Recalling (2.7), we have

$$\partial_i\psi_\varepsilon(x,t) = \partial_i\mathbf{u}_\varepsilon(x,t)\cdot D\mathrm{d}^F(\mathbf{u}_\varepsilon(x,t)) \quad \text{for any } (x,t)\in\Omega\times(0,T], \tag{2.19a}$$

$$\nabla\psi_\varepsilon(x,t) = \nabla|\mathbf{u}_\varepsilon(x,t)|\,g(|\mathbf{u}_\varepsilon(x,t)|) \quad \text{if } \mathbf{u}_\varepsilon(x,t)\neq 0. \tag{2.19b}$$

Now we define the phase-field analogues of the normal vector and the mean curvature vector respectively by

$$\mathbf{n}_\varepsilon(x,t) := \begin{cases}\frac{\nabla\psi_\varepsilon}{|\nabla\psi_\varepsilon|}(x,t) & \text{if } \nabla\psi_\varepsilon(x,t)\neq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{2.20a}$$

$$\mathbf{H}_\varepsilon(x,t) := \begin{cases}-\left(\varepsilon\Delta\mathbf{u}_\varepsilon - \frac{1}{\varepsilon}DF(\mathbf{u}_\varepsilon)\right)\cdot\frac{\nabla\mathbf{u}_\varepsilon}{|\nabla\mathbf{u}_\varepsilon|} & \text{if } \nabla\mathbf{u}_\varepsilon\neq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{2.20b}$$

Note that in (2.20b), the inner product is made with the column vectors of $\nabla\mathbf{u}_\varepsilon = (\partial_1\mathbf{u}_\varepsilon, \partial_2\mathbf{u}_\varepsilon, \partial_3\mathbf{u}_\varepsilon)$. We deduce from (2.20a) that

$$\nabla\psi_\varepsilon = |\nabla\psi_\varepsilon|\mathbf{n}_\varepsilon \quad \text{for any } (x,t). \tag{2.21}$$

Define also the orthogonal projection $\Pi_{\mathbf{u}_\varepsilon}$ by

$$\Pi_{\mathbf{u}_\varepsilon}\partial_i\mathbf{u}_\varepsilon := \begin{cases}\left(\partial_i\mathbf{u}_\varepsilon\cdot\frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|}\right)\frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|} & \text{if } \mathbf{u}_\varepsilon\neq 0,\\ 0, & \text{otherwise.}\end{cases} \tag{2.22}$$

Lemma 2.2 *The following equations hold:*

$$|\nabla\psi_\varepsilon| = |\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon||D\mathrm{d}^F(\mathbf{u}_\varepsilon)| \quad \textit{for any } (x,t), \tag{2.23a}$$

$$\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon = \frac{|\nabla\psi_\varepsilon|}{|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2}D\mathrm{d}^F(\mathbf{u}_\varepsilon)\otimes\mathbf{n}_\varepsilon \quad \textit{on } \{x \mid |\mathbf{u}_\varepsilon|\notin\{0,1\}\}. \tag{2.23b}$$

Proof Concerning (2.23a), it suffices to work with the set $\{x \mid |\mathbf{u}_\varepsilon| \notin \{0,1\}\}$ where $g(|\mathbf{u}_\varepsilon|) > 0$ (cf. (1.10)), for otherwise the equation will follow from (2.17) and (2.19a). On this set we deduce from (2.17) that $D\mathrm{d}^F(\mathbf{u}_\varepsilon) = \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|} g(|\mathbf{u}_\varepsilon|) \neq 0$, and we can rewrite (2.19a) as

$$\partial_i \psi_\varepsilon = \partial_i \mathbf{u}_\varepsilon \cdot \frac{D\mathrm{d}^F(\mathbf{u}_\varepsilon)}{|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|} |D\mathrm{d}^F(\mathbf{u}_\varepsilon)| = \partial_i \mathbf{u}_\varepsilon \cdot \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|} |D\mathrm{d}^F(\mathbf{u}_\varepsilon)|. \tag{2.24}$$

This combined with (2.22) implies (2.23a).

Now we turn to the proof of (2.23b). On the set $\{x \mid |\mathbf{u}_\varepsilon| \notin \{0,1\}\}$, we have

$$\frac{|\nabla \psi_\varepsilon|}{|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2} D\mathrm{d}^F(\mathbf{u}_\varepsilon) \otimes \mathbf{n}_\varepsilon \overset{(2.21)}{=} \frac{D\mathrm{d}^F(\mathbf{u}_\varepsilon)}{|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2} \otimes \nabla \psi_\varepsilon \overset{(2.19b)}{=} \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|} \otimes \nabla |\mathbf{u}_\varepsilon|, \tag{2.25}$$

and this implies (2.23b) in view of (2.22). □

The following lemma establishes coercivity properties of the modulated energy (1.7).

Lemma 2.3 *The following estimates hold for every $t \in [0,T]$:*

$$\int_\Omega \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - |\nabla \psi_\varepsilon| \right) dx \leqslant E_\varepsilon[\mathbf{u}_\varepsilon | I], \tag{2.26a}$$

$$\varepsilon \int_\Omega \left(\mu |\operatorname{div} \mathbf{u}_\varepsilon|^2 + \left| \nabla \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon \right|^2 \right) dx \leqslant 2 E_\varepsilon[\mathbf{u}_\varepsilon | I], \tag{2.26b}$$

$$\int_\Omega \left(\sqrt{\varepsilon} \left| \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon \right| - \frac{1}{\sqrt{\varepsilon}} \left| D\mathrm{d}^F(\mathbf{u}_\varepsilon) \right| \right)^2 dx \leqslant 2 E_\varepsilon[\mathbf{u}_\varepsilon | I], \tag{2.26c}$$

$$\int_\Omega \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) + |\nabla \psi_\varepsilon| \right) (1 - \boldsymbol{\xi} \cdot \mathbf{n}_\varepsilon) \, dx \leqslant 4 E_\varepsilon[\mathbf{u}_\varepsilon | I], \tag{2.26d}$$

$$\int_\Omega \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) + |\nabla \psi_\varepsilon| \right) \min\left(d_I^2, 1\right) dx \leqslant C E_\varepsilon[\mathbf{u}_\varepsilon | I] \tag{2.26e}$$

where $C = C(\delta_0, \phi)$.

Proof The case when $\mu = 0$ has been done in [40], and the proof carries over to the present case. First, it follows from (2.22) that

$$\left| \nabla \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon \right|^2 + \left| \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon \right|^2 = |\nabla \mathbf{u}_\varepsilon|^2. \tag{2.27}$$

Combining this with (2.21), we can write

$$\begin{aligned}
&\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \\
=&\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - |\nabla \psi_\varepsilon| + |\nabla \psi_\varepsilon| (1 - \boldsymbol{\xi} \cdot \mathbf{n}_\varepsilon) \\
=&\frac{\varepsilon}{2} \left| \nabla \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon \right|^2 + \left(\frac{\varepsilon}{2} \left| \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon \right|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - |\nabla \psi_\varepsilon| \right) \\
&+ |\nabla \psi_\varepsilon| (1 - \boldsymbol{\xi} \cdot \mathbf{n}_\varepsilon).
\end{aligned} \tag{2.28}$$

By (2.18) and (2.23a), the second term in the last display is non-negative. Since $|\boldsymbol{\xi}| \leqslant 1$, we also have (2.26a), (2.26b), (2.26c) and

$$E_\varepsilon[\mathbf{u}_\varepsilon | I] \geqslant \int_\Omega (1 - \boldsymbol{\xi} \cdot \mathbf{n}_\varepsilon) |\nabla \psi_\varepsilon| \, dx. \tag{2.29}$$

Combining (2.29) with (2.26a) and the inequality $1 - \boldsymbol{\xi} \cdot \mathbf{n}_\varepsilon \leqslant 2$, we obtain (2.26d). Finally, by (2.11) and $\delta_0 \in (0, 1)$ we have

$$1 - \boldsymbol{\xi} \cdot \mathbf{n}_\varepsilon \geqslant 1 - \phi\left(\frac{d_I}{\delta_0}\right) \geqslant \min\left(\frac{d_I^2}{2\delta_0^2}, 1 - \phi(\tfrac{1}{2})\right) \geqslant C \min(d_I^2, 1). \tag{2.30}$$

This together with (2.26d) implies (2.26e). □

The following result was first proved in [24] for the scalar Allen-Cahn equation, and was generalized to the vectorial case in [40].

Proposition 2.1 *There exists a generic constant $C > 0$ depending only on the geometry of the interface (1.4) such that*

$$\frac{d}{dt} E_\varepsilon[\mathbf{u}_\varepsilon|I] + \frac{1}{2\varepsilon}\int_\Omega \left(\varepsilon^2 |\partial_t \mathbf{u}_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2\right) dx + \frac{1}{2\varepsilon}\int_\Omega \left|\varepsilon \partial_t \mathbf{u}_\varepsilon - (\nabla \cdot \boldsymbol{\xi}) D\mathrm{d}^F(\mathbf{u}_\varepsilon)\right|^2 dx$$
$$+ \frac{1}{2\varepsilon}\int_\Omega \left|\mathbf{H}_\varepsilon - \varepsilon|\nabla \mathbf{u}_\varepsilon|\mathbf{H}\right|^2 dx \leqslant C E_\varepsilon[\mathbf{u}_\varepsilon|I] \quad \text{for } t \in (0, T]. \tag{2.31}$$

We present a proof of (2.31) in Appendix B for the convenience of the readers.

3 Uniform estimates of solutions

Observe that the second term on the left-hand side of (2.31) does not have an obvious sign. However, we have the following theorem.

Theorem 3.1 *Under the assumptions of Theorem 1.1, there exists a constant $C_0 > 0$, which depends only on the geometry of the interface (1.4) and c_1 (cf. (1.14c)), such that*

$$\sup_{t\in[0,T]} \frac{1}{\varepsilon} E_\varepsilon[\mathbf{u}_\varepsilon|I] + \int_0^T \int_\Omega \left(\left|\partial_t \mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\right|^2 + \left|\partial_t \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon}\partial_t \mathbf{u}_\varepsilon\right|^2\right) dxdt \leqslant C_0. \tag{3.1}$$

It is worth mentioning that C_0 is independent of μ. The proof of (3.1) relies on the following lemma.

Lemma 3.2 *For any function η_1 with $\eta_1(\cdot, t) \in C_c(B_{4\delta_0}(I_t); \mathbb{R}_{\geqslant 0})$, there exists a universal constant $C > 0$ which is independent of t and ε such that*

$$\int_\Omega \eta_1 \left|\nabla \mathbf{u}_\varepsilon (\mathbb{I}_3 - \mathbf{n}\otimes\mathbf{n})\right|^2 dx \leqslant C\varepsilon^{-1} E_\varepsilon[\mathbf{u}_\varepsilon|I](t) \qquad \forall t \in [0, T]. \tag{3.2}$$

Proof On the set $\{x \mid g(|\mathbf{u}_\varepsilon|) > 0\} = \{x \mid |\mathbf{u}_\varepsilon| \notin \{0, 1\}\}$ we can use (2.23b) and (2.23a) to estimate

$$\begin{aligned}
&\left|\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon (\mathbb{I}_3 - \mathbf{n}_\varepsilon \otimes \boldsymbol{\xi})\right|^2 \\
&= \left|\frac{|\nabla \psi_\varepsilon|}{|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2} D\mathrm{d}^F(\mathbf{u}_\varepsilon) \otimes (\mathbf{n}_\varepsilon - \boldsymbol{\xi})\right|^2 \\
&\leqslant |\mathbf{n}_\varepsilon - \boldsymbol{\xi}|^2 \left|\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon\right|^2 \\
&\leqslant 2(1 - \boldsymbol{\xi}\cdot\mathbf{n}_\varepsilon)\, |\nabla \mathbf{u}_\varepsilon|^2 .
\end{aligned}$$

On the set $\{x \mid |\mathbf{u}_\varepsilon| = 0\}$ we have $\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon = 0$ by the second case in (2.22). On the open set $\{x \mid |\mathbf{u}_\varepsilon| > 0\} \supset \{x \mid |\mathbf{u}_\varepsilon| = 1\}$ we can write $\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon = \nabla |\mathbf{u}_\varepsilon| \otimes \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|}$ by the first case in (2.22). This combined with [18, Theorem 4.4] implies that $\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon = 0$ for a.e. $x \in \{x \mid |\mathbf{u}_\varepsilon| = 1\}$. Altogether we have shown that

$$\left|\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon (\mathbb{I}_3 - \mathbf{n}_\varepsilon \otimes \boldsymbol{\xi})\right|^2 \leqslant 2(1 - \boldsymbol{\xi} \cdot \mathbf{n}_\varepsilon) \, |\nabla \mathbf{u}_\varepsilon|^2 \quad \text{a.e. in } \Omega. \tag{3.3}$$

This together with (2.26d) implies

$$\int_\Omega \left|\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon (\mathbb{I}_3 - \mathbf{n}_\varepsilon \otimes \boldsymbol{\xi})\right|^2 dx \leqslant C\varepsilon^{-1} E_\varepsilon[\mathbf{u}_\varepsilon | I]. \tag{3.4}$$

In $B_{4\delta_0}(I_t)$ where $\mathbf{n} = \nabla d_I$, we have the decomposition

$$\mathbb{I}_3 - \mathbf{n}_\varepsilon \otimes \mathbf{n} = \mathbb{I}_3 - \mathbf{n}_\varepsilon \otimes \boldsymbol{\xi} + \mathbf{n}_\varepsilon \otimes (\boldsymbol{\xi} - \mathbf{n}). \tag{3.5}$$

Using (2.10) and (2.11), we can estimate the last term by

$$\begin{aligned} |\boldsymbol{\xi} - \mathbf{n}|^2 &= |\mathbf{n}_\varepsilon \otimes (\boldsymbol{\xi} - \mathbf{n})|^2 \\ &\leqslant 2|\boldsymbol{\xi} - \mathbf{n}| = 2\left(1 - \phi(\tfrac{d_I}{\delta_0})\right) \leqslant C \min\left(d_I^2, 1\right). \end{aligned} \tag{3.6}$$

These inequalities and (2.26e) lead to

$$\int_\Omega \eta_1 \left|\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon (\mathbb{I}_3 - \mathbf{n}_\varepsilon \otimes \mathbf{n})\right|^2 dx \leqslant C\varepsilon^{-1} E_\varepsilon[\mathbf{u}_\varepsilon | I]. \tag{3.7}$$

Now using (3.6), (2.26d) and (2.26e) we find

$$\int_\Omega \eta_1 |\nabla \mathbf{u}_\varepsilon|^2 \left(|\mathbf{n}_\varepsilon - \boldsymbol{\xi}|^2 + |\boldsymbol{\xi} - \mathbf{n}|^2\right) dx \leqslant C\varepsilon^{-1} E_\varepsilon[\mathbf{u}_\varepsilon | I].$$

The above two estimates together with the formula

$$(\mathbb{I}_3 - \mathbf{n} \otimes \mathbf{n}) - (\mathbb{I}_3 - \mathbf{n}_\varepsilon \otimes \mathbf{n}) = (\mathbf{n}_\varepsilon - \boldsymbol{\xi}) \otimes \mathbf{n} + (\boldsymbol{\xi} - \mathbf{n}) \otimes \mathbf{n}$$

yield (3.2). □

To proceed we need an L^3-estimate of $\mathbf{u}_\varepsilon$.

Lemma 3.3 *Under the assumption* (1.14b), *there exists a constant* $C = C(c_1) > 0$ *such that*

$$\sup_{t \in [0,T]} A_\varepsilon(\mathbf{u}_\varepsilon(\cdot, t)) + \sup_{t \in [0,T]} \|\nabla \psi_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leqslant C, \tag{3.8a}$$

$$\sup_{t \in [0,T]} \|\mathbf{u}_\varepsilon(\cdot, t)\|_{L^3(\Omega)} \leqslant C. \tag{3.8b}$$

Proof It follows from (2.18), (2.23a) and the Cauchy–Schwarz inequality that

$$A_\varepsilon(\mathbf{u}_\varepsilon) \geqslant \int_\Omega \left(\frac{\varepsilon}{2} |\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{2\varepsilon} |D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2\right) dx \geqslant \int_\Omega |\nabla \psi_\varepsilon| \, dx.$$

This and (2.6) lead to (3.8a). To prove (3.8b), we first note that if $|\mathbf{u}_\varepsilon| > 2$, then

$$\psi_\varepsilon = \int_0^2 g(z) \, dz + \int_2^{|\mathbf{u}_\varepsilon|} g(z) \, dz \overset{(1.11c)}{\geqslant} c_0(|\mathbf{u}_\varepsilon|^2 - 4).$$

This combined with Sobolev's embedding and $\psi_\varepsilon|_{\partial\Omega} = 0$ (cf. (1.2c)) leads to

$$\int_\Omega |\mathbf{u}_\varepsilon|^3\,dx \leqslant C + \int_{\{x\in\Omega||\mathbf{u}_\varepsilon|>2\}} |\mathbf{u}_\varepsilon|^3\,dx \\ \leqslant C\left(1 + \|\psi_\varepsilon\|^{3/2}_{L^{3/2}(\Omega)}\right) \\ \leqslant C\left(1 + \|\nabla\psi_\varepsilon\|^{3/2}_{L^1(\Omega)}\right).$$

□

Proof of Theorem 3.1 We shall only present the proof in 3D because the 2D case is analogous under the conventions made in Sect. 2.1. We shall employ Einstein summation notation by summing over repeated Latin indices.

We first use (2.31) to get

$$\frac{2}{\varepsilon}\frac{d}{dt}E_\varepsilon[\mathbf{u}_\varepsilon|I] + \frac{1}{\varepsilon^2}\int_\Omega\left[\left(\varepsilon^2\,|\partial_t\mathbf{u}_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2\right) + \left|\mathbf{H}_\varepsilon - \varepsilon|\nabla\mathbf{u}_\varepsilon|\mathbf{H}\right|^2\right]dx \\ + \frac{1}{\varepsilon^2}\int_\Omega\left|\varepsilon\partial_t\mathbf{u}_\varepsilon - D\mathrm{d}^F(\mathbf{u}_\varepsilon)(\nabla\cdot\boldsymbol{\xi})\right|^2 dx \leqslant \frac{C}{\varepsilon}E_\varepsilon[\mathbf{u}_\varepsilon|I]. \tag{3.9}$$

Observe that the orthogonal projection (2.22) is parallel to $D\mathrm{d}^F(\mathbf{u}_\varepsilon)$ when it does not vanish. So we can write

$$\left|\varepsilon\partial_t\mathbf{u}_\varepsilon - D\mathrm{d}^F(\mathbf{u}_\varepsilon)(\nabla\cdot\boldsymbol{\xi})\right|^2 \\ = \left|\varepsilon\partial_t\mathbf{u}_\varepsilon - \varepsilon\Pi_{\mathbf{u}_\varepsilon}\partial_t\mathbf{u}_\varepsilon\right|^2 + \left|\varepsilon\Pi_{\mathbf{u}_\varepsilon}\partial_t\mathbf{u}_\varepsilon - D\mathrm{d}^F(\mathbf{u}_\varepsilon)(\nabla\cdot\boldsymbol{\xi})\right|^2.$$

Substituting this identity into (3.9) we find

$$\frac{2}{\varepsilon}\frac{d}{dt}E_\varepsilon[\mathbf{u}_\varepsilon|I] + \frac{1}{\varepsilon^2}\int_\Omega\left[\left(\varepsilon^2\,|\partial_t\mathbf{u}_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2\right) + \left|\mathbf{H}_\varepsilon - \varepsilon|\nabla\mathbf{u}_\varepsilon|\mathbf{H}\right|^2\right]dx \\ + \int_\Omega\left|\partial_t\mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon}\partial_t\mathbf{u}_\varepsilon\right|^2 dx \leqslant \frac{C}{\varepsilon}E_\varepsilon[\mathbf{u}_\varepsilon|I]. \tag{3.10}$$

To estimate the second term on the left-hand side, we use (1.2a) and (2.20b) to write

$$\mathbf{H}_\varepsilon = -\varepsilon\Big(\partial_t\mathbf{u}_\varepsilon - \mu\nabla\,\mathrm{div}\,\mathbf{u}_\varepsilon\Big)\cdot\frac{\nabla\mathbf{u}_\varepsilon}{|\nabla\mathbf{u}_\varepsilon|} \quad \text{if } \nabla\mathbf{u}_\varepsilon \neq 0. \tag{3.11}$$

