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Calculus of Variations

Regularity for quasi-minima of the Alt–Caffarelli functional

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Abstract
We investigate regularity estimates of quasi-minima of the Alt–Caffarelli energy functional. We prove universal Hölder continuity of quasi-minima and optimal Lipchitz regularity along their free boundaries.

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1 Introduction

The celebrated Alt–Caffarelli functional

$$J(u, O) = \int_O |Du|^2 + \chi_{\{u>0\}} dx, \tag{1.1}$$

has been a focal point of scholarly exploration ever since its introduction in the seminal paper [3], given rise to a substantial body of literature and sparked substantial research endeavors. In its one-phase version, i.e. nonnegative minimizers of the Alt–Caffarelli functional are intricately connected to the Bernoulli problem and have significant implications in the realm of optimal design problems with volume constraints, see for instance [1, 2, 15, 16]. For a comprehensive and in-depth exposition of the regularity theory related to the Alt–Caffarelli functional, we recommend consulting the excellent, recent book [17].

In response to challenges originating in the field of material sciences, this paper starts the analysis of quasi-minimizers within the framework of the Alt–Caffarelli functional. Our investigation takes inspiration from the pioneering work of Giaquinta and Giusti, who introduced the concept of quasi-minimizers in their influential papers [12, 13].

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A fundamental aspect of quasi-minimizer theory lies in its capacity to offer a unified framework for examining solutions of elliptic equations and systems, as well as for exploring minima of general functionals within the realm of the Calculus of Variations. This unified approach provides a versatile and powerful analytical tool for understanding a wide range of mathematical phenomena, including problems ruled by elliptic operators with no continuity assumptions on their coefficients.

Given a functional $\mathcal{F}$, defined on an appropriate functional space $X(\Omega)$, and a real number $Q \geq 1$, we say $u \in X$ is a quasi-minimizer (or Q-minimum for short) of $\mathcal{F}$ if

$$\mathcal{F}(u, K) \leq Q \cdot \mathcal{F}(v, K),$$

for all $v \in X(\Omega)$ such that $K := \text{Supp}(u - v)$ is a compact subset of Ω. Typically $X(\Omega)$ is a Sobolev space of functions possessing weak derivatives in a Lebesgue space.

Our primary objective in this study is to derive universal regularity estimates for Q-minimizers governed by the functional (1.1). It's worth noting that the Q-minima class encompasses all minimizers associated with cavitation problems ruled by bounded measurable coefficients, which have direct relevance to issues in material sciences. However, this problem presents a significantly greater level of complexity, primarily due to the absence of a partial differential equation (PDE) governing the problem within the non-coincidence set, $\{u > 0\}$. This lack of a suitable PDE within this set introduces substantial challenges to the analysis.

It is worth mentioning that the concept of almost minimizers for the Alt–Caffarelli functional,

$$J(u, B_r) \leq (1 + Cr^{\alpha}) J(v, B_r), \tag{1.2}$$

has been recently developed by David and Toro in [9], David, Engelstein, and Toro in [10], and De Silva and Savin in [6], with related analysis covered in [7]. This emerging field of almost minimizers presents a fruitful complement to the estimates established in this paper. Our work contributes by providing universal estimates that remain independent of the modulus of continuity involved in the parameters of the problem, i.e. as $\alpha \searrow 0$ in (1.2).

In this paper, we have achieved two significant results. Firstly, we establish a universal Hölder continuity estimate that applies to sign-changing Q-minima of the Alt–Caffarelli functional, as in [4]. More remarkably, we obtain the optimal Lipschitz regularity for non-negative Q-minima along their free boundaries.

Similar results, with nearly verbatim proofs, can be established for Q-minima of the p-Alt–Caffarelli functional, acting on the space $W^{1,p}$, as introduced in [8]. We have decided to present the theorems for the original Alt–Caffarelli functional, i.e. for $p = 2$, to ease the presentation of the main ideas.

Finally, as a continuation of this research, we intend to explore additional geometric measure properties of the free boundary in forthcoming works.

2 Preliminary results

Hereafter in this paper, Ω denotes a fixed open set of $\mathbb{R}^n$ and $u \in H^1(\Omega)$ a generic Q-minimizer of the functional J defined in (1.1). That is,

$$J(u, K) \le Q \cdot J(v, K),$$

for all $v \in H^1(\Omega)$ such that $K := \mathrm{Supp}(u - v) \Subset \Omega$.