Note that the inner product is made with the column vectors of $\nabla\mathbf{u}_\varepsilon = (\partial_1\mathbf{u}_\varepsilon, \partial_2\mathbf{u}_\varepsilon, \partial_3\mathbf{u}_\varepsilon)$. Using the above formula, we expand the integrands of (3.10) and find

$$\varepsilon^2\,|\partial_t\mathbf{u}_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2 + \left|\mathbf{H}_\varepsilon - \varepsilon|\nabla\mathbf{u}_\varepsilon|\mathbf{H}\right|^2 \\ = \varepsilon^2\,|\partial_t\mathbf{u}_\varepsilon|^2 + \varepsilon^2|\mathbf{H}|^2|\nabla\mathbf{u}_\varepsilon|^2 + 2\varepsilon^2\partial_t\mathbf{u}_\varepsilon\cdot(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon \\ - 2\varepsilon^2\mu\,\nabla(\mathrm{div}\,\mathbf{u}_\varepsilon)\cdot(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon \\ = \varepsilon^2|\partial_t\mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon|^2 + \varepsilon^2\left(|\mathbf{H}|^2|\nabla\mathbf{u}_\varepsilon|^2 - |(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon|^2\right) \\ - 2\varepsilon^2\mu\,\nabla(\mathrm{div}\,\mathbf{u}_\varepsilon)\cdot(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon.$$

Note that the second term in the last display is non-negative due to Cauchy-Schwarz's inequality, and this implies that

$$\begin{aligned}&\int_\Omega |\partial_t \mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon|^2\,dx\\&\quad\leqslant \frac{1}{\varepsilon^2}\int_\Omega\left[\left(\varepsilon^2\,|\partial_t\mathbf{u}_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2\right) + \Big|\mathbf{H}_\varepsilon - \varepsilon|\nabla\mathbf{u}_\varepsilon|\mathbf{H}\Big|^2\right]dx\\&\quad\quad + 2\mu\int_\Omega \nabla(\operatorname{div}\mathbf{u}_\varepsilon)\cdot(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\,dx.\end{aligned}$$

Adding the above inequality to (3.10) leads to

$$\begin{aligned}2\varepsilon^{-1}\frac{d}{dt}E_\varepsilon[\mathbf{u}_\varepsilon|I] + \int_\Omega \Big|\partial_t\mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\Big|^2 dx + \int_\Omega\Big|\partial_t\mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon}\partial_t\mathbf{u}_\varepsilon\Big|^2 dx\\ \leqslant C\varepsilon^{-1}E_\varepsilon[\mathbf{u}_\varepsilon|I] + 2\mu\int_\Omega\nabla(\operatorname{div}\mathbf{u}_\varepsilon)\cdot(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\,dx.\end{aligned}\tag{3.12}$$

To estimate the last term, we write $\mathbf{u}_\varepsilon = (u_i^\varepsilon)_{1\leqslant i\leqslant 3}$ and $\mathbf{H} = (H_i)_{1\leqslant i\leqslant 3}$. Using integration by parts and (2.14c), we obtain

$$\begin{aligned}&\int_\Omega \nabla(\operatorname{div}\mathbf{u}_\varepsilon)\cdot(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\,dx\\&\quad= -\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)(\mathbf{H}\cdot\nabla)\operatorname{div}\mathbf{u}_\varepsilon\,dx - \int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)(\partial_j\mathbf{H}\cdot\nabla)u_j^\varepsilon\,dx\\&\quad= \frac{1}{2}\int_\Omega(\operatorname{div}\mathbf{H})(\operatorname{div}\mathbf{u}_\varepsilon)^2\,dx - \int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)\partial_k H_j\partial_k u_j^\varepsilon\,dx\\&\quad\quad-\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)(\partial_j H_k - \partial_k H_j)\partial_k u_j^\varepsilon\,dx.\end{aligned}\tag{3.13}$$

In view of (2.26b), the first integral in the last display of (3.13) is bounded by

$$\mu^{-1}\varepsilon^{-1}\|\operatorname{div}\mathbf{H}\|_{L^\infty_{t,x}}E_\varepsilon[\mathbf{u}_\varepsilon|I].$$

The second integral can be estimated by decomposing ∇u_j^ε and by using (2.14a):

$$\begin{aligned}&-\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)\,\nabla H_j\cdot\nabla u_j^\varepsilon\,dx\\&\quad= -\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)\,\nabla H_j\cdot\left((\mathbb{I}_3 - \mathbf{n}\otimes\mathbf{n})\nabla u_j^\varepsilon\right)dx - \int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)\,(\mathbf{n}\cdot\nabla H_j)\left(\mathbf{n}\cdot\nabla u_j^\varepsilon\right)dx\\&\quad\leqslant \int_\Omega|\operatorname{div}\mathbf{u}_\varepsilon|^2\,dx + \int_\Omega|\nabla\mathbf{H}|^2\Big|(\mathbb{I}_3 - \mathbf{n}\otimes\mathbf{n})\nabla\mathbf{u}_\varepsilon\Big|^2 dx\\&\quad\quad + C\int_\Omega|\nabla\mathbf{u}_\varepsilon|^2\min\left(d_I^2, 1\right)\,dx.\end{aligned}\tag{3.14}$$

By (2.13) and (2.12), the second integral in the last display can be estimated using (3.2) with $\eta_1 := |\nabla\mathbf{H}|^2$. The other two terms can be controlled by $(\mu^{-1}+1)C\varepsilon^{-1}E_\varepsilon[\mathbf{u}_\varepsilon|I]$ using

(2.26b) and (2.26e) respectively. To summarize we deduce from (3.13) and (3.14) that

$$
\begin{aligned}
&\int_\Omega \nabla(\operatorname{div}\mathbf{u}_\varepsilon)\cdot(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\,dx\\
&\quad\leqslant \mu^{-1}\varepsilon^{-1}\|\operatorname{div}\mathbf{H}\|_{L^\infty_{t,x}}E_\varepsilon[\mathbf{u}_\varepsilon|I]+(\mu^{-1}+1)C\varepsilon^{-1}E_\varepsilon[\mathbf{u}_\varepsilon|I]\\
&\qquad-\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)(\partial_j H_k-\partial_k H_j)\partial_k u_j^\varepsilon\,dx.
\end{aligned}
$$

Combining this with (3.12), we find

$$
\begin{aligned}
2\varepsilon^{-1}\frac{d}{dt}E_\varepsilon[\mathbf{u}_\varepsilon|I]+\int_\Omega\left|\partial_t\mathbf{u}_\varepsilon+(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\right|^2dx+\int_\Omega\left|\partial_t\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\partial_t\mathbf{u}_\varepsilon\right|^2dx\\
\leqslant C\varepsilon^{-1}E_\varepsilon[\mathbf{u}_\varepsilon|I]-2\mu\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)(\partial_j H_k-\partial_k H_j)\partial_k u_j^\varepsilon\,dx. \qquad (3.15)
\end{aligned}
$$

Note that due to (2.4) the constant C above can be made independent of μ. It remains to estimate the last integral in (3.15). By orthogonal decompositions[1],

$$
(\partial_j H_k-\partial_k H_j)\partial_k u_j^\varepsilon=-(\operatorname{rot}\mathbf{u}_\varepsilon)\cdot(\operatorname{rot}\mathbf{H}).
$$

We also need the following identity which follows by taking the wedge product of (1.2a) with $\mathbf{u}_\varepsilon$.

$$
\mu(\nabla\operatorname{div}\mathbf{u}_\varepsilon)\wedge\mathbf{u}_\varepsilon=(\partial_t\mathbf{u}_\varepsilon-\Delta\mathbf{u}_\varepsilon)\wedge\mathbf{u}_\varepsilon.
$$

Using the above two identities, we integrate by parts to obtain

$$
\begin{aligned}
&-\mu\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)(\partial_j H_k-\partial_k H_j)\partial_k u_j^\varepsilon\,dx\\
&=\mu\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)(\operatorname{rot}\mathbf{u}_\varepsilon)\cdot(\operatorname{rot}\mathbf{H})\,dx\\
&=\mu\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\operatorname{rot}\mathbf{H})\,dx-\int_\Omega\mu(\nabla\operatorname{div}\mathbf{u}_\varepsilon)\wedge\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\mathbf{H})\,dx\\
&=\mu\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\operatorname{rot}\mathbf{H})\,dx-\int_\Omega(\partial_t\mathbf{u}_\varepsilon-\Delta\mathbf{u}_\varepsilon)\wedge\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\mathbf{H})\,dx\\
&=\mu\int_\Omega(\operatorname{div}\mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\operatorname{rot}\mathbf{H})\,dx-\int_\Omega\Big(\partial_t\mathbf{u}_\varepsilon+(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\Big)\wedge\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\mathbf{H})\,dx\\
&\quad+\int_\Omega(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\wedge\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\mathbf{H})\,dx+\int_\Omega\Delta\mathbf{u}_\varepsilon\wedge\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\mathbf{H})\,dx.
\end{aligned}
$$

[1] For a square matrix A, the decomposition $A=\frac{A+A^\mathsf{T}}{2}+\frac{A-A^\mathsf{T}}{2}$ is orthogonal under the Frobenius inner product $A:B\triangleq\operatorname{tr}(A^\mathsf{T}B)$.

Inserting this identity into (3.15), and using the Cauchy–Schwarz inequality, (3.8b) and (2.26b), we find

$$\begin{aligned}
&2\varepsilon^{-1}\frac{d}{dt}E_\varepsilon[\mathbf{u}_\varepsilon|I]+\frac{1}{2}\int_\Omega\left|\partial_t\mathbf{u}_\varepsilon+(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\right|^2dx+\int_\Omega\left|\partial_t\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\partial_t\mathbf{u}_\varepsilon\right|^2dx\\
&\quad\leqslant C\Big(1+\varepsilon^{-1}E_\varepsilon[\mathbf{u}_\varepsilon|I]\Big)+2\int_\Omega(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\wedge\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\mathbf{H})\,dx+2\int_\Omega\Delta\mathbf{u}_\varepsilon\wedge\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\mathbf{H})\,dx\\
&\quad=C\Big(1+\varepsilon^{-1}E_\varepsilon[\mathbf{u}_\varepsilon|I]\Big)+2\int_\Omega H_k\Big(\partial_k\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\partial_k\mathbf{u}_\varepsilon\Big)\wedge\mathbf{u}_\varepsilon\cdot(\operatorname{rot}\mathbf{H})\,dx\\
&\qquad-2\int_\Omega\Big(\partial_k\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\partial_k\mathbf{u}_\varepsilon\Big)\wedge\mathbf{u}_\varepsilon\cdot\Big(\partial_k\operatorname{rot}\mathbf{H}\Big)\,dx.
\end{aligned}\tag{3.16}$$

Note that in the last step we used integration by parts, the identity

$$(\Pi_{\mathbf{u}_\varepsilon}\partial_k\mathbf{u}_\varepsilon)\wedge\mathbf{u}_\varepsilon=0\tag{3.17}$$

which follows from (2.22), and the identities $(\partial_k\mathbf{u}_\varepsilon)\wedge(\partial_k\mathbf{u}_\varepsilon)=0$ for each fixed $k\in\{1,2,3\}$. Finally, applying the Cauchy–Schwarz inequality and then (2.26b) and (3.8b) in the last two integrals of (3.16), we find

$$\begin{aligned}
&2\varepsilon^{-1}\frac{d}{dt}E_\varepsilon[\mathbf{u}_\varepsilon|I]+\frac{1}{2}\int_\Omega\left|\partial_t\mathbf{u}_\varepsilon+(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\right|^2dx+\int_\Omega\left|\partial_t\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\partial_t\mathbf{u}_\varepsilon\right|^2dx\\
&\qquad\leqslant C\Big(1+\varepsilon^{-1}E_\varepsilon[\mathbf{u}_\varepsilon|I]\Big).
\end{aligned}\tag{3.18}$$

This combined with (1.14c) and Grönwall's inequality leads to (3.1). □

Using (2.26e) and (3.1), we readily obtain the following corollary.

Corollary 3.4 *Under the assumptions of Theorem 1.1, there exists a constant $C>0$, which depends only on the geometry of the interface (1.4) and c_1, such that*

$$\sup_{t\in[0,T]}\int_{\Omega_t^\pm\backslash B_\delta(I_t)}\left(|\nabla\mathbf{u}_\varepsilon|^2+\frac{1}{\varepsilon^2}F(\mathbf{u}_\varepsilon)+\frac{1}{\varepsilon}|\nabla\psi_\varepsilon|\right)dx\leqslant C\delta^{-2},\tag{3.19a}$$

$$\int_0^T\int_{\Omega_t^\pm\backslash B_\delta(I_t)}|\partial_t\mathbf{u}_\varepsilon|^2\,dxdt\leqslant C\delta^{-2},\tag{3.19b}$$

hold for each fixed $\delta\in(0,\delta_0)$.

Indeed, (3.19b) follows from (3.19a) and the inequality

$$\int_0^T\int_\Omega\left|\partial_t\mathbf{u}_\varepsilon+(\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\right|^2dxdt\leqslant C,\tag{3.20}$$

which is a consequence of (3.1). Another consequence of (3.1) is the following lemma concerning

$$\widehat{\mathbf{u}}_\varepsilon:=\begin{cases}\frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|} & \text{if }\mathbf{u}_\varepsilon\neq0,\\ 0 & \text{otherwise.}\end{cases}\tag{3.21}$$

Lemma 3.5 *Under the assumptions of Theorem 1.1, there exists a constant $C > 0$, which depends only on the geometry of the interface (1.4) and c_1, such that*

$$\sup_{t\in[0,T]}\int_\Omega |\mathbf{u}_\varepsilon|^2\,|\nabla\widehat{\mathbf{u}}_\varepsilon|^2\,dx+\sup_{t\in[0,T]}\int_\Omega\left|\widehat{\mathbf{u}}_\varepsilon\cdot\nabla|\mathbf{u}_\varepsilon|\right|^2dx\leqslant(1+\mu^{-1})C,\tag{3.22a}$$

$$\sup_{t\in[0,T]}\int_\Omega(\widehat{\mathbf{u}}_\varepsilon\cdot\mathbf{n}_\varepsilon)^2\,|\nabla\psi_\varepsilon|\,dx\leqslant(1+\mu^{-1})(1+\sqrt{\mu+1})C\varepsilon.\tag{3.22b}$$

Proof We first deduce from (3.1) and (2.26b) that

$$\sup_{t\in[0,T]}\int_\Omega\left(\mu|\operatorname{div}\mathbf{u}_\varepsilon|^2+\left|\nabla\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon\right|^2\right)dx\leqslant C.\tag{3.23}$$

By (3.21) we have the identity $\mathbf{u}_\varepsilon=|\mathbf{u}_\varepsilon|\widehat{\mathbf{u}}_\varepsilon$. Using this and (2.22), we can write

$$\nabla\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon=|\mathbf{u}_\varepsilon|\nabla\widehat{\mathbf{u}}_\varepsilon\ \text{ if }\mathbf{u}_\varepsilon\neq0.\tag{3.24}$$

Substituting this formula into (3.23), we obtain the estimate of the first integral on the left-hand side of (3.22a). To control the second one, we use the following formula which follows from (2.22):

$$\operatorname{tr}\nabla\mathbf{u}_\varepsilon-\operatorname{tr}\left(\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon\right)=\operatorname{div}\mathbf{u}_\varepsilon-\widehat{\mathbf{u}}_\varepsilon\cdot\nabla|\mathbf{u}_\varepsilon|\ \text{ if }\mathbf{u}_\varepsilon\neq0.\tag{3.25}$$

Note that on the set $\{x\mid|\mathbf{u}_\varepsilon|=0\}$, we have $\nabla|\mathbf{u}_\varepsilon|=0$ a.e., and thus the above formula is still valid. This and (3.23) yield the estimate of $\widehat{\mathbf{u}}_\varepsilon\cdot\nabla|\mathbf{u}_\varepsilon|$ and (3.22a) is proved.

Regarding (3.22b), it suffices to estimate over the set

$$\{x\mid\nabla\psi_\varepsilon\neq0\}=:U_\varepsilon$$

because the integral over its complement vanishes. By (2.17) and (2.19a), we have $U_\varepsilon\subset\{x\mid|\mathbf{u}_\varepsilon|\notin\{0,1\}\}$ where $g(|\mathbf{u}_\varepsilon|)=|D\mathrm{d}^F|(\mathbf{u}_\varepsilon)>0$. This combined with (2.19b) and (2.20a) implies that

$$\mathbf{n}_\varepsilon=\frac{\nabla\psi_\varepsilon}{|\nabla\psi_\varepsilon|}=\frac{\nabla|\mathbf{u}_\varepsilon|}{|\nabla|\mathbf{u}_\varepsilon||}\quad\text{on }\ U_\varepsilon.$$

On the other hand, by the polar decomposition $\mathbf{u}_\varepsilon=|\mathbf{u}_\varepsilon|\widehat{\mathbf{u}}_\varepsilon$ and orthogonality $\hat{\mathbf{u}}_\varepsilon\perp\partial_{x_j}\hat{\mathbf{u}}_\varepsilon$, we have

$$|\nabla\mathbf{u}_\varepsilon|^2=|\nabla|\mathbf{u}_\varepsilon||^2+|\mathbf{u}_\varepsilon|^2|\nabla\hat{\mathbf{u}}_\varepsilon|^2\geqslant|\nabla|\mathbf{u}_\varepsilon||^2\quad\text{on }\ U_\varepsilon.\tag{3.26}$$

Setting $\widehat{\mathbf{u}}_\varepsilon\cdot\mathbf{n}_\varepsilon=:\cos\theta_\varepsilon$, we have

$$\mu\int_{U_\varepsilon}\cos^2\theta_\varepsilon\left|\nabla|\mathbf{u}_\varepsilon|\right|^2dx=\mu\int_{U_\varepsilon}|\widehat{\mathbf{u}}_\varepsilon\cdot\mathbf{n}_\varepsilon|^2\left|\nabla|\mathbf{u}_\varepsilon|\right|^2dx\overset{(3.22a)}{\leqslant}(1+\mu)C.\tag{3.27}$$

This inequality, (2.26a) and (3.26) together imply that

$$\begin{aligned}(1+\mu)C&\geqslant\int_{U_\varepsilon}\frac{\mu}{2}\cos^2\theta_\varepsilon\left|\nabla|\mathbf{u}_\varepsilon|\right|^2dx+\int_{U_\varepsilon}\left(\frac12\left|\nabla|\mathbf{u}_\varepsilon|\right|^2+\frac{1}{\varepsilon^2}F(\mathbf{u}_\varepsilon)-\frac1\varepsilon|\nabla\psi_\varepsilon|\right)dx\\&\geqslant\frac1\varepsilon\int_{U_\varepsilon}\left(\sqrt{\mu\cos^2\theta_\varepsilon+1}\,\left|\nabla|\mathbf{u}_\varepsilon|\right|\sqrt{2F(\mathbf{u}_\varepsilon)}-|\nabla\psi_\varepsilon|\right)dx\\&=\frac1\varepsilon\int_{U_\varepsilon}\left(\sqrt{\mu\cos^2\theta_\varepsilon+1}-1\right)|\nabla\psi_\varepsilon|\,dx.\end{aligned}$$

Note that in the last step we have used the identity $\left|\nabla|\mathbf{u}_\varepsilon|\right|\sqrt{2F(\mathbf{u}_\varepsilon)}=|\nabla\psi_\varepsilon|$, which holds on U_ε. So (3.22b) follows from conjugation. □

4 Estimates of level sets

Recalling (2.5), the main result of this section is the following L^1-estimate of ψ_ε.

Theorem 4.1 *Under the assumptions of Theorem 1.1, there exists $C > 0$ independent of ε such that*

$$\sup_{t\in[0,T]} B[\mathbf{u}_\varepsilon|I](t) \leqslant C\varepsilon, \tag{4.1}$$

$$\sup_{t\in[0,T]} \int_\Omega |\psi_\varepsilon - \mathbf{1}_{\Omega_t^+}|\, dx \leqslant C\varepsilon^{1/4}. \tag{4.2}$$

Proof We shall denote the positive and negative parts of a function h by h^+ and h^- respectively. For simplicity we shall suppress dx in a volume integral. By [18, pp. 153], for any $h \in W^{1,1}(\Omega)$, we have

$$\partial_i (h(x))^+ = (\partial_i h(x))\mathbf{1}_{\{x|h(x)>0\}}(x) \quad \text{for } a.e.\ x \in \Omega. \tag{4.3}$$

Our goal is to estimate $2\psi_\varepsilon - 1 - \chi$ where $\chi(x,t) = \pm 1$ in $\Omega_t^\pm$. Using the formula $h = h^+ - h^-$, we can write

$$2\psi_\varepsilon - 1 = 2(\psi_\varepsilon - 1)^+ + \left(1 - 2(\psi_\varepsilon - 1)^-\right), \tag{4.4}$$

and we shall estimate its difference with χ. This will be done by establishing differential inequalities for the following energies which add up to (1.12):

$$g_\varepsilon(t) := \int_\Omega (\psi_\varepsilon - 1)^+ \zeta \circ d_I, \tag{4.5a}$$

$$h_\varepsilon(t) := \int_\Omega \Big(\chi - [1 - 2(\psi_\varepsilon - 1)^-]\Big)\eta \circ d_I, \tag{4.5b}$$

where $\eta(z)$ is defined by (1.13) and $|\eta|(z) =: \zeta(z)$. It is obvious that the integrand of (4.5a) is non-negative. Since $\psi_\varepsilon \geqslant 0$, we have $(\psi_\varepsilon - 1)^- \in [0, 1]$ and thus $[1 - 2(\psi_\varepsilon - 1)^-]$ ranges in $[-1, 1]$. Using the identity $(\eta \circ d_I)\,\chi = |\eta \circ d_I|$, we deduce that the integrand of (4.5b) is also non-negative and

$$h_\varepsilon(t) = \int_\Omega \Big|1 - 2(\psi_\varepsilon - 1)^- - \chi\Big|\, \zeta \circ d_I. \tag{4.6}$$

Finally, we deduce from (1.14c) that

$$g_\varepsilon(0) + h_\varepsilon(0) \leqslant c_1\varepsilon. \tag{4.7}$$

Step 1: estimates of weighted errors. Using (1.8) and (1.9), we have

$$\partial_t \psi_\varepsilon = \Big(\partial_t \mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\Big) \cdot \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|}\sqrt{2F(\mathbf{u}_\varepsilon)} - \mathbf{H}\cdot\nabla\psi_\varepsilon. \tag{4.8}$$

Using this and (4.3) we can calculate

$$\begin{aligned}
g_\varepsilon'(t) =& \int_{\{\psi_\varepsilon>1\}} (\partial_t \mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon)\cdot \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|}\sqrt{2F(\mathbf{u}_\varepsilon)}\,\zeta\circ d_I \\
&- \int_{\{\psi_\varepsilon>1\}} \mathbf{H}\cdot\nabla\psi_\varepsilon\,\zeta\circ d_I + \int_\Omega (\psi_\varepsilon-1)^+\partial_t(\zeta\circ d_I) \\
=& \int_{\{\psi_\varepsilon>1\}} (\partial_t \mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon)\cdot \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|}\sqrt{2F(\mathbf{u}_\varepsilon)}\,\zeta\circ d_I \\
&- \int_\Omega \mathbf{H}\cdot\nabla(\psi_\varepsilon-1)^+\,\zeta\circ d_I - \int_\Omega (\psi_\varepsilon-1)^+\mathbf{H}\cdot\nabla(\zeta\circ d_I) \\
&+ \int_\Omega \Big(\partial_t(\zeta\circ d_I) + \mathbf{H}\cdot\nabla(\zeta\circ d_I)\Big)(\psi_\varepsilon-1)^+.
\end{aligned}$$

By (2.15b), the integrand of the last integral vanishes on $B_{\delta_0}(I_t)$. Moreover, we can combine the second and the third integrals in the last display using integration by parts. Using also that $\|\operatorname{div}\mathbf{H}\|_{L^\infty_{x,t}} \leqslant C$ and (2.26e), we find

$$\begin{aligned}
g_\varepsilon'(t) \leqslant& \int_{\{\psi_\varepsilon>1\}} (\partial_t \mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon)\cdot \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|}\sqrt{2F(\mathbf{u}_\varepsilon)}\,\zeta\circ d_I \\
&+ \int_\Omega (\operatorname{div}\mathbf{H})(\psi_\varepsilon-1)^+\,\zeta\circ d_I + C\int_{\Omega\backslash B_{\delta_0}(I_t)} (\psi_\varepsilon-1)^+ \\
\leqslant& \int_\Omega \varepsilon\Big|\partial_t\mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\Big|^2 + \left(\int_\Omega \frac{1}{\varepsilon}F(\mathbf{u}_\varepsilon)\zeta^2\circ d_I\right) + Cg_\varepsilon \\
\leqslant& CE_\varepsilon[\mathbf{u}_\varepsilon|I] + Cg_\varepsilon + \int_\Omega \varepsilon\Big|\partial_t\mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\Big|^2. \qquad (4.9)
\end{aligned}$$

Now using (4.7), (3.20) and (3.1), we can apply the Grönwall lemma and obtain $\sup_{t\in[0,T]} g_\varepsilon(t) \leqslant C\varepsilon$ for some C which is independent of ε. Concerning h_ε, for simplicity we introduce $w_\varepsilon := \chi - [1-2(\psi_\varepsilon-1)^-]$. Using the identity $(\partial_i\chi)\,\eta\circ d_I \equiv 0$ (in the sense of distribution), we find

$$(\partial_i w_\varepsilon)\,\eta\circ d_I = (2\partial_i\psi_\varepsilon)\,\mathbf{1}_{\{\psi_\varepsilon<1\}}\,\eta\circ d_I. \qquad (4.10)$$

So by the same calculation for g_ε we obtain

$$\begin{aligned}
h_\varepsilon'(t) =& \int_{\{\psi_\varepsilon<1\}} 2(\partial_t \mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon)\cdot \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|}\sqrt{2F(\mathbf{u}_\varepsilon)}\,\eta\circ d_I \\
&+ \int_\Omega (\operatorname{div}\mathbf{H})w_\varepsilon\,\eta\circ d_I + \int_\Omega \Big(\partial_t(\eta\circ d_I) + (\mathbf{H}\cdot\nabla)\eta\circ d_I\Big)w_\varepsilon \\
\leqslant& CE_\varepsilon[\mathbf{u}_\varepsilon|I] + Ch_\varepsilon(t) + \int_\Omega \varepsilon\Big|\partial_t\mathbf{u}_\varepsilon + (\mathbf{H}\cdot\nabla)\mathbf{u}_\varepsilon\Big|^2.
\end{aligned}$$

Using (4.7) and (3.20), we can apply the Grönwall lemma and obtain $\sup_{t\in[0,T]} h_\varepsilon(t) \leqslant C\varepsilon$. Finally, by (4.4) and (4.6), we find

$$\begin{aligned}
&\int_\Omega |2\psi_\varepsilon - 1 - \chi|\zeta\circ d_I \\
&\leqslant \int_\Omega 2(\psi_\varepsilon-1)^+\zeta\circ d_I + \int_\Omega \Big|1-2(\psi_\varepsilon-1)^- - \chi\Big|\zeta\circ d_I \\
&= 2g_\varepsilon(t) + h_\varepsilon(t) \leqslant C\varepsilon \quad \text{for all } t\in[0,T], \qquad (4.11)
\end{aligned}$$

and this proves (4.1).

Step 2: remove the weight. First note that (4.11) implies (4.2) with Ω replaced by $\Omega\backslash B_{\delta_0}(I_t)$. So we shall focus on the estimate on $B_{\delta_0}(I_t)$. We set $\chi_\varepsilon := 2\psi_\varepsilon - 1$ and abbreviate δ_0 by δ. For fixed $t \in [0, T]$ and $p \in I_t$ with normal vector $\mathbf{n} = \mathbf{n}(p)$, applying Hölder's inequality and Lemma 4.2 below with $f(r, p, t) = |\chi\,(p + r\mathbf{n}, t) - \chi_\varepsilon(p + r\mathbf{n}, t)|$, we find

$$
\begin{aligned}
&\left(\int_{B_\delta(I_t)} |\chi(x,t) - \chi_\varepsilon(x,t)|\, dx\right)^{4/3} \\
&= \left(\int_{I_t}\int_{-\delta}^{\delta} f(r,p,t)\, dr\, d\mathcal{H}^{d-1}(p)\right)^{4/3} \\
&\leqslant C \int_{I_t} \left(\int_{-\delta}^{\delta} f(r,p,t)\, dr\right)^{4/3} d\mathcal{H}^{d-1}(p) \\
&\overset{(4.12)}{\leqslant} C \int_{I_t} \|f(\cdot,p,t)\|_{L^{3/2}(-\delta,\delta)} \left(\int_{-\delta}^{\delta} f(r,p,t)|r|\, dr\right)^{1/3} d\mathcal{H}^{d-1}(p) \\
&= C\|f(\cdot,t)\|_{L^{3/2}(B_\delta(I_t))} \left(\int_{I_t}\int_{-\delta}^{\delta} f(r,p,t)|r|\, dr\, d\mathcal{H}^{d-1}(p))\right)^{1/3}.
\end{aligned}
$$

In view of (1.8) and (1.2c), we have $\psi_\varepsilon = 0$ on $\partial\Omega$. So by Sobolev's embedding $W^{1,1} \hookrightarrow L^{3/2}$ we obtain

$$
\begin{aligned}
&\left(\int_{B_\delta(I_t)} |\chi(x,t) - \chi_\varepsilon(x,t)|\, dx\right)^{4} \\
&\leqslant C\left(\|\chi\|^3_{L^{3/2}(\Omega)} + \|\chi_\varepsilon\|^3_{L^{3/2}(\Omega)}\right) \int_\Omega \zeta \circ d_I\, |\chi_\varepsilon - \chi|\, dx \\
&\leqslant C(1 + \|\nabla\psi_\varepsilon\|^3_{L^1(\Omega)}) \int_\Omega \zeta \circ d_I\, |\chi_\varepsilon - \chi|\, dx \leqslant C\varepsilon.
\end{aligned}
$$

Note that in the last step we employed (3.8a) and (4.11). This gives the desired estimate in $B_{\delta_0}(I_t)$ and thus the proof of (4.2) is finished. □

Lemma 4.2 *For any integrable function $f : [-\delta, \delta] \to \mathbb{R}_{\geqslant 0}$, we have*

$$
\left(\int_{-\delta}^{\delta} f(r)\, dr\right)^4 \leqslant 6\|f\|^3_{L^{3/2}(-\delta,\delta)} \int_{-\delta}^{\delta} |r| f(r)\, dr. \tag{4.12}
$$

Proof We write $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $F(x) = f(x_1)f(x_2)f(x_3)$. By symmetry and the Hölder inequality, we find

$$
\begin{aligned}
\|f\|^6_{L^1(0,\delta)} &= \int_{[0,\delta]^6} F(x)F(y)\, dxdy \\
&= 2\int_{[0,\delta]^6\cap\{(x,y)\ |x_1+x_2+x_3\leqslant y_1+y_2+y_3\}} F(x)F(y)\, dxdy \\
&= 2\int_{[0,\delta]^3} \left(\int_{[0,\delta]^3\cap\{x\ |x_1+x_2+x_3\leqslant y_1+y_2+y_3\}} 1\cdot F(x)\, dx\right) F(y)\, dy \\
&\leqslant 2\int_{[0,\delta]^3} (y_1+y_2+y_3) \left(\int_{[0,\delta]^3} F^{3/2}(x)\, dx\right)^{2/3} F(y)\, dy
\end{aligned}
$$

$$= 6\|f\|^3_{L^{3/2}(0,\delta)}\|f\|^2_{L^1(0,\delta)}\int_0^\delta rf(r)\,dr.$$

□

Now we turn to the study of the level sets of ψ_ε. The main tool is the following estimate, which is a consequence of (2.20a), (2.26d) and (3.1).

$$\begin{aligned}&\sup_{t\in[0,T]}\int_U\Big(|\nabla\psi_\varepsilon|-\boldsymbol{\xi}\cdot\nabla\psi_\varepsilon\Big)\,dx\\&=\sup_{t\in[0,T]}\int_U\Big(|\nabla\psi_\varepsilon|-\boldsymbol{\xi}\cdot\mathbf{n}_\varepsilon|\nabla\psi_\varepsilon|\Big)\,dx\leqslant C\varepsilon,\quad\forall U\text{ measurable in }\Omega.\end{aligned}\tag{4.13}$$

Lemma 4.3 *For each $t\in[0,T]$ there exists a null set $\mathcal{N}_t^\varepsilon\subset(0,1/8)$ such that the following holds: for every $\alpha\in(0,1/8)\backslash\mathcal{N}_t^\varepsilon$, there exist*

$$b_{\varepsilon,\alpha}(t)\in[1/2-\alpha,1/2+\alpha]\quad\textit{and}\quad q_{\varepsilon,\alpha}(t)\in[2-\alpha,2+\alpha]\tag{4.14}$$

such that the sets

$$\{x\mid\psi_\varepsilon(x,t)>b_{\varepsilon,\alpha}(t)\}\text{ and }\{x\mid\psi_\varepsilon(x,t)<q_{\varepsilon,\alpha}(t)\}\tag{4.15}$$

are of finite perimeter and

$$\Big|\mathcal{H}^{d-1}(\{x\mid\psi_\varepsilon(x,t)=b_{\varepsilon,\alpha}(t)\})-\mathcal{H}^{d-1}(I_t)\Big|\leqslant C\varepsilon^{1/4}\alpha^{-1},\tag{4.16a}$$

$$\mathcal{H}^{d-1}(\{x\mid\psi_\varepsilon(x,t)=q_{\varepsilon,\alpha}(t)\})\leqslant C\varepsilon^{1/4}\alpha^{-1},\tag{4.16b}$$

where $C>0$ is independent of t, ε and α.

Proof To prove (4.16a), we consider the set

$$S_t^{\varepsilon,\alpha}=\{x\in\Omega\mid|2\psi_\varepsilon(x,t)-1|\leqslant2\alpha\},\qquad\forall\alpha\in(0,1/8).\tag{4.17}$$

It follows from the co-area formula of BV function [18, section 5.5] that $S_t^{\varepsilon,\alpha}$ has finite perimeter for every $\alpha\in(0,1/8)\backslash\widetilde{\mathcal{N}}_t^\varepsilon$ for some null set $\widetilde{\mathcal{N}}_t^\varepsilon\subset(0,1/8)$. Moreover, by (4.13), we have for every $\alpha\in(0,1/8)\backslash\widetilde{\mathcal{N}}_t^\varepsilon$ that

$$\begin{aligned}C\varepsilon&\geqslant\int_{S_t^{\varepsilon,\alpha}}\Big(|\nabla\psi_\varepsilon|-\boldsymbol{\xi}\cdot\nabla\psi_\varepsilon\Big)\,dx\\&=\int_{\frac12-\alpha}^{\frac12+\alpha}\mathcal{H}^{d-1}\left(\{x\mid\psi_\varepsilon=s\}\right)\,ds-\int_{\partial S_t^{\varepsilon,\alpha}}\boldsymbol{\xi}\cdot\boldsymbol{\nu}\psi_\varepsilon\,d\mathcal{H}^{d-1}+\int_{S_t^{\varepsilon,\alpha}}(\operatorname{div}\boldsymbol{\xi})\psi_\varepsilon\,dx,\end{aligned}\tag{4.18}$$

where $\boldsymbol{\nu}$ is the outward normal of the set $S_t^{\varepsilon,\alpha}$, defined on its (measure-theoretic) boundary. Since $|\boldsymbol{\xi}|\leqslant1$ on Ω and $\psi_\varepsilon\leqslant1$ on $S_t^{\varepsilon,\alpha}$, we have

$$\left|\int_{S_t^{\varepsilon,\alpha}}(\operatorname{div}\boldsymbol{\xi})\psi_\varepsilon\,dx\right|\leqslant C|S_t^{\varepsilon,\alpha}|,$$

where $|A|=\mathcal{L}^d(A)$ is the d-Lebesgue measure of a set A. Combining this with (4.18), we find

$$\left|\int_{\frac12-\alpha}^{\frac12+\alpha}\mathcal{H}^{d-1}\left(\{x\mid\psi_\varepsilon=s\}\right)\,ds-\int_{\partial S_t^{\varepsilon,\alpha}}\boldsymbol{\xi}\cdot\boldsymbol{\nu}\psi_\varepsilon\,d\mathcal{H}^{d-1}\right|\leqslant C(\varepsilon+|S_t^{\varepsilon,\alpha}|).\tag{4.19}$$

By the divergence theorem, we have

$$\int_{\partial S_t^{\varepsilon,\alpha}} \boldsymbol{\xi} \cdot \boldsymbol{\nu} \psi_\varepsilon \, d\mathcal{H}^{d-1} = -\left(\tfrac{1}{2}-\alpha\right) \int_{\{x|\psi_\varepsilon < \frac{1}{2}-\alpha\}} (\operatorname{div} \boldsymbol{\xi})\, dx - \left(\tfrac{1}{2}+\alpha\right) \int_{\{x|\psi_\varepsilon > \frac{1}{2}+\alpha\}} (\operatorname{div} \boldsymbol{\xi})\, dx,$$

$$-2\alpha \mathcal{H}^{d-1}(I_t) \overset{(2.10)}{=} \left(\tfrac{1}{2}-\alpha\right) \int_{\Omega_t^-} (\operatorname{div} \boldsymbol{\xi})\, dx + \left(\tfrac{1}{2}+\alpha\right) \int_{\Omega_t^+} (\operatorname{div} \boldsymbol{\xi})\, dx.$$

Inserting these two equations into (4.19), we find

$$\begin{aligned} &\left| \int_{\frac{1}{2}-\alpha}^{\frac{1}{2}+\alpha} \mathcal{H}^{d-1}\left(\{x \mid \psi_\varepsilon = s\}\right) ds - 2\alpha \mathcal{H}^{d-1}(I_t) \right| \\ &\quad \leqslant C\left(\varepsilon + |S_t^{\varepsilon,\alpha}| + \left|\Omega_t^- \triangle \{x \mid \psi_\varepsilon < \tfrac{1}{2}-\alpha\}\right| + \left|\Omega_t^+ \triangle \{x \mid \psi_\varepsilon > \tfrac{1}{2}+\alpha\}\right| \right), \end{aligned} \tag{4.20}$$

where $A \triangle B := (A - B) \cup (B - A)$ is the symmetric difference of two sets A and B. We first estimate $r_\varepsilon^+ := \left|\Omega_t^+ \triangle \{x \mid \psi_\varepsilon > \frac{1}{2}+\alpha\}\right|$.

$$\begin{aligned} r_\varepsilon^+ &= \left|\Omega_t^+ - \{x \mid \psi_\varepsilon > \tfrac{1}{2}+\alpha\}\right| + \left|\{x \mid \psi_\varepsilon > \tfrac{1}{2}+\alpha\} - \Omega_t^+\right| \\ &= \left| \left(\Omega_t^+ - \{x \in \Omega_t^+ \mid \psi_\varepsilon > \tfrac{1}{2}+\alpha\}\right) - \{x \in \Omega_t^- \mid \psi_\varepsilon > \tfrac{1}{2}+\alpha\} \right| \\ &\quad + \left|\{x \in \Omega_t^- \mid \psi_\varepsilon > \tfrac{1}{2}+\alpha\}\right| \\ &\leqslant \left|\{x \in \Omega_t^+ \mid \psi_\varepsilon \leqslant \tfrac{1}{2}+\alpha\}\right| + \left|\{x \in \Omega_t^- \mid \psi_\varepsilon > \tfrac{1}{2}+\alpha\}\right|. \end{aligned}$$

Now using Chebyshev's inequality and (4.2), we find $r_\varepsilon^+ \leqslant C\varepsilon^{1/4}$. Similar estimates apply to $|S_t^{\varepsilon,\alpha}|$ and $r_\varepsilon^- := |\Omega_t^- \triangle \{x \mid \psi_\varepsilon < \frac{1}{2}-\alpha\}|$. Substituting these estimates into (4.20), we find

$$\left| \frac{1}{2\alpha} \int_{\frac{1}{2}-\alpha}^{\frac{1}{2}+\alpha} \left(\mathcal{H}^{d-1}\left(\{x \mid \psi_\varepsilon = s\}\right) - \mathcal{H}^{d-1}(I_t) \right) ds \right| \leqslant C\varepsilon^{1/4}\alpha^{-1}. \tag{4.21}$$

So (4.16a) follows from Fubini's theorem.