Lemma 1 (Scaling) *Let $u \in H^1(B_1)$ be a Q-minimizer of J, $x_0 \in B_1$, $0 < r < dist(x_0, \partial B_1)$, and $\mu \in \mathbb{R}$. The scaled function,*

$$u_{r,s;\mu}(z) := \frac{u(x_0 + rz) - \mu}{s},$$

defined over B_1 is a Q-minimizer of the functional $J_{r,s;\mu}$ given as:

$$J_{r,s;\mu}(u, O) = \int_O |Du|^2 + (s^{-1}r)^2 \cdot \chi_{\{u>\mu\}} dx.$$

Proof Let $v \in H^1(B_1)$ be such that $K := \mathrm{Supp}\left(u_{r,s;\mu} - v\right) \Subset B_1$. Define $\tilde{v} \colon B_r(x_0) \to \mathbb{R}$ by

$$\tilde{v}(y) = s \cdot v\left(\frac{y - x_0}{r}\right) + \mu.$$

Easily one checks that $\tilde{K} := \mathrm{Supp}\,(u - \tilde{v}) \Subset B_r(x_0)$. Thus, by the Q-minimality of u, we can write

$$J(u, B_r(x_0)) \le Q \cdot J(\tilde{v}, B_r(x_0)). \tag{2.1}$$

The change of variables, $x_0 + rz = x$, yields:

$$\begin{aligned} J(u, B_r(x_0)) &= \int_{B_r(x_0)} |Du(x)|^2 + \chi_{\{u>0\}} dx \\ &= \int_{B_1} \left(|Du(x_0 + rz)|^2 + \chi_{\{u(x_0+rz)>0\}}\right) r^n dz \\ &= \int_{B_1} \left(|(sr^{-1} Du_{r,s}(z)|^2 + \chi_{\{u(x_0+rz)>0\}}\right) r^n dz \\ &= s^2 r^{n-2} \cdot \left(\int_{B_1} |Du_{r,s}(z)|^2 + (s^{-1}r)^2 \chi_{\{u_{r,s}>\mu\}} dz\right). \end{aligned} \tag{2.2}$$

Similarly, making the change of variables, $\frac{y-x_0}{r} = z$, one finds

$$\begin{aligned} J(\tilde{v}, B_r(x_0)) &= \int_{B_r(x_0)} |D\tilde{v}(y)|^2 + \chi_{\{\tilde{v}>\mu\}} dy \\ &= \int_{B_r(x_0)} \left|sr^{-1} Dv\left(\frac{y - x_0}{r}\right)\right|^2 + \chi_{\left\{v\left(\frac{y-x_0}{r}\right)>0\right\}} dy \\ &= \int_{B_1} \left(|sr^{-1} Dv(z)|^2 + \chi_{\{v>0\}}\right) r^n dz \\ &= s^2 r^{n-2} \cdot \left(\int_{B_1} |Dv(z)|^2 + (s^{-1}r)^2 \chi_{\{v>0\}} dz\right). \end{aligned} \tag{2.3}$$

Combining (2.1) with (2.2) and (2.3), and cancelling the common term $s^2 r^{n-2} > 0$, we finally reach:

$$\left(\int_{B_1} |Du_{r,s}(z)|^2 + (s^{-1}r)^2 \cdot \chi_{\{u_{r,s;\mu}>\mu\}} dz\right) \le Q \cdot \left(\int_{B_1} |Dv(z)|^2 + (s^{-1}r)^2 \cdot \chi_{\{v>\mu\}} dz\right),$$

and thus the lemma is proven. □

We proceed to establish a version of the Cacciopoli inequality tailored to Q-minima, which can be readily adapted to our specific case. The proof closely follows the approach presented in [14, Theorem 6.5], and for brevity, we omit it here.

Lemma 2 *Let $u \in H^1(B_1)$ be a Q-minimizer of J. There exists a constant $C_1 > 0$, depending only on dimension and Q, such that:*

$$\int_{B_{1/2}} |Du|^2\, dx \le C_1 \cdot \left(1 + \int_{B_1} |u|^2\, dx\right).$$

The following lemma plays a pivotal role in deriving the regularity estimates of this paper, in particular in obtaining the optimal Lipchitz regularity of Q-minima along their free boundaries.