To prove (4.16b), we consider the set

$$Q_t^{\varepsilon,\alpha} = \{x \in \Omega \mid |\psi_\varepsilon(x,t) - 2| \leqslant \alpha\}, \qquad \forall \alpha \in (0, 1/8), \tag{4.22}$$

Using (4.13) and the co-area formula, we have for every $\alpha \in (0, 1/8) \backslash \mathcal{N}_t^\varepsilon$ that

$$\begin{aligned} C\varepsilon &\geqslant \int_{Q_t^{\varepsilon,\alpha}} \left(|\nabla \psi_\varepsilon| - \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon\right) dx \\ &= \int_{2-\alpha}^{2+\alpha} \mathcal{H}^{d-1}\left(\{x \mid \psi_\varepsilon = s\}\right) ds - \int_{\partial Q_t^{\varepsilon,\alpha}} \boldsymbol{\xi} \cdot \boldsymbol{\nu} \psi_\varepsilon \, d\mathcal{H}^{d-1} + \int_{Q_t^{\varepsilon,\alpha}} (\operatorname{div} \boldsymbol{\xi}) \psi_\varepsilon \, dx, \end{aligned}$$

where $\mathcal{N}_t^\varepsilon \supset \widetilde{\mathcal{N}}_t^\varepsilon$ is a null set in $(0, 1/8)$ and $\boldsymbol{\nu}$ is the outward normal of $\partial Q_t^{\varepsilon,\alpha}$. Since $\psi_\varepsilon \leqslant 3$ on $Q_t^{\varepsilon,\alpha}$, we have $\left|\int_{Q_t^{\varepsilon,\alpha}} (\operatorname{div} \boldsymbol{\xi}) \psi_\varepsilon \, dx\right| \leqslant C|Q_t^{\varepsilon,\alpha}|$, and thus

$$\int_{2-\alpha}^{2+\alpha} \mathcal{H}^{d-1}\left(\{x \mid \psi_\varepsilon = s\}\right) ds \leqslant \left| \int_{\partial Q_t^{\varepsilon,\alpha}} \boldsymbol{\xi} \cdot \boldsymbol{\nu} \psi_\varepsilon \, d\mathcal{H}^{d-1} \right| + C\varepsilon + C|Q_t^{\varepsilon,\alpha}|. \tag{4.23}$$

Using (2.14c), we have $\int_\Omega (\operatorname{div} \boldsymbol{\xi})\, dx = 0$, and thus

$$\begin{aligned}
&\int_{\partial Q_t^{\varepsilon,\alpha}} \boldsymbol{\xi} \cdot \boldsymbol{\nu} \psi_\varepsilon \, d\mathcal{H}^{d-1} \\
&= (2-\alpha) \int_{\{x|\psi_\varepsilon \geqslant 2-\alpha\}} (\operatorname{div} \boldsymbol{\xi})\, dx + (2+\alpha) \int_{\{x|\psi_\varepsilon \leqslant 2+\alpha\}} (\operatorname{div} \boldsymbol{\xi})\, dx \\
&= (2-\alpha) \int_{\{x:\psi_\varepsilon \geqslant 2-\alpha\}} (\operatorname{div} \boldsymbol{\xi})\, dx - (2+\alpha) \int_{\{x|\psi_\varepsilon > 2+\alpha\}} (\operatorname{div} \boldsymbol{\xi})\, dx.
\end{aligned}$$

This combined with Chebyshev's inequality and (4.2) implies that

$$|Q_t^{\varepsilon,\alpha}| + \left| \int_{\partial Q_t^{\varepsilon,\alpha}} \boldsymbol{\xi} \cdot \boldsymbol{\nu} \psi_\varepsilon \, d\mathcal{H}^{d-1} \right| \leqslant C\varepsilon^{1/4}.$$

Substituting this in (4.23) leads to

$$\frac{1}{2\alpha} \int_{2-\alpha}^{2+\alpha} \mathcal{H}^{d-1} \left(\{x \mid \psi_\varepsilon = s\} \right) ds \leqslant C\varepsilon^{1/4}\alpha^{-1}. \tag{4.24}$$

So (4.16b) follows from Fubini's theorem. □

We end this section with the following result concerning the convergence of $\mathbf{u}_\varepsilon$.

Proposition 4.1 *For every sequence $\varepsilon_k \downarrow 0$ there exists a subsequence, which we will not relabel, such that $\mathbf{u}_k := \mathbf{u}_{\varepsilon_k}$ satisfies*

$$\partial_t \mathbf{u}_k \wedge \mathbf{u}_k \xrightarrow{k\to\infty} \partial_t \mathbf{u} \wedge \mathbf{u} \text{ weakly in } L^2(0,T; L^{6/5}(\Omega)), \tag{4.25a}$$

$$\partial_i \mathbf{u}_k \wedge \mathbf{u}_k \xrightarrow{k\to\infty} \partial_i \mathbf{u} \wedge \mathbf{u} \text{ weakly-star in } L^\infty(0,T; L^{6/5}(\Omega)),\ 1 \leqslant i \leqslant 3, \tag{4.25b}$$

where $\mathbf{u} = \mathbf{u}(x,t)$ satisfies

$$\mathbf{u} \in L^\infty(0,T; W^{1,2}_{loc} \cap W^{1,6/5}(\Omega_t^+; \mathbb{S}^2)), \tag{4.26a}$$

$$\partial_t \mathbf{u} \in L^2(0,T; L^2_{loc} \cap L^{6/5}(\Omega_t^+)), \tag{4.26b}$$

$$\mathbf{u}(x,t) = 0 \text{ for every } t \in [0,T] \text{ and for a.e. } x \in \Omega_t^-. \tag{4.26c}$$

Furthermore,

$$\partial_t \mathbf{u}_k \xrightarrow{k\to\infty} \partial_t \mathbf{u} \text{ weakly in } L^2(0,T; L^2_{loc}(\Omega_t^\pm)), \tag{4.27a}$$

$$\nabla \mathbf{u}_k \xrightarrow{k\to\infty} \nabla \mathbf{u} \text{ weakly-star in } L^\infty(0,T; L^2_{loc}(\Omega_t^\pm)), \tag{4.27b}$$

$$\mathbf{u}_k \xrightarrow{k\to\infty} \mathbf{u} \text{ strongly in } C([0,T]; L^2_{loc}(\Omega_t^\pm)). \tag{4.27c}$$

Before proving this result, we state the Aubin–Lions–Simon lemma. See [41, Theorem 8.62, Exercise 8.63] or [52, Corollary 8] for the proof.

Lemma 4.4 *Let $I \subset \mathbb{R}$ be an open bounded interval, let $(Y_0, \|\cdot\|_{Y_0})$, $(Y_1, \|\cdot\|_{Y_1})$, and $(Y_2, \|\cdot\|_{Y_2})$ be Banach spaces with $Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2$. Assume that the embedding $Y_0 \hookrightarrow Y_1$ is compact. Let $\mathcal{V}$ be the Banach space of all functions $u \in L^\infty(I; Y_0)$ whose distributional derivative $\partial_t u$ belongs to $L^2(I; Y_2)$ endowed with the norm*

$$\|u\|_{\mathcal{V}} := \|u\|_{L^\infty(I;Y_0)} + \|\partial_t u\|_{L^2(I;Y_2)}.$$

Then the embedding $\mathcal{V} \hookrightarrow C(\bar{I}; Y_1)$ is compact.

Proof of Proposition 4.1 Define $\Omega^\pm := \bigcup_{t\in(0,T]} \Omega_t^\pm \times \{t\}$. We first deduce from (3.1) and (2.26b) that

$$\|\partial_t \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \partial_t \mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \|\nabla \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C \tag{4.28}$$

for some C independent of ε. On the other hand, by (2.22) we find

$$\Pi_{\mathbf{u}_\varepsilon} \partial_i \mathbf{u}_\varepsilon(x,t) \wedge \mathbf{u}_\varepsilon(x,t) = 0 \quad \forall (x,t) \in \Omega \times (0,T) \tag{4.29}$$

for $0 \leqslant i \leqslant 3$ where $\partial_0 := \partial_t$. Combining (4.29) and (4.28) with (3.8b), we deduce that

$$\begin{aligned} &\|\partial_t \mathbf{u}_\varepsilon \wedge \mathbf{u}_\varepsilon\|_{L^2(0,T;L^{6/5}(\Omega))} + \|\nabla \mathbf{u}_\varepsilon \wedge \mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^{6/5}(\Omega))} \\ &\quad = \|(\partial_t \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \partial_t \mathbf{u}_\varepsilon) \wedge \mathbf{u}_\varepsilon\|_{L^2(0,T;L^{6/5}(\Omega))} \\ &\qquad + \|(\nabla \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon) \wedge \mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^{6/5}(\Omega))} \leqslant C. \end{aligned} \tag{4.30}$$

So it follows from the Banach–Alaoglu theorem (cf. [41, A.5.]) that

$$\begin{aligned} &\partial_t \mathbf{u}_k \wedge \mathbf{u}_k \xrightarrow{k\to\infty} \mathbf{g}_0 \text{ weakly in } L^2(0,T;L^{6/5}(\Omega)), \\ &\partial_i \mathbf{u}_k \wedge \mathbf{u}_k \xrightarrow{k\to\infty} \mathbf{g}_i \text{ weakly-star in } L^\infty(0,T;L^{6/5}(\Omega)) \end{aligned} \tag{4.31}$$

where

$$\mathbf{g}_0 \in L^2(0,T;L^{6/5}(\Omega)) \text{ and } \{\mathbf{g}_i\}_{1\leqslant i\leqslant 3} \subset L^\infty(0,T;L^{6/5}(\Omega)). \tag{4.32}$$

It follows from (3.8b), (3.19a) and (3.19b) that, for any fixed $\delta \in (0,\delta_0)$, up to extraction of subsequences there exists $\varepsilon_k = \varepsilon_k(\delta) \xrightarrow{k\to\infty} 0$ such that

$$\mathbf{u}_{\varepsilon_k} \xrightarrow{k\to\infty} \mathbf{u} \text{ weakly-star in } L^\infty(0,T;L^3(\Omega)), \tag{4.33a}$$

$$\mathbf{u}_{\varepsilon_k} \xrightarrow{k\to\infty} \bar{\mathbf{u}}_\delta \text{ weakly-star in } L^\infty(0,T;L^3(\Omega_t^\pm \backslash B_\delta(I_t))), \tag{4.33b}$$

$$\partial_t \mathbf{u}_{\varepsilon_k} \xrightarrow{k\to\infty} \partial_t \bar{\mathbf{u}}_\delta \text{ weakly in } L^2\Big(0,T;L^2(\Omega_t^\pm \backslash B_\delta(I_t))\Big), \tag{4.33c}$$

$$\nabla \mathbf{u}_{\varepsilon_k} \xrightarrow{k\to\infty} \nabla \bar{\mathbf{u}}_\delta \text{ weakly-star in } L^\infty\Big(0,T;L^2(\Omega_t^\pm \backslash B_\delta(I_t))\Big). \tag{4.33d}$$

By (4.33a) and (4.33b), we have $\mathbf{u} = \bar{\mathbf{u}}_\delta$ a.e. in $U^\pm(\delta) := \cup_{t\in[0,T]} \big(\Omega_t^\pm \backslash B_\delta(I_t)\big) \times \{t\}$. This combined with (4.33c) and (4.33d) leads to

$$\mathbf{u} \in L^\infty(0,T;W^{1,2}_{loc}(\Omega_t^\pm)) \text{ with } \partial_t \mathbf{u} \in L^2(0,T;L^2_{loc}(\Omega_t^\pm)). \tag{4.34}$$

Furthermore, employing (4.33b)–(4.33d) and Lemma 4.4, we obtain

$$\mathbf{u}_{\varepsilon_k} \xrightarrow{k\to\infty} \bar{\mathbf{u}}_\delta = \mathbf{u} \text{ strongly in } C([0,T];L^2(\Omega_t^\pm \backslash B_\delta(I_t))). \tag{4.35}$$

By passing to a sequential limit $\delta = \delta_\ell \xrightarrow{\ell\to 0} 0$ and by a diagonal argument we obtain (4.27) up to extraction of subsequences.

Now we turn to the proof of (4.26). Using (3.19a), (4.35) and Fatou's lemma, we deduce that

$$f(|\mathbf{u}|) = F(\mathbf{u}) = F(\bar{\mathbf{u}}_\delta) = 0 \text{ a.e. in } U^\pm(\delta)$$

for any fixed $\delta \in (0,\delta_0)$. This together with (1.10) implies that $\mathbf{u}$ ranges in $\{0\} \cup \mathbb{S}^2$ a.e. in $\Omega \times (0,T)$. This combined with (4.2) and (4.34) yields (4.26c) and

$$\mathbf{u} \in L^\infty(0,T;W^{1,2}_{loc}(\Omega_t^+;\mathbb{S}^2)) \text{ with } \partial_t \mathbf{u} \in L^2(0,T;L^2_{loc}(\Omega_t^+)). \tag{4.36}$$

Now we show the integrability of $\nabla_{x,t}\mathbf{u}$ up to the boundary. To this aim, we choose a sequence of functions

$$\{\eta_k(\cdot,t)\}_{k\geqslant 1}\subset C_c^\infty(\Omega_t^+) \text{ with } \eta_k(\cdot,t)\xrightarrow{k\to\infty}\mathbf{1}_{\Omega_t^+} \text{ in } L^\infty(\Omega). \tag{4.37}$$

By (4.31) and (4.27), we deduce that for $0\leqslant i\leqslant 3$,

$$\eta_k\mathbf{g}_i=\eta_k\partial_i\mathbf{u}\wedge\mathbf{u} \text{ a.e. in } \Omega\times(0,T). \tag{4.38}$$

By (4.32) and the dominated convergence theorem, we can take $k\to\infty$ and get

$$\mathbf{g}_i=\partial_i\mathbf{u}\wedge\mathbf{u} \text{ a.e. in } \Omega\times(0,T),\ 0\leqslant i\leqslant 3. \tag{4.39}$$

This and (4.31) lead to (4.25a) and (4.25b). Since $\mathbf{u}$ maps Ω^+ into $\mathbb{S}^2$, we have

$$|\partial_i\mathbf{u}|^2=|\partial_i\mathbf{u}\wedge\mathbf{u}|^2=|\mathbf{g}_i|^2 \ a.e. \text{ in } \Omega^+,\ 0\leqslant i\leqslant 3. \tag{4.40}$$

This and (4.32) improve (4.36) and yield (4.26a) and (4.26b). □

5 Proof of Theorem 1.1: anchoring boundary condition

The inequalities (1.15) and (1.16) have been proved in Theorem 3.1 and in Theorem 4.1. The assertions (1.17), (1.18a) and (1.18b) have been proved in Proposition 4.1 (cf. (4.27c) and (4.26)). It remains to verify (1.18c), and this will be done by applying Lemma 4.3 for every $t\in[0,T]$ and by choosing an appropriate α outside the null set $\mathcal{N}_t^{\varepsilon_k}\subset(0,1/8)$. For simplicity we shall abbreviate ψ_{ε_k} and $\mathbf{u}_{\varepsilon_k}$ by ψ_k and $\mathbf{u}_k$ respectively. For any $k\geqslant 1$ we can choose $\beta_k\in[1/2,1]$ such that $\alpha=\alpha_k:=\beta_k\varepsilon_k^{1/8}\notin\mathcal{N}_t^{\varepsilon_k}$. Then by Lemma 4.3 there exist

$$b_{\varepsilon_k,\alpha_k}(t)=:b_k\in[\tfrac12-\alpha_k,\tfrac12+\alpha_k],\quad q_{\varepsilon_k,\alpha_k}(t)=:q_k\in[2-\alpha_k,2+\alpha_k] \tag{5.1}$$

such that

$$(b_k,q_k)\xrightarrow{k\to\infty}(\tfrac12,2), \tag{5.2}$$

and such that the set

$$\Omega_t^k:=\{x\in\Omega\mid b_k<\psi_k(x,t)<q_k\} \text{ has finite perimeter.} \tag{5.3}$$

Moreover, there exists $C>0$ which is independent of t and the particular choice of the subsequence ε_k such that

$$\mathcal{H}^{d-1}(\{x\mid\psi_k(x,t)=q_k\})\leqslant C\varepsilon_k^{1/8}, \tag{5.4a}$$

$$\left|\mathcal{H}^{d-1}(\partial\Omega_t^k)-\mathcal{H}^{d-1}(I_t)\right|\leqslant 2C\varepsilon_k^{1/8}. \tag{5.4b}$$

Using these level sets, we can prove the following proposition which improves (4.27) to the convergence of $\mathbf{u}_k$ up to the boundary I_t.

Proposition 5.1 *Let* $\mathbf{u}$ *be the limit vector field in Proposition* 4.1. *For a.e.* $t\in[0,T]$, *up to extraction of subsequences which we will not relabel, we have*

$$\mathbf{1}_{\bar\Omega_t^k}\widehat{\mathbf{u}}_k\xrightarrow{k\to\infty}\mathbf{1}_{\Omega_t^+}\mathbf{u} \text{ weakly-star in } BV(\Omega), \tag{5.5a}$$

$$\mathbf{1}_{\bar\Omega_t^k}\nabla\widehat{\mathbf{u}}_k\xrightarrow{k\to\infty}\mathbf{1}_{\Omega_t^+}\nabla\mathbf{u} \text{ weakly in } L^1(\Omega), \tag{5.5b}$$

$$\mathbf{1}_{\bar\Omega_t^k}\widehat{\mathbf{u}}_k\xrightarrow{k\to\infty}\mathbf{1}_{\Omega_t^+}\mathbf{u} \text{ strongly in } L^p(\Omega), \text{ for any fixed } p\in[1,\infty), \tag{5.5c}$$

where $\widehat{\mathbf{u}}_k = \widehat{\mathbf{u}}_{\varepsilon_k}$ *is defined in* (3.21).

Proof We first claim that there exists a positive constant C_3 depending only on f (cf. (1.11)) such that the following statement holds for any $\delta \in (0, 1/8)$:

$$|\mathbf{u}_\varepsilon(x,t)| \geqslant C_3\delta \qquad \forall x \in \{x : \psi_\varepsilon \geqslant \delta\}. \tag{5.6}$$

Indeed, by (1.11a), f (and also g) is increasing on $(0, s_0)$. If $|\mathbf{u}_\varepsilon| \geqslant s_0$, we are done. Otherwise,

$$\delta \leqslant \psi_\varepsilon = \int_0^{|\mathbf{u}_\varepsilon|} g(s)\, ds \leqslant |\mathbf{u}_\varepsilon| g(s_0),$$

which implies (5.6). This combined with (3.22a) and (5.3) implies

$$\sup_{t\in[0,T]} \int_{\Omega_t^k} |\nabla \widehat{\mathbf{u}}_k|^2\, dx \leqslant C \tag{5.7}$$

for k sufficiently large. This and (5.3) imply that the distributional derivatives of $\mathbf{v}_k(\cdot,t) := \mathbf{1}_{\bar{\Omega}_t^k} \widehat{\mathbf{u}}_k(\cdot,t)$ have no Cantor parts, and the absolute continuous parts $\{\mathbf{1}_{\bar{\Omega}_t^k} \nabla \widehat{\mathbf{u}}_k\}_{k\geqslant 1}$ is bounded in $L^2(\Omega)$. Moreover, their jump parts enjoy the estimate

$$\int_{\partial\Omega_t^k} |\mathbf{v}_k(\cdot,t) - 0|^2\, d\mathcal{H}^{d-1} \overset{(5.4b).}{\leqslant} C,$$

and $\{\mathbf{v}_k(\cdot,t)\}_{k\geqslant 1}$ is bounded in $L^\infty(\Omega)$. With these properties, it follows from [2] (or [3, Section 4.1]) that $\{\mathbf{v}_k(\cdot,t)\}_{k\geqslant 1}$ is compact in $SBV(\Omega)$, the class of special functions of bounded variation on Ω. More precisely, there exists $\mathbf{v}(\cdot,t) \in SBV(\Omega)$ s.t. $\mathbf{v}_k \to \mathbf{v}$ weakly-star in $BV(\Omega)$ as $k \to \infty$, and the absolute continuous part of the gradient $\nabla^a \mathbf{v}_k = \mathbf{1}_{\bar{\Omega}_t^k} \nabla \widehat{\mathbf{u}}_k$ converges weakly in $L^1(\Omega)$ to $\nabla^a \mathbf{v}$. To identify $\mathbf{v}$, we use (4.2) to deduce that $\mathbf{1}_{\bar{\Omega}_t^k} \to \mathbf{1}_{\Omega_t^+}$ in $L^1(\Omega)$ as $k \to \infty$. This and (4.27c) yield $\mathbf{v}(\cdot,t) = \mathbf{1}_{\Omega_t^+} \mathbf{u}(\cdot,t)$ a.e. in Ω, and thus (5.5a) and (5.5b) are proved. Finally by (5.5a), the compact embedding of BV functions and the L^∞ bound we get (5.5c). □

To proceed we define the following measures for Borel sets $A \subset \Omega$:

$$\theta(A) = \mathcal{H}^{d-1}(A \cap I_t), \tag{5.8a}$$

$$\theta_k(A) = \int_{A\cap\Omega_t^k} |\nabla\psi_k|\, dx. \tag{5.8b}$$

Lemma 5.1 *For a.e.* $t \in [0,T]$,

$$\theta_k \xrightarrow{k\to\infty} \frac{1}{2}\theta \quad \textit{weakly-star as Radon measures.} \tag{5.9}$$

Proof We define truncation functions

$$T_k(s) = \begin{cases} 0 & \text{when } s \leqslant b_k, \\ s - b_k & \text{when } b_k \leqslant s \leqslant q_k, \\ q_k - b_k & \text{when } s \geqslant q_k, \end{cases} \tag{5.10}$$

$$T(s) = \begin{cases} 0 & \text{when } s \leqslant 1/2, \\ s - 1/2 & \text{when } 1/2 \leqslant s \leqslant 2, \\ 3/2 & \text{when } s \geqslant 2. \end{cases} \tag{5.11}$$

By (5.2), we have $T_k \xrightarrow{k\to\infty} T$ uniformly on $\mathbb{R}$. Moreover,

$$\nabla(T_k \circ \psi_k) = \nabla\psi_k \, \mathbf{1}_{\Omega_t^k} \qquad \text{a.e. in } \Omega, \tag{5.12a}$$

$$T_k \circ \psi_k \xrightarrow{k\to\infty} \tfrac{1}{2}\mathbf{1}_{\Omega_t^+} \text{ strongly in } L^p(\Omega) \qquad \text{for any fixed } p \in [1,\infty). \tag{5.12b}$$

Indeed, by (2.7) and (2.17) we know that $\psi_k(\cdot,t) \in C^1(\Omega)$. Also by (5.3) we have $T_k' \circ \psi_k = \mathbf{1}_{\Omega_t^k}$ for a.e. $x \in \Omega$. Therefore, (5.12a) follows from the chain rule (cf. [26, Proposition 3.24]), while (5.12b) follows from (4.2) and the dominated convergence theorem. By (4.13) we have for any $g \in C_c^1(\Omega)$ that

$$\begin{aligned}\int_\Omega g\, d\theta_k &\overset{(5.8b)}{=} \int_{\Omega_t^k} g|\nabla\psi_k|\, dx \overset{(4.13)}{=} O(\varepsilon_k) + \int_{\Omega_t^k} g\, \boldsymbol{\xi}\cdot\nabla\psi_k\, dx \\ &\overset{(5.12a)}{=} O(\varepsilon_k) + \int_\Omega g\,\boldsymbol{\xi}\cdot\nabla(T_k\circ\psi_k)\, dx \\ &= O(\varepsilon_k) - \int_\Omega \operatorname{div}(g\boldsymbol{\xi})\, T_k\circ\psi_k\, dx.\end{aligned}$$

Recalling that $\boldsymbol{\xi}$ is the inward normal of I_t, we use (5.12b) to pass to the limit in the above equations and obtain

$$\lim_{k\to\infty}\int_\Omega g\, d\theta_k \overset{(5.12b)}{=} -\frac{1}{2}\int_{\Omega_t^+}\operatorname{div}(g\boldsymbol{\xi})\, dx = \frac{1}{2}\int_{I_t} g\, d\mathcal{H}^{d-1} \overset{(5.8a)}{=} \frac{1}{2}\int_\Omega g\, d\theta,$$

for any $g \in C_c^1(\Omega)$. By approximation, one can pass from $C_c^1(\Omega)$ to $C_c^0(\Omega)$, and this proves (5.9). □

Now we finish the proof of Theorem 1.1 by verifying (1.18c). The proof here is inspired by the blow-up argument in [43]. See also [25] for the applications of such a method in the study of quasi-convex functionals.

Proof of (1.18c) For any $x_0 \in I_t$ and any $R > 0$, it follows from (5.5c), (5.12b) and the dominated convergence theorem that

$$\lim_{k\to\infty}\int_{B_R(x_0)} \mathbf{1}_{\hat\Omega_t^k}\widehat{\mathbf{u}}_k\cdot\frac{x-x_0}{|x-x_0|}T_k\circ\psi_k\, dx = \frac{1}{2}\int_{B_R(x_0)}\mathbf{1}_{\Omega_t^+}\mathbf{u}\cdot\frac{x-x_0}{|x-x_0|}\, dx.$$

We can use spherical coordinate to rewrite the above two integrals in the form of $\int_0^R\int_{\partial B_r(x_0)}(\cdot)\, d\mathcal{H}^{d-1}dr$, and then apply Fubini's theorem. Therefore, there exists $r_j \downarrow 0$ such that for each j we have

$$\lim_{k\to\infty}\int_{\partial B_{r_j}(x_0)\cap\Omega_t^k}\widehat{\mathbf{u}}_k\cdot\boldsymbol{\nu}\, T_k\circ\psi_k\, d\mathcal{H}^{d-1} = \frac{1}{2}\int_{\partial B_{r_j}(x_0)\cap\Omega_t^+}\mathbf{u}\cdot\boldsymbol{\nu}\, d\mathcal{H}^{d-1} \tag{5.13}$$

where $\boldsymbol{\nu}$ is the outward normal of $\partial B_{r_j}(x_0)$. Moreover, we can arrange r_j such that $\theta(\partial B_{r_j}(x_0)) = 0$. This combined with (5.9) implies that

$$\lim_{k\to\infty}\theta_k(B_{r_j}(x_0)) = \frac{1}{2}\theta(B_{r_j}(x_0)). \tag{5.14}$$

To proceed, we use convexity to write, for some $a_m, c_m \in \mathbb{R}$, that

$$s^2 = \sup_{m\in\mathbb{N}^+}(a_m s + c_m), \quad \forall s \in \mathbb{R}. \tag{5.15}$$

(cf. [3, Proposition 2.31]). For $\theta - a.e.\ x_0 \in \operatorname{supp}(\theta) = I_t$, we have for each $m \geqslant 1$ that

$$
\begin{aligned}
0 &\overset{(3.22b)}{=} \lim_{k\to\infty} \int_{B_{r_j}(x_0)} (\widehat{\mathbf{u}}_k \cdot \mathbf{n}_k)^2 \, d\theta_k \\
&\overset{(5.15)}{\geqslant} \lim_{k\to\infty} \int_{B_{r_j}(x_0)} (a_m \widehat{\mathbf{u}}_k \cdot \mathbf{n}_k + c_m) \, d\theta_k \\
&\overset{(2.20a)}{=} \lim_{k\to\infty} a_m \int_{B_{r_j}(x_0)} \mathbf{1}_{\Omega_t^k} \widehat{\mathbf{u}}_k \cdot \nabla \psi_k \, dx + c_m \theta_k(B_{r_j}(x_0)) \\
&\overset{(5.12a)}{=} a_m \lim_{k\to\infty} \int_{B_{r_j}(x_0)\cap\Omega_t^k} \widehat{\mathbf{u}}_k \cdot \nabla (T_k \circ \psi_k) \, dx + \theta\left(B_{r_j}(x_0)\right) \frac{c_m}{2}.
\end{aligned} \tag{5.16}
$$

Note that in the last step we also used (5.14). It remains to compute the integral in the last display of (5.16) under the limit $k \to \infty$ for fixed j, m. To this aim, we use (5.12a) and integration by parts to find

$$
\begin{aligned}
&\int_{B_{r_j}(x_0)\cap\Omega_t^k} \widehat{\mathbf{u}}_k \cdot \nabla (T_k \circ \psi_k) \, dx \\
&= \int_{\partial\left(B_{r_j}(x_0)\cap\Omega_t^k\right)} (\widehat{\mathbf{u}}_k \cdot \boldsymbol{\nu}) \; T_k \circ \psi_k \, d\mathcal{H}^{d-1} - \int_{B_{r_j}(x_0)} \mathbf{1}_{\bar{\Omega}_t^k} (\operatorname{div} \widehat{\mathbf{u}}_k) \; T_k \circ \psi_k \, dx \\
&=: A_k - B_k.
\end{aligned} \tag{5.17}
$$

Note that the integrand of A_k is uniformly bounded in L^∞. To compute the limit of A_k, we first deduce from (5.10) that $T_k \circ \psi_k = 0$ on the set $\{x \in \Omega \mid \psi_k(x,t) = b_k\}$ which has finite perimeter (cf. (5.4b)). So we employ (5.3) to find

$$
A_k = \int_{\partial B_{r_j}(x_0)\cap\Omega_t^k} (\widehat{\mathbf{u}}_k \cdot \boldsymbol{\nu}) \; T_k \circ \psi_k \, d\mathcal{H}^{d-1} + \int_{B_{r_j}(x_0)\cap\{x|\psi_k=q_k\}} (\widehat{\mathbf{u}}_k \cdot \boldsymbol{\nu}) \; T_k \circ \psi_k \, d\mathcal{H}^{d-1}. \tag{5.18}
$$

The limit of the first integral is given in (5.13), and that of the second vanishes in the limit $k \to \infty$ by (5.4a). So we conclude that

$$
\lim_{k\to\infty} A_k = \frac{1}{2} \int_{\partial B_{r_j}(x_0)\cap\Omega_t^+} \mathbf{u} \cdot \boldsymbol{\nu} \, d\mathcal{H}^{d-1}. \tag{5.19}
$$

Concerning the integral B_k, by (5.5b) the sequence $\{\mathbf{1}_{\bar{\Omega}_t^k} \operatorname{div} \widehat{\mathbf{u}}_k\}_{k\geqslant 1}$ converges weakly in $L^1(\Omega)$. Moreover, $\{T_k \circ \psi_k\}_{k\geqslant 1}$ is uniformly bounded in L^∞, and converges a.e. in Ω to $\frac{1}{2}\mathbf{1}_{\Omega_t^+}$, due to (5.12b). Therefore, applying the Product Limit Theorem (cf. [16] or [49, pp. 169]), we obtain

$$
\lim_{k\to\infty} B_k = \frac{1}{2} \int_{B_{r_j}(x_0)\cap\Omega_t^+} (\operatorname{div} \mathbf{u}) \, dx. \tag{5.20}
$$

Using (5.19) and (5.20), we can compute the limit in (5.17) and find

$$\begin{aligned}\lim_{k\to\infty}&\int_{B_{r_j}(x_0)\cap\Omega_t^k}\widehat{\mathbf{u}}_k\cdot\nabla(T_k\circ\psi_k)\,dx\\ =&\frac{1}{2}\int_{\partial B_{r_j}(x_0)\cap\Omega_t^+}\mathbf{u}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{d-1}-\frac{1}{2}\int_{\partial\left(B_{r_j}(x_0)\cap\Omega_t^+\right)}\mathbf{u}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{d-1}\\ =&\frac{1}{2}\int_{B_{r_j}(x_0)\cap\partial\Omega_t^+}\mathbf{u}\cdot\boldsymbol{\xi}\,d\mathcal{H}^{d-1}\end{aligned}\tag{5.21}$$

where in the last step we used $\boldsymbol{\xi}=-\boldsymbol{\nu}$ on $\partial\Omega_t^+$. Note that $\boldsymbol{\xi}$ is the inward normal of I_t according to (2.10), and Ω_t^+ is the region enclosed by I_t with outward normal $\boldsymbol{\nu}$. Substituting (5.21) into (5.16) and then dividing the resulting inequality by $\theta\left(B_{r_j}(x_0)\right)$ and taking $j\to\infty$, we find

$$\begin{aligned}0\geqslant&\lim_{j\to\infty}\frac{a_m}{\theta\left(B_{r_j}(x_0)\right)}\frac{1}{2}\int_{B_{r_j}(x_0)\cap I_t}\mathbf{u}\cdot\boldsymbol{\xi}\,d\mathcal{H}^{d-1}+\frac{c_m}{2}\\ \overset{(5.8a)}{=}&\frac{a_m}{2}(\mathbf{u}\cdot\boldsymbol{\xi})(x_0)+\frac{c_m}{2},\qquad\forall m\in\mathbb{N}^+.\end{aligned}\tag{5.22}$$

This together with (5.15) implies that $(\mathbf{u}\cdot\boldsymbol{\xi})^2(x_0)=0$ for $\mathcal{H}^{d-1}$-a.e. $x_0\in I_t$. □

6 Proof of Theorem 1.2: Oseen–Frank limit in the bulk

The method here is inspired by [17, 38], which has a 2D nature. We set $\boldsymbol{\tau}_\varepsilon:=\partial_t\mathbf{u}_\varepsilon$ and write (1.2a) as

$$\boldsymbol{\tau}_\varepsilon=\mu\nabla(\operatorname{div}\mathbf{u}_\varepsilon)+\Delta\mathbf{u}_\varepsilon-\varepsilon^{-2}DF(\mathbf{u}_\varepsilon)\ \text{ in }\Omega\times(0,T).\tag{6.1}$$

By Corollary 3.4 and Proposition 4.1 (cf. (4.27c)), for a.e. $t_0\in(0,T)$ and for any compact set $K\subset\subset\Omega_{t_0}^+$, we have

$$\int_K|\boldsymbol{\tau}_\varepsilon|^2\,dx+\int_K\left(\frac{1}{2}|\nabla\mathbf{u}_\varepsilon|^2+\frac{1}{\varepsilon^2}F(\mathbf{u}_\varepsilon)\right)dx\leqslant\hat{c}^2\quad\text{at }\ t=t_0,\tag{6.2a}$$

$$\mathbf{u}_{\varepsilon_k}(\cdot,t_0)\xrightarrow{k\to\infty}\mathbf{u}(\cdot,t_0)\ \text{ strongly in }L^2(K),\tag{6.2b}$$

where $\hat{c}=\hat{c}(K,t_0)>1$ is independent of μ and ε.

Proposition 6.1 *Let K be a compact set of $\Omega_{t_0}^+$ and assume that (6.2a) and (6.2b) hold. There exists an absolute constant $\Lambda\in(0,1)$ with the following property: under the assumptions*

$$\hat{\epsilon}<\Lambda/\hat{c}^2\ \textit{ and }\ \mu<\Lambda,\tag{6.3a}$$

$$B_{2r}(x_0)\subset K\ \textit{ with }\ r<1\ \textit{ and}\tag{6.3b}$$

$$\int_{B_{2r}(x_0)}\left(\frac{1}{2}|\nabla\mathbf{u}_\varepsilon|^2+\frac{1}{\varepsilon^2}F(\mathbf{u}_\varepsilon)\right)dx\leqslant\hat{\epsilon}^2\ \textit{ at }\ t=t_0,\tag{6.3c}$$

there exists a subsequence $\varepsilon_k\downarrow 0$, which we will not relabel, such that

$$\nabla\mathbf{u}_{\varepsilon_k}(\cdot,t_0)\xrightarrow{k\to\infty}\nabla\mathbf{u}(\cdot,t_0)\ \textit{strongly in }L^2(B_{r/2}(x_0)).\tag{6.4}$$

We shall need the following inequality due to the special choice of f in (1.22):

$$|f'(s)|^2 \leqslant C_4 f(s), \qquad \forall s \geqslant 0, \tag{6.5}$$

for some $C_4 > 1$. In the sequel $C_4 > 1$ will also be used as a generic constant that might change from line to line due to the use of the Sobolev embeddings and elliptic estimates. Note that C_4 is independent of μ and r.

Lemma 6.1 *Under the assumptions* (6.2a) *and* (6.3b) *with a sufficiently small* $\hat{\epsilon}$ *(defined in* (6.11) *below), we have*

$$3/4 \leqslant |\mathbf{u}_\varepsilon(\cdot, t_0)| \leqslant 5/4 \text{ on } B_r(x_0) \quad \text{for } \varepsilon \leqslant r/4. \tag{6.6}$$

Proof Without loss of generality, we assume $x_0 = 0$. For brevity we write $B_r(0)$ as B_r. Since all arguments are made at $t = t_0$, we shall suppress the time dependence of $\mathbf{u}_\varepsilon$.

Step 1. There exists $\hat{C} > 1$ depending on $\hat{c}$ such that for any $x_1 \in B_r$ we have

$$|\mathbf{u}_\varepsilon(x) - \mathbf{u}_\varepsilon(y)| \leqslant \hat{C}\sqrt{\frac{|x-y|}{\varepsilon}}, \quad \forall x, y \in B_\varepsilon(x_1). \tag{6.7}$$

To prove (6.7), let $\hat{\mathbf{u}}_\varepsilon(z) = \mathbf{u}_\varepsilon(x_1 + \varepsilon z) : B_2 \to \mathbb{R}^3$. Then we can write (6.1) as

$$\mu \nabla \operatorname{div} \hat{\mathbf{u}}_\varepsilon(z) + \Delta \hat{\mathbf{u}}_\varepsilon(z) = \varepsilon^2 \boldsymbol{\tau}_\varepsilon(x_1 + \varepsilon z) + DF(\hat{\mathbf{u}}_\varepsilon(z)), \quad z \in B_2. \tag{6.8}$$

It follows from (6.2a) and a change of variable that $\{\varepsilon^2 \boldsymbol{\tau}_\varepsilon(x_1 + \varepsilon \cdot)\}_{\varepsilon > 0}$ is uniformly bounded in $L^2(B_2)$. Using (6.5), we can estimate

$$\left\| DF(\hat{\mathbf{u}}_\varepsilon) \right\|^2_{L^2(B_2)} \overset{(1.9)}{=} \varepsilon^{-2} \left\| f'(|\mathbf{u}_\varepsilon|) \right\|^2_{L^2(B_{2\varepsilon}(x_1))} \leqslant \varepsilon^{-2} C_4 \int_{B_{2\varepsilon}(x_1)} F(\mathbf{u}_\varepsilon)\, dx \overset{(6.2a)}{\leqslant} \hat{c}^2 C_4. \tag{6.9}$$

Altogether, we prove that the terms on the right-hand side of (6.8) is bounded in $L^2(B_2)$. Invoking the interior estimate for elliptic system (cf. [26, Theorem 4.9]), we obtain

$$\|\hat{\mathbf{u}}_\varepsilon\|_{W^{2,2}(B_1)} \leqslant C_4(\hat{c} + \|\hat{\mathbf{u}}_\varepsilon\|_{L^2(B_2)}). \tag{6.10}$$

Note that C_4 is independent of μ. Now we estimate the last term by

$$\begin{aligned} \|\hat{\mathbf{u}}_\varepsilon\|^2_{L^2(B_2)} &\leqslant C_4 \left(1 + \varepsilon^{-2} \int_{B_{2\varepsilon}(x_1) \cap \{x | |\mathbf{u}_\varepsilon(x)| \geqslant 2\}} (|\mathbf{u}_\varepsilon| - 1)^2 \right) \\ &\overset{(1.22)}{\leqslant} C_4 \left(1 + \varepsilon^{-2} \int_{B_{2\varepsilon}(x_1)} f(|\mathbf{u}_\varepsilon|) \right) \overset{(6.2a)}{\leqslant} (1 + \hat{c}^2) C_4. \end{aligned}$$

Substituting this estimate in (6.10) and using Morrey's embedding $W^{2,2} \hookrightarrow C^{1/2}$, we obtain $\|\hat{\mathbf{u}}_\varepsilon\|_{C^{1/2}(\bar{B}_1)} \leqslant C_4 \hat{c}$. Rescaling back, we find (6.7) with

$$\hat{C} := C_4 \hat{c}.$$

Step 2: We claim that with the choice

$$\hat{\epsilon} < 16^{-8} C_4^{-2} \hat{c}^{-2} = 16^{-8} \hat{C}^{-2}, \tag{6.11}$$

we have either (6.6) or

$$|\mathbf{u}_\varepsilon| \leqslant 1/4 \text{ on } B_r \quad \text{for } \varepsilon \leqslant r/4. \tag{6.12}$$

Indeed, if neither of them were valid, then

$$\exists\, \varepsilon \in (0, r/4) \text{ and } x_1 \in B_r \text{ s.t. } |\mathbf{u}_\varepsilon(x_1)| \in (1/4, 3/4) \cup (5/4, +\infty). \tag{6.13}$$

Since $16\hat{\epsilon} < 1$, it follows from (6.7) that

$$|\mathbf{u}_\varepsilon(x_1) - \mathbf{u}_\varepsilon(x)| < 4^{-7} \qquad \text{for } x \in B_{16\hat{\epsilon}\varepsilon}(x_1). \tag{6.14}$$

Using this and (1.22), we deduce one of the following two cases for $x \in B_{16\hat{\epsilon}\varepsilon}(x_1)$:

a) If $|\mathbf{u}_\varepsilon(x_1)| > 3$, then $|\mathbf{u}_\varepsilon(x)| > 2$ and thus $f(|\mathbf{u}_\varepsilon(x)|) \geqslant 1$.

b) If $|\mathbf{u}_\varepsilon(x_1)| \in (1/4, 3/4) \cup (5/4, 3]$, then $f(|\mathbf{u}_\varepsilon(x_1)|) \geqslant 1/16$. By the third condition in (1.22) and (6.14), we have $f(|\mathbf{u}_\varepsilon(x)|) > 1/32$.