Lemma 3 (Weak closeness) *Let $\{u_n\}_{n=1}^{\infty} \subset H^1(\Omega)$ be a sequence of Q-minima of energy functionals*

$$J_n(v, O) := \int_O |Dv|^2 + a_n(x)\chi_{\{v>0\}} dx.$$

Assume $u_n \to u$ weakly in $H^1_{loc}(B_1)$ and $a_n(x) \to a_\infty(x)$ strongly in L^1. Then u is a $(Q+2)$–minimizer of the energy functional

$$J_\infty(v, O) := \int_O |Dv|^2 + |a_\infty(x)| dx.$$

Proof Let $O \subset B_1$ be a regular set and v a function satisfying $v = u$ outside O. Define φ_n to be the harmonic function in O whose boundary data equals $u_n - u$. Extend it to agree with $u_n - u$ outside of O too. Define

$$v_n := v + \varphi_n,$$

and note that v_n agrees with u_n outside O. Using the Q-minimality of u_n, we obtain

$$\int_O |Du_n|^2 + a_n(x)\chi_{\{u_n>0\}} dx \le Q \cdot \left(\int_O |Dv_n|^2 + a_n(x)\chi_{\{v_n>0\}} dx\right). \tag{2.4}$$

By weak convergence of u_n to u, there readily holds:

$$\int_O |Du|^2 dx \le \liminf_{n\to\infty} \int_O |Du_n|^2 dx. \tag{2.5}$$

We can further estimate

$$\int_O a_n(x)\chi_{\{u_n>0\}} dx \ge -\int_O |a_n(x)| dx \to -\int_O |a_\infty(x)| dx. \tag{2.6}$$

Combining (2.5) and (2.6), and taking the lim inf as $n \to \infty$, we reach:

$$\int_O |Du|^2 dx - |a_\infty(x)| dx \le \liminf_{n\to\infty} \int_O |Du_n|^2 + a_n(x)\chi_{\{u_n>0\}} dx. \tag{2.7}$$

Next, keeping in mind that, by the trace theorem, $\varphi_n \to 0$ strongly in H^1, we compute:

$$\begin{aligned} \int_O |Dv_n|^2 + a_n(x)\chi_{\{v_n>0\}} dx &= \int_O |Dv|^2 + 2Dv \cdot D\varphi_n + |D\varphi_n|^2 + a_n(x)\chi_{\{v_n>0\}} dx \\ &\le \int_O |Dv|^2 + |a_\infty(x)| + \mathrm{o}(1). \end{aligned} \tag{2.8}$$

Combining (2.4), (2.7), and (2.8) we obtain

$$\int_O |Du|^2 dx - |a_\infty(x)| dx \le Q \cdot \left(\int_O |Dv|^2 + |a_\infty(x)| dx \right). \tag{2.9}$$

Next, adding $2\int_O |a_\infty(x)| dx$ both sides of the inequality above finally yields

$$\begin{aligned} \int_O |Du|^2 dx + |a_\infty(x)| dx &\le Q \cdot \left(\int_O |Dv|^2 + |a_\infty(x)| dx \right) + 2\int_O |a_\infty(x)| dx \\ &\le (Q+2) \cdot \left(\int_O |Dv|^2 + |a_\infty(x)| dx \right), \end{aligned} \tag{2.10}$$

and the lemma is proven. □

Remark 1 A quick inspection of the proof of Lemma 3 reveals that if $a_\infty = 0$, then the H^1 weak limit is actually a Q-minimum of the Dirichlet integral,

$$\mathscr{D}(u, O) = \int_O |Du|^2 dx.$$

That is, we do not need to increase the parameter Q. This applies to the compactness argument used in the key Lemma 4. However, while this remark is interesting, it will not play a role in the proofs presented in this paper, as no quantitative estimates on the Hölder exponent of Q-minima of the Dirichlet integral are, after all, available.

We conclude this section with some known facts about Q-minima. The proofs can be found in [14]; see for instance [14, Theorem 7.6] and [14, Theorem 7.10].

Theorem 1 (DeGiorgi class) *Let $u \in H^1(B_1)$ be a Q-minimizer of the Dirichlet energy functional,*

$$\mathscr{D}(v, O) := \int_O |Dv|^2 dx.$$

Then $v \in C^{0,\alpha}(B_{1/2})$, for some $0 < \alpha < 1$ that depends only on Q and dimension.