To summarize, we have the following inequality:

$$F(\mathbf{u}_\varepsilon(x)) = f(|\mathbf{u}_\varepsilon(x)|) > 1/32 \qquad \forall x \in B_{16\hat{\epsilon}\varepsilon}(x_1). \tag{6.15}$$

Integrating this inequality over $B_{16\hat{\epsilon}\varepsilon}(x_1)$ and using the assumption $\varepsilon < r/4$, we find

$$\varepsilon^{-2} \int_{B_{16\hat{\epsilon}\varepsilon}(x_1)} F(\mathbf{u}_\varepsilon(x))\, dx > 8\pi \hat{\epsilon}^2.$$

However, this would contradict (6.3b) since $B_{16\hat{\epsilon}\varepsilon}(x_1) \subset B_{2r}(x_0)$. So (6.13) is not valid and the claim is proved.

Step 3: We shall rule out (6.12).

Assuming (6.12), we deduce from (1.22) that $F(\mathbf{u}_\varepsilon) = |\mathbf{u}_\varepsilon|^2$. By (6.1) we have

$$\mu \nabla(\operatorname{div} \mathbf{u}_\varepsilon) + \Delta \mathbf{u}_\varepsilon - 2\varepsilon^{-2}\mathbf{u}_\varepsilon = \boldsymbol{\tau}_\varepsilon \ \text{ in } B_r. \tag{6.16}$$

For $z \in B_1$, we introduce $\widetilde{\mathbf{u}}_\varepsilon(z) := \mathbf{u}_\varepsilon(rz)$ and $\widetilde{\boldsymbol{\tau}}_\varepsilon(z) := \boldsymbol{\tau}_\varepsilon(rz)$. Then

$$\mu \nabla(\operatorname{div} \widetilde{\mathbf{u}}_\varepsilon) + \Delta \widetilde{\mathbf{u}}_\varepsilon - 2r^2\varepsilon^{-2}\widetilde{\mathbf{u}}_\varepsilon = r^2 \widetilde{\boldsymbol{\tau}}_\varepsilon \ \text{ in } B_1. \tag{6.17}$$

By the interior estimate for elliptic system, we have

$$\|\widetilde{\mathbf{u}}_\varepsilon\|_{W^{2,2}(B_{1/2})} + r^2\varepsilon^{-2}\|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(B_{1/2})} \leqslant C_4 \left(\|\widetilde{\boldsymbol{\tau}}_\varepsilon\|_{L^2(B_1)} + \|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(B_1)} \right). \tag{6.18}$$

Indeed, one can adapt the proof of [26, Theorem 4.9] to gain the term $r^2\varepsilon^{-2}\|\widetilde{\mathbf{u}}_\varepsilon\|_{L^2(B_{1/2})}$. By (6.18), (6.2a) and the conclusion in step 2, we find

$$r^2\varepsilon^{-2}\|\mathbf{u}_\varepsilon\|_{L^2(B_{r/2})} \leqslant C_4 \left(\|\boldsymbol{\tau}_\varepsilon\|_{L^2(B_r)} + \|\mathbf{u}_\varepsilon\|_{L^2(B_r)} \right) \leqslant C_4(\hat{c} + 1).$$

This implies that $\mathbf{u}_\varepsilon \to 0$ strongly in $L^2(B_{r/2})$, which contradicts (4.2) since $B_{r/2} \subset K \subset\subset \Omega_t^+$. Therefore, we rule out (6.12) and obtain (6.6). □

By (6.6), we have polar decomposition $\mathbf{u}_\varepsilon = \rho_\varepsilon \mathbf{v}_\varepsilon$ where

$$\rho_\varepsilon = |\mathbf{u}_\varepsilon|, \quad \mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon / |\mathbf{u}_\varepsilon| \ \text{ in } B_r(x_0). \tag{6.19}$$

We set

$$\mathbf{w}_\varepsilon := (\mathbf{v}_\varepsilon, \rho_\varepsilon), \tag{6.20}$$

and define the projection

$$\mathbf{a}_\parallel := (\mathbb{I}_3 - \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon)\mathbf{a} \tag{6.21}$$

for a vector field $\mathbf{a}$.

Lemma 6.2 *Under the assumptions $\varepsilon \leqslant r/4$, (6.2a) and (6.3b) for $\hat{\epsilon}$ defined in (6.11), ρ_ε satisfies the following equation in $B_r(x_0)$.*

$$\Delta\rho_\varepsilon - \varepsilon^{-2} f'(\rho_\varepsilon) + \mu\nabla^2\rho_\varepsilon : (\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) + \mu\rho_\varepsilon(\mathbf{v}_\varepsilon \cdot \nabla)\operatorname{div}\mathbf{v}_\varepsilon \\ = \boldsymbol{\tau}_\varepsilon \cdot \mathbf{v}_\varepsilon + \mathcal{N}_{1,\varepsilon}(\nabla\mathbf{w}_\varepsilon, \nabla\mathbf{w}_\varepsilon), \tag{6.22}$$

where $\mathcal{N}_{1,\varepsilon}(\cdot,\cdot) : \mathbb{R}^{4\times3} \times \mathbb{R}^{4\times3} \mapsto \mathbb{R}$ is bilinear with uniformly bounded coefficients. Also, $\mathbf{v}_\varepsilon$ satisfies the following equation in $B_r(x_0)$.

$$\rho_\varepsilon\Delta\mathbf{v}_\varepsilon + \mu\,((\nabla^2\rho_\varepsilon)\mathbf{v}_\varepsilon)_\parallel + \mu\rho_\varepsilon\,(\nabla(\operatorname{div}\mathbf{v}_\varepsilon))_\parallel \\ = (\boldsymbol{\tau}_\varepsilon)_\parallel + \mathcal{N}_{2,\varepsilon}(\nabla\mathbf{w}_\varepsilon, \nabla\mathbf{w}_\varepsilon), \tag{6.23}$$

where $\mathcal{N}_{2,\varepsilon}(\cdot,\cdot) : \mathbb{R}^{4\times3} \times \mathbb{R}^{4\times3} \mapsto \mathbb{R}^3$ is bilinear with uniformly bounded coefficients.

Proof To simplify the presentation we will suppress the subscript ε. By (6.19) we have $|\mathbf{v}|^2 \equiv 1$ and thus

$$\Delta\mathbf{v} \cdot \mathbf{v} = -|\nabla\mathbf{v}|^2. \tag{6.24}$$

Substituting (6.19) into (6.1), we find

$$\boldsymbol{\tau} = (\Delta\rho)\mathbf{v} + 2(\nabla\rho \cdot \nabla)\mathbf{v} + \rho\Delta\mathbf{v} - \varepsilon^{-2} f'(\rho)\mathbf{v} \\ + \mu\,(\nabla^2\rho)\mathbf{v} + \mu\rho\nabla(\operatorname{div}\mathbf{v}) + \mu\,(\nabla\rho \cdot \partial_i\mathbf{v})_{1\leqslant i\leqslant 3} + \mu\nabla\rho(\operatorname{div}\mathbf{v}). \tag{6.25}$$

Testing (6.25) with $\mathbf{v}$ and using (6.24), we obtain

$$-\Delta\rho + \varepsilon^{-2} f'(\rho) \\ = -\boldsymbol{\tau} \cdot \mathbf{v} + \mu\nabla^2\rho : (\mathbf{v} \otimes \mathbf{v}) + \mu\rho(\mathbf{v} \cdot \nabla)\operatorname{div}\mathbf{v} \\ + \mu(\nabla\rho \cdot \partial_i\mathbf{v})v_i + \mu(\mathbf{v} \cdot \nabla\rho)\operatorname{div}\mathbf{v} - \rho|\nabla\mathbf{v}|^2. \tag{6.26}$$

The terms in the last line are bilinear with respect to $\nabla\mathbf{w} = (\nabla\mathbf{v}, \nabla\rho)$, and we denote their sum by $-\mathcal{N}_{1,\varepsilon}(\nabla\mathbf{w}, \nabla\mathbf{w})$. By (6.6), it has bounded coefficients and thus (6.22) is proved.

To derive (6.23), we shall use the following identities.

$$\mathbf{v}_\parallel = 0 \text{ and } (\partial_i\mathbf{v})_\parallel = \partial_i\mathbf{v}. \tag{6.27}$$

These combined with (6.24) lead to

$$(\Delta\mathbf{v})_\parallel = \Delta\mathbf{v} + |\nabla\mathbf{v}|^2\mathbf{v}. \tag{6.28}$$

Now applying $(\cdot)_\parallel$ to the equation in (6.25), and using (6.27) and (6.28), we obtain

$$\boldsymbol{\tau}_\parallel = \rho\Delta\mathbf{v} + \mu\,((\nabla^2\rho)\mathbf{v})_\parallel + \mu\rho\,(\nabla(\operatorname{div}\mathbf{v}))_\parallel \\ + 2((\nabla\rho \cdot \nabla)\mathbf{v})_\parallel + \rho\mathbf{v}|\nabla\mathbf{v}|^2 + \mu\Big((\nabla\rho \cdot \partial_i\mathbf{v})_{1\leqslant i\leqslant 3}\Big)_\parallel + \mu(\nabla\rho)_\parallel(\operatorname{div}\mathbf{v}). \tag{6.29}$$

The terms in the second line of the above equation are bilinear with respect to $\nabla\mathbf{w}$, and we denote their sum by $-\mathcal{N}_{2,\varepsilon}(\nabla\mathbf{w}, \nabla\mathbf{w})$. By (6.6), it has bounded coefficients, and thus (6.23) is proved. □

Proof of Proposition 6.1 We first show that, by choosing $\hat{\epsilon}$ and μ sufficiently small, we have

$$\|\nabla^2(\mathbf{v}_\varepsilon, \rho_\varepsilon)\|_{L^{4/3}(B_{r/2}(x_0))} \leqslant 2C_4 r^{-2}. \tag{6.30}$$

Recalling (6.20), we deduce from (6.3b) and (6.6) that

$$\|\nabla \mathbf{w}_\varepsilon\|_{L^2(B_r(x_0))} \leqslant 4\hat{\epsilon} \;\text{ on } B_r(x_0) \;\text{ for } \varepsilon \leqslant r/4. \tag{6.31}$$

Recalling that $r < 1$, let χ be a C^2 cut-off function such that

$$\chi \equiv \begin{cases} 1 \text{ in } B_{r/2}(x_0) \\ 0 \text{ in } B_1(x_0)\backslash B_r(x_0) \end{cases} \quad \text{and } |\nabla^\ell \chi| \leqslant 8r^{-\ell} \text{ in } B_1(x_0) \text{ for } \ell \in \{1,2\}. \tag{6.32}$$

and let $\bar{\mathbf{w}}_\varepsilon := (\bar{\mathbf{v}}_\varepsilon, \bar{\rho}_\varepsilon)$ with

$$\bar{\rho}_\varepsilon = \chi(\rho_\varepsilon - 1) \quad \text{and} \quad \bar{\mathbf{v}}_\varepsilon = \chi \mathbf{v}_\varepsilon. \tag{6.33}$$

Multiplying (6.23) by χ and using the linearity of $\mathbf{a}_\parallel$ about $\mathbf{a}$ (cf. (6.21)), we find

$$\begin{aligned} &\rho_\varepsilon \Delta \bar{\mathbf{v}}_\varepsilon + \rho_\varepsilon[\chi, \Delta]\mathbf{v}_\varepsilon + \mu\Big([\chi, \nabla^2](\rho_\varepsilon - 1)\mathbf{v}_\varepsilon\Big)_\parallel + \mu\Big(\nabla^2 \bar{\rho}_\varepsilon \mathbf{v}_\varepsilon\Big)_\parallel \\ &\quad + \mu\rho_\varepsilon \Big([\chi, \nabla \operatorname{div}]\mathbf{v}_\varepsilon\Big)_\parallel + \mu\rho_\varepsilon \Big(\nabla(\operatorname{div} \bar{\mathbf{v}}_\varepsilon)\Big)_\parallel \\ &\quad = (\boldsymbol{\tau}_\varepsilon)_\parallel \chi + \mathcal{N}_{2,\varepsilon}(\chi \nabla \mathbf{w}_\varepsilon, \nabla \mathbf{w}_\varepsilon) \qquad \text{in } B_1(x_0), \end{aligned} \tag{6.34}$$

and $\bar{\mathbf{v}}_\varepsilon|_{\partial B_1(x_0)} = 0$. For brevity we denote $L^p(B_1(x_0))$ by L^p. Note that the commutators in (6.34) involve at most first order derivatives of $\mathbf{w}_\varepsilon = (\mathbf{v}_\varepsilon, \rho_\varepsilon)$, which satisfies (6.31). Now applying the L^p-estimate for elliptic equation [39, pp. 109] (componentwise) in (6.34), and invoking (6.31) and (6.6), we have

$$\|\nabla^2 \bar{\mathbf{v}}_\varepsilon\|_{L^{4/3}} \leqslant C_4\Big(r^{-2} + r^{-1} + \mu\|\nabla^2 \bar{\mathbf{w}}_\varepsilon\|_{L^{4/3}} + \big\|\mathcal{N}_{2,\varepsilon}(\chi \nabla \mathbf{w}_\varepsilon, \nabla \mathbf{w}_\varepsilon)\big\|_{L^{4/3}}\Big). \tag{6.35}$$

Note that the prefactors r^{-1} and r^{-2} are due to the differentiation of χ (cf. (6.32)), and that C_4 is independent of r. To estimate the last term, we employ the bi-linearity of $\mathcal{N}_{2,\varepsilon}$, (6.31) and (6.6):

$$\begin{aligned} &\big\|\mathcal{N}_{2,\varepsilon}(\chi \nabla \mathbf{w}_\varepsilon, \nabla \mathbf{w}_\varepsilon)\big\|_{L^{4/3}} \\ &\quad \leqslant \big\|\mathcal{N}_{2,\varepsilon}(\nabla \bar{\mathbf{w}}_\varepsilon, \nabla \mathbf{w}_\varepsilon)\big\|_{L^{4/3}} + \big\|\mathcal{N}_{2,\varepsilon}(\nabla \chi \otimes \mathbf{w}_\varepsilon, \nabla \mathbf{w}_\varepsilon)\big\|_{L^{4/3}} + C_4 r^{-1} \\ &\quad \leqslant C_4\left(\|\nabla \bar{\mathbf{w}}_\varepsilon\|_{L^4}\|\nabla \mathbf{w}_\varepsilon\|_{L^2} + r^{-1}\right) \\ &\quad \leqslant C_4\left(\|\nabla^2 \bar{\mathbf{w}}_\varepsilon\|_{L^{4/3}}\|\nabla \mathbf{w}_\varepsilon\|_{L^2} + r^{-1}\right). \end{aligned} \tag{6.36}$$

Note that in the last step we used the Sobolev's embedding $W^{1,4/3}(B_1) \subset L^4(B_1)$. Combining (6.36) with (6.35), we obtain

$$\|\nabla^2 \bar{\mathbf{v}}_\varepsilon\|_{L^{4/3}} \leqslant C_4\left(r^{-2} + \mu\|\nabla^2(\bar{\mathbf{v}}_\varepsilon, \bar{\rho}_\varepsilon)\|_{L^{4/3}} + \|\nabla^2 \bar{\mathbf{w}}_\varepsilon\|_{L^{4/3}}\|\nabla \mathbf{w}_\varepsilon\|_{L^2}\right). \tag{6.37}$$

Now we turn to the estimate of ρ_ε. Using (6.6) and (1.22), we have $f'(\rho_\varepsilon) = 2(\rho_\varepsilon - 1)$ in $B_r(x_0)$. Now multiplying (6.22) by χ and using the linearity of (6.21), we find

$$\begin{aligned} &-2\varepsilon^{-2}\bar{\rho}_\varepsilon + \Delta\bar{\rho}_\varepsilon + [\chi, \Delta](\rho_\varepsilon - 1) + \mu(\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) : \nabla^2 \bar{\rho}_\varepsilon \\ &\quad + \mu(\mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon) : [\chi, \nabla^2](\rho_\varepsilon - 1) + \mu\rho_\varepsilon \mathbf{v}_\varepsilon \cdot (\nabla \operatorname{div} \bar{\mathbf{v}}_\varepsilon) + \mu\rho_\varepsilon \mathbf{v}_\varepsilon \cdot ([\chi, \nabla \operatorname{div}]\mathbf{v}_\varepsilon) \\ &= \chi \boldsymbol{\tau}_\varepsilon \cdot \mathbf{v}_\varepsilon + \mathcal{N}_{1,\varepsilon}(\chi \nabla \mathbf{w}_\varepsilon, \nabla \mathbf{w}_\varepsilon). \end{aligned}$$

In the same way as we did for (6.37), we find

$$\begin{aligned} \|\nabla^2 \bar{\rho}_\varepsilon\|_{L^{4/3}} &\leqslant C_4\left(r^{-2} + \mu\|\nabla^2(\bar{\mathbf{v}}_\varepsilon, \bar{\rho}_\varepsilon)\|_{L^{4/3}} + \big\|\mathcal{N}_{1,\varepsilon}(\chi \nabla \mathbf{w}_\varepsilon, \nabla \mathbf{w}_\varepsilon)\big\|_{L^{4/3}}\right) \\ &\leqslant C_4\left(r^{-2} + \mu\|\nabla^2(\bar{\mathbf{v}}_\varepsilon, \bar{\rho}_\varepsilon)\|_{L^{4/3}} + \|\nabla^2 \bar{\mathbf{w}}_\varepsilon\|_{L^{4/3}}\|\nabla \mathbf{w}_\varepsilon\|_{L^2}\right). \end{aligned}$$

Combining this with (6.37) and (6.31) we discover

$$\|\nabla^2(\bar{\mathbf{v}}_\varepsilon, \bar{\rho}_\varepsilon)\|_{L^{4/3}(B_r(x_0))} \leqslant C_4\left(r^{-2} + \max\{\hat{\epsilon}, \mu\}\|\nabla^2(\bar{\mathbf{v}}_\varepsilon, \bar{\rho}_\varepsilon)\|_{L^{4/3}(B_r(x_0))}\right). \tag{6.38}$$

Note that before Lemma 6.1, we have assumed that $C_4 > 1$ and $\hat{c} > 1$. Recall also the choice of $\hat{\epsilon}$ in (6.11). By choosing

$$\Lambda = 16^{-8}C_4^{-2} \text{ in(6.3a)},$$

we find $C_4 \max\{\hat{\epsilon}, \mu\} < 1/2$. This combined with (6.38) yields

$$\|\nabla^2(\bar{\mathbf{v}}_\varepsilon, \bar{\rho}_\varepsilon)\|_{L^{4/3}(B_r(x_0))} \leqslant 2C_4 r^{-2}.$$

In view of (6.32) and (6.33), this implies (6.30).

Now using (6.2b), we have $\rho_{\varepsilon_k}(\cdot, t_0) \xrightarrow{k\to\infty} |\mathbf{u}|(\cdot, t_0) = 1$ strongly in $L^2(B_r(x_0))$. Thus, using (6.6) we find

$$\|\mathbf{v}_{\varepsilon_k} - \mathbf{u}\|^2_{L^2(B_r(x_0))} \leqslant 2\|\mathbf{u}_{\varepsilon_k} - \mathbf{u}\rho_{\varepsilon_k}\|^2_{L^2(B_r(x_0))} \xrightarrow{k\to\infty} 0.$$

These together with (6.30) and the Gagliardo-Nirenberg interpolation inequality yield

$$\left(\mathbf{v}_{\varepsilon_k}, \rho_{\varepsilon_k}\right) \xrightarrow{k\to\infty} (\mathbf{u}, 1) \text{ strongly in } W^{1,2}(B_{r/2}(x_0)). \tag{6.39}$$

Finally, using (6.6) and (6.39) we find

$$\nabla\mathbf{u}_{\varepsilon_k} = \rho_{\varepsilon_k}\nabla\mathbf{v}_{\varepsilon_k} + \mathbf{v}_{\varepsilon_k}\nabla\rho_{\varepsilon_k} \xrightarrow{k\to\infty} \nabla\mathbf{u} \text{ strongly in } L^2(B_{r/2}(x_0)),$$

and finish the proof of (6.4). □

Proof of Theorem 1.2 We employ the covering argument in [14]. For any test function $\mathbf{\Psi} \in C_c^1(\Omega_t^+; \mathbb{R}^3)$, we choose $K = \overline{\operatorname{supp}(\mathbf{\Psi})} \subset\subset \Omega_t^+$, and we define the singular set at time $t \in (0, T]$ by

$$\Sigma_t := \bigcap_{0<r<1}\left\{x \in K \mid B_{2r}(x) \subset K, \varliminf_{k\to\infty}\int_{B_{2r}(x)}\left(\frac{1}{2}\left|\nabla\mathbf{u}_{\varepsilon_k}\right|^2 + \frac{F(\mathbf{u}_{\varepsilon_k})}{\varepsilon_k^2}\right)dx > \frac{\hat{\epsilon}^2}{2}\right\}. \tag{6.40}$$

We claim that Σ_t is discrete. Indeed, choose an arbitrary finite set $\{y_j\}_{j=1}^J \subset \Sigma_t$ with mutually disjoint balls $\{B_{2r_j}(y_j)\}_{j=1}^J$ inside K with $r_j < 1/2$. Since J is finite, there exists $k_J > 0$ such that for any $k \geqslant k_J$ we have

$$\int_{B_{2r_j}(y_j)}\left(\frac{1}{2}\left|\nabla\mathbf{u}_{\varepsilon_k}\right|^2 + \frac{F(\mathbf{u}_{\varepsilon_k})}{\varepsilon_k^2}\right)dx > \frac{\hat{\epsilon}^2}{4} \quad \text{for all } j \in \{1, \cdots, J\}. \tag{6.41}$$

Combined with (6.2a), this implies

$$\hat{c}^2 \geqslant \int_{\bigsqcup_{j=1}^J B_{2r_j}(y_j)}\left(\frac{1}{2}\left|\nabla\mathbf{u}_{\varepsilon_k}\right|^2 + \frac{F(\mathbf{u}_{\varepsilon_k})}{\varepsilon_k^2}\right)dx > \frac{\hat{\epsilon}^2}{4}J. \tag{6.42}$$

As a result, $J \leqslant 4\hat{c}^2\hat{\epsilon}^{-2}$ and thus Σ_t is discrete. Therefore w.l.o.g. we can assume that $\Sigma_t = \{x_0\}$ and $B_{2r}(x_0) \subset K$. Let $\eta \in C_c^1(B_2(0))$ be a cut-off function which $\equiv 1$ in $B_1(0)$. Then

$$\mathbf{\Psi}_\delta(x) := \mathbf{\Psi}(x)\left(1 - \eta(\tfrac{x-x_0}{\delta})\right) \xrightarrow{\delta\to 0} \mathbf{\Psi}(x) \text{ for any } x \neq x_0. \tag{6.43}$$

It is obvious that $\mathbf{\Psi}_\delta = 0$ in $B_\delta(x_0)$. By (6.40) and Proposition 6.1, we have

$$\nabla \mathbf{u}_{\varepsilon_k} \xrightarrow{k\to\infty} \nabla \mathbf{u} \text{ strongly in } L^2(K\backslash B_\delta(x_0)). \tag{6.44}$$

Using these properties, we can apply $\wedge \mathbf{u}_{\varepsilon_k} \cdot \mathbf{\Psi}_\delta$ to both sides of (1.2a), integrate by parts and then send $k \to \infty$:

$$\int_\Omega \partial_t \mathbf{u} \wedge \mathbf{u} \cdot \mathbf{\Psi}_\delta \, dx + \mu \int_\Omega (\operatorname{div} \mathbf{u}) (\operatorname{rot} \mathbf{u}) \cdot \mathbf{\Psi}_\delta \, dx$$
$$+ \int_\Omega (\nabla \mathbf{u} \wedge \mathbf{u}) \cdot \nabla \mathbf{\Psi}_\delta \, dx - \mu \int_\Omega (\operatorname{div} \mathbf{u}) (\operatorname{rot} \mathbf{\Psi}_\delta) \cdot \mathbf{u} \, dx = 0. \tag{6.45}$$

Note that we have also used $\partial_t \mathbf{u}_{\varepsilon_k} \wedge \mathbf{u}_{\varepsilon_k} \xrightarrow{k\to\infty} \partial_t \mathbf{u} \wedge \mathbf{u}$ weakly in $L^2(0, T; L^{6/5}(\Omega))$, which is due to Proposition 4.1. By (6.43) and the regularity of $\mathbf{u}$ (cf. (4.26a) and (4.26b)), we can send $\delta \to 0$ in the first and the second integrals in (6.45) using the dominated convergence theorem. Concerning the third one, we have

$$\int_\Omega (\nabla \mathbf{u} \wedge \mathbf{u}) \cdot \nabla \mathbf{\Psi}_\delta \, dx = \int_\Omega \left(1 - \eta(\tfrac{x-x_0}{\delta})\right) (\nabla \mathbf{u} \wedge \mathbf{u}) \cdot \nabla \mathbf{\Psi} \, dx$$
$$- \sum_{i=1}^{3} \int_{B_{2\delta}(x_0)} \frac{1}{\delta} (\partial_i \eta)(\tfrac{x-x_0}{\delta}) \, \partial_i \mathbf{u} \wedge \mathbf{u} \cdot \mathbf{\Psi} \, dx. \tag{6.46}$$

We claim that the second integral on the right-hand side vanishes as $\delta \to 0$. Indeed, by the Cauchy–Schwarz inequality we have

$$\left| \sum_{i=1}^{3} \int_{B_{2\delta}(x_0)} \frac{1}{\delta} (\partial_i \eta)(\tfrac{x-x_0}{\delta}) \, \partial_i \mathbf{u} \wedge \mathbf{u} \cdot \mathbf{\Psi} \, dx \right|$$
$$\leqslant C \|\mathbf{\Psi}\|_{L^\infty} \|\nabla \eta\|_{L^2(B_2)} \|\nabla \mathbf{u}\|_{L^2(B_{2\delta}(x_0))} \xrightarrow{\delta\to 0} 0. \tag{6.47}$$

Now using $\lim_{\delta\to 0} \eta(\frac{x-x_0}{\delta}) = 0$ for any $x \neq x_0$, we can send $\delta \to 0$ in (6.46) and obtain

$$\int_\Omega (\nabla \mathbf{u} \wedge \mathbf{u}) \cdot \nabla \mathbf{\Psi}_\delta \, dx \xrightarrow{\delta\to 0} \int_\Omega (\nabla \mathbf{u} \wedge \mathbf{u}) \cdot \nabla \mathbf{\Psi} \, dx.$$

By the same argument we can compute the fourth integral in (6.45) and find

$$\int_\Omega (\operatorname{div} \mathbf{u}) (\operatorname{rot} \mathbf{\Psi}_\delta) \cdot \mathbf{u} \, dx \xrightarrow{\delta\to 0} \int_\Omega (\operatorname{div} \mathbf{u}) (\operatorname{rot} \mathbf{\Psi}) \cdot \mathbf{u} \, dx. \tag{6.48}$$

Using the above two formulas, we can send $\delta \to 0$ in (6.45) and obtain (1.23). □

Appendix A: Proof of Proposition 1.1

Proof of Proposition 1.1 We first recall that $\sigma = 1$ (cf. (2.5)), $I_0 \subset \Omega$ is the initial interface and η_0 is the cut-off function in (2.12). Then we define

$$s_\varepsilon(x) := \eta_0(x) \, \theta \left(\frac{d_{I_0}(x)}{\varepsilon} \right) + \Big(1 - \eta_0(x)\Big) \mathbf{1}_{\Omega_0^+}, \tag{A.1}$$

where $\theta(z)$ is the solution of the ODE

$$-\theta''(z) + f'(\theta) = 0, \quad \theta(-\infty) = 0, \ \theta(+\infty) = 1. \tag{A.2}$$

We note that d_{I_0} is Lipschitz continuous in Ω, and thus by Rademacher's theorem we have $|\nabla d_{I_0}| \leqslant 1$ a.e. in Ω. Recalling (1.19), we define

$$\mathbf{u}_\varepsilon^{in}(x) := s_\varepsilon(x)\mathbf{u}^{in}(x). \tag{A.3}$$

One can verify that $\mathbf{u}_\varepsilon^{in} \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\|\mathbf{u}_\varepsilon^{in}\|_{L^\infty(\Omega)} \leqslant 1$, and

$$\mathbf{u}_\varepsilon^{in} = \begin{cases} \mathbf{u}^{in} & \text{if } x \in \Omega_0^+ \backslash B_{2\delta_0}(I_0), \\ \theta\left(\frac{d_{I_0}}{\varepsilon}\right)\mathbf{u}^{in} & \text{if } x \in B_{\delta_0}(I_0), \\ 0 & \text{if } x \in \Omega_0^- \backslash B_{2\delta_0}(I_0). \end{cases} \tag{A.4}$$

So the condition (1.14a) is verified. To verify the others, we first compute the modulated energy in (1.7) for the initial datum $\mathbf{u}_\varepsilon^{in}$. We write (A.1) as

$$s_\varepsilon(x) = \theta\left(\frac{d_{I_0}(x)}{\varepsilon}\right) + \hat{s}_\varepsilon(x), \tag{A.5}$$

where $\hat{s}_\varepsilon(x) := (1 - \eta_0(x))\left(\mathbf{1}_{\Omega_0^+} - \theta\left(\frac{d_{I_0}(x)}{\varepsilon}\right)\right)$. Invoking (2.12) and the exponential convergence of $\theta(z)$ as $z \to \pm\infty$ (cf. (A.2)), we deduce that

$$\|\hat{s}_\varepsilon\|_{L^\infty(\Omega)} + \|\nabla\hat{s}_\varepsilon\|_{L^\infty(\Omega)} \leqslant Ce^{-\frac{C}{\varepsilon}}, \tag{A.6}$$

for some constant $C > 0$ that only depends on I_0. By a Taylor's expansion, we find

$$F(\mathbf{u}_\varepsilon^{in}) = f(\theta + \hat{s}_\varepsilon) = f(\theta) + O(e^{-C/\varepsilon}).$$

Combining (A.3), (A.5) with (A.6), we obtain

$$|\nabla \mathbf{u}_\varepsilon^{in}|^2 = \varepsilon^{-2}\theta'^2 + \theta^2|\nabla\mathbf{u}^{in}|^2 + O(e^{-C/\varepsilon})(|\nabla\mathbf{u}^{in}|^2 + 1).$$

Note that we have also employed the identities $\partial_{x_i}\mathbf{u}^{in} \cdot \mathbf{u}^{in} = 0$ a.e. in Ω. Recalling (1.8), we have

$$\psi_\varepsilon\Big|_{t=0} = \int_0^{\theta+\hat{s}_\varepsilon} \sqrt{2f(s)}\,ds : \Omega \mapsto [0, 1]. \tag{A.7}$$

So we can compute

$$\begin{aligned} &\frac{\varepsilon}{2}\left|\nabla\mathbf{u}_\varepsilon^{in}\right|^2 + \frac{1}{\varepsilon}F(\mathbf{u}_\varepsilon^{in}) - \boldsymbol{\xi}\cdot\nabla\psi_\varepsilon\Big|_{t=0} \\ &\quad= \frac{1}{2\varepsilon}\theta'^2 + \frac{1}{\varepsilon}f(\theta) - \varepsilon^{-1}\boldsymbol{\xi}\cdot\mathbf{n}_{I_0}\theta'\sqrt{2f(\theta)} + \frac{\varepsilon}{2}\theta^2|\nabla\mathbf{u}^{in}|^2 + O(e^{-C/\varepsilon})(|\nabla\mathbf{u}^{in}|^2 + 1). \end{aligned} \tag{A.8}$$

It follows from (2.10) that $1 - \boldsymbol{\xi}\cdot\mathbf{n}_{I_0} = O(d_I^2)$. So we have

$$\varepsilon^{-1}\boldsymbol{\xi}\cdot\mathbf{n}_{I_0}\theta'\sqrt{2f(\theta)} = \varepsilon^{-1}\theta'\sqrt{2f(\theta)} + O(e^{-C/\varepsilon}) + \varepsilon^{-1}O(d_{I_0}^2)\theta'\sqrt{2f(\theta)}.$$

Note that the last term can be written as

$$\varepsilon^{-1}O(d_{I_0}^2)\theta'\sqrt{2f(\theta)} = O(\varepsilon)z^2\theta'(z)\sqrt{2f(\theta(z))}\Big|_{z=\frac{d_{I_0}(x)}{\varepsilon}}.$$

Substituting the above two equations into (A.8), we find

$$\int_\Omega \left(\frac{\varepsilon}{2}\left|\nabla \mathbf{u}_\varepsilon^{in}\right|^2 + \frac{1}{\varepsilon}F(\mathbf{u}_\varepsilon^{in}) - \boldsymbol{\xi}\cdot\nabla\psi_\varepsilon\right)dx$$
$$= \underbrace{\int_\Omega \left(\frac{1}{2\varepsilon}\theta'^2 + \frac{1}{\varepsilon}f(\theta) - \frac{1}{\varepsilon}\theta'\sqrt{2f(\theta)}\right)dx}_{=0}$$
$$+\int_\Omega \frac{\varepsilon}{2}\theta^2|\nabla \mathbf{u}^{in}|^2\,dx + O(e^{-C/\varepsilon})\int_\Omega(|\nabla \mathbf{u}^{in}|^2+1)\,dx + O(\varepsilon) \qquad \text{at } t=0. \tag{A.9}$$

Note that the integrand of the first integral on the right-hand side of (A.9) vanishes due to the identity $\theta'^2(z) = 2f(\theta(z))$, which follows from (A.2). Now we turn to the first term in (1.7). Using (A.6) we can estimate

$$|\operatorname{div}\mathbf{u}_\varepsilon^{in}|^2 \leqslant 2|\nabla\theta\cdot\mathbf{u}^{in}|^2 + 2\theta^2|\operatorname{div}\mathbf{u}^{in}|^2 + O(e^{-C/\varepsilon})(1+|\operatorname{div}\mathbf{u}^{in}|^2). \tag{A.10}$$

By the exponential decay of $\theta'(z)$ as $z\to\pm\infty$, we deduce that

$$|\nabla\theta\cdot\mathbf{u}^{in}| = \begin{cases} \left|\frac{d_{I_0}}{\varepsilon}\theta'\left(\frac{d_{I_0}}{\varepsilon}\right)\frac{\mathbf{u}^{in}\cdot\mathbf{n}_{I_0}}{d_{I_0}}\right| \leqslant C\left|\frac{\mathbf{u}^{in}\cdot\mathbf{n}_{I_0}}{d_{I_0}}\right| & \text{in } B_{\delta_0}(I_0)\backslash I_0,\\ \left|\frac{1}{\varepsilon}\theta'\left(\frac{d_{I_0}}{\varepsilon}\right)\mathbf{u}^{in}\cdot\mathbf{n}_{I_0}\right| \leqslant e^{-\frac{C}{\varepsilon}} & \text{in } \Omega\backslash B_{\delta_0}(I_0). \end{cases} \tag{A.11}$$

Using this, (1.19) and Hardy's inequality (cf. [7]), we find

$$\int_\Omega |\nabla\theta\cdot\mathbf{u}^{in}|^2\,dx \leqslant C\int_{I_0}\int_{-\delta_0}^{\delta_0}\left|\frac{\mathbf{u}^{in}\cdot\mathbf{n}_{I_0}}{d_{I_0}}\right|^2 dr\,d\mathcal{H}^{d-1} + C$$
$$\leqslant C\left(\int_\Omega |\nabla\mathbf{u}^{in}|^2\,dx + 1\right). \tag{A.12}$$

Combining this with (A.10) and (A.9) we derive $E_\varepsilon[\mathbf{u}_\varepsilon^{in}|I_0] \leqslant C\varepsilon$. Recalling (1.21), we have also obtained (1.14b). To verify (1.14c), we shall compute (1.12) at $t=0$. By (A.7), we see that

$$B[\mathbf{u}_\varepsilon^{in}|I_0] = 2\int_\Omega\left(\tfrac{\chi+1}{2} - \psi_\varepsilon\right)\eta\circ d_I\,dx.$$

We shall only give the estimate in $B_{\delta_0}(I_0)\cap\Omega_0^+$ because the one in $B_{\delta_0}(I_0)\cap\Omega_0^-$ follows in the same way, and the one in $\Omega\backslash B_{\delta_0}(I_0)$ is due to (A.6) and the exponential convergence of $\theta(z)$ at $\pm\infty$.

$$\int_{B_{\delta_0}(I_0)\cap\Omega_0^+}|\psi_\varepsilon - 1|d_I(x)\,dx\Big|_{t=0}$$
$$\overset{(A.7)}{=}\int_{B_{\delta_0}(I_0)\cap\Omega_0^+}\left(\int_{s_\varepsilon(x)}^1\sqrt{2f(s)}\,ds\right)d_I(x)\,dx\Big|_{t=0}$$
$$\overset{(A.6)}{=}\varepsilon\int_{B_{\delta_0}(I_0)\cap\Omega_0^+}\left(\int_{\theta(\frac{d_I(x)}{\varepsilon})}^1\sqrt{2f(s)}\,ds\right)\frac{d_I(x)}{\varepsilon}\,dx\Big|_{t=0} + O(e^{-C/\varepsilon}) \leqslant C\varepsilon^2, \tag{A.13}$$

where the last step is due to the exponential decay of $Q(z) := z\int_{\theta(z)}^1\sqrt{2f(s)}\,ds$ as $z\uparrow\infty$.
□

Appendix B: Proof of Proposition 2.1

Lemma B.1 *The following identity holds:*

$$
\begin{aligned}
&\int \nabla \mathbf{H} : (\boldsymbol{\xi} \otimes \mathbf{n}_\varepsilon)\, |\nabla \psi_\varepsilon|\; dx - \int (\nabla \cdot \mathbf{H})\, \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon\, dx \\
&= \int \nabla \mathbf{H} : (\boldsymbol{\xi} - \mathbf{n}_\varepsilon) \otimes \mathbf{n}_\varepsilon\, |\nabla \psi_\varepsilon|\; dx + \int \mathbf{H}_\varepsilon \cdot \mathbf{H} |\nabla \mathbf{u}_\varepsilon|\, dx \\
&\quad + \int \nabla \cdot \mathbf{H} \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - |\nabla \psi_\varepsilon| \right) dx + \int \nabla \cdot \mathbf{H} (|\nabla \psi_\varepsilon| - \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon)\, dx \\
&\quad - \int (\nabla \mathbf{H})_{ij}\, \varepsilon \left(\partial_i \mathbf{u}_\varepsilon \cdot \partial_j \mathbf{u}_\varepsilon \right) dx + \int \nabla \mathbf{H} : (\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon)\, |\nabla \psi_\varepsilon|\; dx. \qquad \text{(B.1)}
\end{aligned}
$$

Proof We introduce the stress tensor $(T_\varepsilon)_{ij} := \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) \right) \delta_{ij} - \varepsilon \partial_i \mathbf{u}_\varepsilon \cdot \partial_j \mathbf{u}_\varepsilon$. By (2.20b), we have the identity $\nabla \cdot T_\varepsilon = \mathbf{H}_\varepsilon |\nabla \mathbf{u}_\varepsilon|$. Testing this identity with $\mathbf{H}$, integrating by parts and using (2.14c), we obtain

$$
\begin{aligned}
\int \mathbf{H}_\varepsilon \cdot \mathbf{H} |\nabla \mathbf{u}_\varepsilon|\, dx &= - \int \nabla \mathbf{H} \colon T_\varepsilon\, dx \\
&= - \int \nabla \cdot \mathbf{H} \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) \right) dx + \int (\nabla \mathbf{H})_{ij}\, \varepsilon \left(\partial_i \mathbf{u}_\varepsilon \cdot \partial_j \mathbf{u}_\varepsilon \right) dx.
\end{aligned}
$$

So adding zero leads to

$$
\begin{aligned}
&\int \nabla \mathbf{H} : \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon\, |\nabla \psi_\varepsilon|\, dx \\
&= \int \mathbf{H}_\varepsilon \cdot \mathbf{H} |\nabla \mathbf{u}_\varepsilon|\, dx + \int \nabla \cdot \mathbf{H} \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - |\nabla \psi_\varepsilon| \right) dx + \int \nabla \cdot \mathbf{H} |\nabla \psi_\varepsilon|\, dx \\
&- \int (\nabla \mathbf{H})_{ij}\, \varepsilon \left(\partial_i \mathbf{u}_\varepsilon \cdot \partial_j \mathbf{u}_\varepsilon \right) dx + \int (\nabla \mathbf{H}) : (\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon)\, |\nabla \psi_\varepsilon|\, dx,
\end{aligned}
$$

which yields (B.1). □

Lemma B.2 *Under the assumptions of Theorem 1.1, the following identity holds:*

$$
\begin{aligned}
&\frac{d}{dt} E\left[\mathbf{u}_\varepsilon | I\right] + \frac{1}{2\varepsilon} \int \left(\varepsilon^2 \left|\partial_t \mathbf{u}_\varepsilon\right|^2 - |\mathbf{H}_\varepsilon|^2 \right) dx \\
&\quad + \frac{1}{2\varepsilon} \int \left| \varepsilon \partial_t \mathbf{u}_\varepsilon - (\nabla \cdot \boldsymbol{\xi}) D\mathrm{d}^F(\mathbf{u}_\varepsilon) \right|^2 dx + \frac{1}{2\varepsilon} \int \left| \mathbf{H}_\varepsilon - \varepsilon |\nabla \mathbf{u}_\varepsilon| \mathbf{H} \right|^2 dx \\
&= \frac{1}{2\varepsilon} \int \left| (\nabla \cdot \boldsymbol{\xi}) |D\mathrm{d}^F(\mathbf{u}_\varepsilon)| \mathbf{n}_\varepsilon + \varepsilon |\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon| \mathbf{H} \right|^2 dx \qquad \text{(B.2a)} \\
&\quad + \frac{\varepsilon}{2} \int |\mathbf{H}|^2 \left(|\nabla \mathbf{u}_\varepsilon|^2 - |\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon|^2 \right) dx - \int \nabla \mathbf{H} \cdot (\boldsymbol{\xi} - \mathbf{n}_\varepsilon)^{\otimes 2}\, |\nabla \psi_\varepsilon|\; dx \qquad \text{(B.2b)} \\
&\quad + \int (\nabla \cdot \mathbf{H}) \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - |\nabla \psi_\varepsilon| \right) dx \qquad \text{(B.2c)} \\
&\quad + \int (\nabla \cdot \mathbf{H})\, (1 - \boldsymbol{\xi} \cdot \mathbf{n}_\varepsilon)\, |\nabla \psi_\varepsilon|\, dx + \int (J_\varepsilon^1 + J_\varepsilon^2)\, dx, \qquad \text{(B.2d)}
\end{aligned}
$$

where

$$
\begin{aligned}
J_\varepsilon^1 := &\nabla \mathbf{H} : \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon \left(|\nabla \psi_\varepsilon| - \varepsilon |\nabla \mathbf{u}_\varepsilon|^2\right) \\
&+ \varepsilon \nabla \mathbf{H} : (\mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon) \left(|\nabla \mathbf{u}_\varepsilon|^2 - |\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon|^2\right) \\
&- \sum_{ij} \varepsilon (\nabla \mathbf{H})_{ij} \Big((\partial_i \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \partial_i \mathbf{u}_\varepsilon) \cdot (\partial_j \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \partial_j \mathbf{u}_\varepsilon) \Big),
\end{aligned} \tag{B.3}
$$

$$
J_\varepsilon^2 := - \left(\partial_t \boldsymbol{\xi} + (\mathbf{H} \cdot \nabla) \boldsymbol{\xi} + (\nabla \mathbf{H})^{\mathsf{T}} \boldsymbol{\xi} \right) \cdot \nabla \psi_\varepsilon. \tag{B.4}
$$

Proof We shall employ the Einstein summation convention by summing over repeated indices. Using the energy dissipation law in (2.6) and adding zero, we find

$$
\begin{aligned}
&\frac{d}{dt} E_\varepsilon[\mathbf{u}_\varepsilon | I] + \varepsilon \int |\partial_t \mathbf{u}_\varepsilon|^2 \, dx - \int (\nabla \cdot \boldsymbol{\xi}) D\mathrm{d}^F(\mathbf{u}_\varepsilon) \cdot \partial_t \mathbf{u}_\varepsilon \, dx \\
&= \int (\mathbf{H} \cdot \nabla) \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \, dx + \int (\nabla \mathbf{H})^{\mathsf{T}} \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \, dx + \int J_\varepsilon^2 \, dx.
\end{aligned} \tag{B.5}
$$

By the symmetry of $\nabla^2 \psi_\varepsilon$ and the boundary conditions in (2.14c), we have

$$
\int \nabla \cdot (\boldsymbol{\xi} \otimes \mathbf{H}) \cdot \nabla \psi_\varepsilon \, dx = \int \nabla \cdot (\mathbf{H} \otimes \boldsymbol{\xi}) \cdot \nabla \psi_\varepsilon \, dx.
$$

Hence, the first integral on the right-hand side of (B.5) can be rewritten as

$$
\begin{aligned}
&\int (\mathbf{H} \cdot \nabla) \, \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \, dx \\
&= \int \nabla \cdot (\boldsymbol{\xi} \otimes \mathbf{H}) \cdot \nabla \psi_\varepsilon \, dx - \int (\nabla \cdot \mathbf{H}) \, \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \, dx \\
&= \int (\nabla \cdot \boldsymbol{\xi}) \, \mathbf{H} \cdot \nabla \psi_\varepsilon \, dx + \int (\boldsymbol{\xi} \cdot \nabla) \, \mathbf{H} \cdot \nabla \psi_\varepsilon \, dx - \int (\nabla \cdot \mathbf{H}) \, \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \, dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt} E_\varepsilon[\mathbf{u}_\varepsilon | I] + \varepsilon \int |\partial_t \mathbf{u}_\varepsilon|^2 \, dx - \int (\nabla \cdot \boldsymbol{\xi}) D\mathrm{d}^F(\mathbf{u}_\varepsilon) \cdot \partial_t \mathbf{u}_\varepsilon \, dx \\
&\quad = \int (\nabla \cdot \boldsymbol{\xi}) \, \mathbf{H} \cdot \nabla \psi_\varepsilon dx + \int (\boldsymbol{\xi} \cdot \nabla) \, \mathbf{H} \cdot \nabla \psi_\varepsilon \, dx - \int (\nabla \cdot \mathbf{H}) \, \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \, dx \\
&\qquad + \int \nabla \mathbf{H} : (\boldsymbol{\xi} \otimes \mathbf{n}_\varepsilon) \, |\nabla \psi_\varepsilon| \, dx + \int J_\varepsilon^2 \, dx.
\end{aligned}
$$

Now using (B.1) to replace the third and the fourth integrals on the right-hand side of the above equation, we find

$$
\begin{aligned}
&\frac{d}{dt} E_\varepsilon[\mathbf{u}_\varepsilon | I] + \varepsilon \int |\partial_t \mathbf{u}_\varepsilon|^2 \, dx - \int (\nabla \cdot \boldsymbol{\xi}) D\mathrm{d}^F(\mathbf{u}_\varepsilon) \cdot \partial_t \mathbf{u}_\varepsilon \, dx \\
&\quad = \int (\nabla \cdot \boldsymbol{\xi}) \, \mathbf{H} \cdot \nabla \psi_\varepsilon \, dx + \int (\boldsymbol{\xi} \cdot \nabla) \, \mathbf{H} \cdot \nabla \psi_\varepsilon \, dx + \int \nabla \mathbf{H} : (\boldsymbol{\xi} - \mathbf{n}_\varepsilon) \otimes \mathbf{n}_\varepsilon \, |\nabla \psi_\varepsilon| \, dx \\
&\qquad + \int \mathbf{H}_\varepsilon \cdot \mathbf{H} |\nabla \mathbf{u}_\varepsilon| \, dx + \int \nabla \cdot \mathbf{H} \left(\frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) - |\nabla \psi_\varepsilon| \right) dx \\
&\qquad + \int \nabla \cdot \mathbf{H} \, (|\nabla \psi_\varepsilon| - \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon) \, dx - \int (\nabla \mathbf{H})_{ij} \, \varepsilon \left(\partial_i \mathbf{u}_\varepsilon \cdot \partial_j \mathbf{u}_\varepsilon \right) \, dx \\
&\qquad + \int \nabla \mathbf{H} : \mathbf{n}_\varepsilon \otimes \mathbf{n}_\varepsilon \, |\nabla \psi_\varepsilon| \, dx + \int J_\varepsilon^2 \, dx.
\end{aligned} \tag{B.6}
$$

We shall show that J_ε^1 arises from the second and the third to last integrals by proving the following identity:

$$\Pi_{\mathbf{u}_\varepsilon}\partial_i\mathbf{u}_\varepsilon\cdot\Pi_{\mathbf{u}_\varepsilon}\partial_j\mathbf{u}_\varepsilon=n_\varepsilon^i n_\varepsilon^j|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|^2\ \text{ a.e. in }\Omega, \tag{B.7}$$

where $(n_\varepsilon^\ell)_{1\leqslant\ell\leqslant3}=\mathbf{n}_\varepsilon$. Such an identity holds obviously on the set $\{x\mid\mathbf{u}_\varepsilon=0\}$ by (2.22). It also holds on $\{x\mid g(|\mathbf{u}_\varepsilon|)>0\}$ due to the following identity which follows from (2.22) and (2.23a):

$$\Pi_{\mathbf{u}_\varepsilon}\partial_i\mathbf{u}_\varepsilon\cdot\Pi_{\mathbf{u}_\varepsilon}\partial_j\mathbf{u}_\varepsilon|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2=\partial_i\psi_\varepsilon\,\partial_j\psi_\varepsilon=n_\varepsilon^i\,n_\varepsilon^j|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|^2|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2.$$

On the open set $\{x\mid|\mathbf{u}_\varepsilon|>0\}$ which includes $\{x\mid|\mathbf{u}_\varepsilon|=1\}$, we deduce from (2.22) and (2.19a) that $\Pi_{\mathbf{u}_\varepsilon}\partial_j\mathbf{u}_\varepsilon=(\partial_j|\mathbf{u}_\varepsilon|)\,\mathbf{u}_\varepsilon$. By [18, Theorem 4.4] we have $\partial_j|\mathbf{u}_\varepsilon|=0$ a.e. on $\{x\mid|\mathbf{u}_\varepsilon|=1\}$. We thus complete the proof of (B.7).

Now by (B.7) and adding zero, we find

$$\begin{aligned}&\nabla\mathbf{H}:\mathbf{n}_\varepsilon\otimes\mathbf{n}_\varepsilon\,|\nabla\psi_\varepsilon|-(\nabla\mathbf{H})_{ij}\,\varepsilon\left(\partial_i\mathbf{u}_\varepsilon\cdot\partial_j\mathbf{u}_\varepsilon\right)\\&\overset{(2.22)}{=}\nabla\mathbf{H}:\mathbf{n}_\varepsilon\otimes\mathbf{n}_\varepsilon\,|\nabla\psi_\varepsilon|-\varepsilon(\nabla\mathbf{H})_{ij}(\Pi_{\mathbf{u}_\varepsilon}\partial_i\mathbf{u}_\varepsilon\cdot\Pi_{\mathbf{u}_\varepsilon}\partial_j\mathbf{u}_\varepsilon)\\&\qquad-(\nabla\mathbf{H})_{ij}\,\varepsilon\Big((\partial_i\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\partial_i\mathbf{u}_\varepsilon)\cdot(\partial_j\mathbf{u}_\varepsilon-\Pi_{\mathbf{u}_\varepsilon}\partial_j\mathbf{u}_\varepsilon)\Big)\overset{(B.3)}{=}J_\varepsilon^1\ \text{ a.e. in }\Omega.\end{aligned}$$

Using the identities $\nabla\psi_\varepsilon=\mathbf{n}_\varepsilon|\nabla\psi_\varepsilon|$ and $\nabla\mathbf{H}:(\boldsymbol{\xi}\otimes\boldsymbol{\xi})=0$ (due to (2.14b)), we merge the second and the third integrals on the right-hand side of (B.6):

$$\begin{aligned}\frac{d}{dt}E_\varepsilon[\mathbf{u}_\varepsilon|I]=&-\varepsilon\int|\partial_t\mathbf{u}_\varepsilon|^2\,dx+\int(\nabla\cdot\boldsymbol{\xi})D\mathrm{d}^F(\mathbf{u}_\varepsilon)\cdot\partial_t\mathbf{u}_\varepsilon\,dx\\&+\int(\nabla\cdot\boldsymbol{\xi})\,\mathbf{H}\cdot\nabla\psi_\varepsilon\,dx+\int\mathbf{H}_\varepsilon\cdot\mathbf{H}|\nabla\mathbf{u}_\varepsilon|\,dx-\int\nabla\mathbf{H}:(\boldsymbol{\xi}-\mathbf{n}_\varepsilon)^{\otimes2}\,|\nabla\psi_\varepsilon|\,dx\\&+\int(\nabla\cdot\mathbf{H})\left(\frac{\varepsilon}{2}|\nabla\mathbf{u}_\varepsilon|^2+\frac{1}{\varepsilon}F(\mathbf{u}_\varepsilon)-|\nabla\psi_\varepsilon|\right)dx\\&+\int(\nabla\cdot\mathbf{H})\,(1-\boldsymbol{\xi}\cdot\mathbf{n}_\varepsilon)\,|\nabla\psi_\varepsilon|\,dx+\int(J_\varepsilon^1+J_\varepsilon^2)\,dx.\end{aligned}\tag{B.8}$$

Now we complete squares for the first four terms on the right-hand side of (B.8). Reordering terms, we have

$$\begin{aligned}&-\varepsilon|\partial_t\mathbf{u}_\varepsilon|^2+(\nabla\cdot\boldsymbol{\xi})D\mathrm{d}^F(\mathbf{u}_\varepsilon)\cdot\partial_t\mathbf{u}_\varepsilon+(\nabla\cdot\boldsymbol{\xi})\mathbf{H}\cdot\nabla\psi_\varepsilon+\mathbf{H}_\varepsilon\cdot\mathbf{H}|\nabla\mathbf{u}_\varepsilon|\\&=-\frac{1}{2\varepsilon}\Big(|\varepsilon\partial_t\mathbf{u}_\varepsilon|^2-2(\nabla\cdot\boldsymbol{\xi})D\mathrm{d}^F(\mathbf{u}_\varepsilon)\cdot\varepsilon\partial_t\mathbf{u}_\varepsilon+(\nabla\cdot\boldsymbol{\xi})^2|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2\Big)\\&\quad-\frac{1}{2\varepsilon}|\varepsilon\partial_t\mathbf{u}_\varepsilon|^2+\frac{1}{2\varepsilon}(\nabla\cdot\boldsymbol{\xi})^2|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2+(\nabla\cdot\boldsymbol{\xi})\mathbf{H}\cdot\nabla\psi_\varepsilon\\&\quad-\frac{1}{2\varepsilon}\Big(|\mathbf{H}_\varepsilon|^2-2\varepsilon|\nabla\mathbf{u}_\varepsilon|\mathbf{H}_\varepsilon\cdot\mathbf{H}+\varepsilon^2|\nabla\mathbf{u}_\varepsilon|^2|\mathbf{H}|^2\Big)+\frac{1}{2\varepsilon}\Big(|\mathbf{H}_\varepsilon|^2+\varepsilon^2|\nabla\mathbf{u}_\varepsilon|^2|\mathbf{H}|^2\Big)\\&=-\frac{1}{2\varepsilon}\Big|\varepsilon\partial_t\mathbf{u}_\varepsilon-(\nabla\cdot\boldsymbol{\xi})D\mathrm{d}^F(\mathbf{u}_\varepsilon)\Big|^2-\frac{1}{2\varepsilon}\Big|\mathbf{H}_\varepsilon-\varepsilon|\nabla\mathbf{u}_\varepsilon|\mathbf{H}\Big|^2-\frac{1}{2\varepsilon}|\varepsilon\partial_t\mathbf{u}_\varepsilon|^2+\frac{1}{2\varepsilon}|\mathbf{H}_\varepsilon|^2\\&\quad+\frac{1}{2\varepsilon}\Big((\nabla\cdot\boldsymbol{\xi})^2|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|^2+2\varepsilon(\nabla\cdot\boldsymbol{\xi})\nabla\psi_\varepsilon\cdot\mathbf{H}+|\varepsilon\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|^2|\mathbf{H}|^2\Big)\\&\quad+\frac{\varepsilon}{2}\left(|\nabla\mathbf{u}_\varepsilon|^2-|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|^2\right)|\mathbf{H}|^2.\end{aligned}$$

Substituting this identity into (B.8), we arrive at (B.2). □

Proof of Proposition 2.1 The proof here is the same as the case $\mu = 0$, done in [40, Lemma 4.4]. This is because the form of the energy dissipation law (2.6) remains unchanged in the presence of the divergence term in (1.2a).

We first estimate the right-hand side of (B.2) by $E_\varepsilon[\mathbf{u}_\varepsilon|I]$ up to a constant that only depends on I_t. Concerning (B.2a), it follows from the triangle inequality that

$$\begin{aligned}
&\int \left|\frac{1}{\sqrt{\varepsilon}}(\nabla\cdot\boldsymbol{\xi})|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|\mathbf{n}_\varepsilon + \sqrt{\varepsilon}|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|\mathbf{H}\right|^2 dx\\
&\leqslant \int \left|(\nabla\cdot\boldsymbol{\xi})\left(\frac{1}{\sqrt{\varepsilon}}|D\mathrm{d}^F(\mathbf{u}_\varepsilon)| - \sqrt{\varepsilon}|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|\right)\mathbf{n}_\varepsilon\right|^2 dx\\
&\quad + \int \left|(\nabla\cdot\boldsymbol{\xi})\sqrt{\varepsilon}|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|(\mathbf{n}_\varepsilon - \boldsymbol{\xi})\right|^2 dx\\
&\quad + \int \left|\big((\nabla\cdot\boldsymbol{\xi})\boldsymbol{\xi} + \mathbf{H}\big)\sqrt{\varepsilon}|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|\right|^2 dx.
\end{aligned}$$

The first integral on the right-hand side of the above inequality is controlled using (2.26c). Due to the elementary inequality $|\boldsymbol{\xi} - \mathbf{n}_\varepsilon|^2 \leqslant 2(1 - \mathbf{n}_\varepsilon\cdot\boldsymbol{\xi})$, the second integral is controlled by (2.26d). The third integral can be treated using the relation $\mathbf{H} = (\mathbf{H}\cdot\boldsymbol{\xi})\boldsymbol{\xi} + O(d_I(x,t))$ and (2.15a). So it can be controlled by (2.26e).

The integrals in (B.2b) can be controlled using (2.26c) and (2.26d). The one in (B.2c) is controlled by (2.26a). The first term in (B.2d) can be controlled using (2.26d). It remains to estimate (B.3) and (B.4). The integrals of the last two terms defining J_ε^1 can be controlled by (2.26b). Therefore,

$$\begin{aligned}
\int J_\varepsilon^1\,dx &\overset{(2.26b)}{\leqslant} \int \nabla\mathbf{H} : (\mathbf{n}_\varepsilon\otimes(\mathbf{n}_\varepsilon - \boldsymbol{\xi}))\left(|\nabla\psi_\varepsilon| - \varepsilon|\nabla\mathbf{u}_\varepsilon|^2\right) dx\\
&\qquad + \int (\boldsymbol{\xi}\cdot\nabla)\mathbf{H}\cdot\mathbf{n}_\varepsilon\left(|\nabla\psi_\varepsilon| - \varepsilon|\nabla\mathbf{u}_\varepsilon|^2\right) dx + CE_\varepsilon[\mathbf{u}_\varepsilon|I]\\
&\overset{(2.14b)}{\leqslant} C\Big(\int |\mathbf{n}_\varepsilon - \boldsymbol{\xi}|\Big(\varepsilon|\nabla\mathbf{u}_\varepsilon|^2 - \varepsilon|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|^2\Big) dx\\
&\qquad + \int |\mathbf{n}_\varepsilon - \boldsymbol{\xi}|\left|\varepsilon|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|^2 - |\nabla\psi_\varepsilon|\right| dx\\
&\qquad + \int \min\left(d_I^2, 1\right)\left(|\nabla\psi_\varepsilon| + \varepsilon|\nabla\mathbf{u}_\varepsilon|^2\right) dx + E_\varepsilon[\mathbf{u}_\varepsilon|I]\Big).
\end{aligned}$$

The first and the third integrals in the last display can be estimated using (2.26b) and (2.26e) respectively. Then we employ (2.23a) to find

$$\begin{aligned}
\int J_\varepsilon^1\,dx &\leqslant C\Big(\int |\mathbf{n}_\varepsilon - \boldsymbol{\xi}|\left|\varepsilon|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|^2 - |\nabla\psi_\varepsilon|\right| dx + E_\varepsilon[\mathbf{u}_\varepsilon|I]\Big)\\
&\overset{(2.23a)}{=} C\Big(\int |\mathbf{n}_\varepsilon - \boldsymbol{\xi}|\sqrt{\varepsilon}|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon|\left|\sqrt{\varepsilon}|\Pi_{\mathbf{u}_\varepsilon}\nabla\mathbf{u}_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}|D\mathrm{d}^F(\mathbf{u}_\varepsilon)|\right| dx + E_\varepsilon[\mathbf{u}_\varepsilon|I]\Big).
\end{aligned}$$

Finally applying the Cauchy-Schwarz inequality and then (2.26c) and (2.26d), we obtain $\int J_\varepsilon^1\,dx \leqslant CE_\varepsilon[\mathbf{u}_\varepsilon|I]$. As for J_ε^2 (B.4), we employ (2.15c) and (2.26e) to obtain $\int J_\varepsilon^2\,dx \leqslant CE_\varepsilon[\mathbf{u}_\varepsilon|I]$. All in all, we have proved that the right-hand side of (B.2) is bounded by $E_\varepsilon[\mathbf{u}_\varepsilon|I]$ up to a multiplicative constant which only depends on I_t. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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