Theorem 2 (Harnack inequality) *Let $u \in H^1(B_1)$ be a non-negative Q-minimizer of the Dirichlet energy functional,*

$$\mathscr{D}(v, O) := \int_O |Dv|^2 dx.$$

Then,

$$\sup_{B_{1/2}} u \le C \inf_{B_{1/2}},$$

where $C > 0$ depends only on dimension and on Q.

3 Hölder continuity

In this section, we employ compactness methods inspired by Caffarelli [5] to establish the universal local regularity of Q-minima within the framework of the Alt–Caffarelli functional (1.1).

We initiate the process by establishing the suitable functional analysis framework, which is essential for applying the compactness method. Given a constant $0 < \alpha < 1$, we denote the space of α-Hölder continuous functions as

$$C^{0,\alpha}(\Omega) := \left\{ u \in C(\Omega) \;\middle|\; \exists C_u > 0 \text{ such that } |u(x) - u(y)| \le C_u |x - y|^\alpha,\ \forall x, y \in \Omega \right\}.$$

The ball of radius R in $C^{0,\alpha}(\Omega)$ will be denoted by $\Xi_R^\alpha(\Omega)$, i.e.

$$\Xi_R^\alpha(\Omega) := \left\{ u \in C^{0,\alpha}(\Omega) \;\middle|\; |u(x) - u(y)| \le R|x - y|^\alpha,\ \forall x, y \in \Omega \right\}.$$

The main theorem of this section is a local universal continuity estimate of Q-minima of the energy functional J. More precisely we will prove:

Theorem 3 *Let $u \in H^1(B_1)$ be a Q-minimizer of J in B_1. There exist constants $0 < \alpha < 1$ and $C_2 > 0$, depending only on dimension and Q such that*

$$|u(x) - u(y)| \le C_2 \|u\|_{L^2(B_1)} |x - y|^\alpha,$$

for all $x, y \in B_{1/2}$.

Theorem 3 will be established with the aid of a series of lemmas.

Lemma 4 *Let $a \in L^1(B_1)$, $\mu \in \mathbb{R}$, and $u \in H^1(B_1)$, verifying,*

$$\fint_{B_1} u^2 dx \le 1,$$

be a Q-minimizer of

$$J_a(v) = \int |Dv|^2 + a(x)\chi_{\{v>\mu\}} dx.$$

There exist universal constants $R > 0$ and $0 < \alpha < 1$ such that, for any given $\epsilon > 0$, one can find a $\delta > 0$, depending only on $\epsilon > 0$ and universal parameters, such that if $\|a\|_{L^1} < \delta$, then

$$\inf \left\{ \fint_{B_{1/2}} |u - h|^2 dx \;\middle|\; h \in \Xi_R^\alpha(B_{1/2}) \right\} < \epsilon.$$

Proof Suppose, seeking a contradiction, the thesis of the lemma fails to hold. This means there exists a constant $\epsilon_0 > 0$ and sequences of functions $a_n \in L^1(B_1)$, $\mu_n \in \mathbb{R}$ (not necessarily bounded), $u_n \in H^1(B_1)$, with

$$\fint_{B_1} u_n^2 dx \le 1 \tag{3.1}$$

and $\|a_n\|_{L^1(B_1)} = \mathrm{o}(1)$, such that u_n is a Q-minimum of

$$J_n(v) := \int |Dv|^2 + a_n(x)\chi_{\{v>\mu_n\}}dx;$$

however

$$\fint_{B_{1/2}} |u-h|^2 dx \geq \epsilon_0, \tag{3.2}$$

for all $h \in \Xi_R^\alpha(B_{1/2})$. From Lemma 2, up to a subsequence, $u_n \to u$ weakly in $H^1(B_1)$ and strongly in $L^2(B_1)$. In particular, from (3.1) there holds

$$\int_{B_1} |u|^2 dx = \int_{B_1} |u_n|^2 dx + \mathrm{o}(1) \leq |B_1| + 1 := C_n;$$

a dimensional constant. It is plan to see from Lemma 3 that u is a Q-minimum of the Dirichlet functional

$$\mathscr{D}(v, O) = \int_O |Dv|^2 dx.$$

However, from Theorem 1 u belongs to $\Xi_R^\alpha(B_{1/2})$, for some universal constants $R > 0$ and $0 < \alpha < 1$. We then obtain a contradiction by taking n large enough in (3.2). □

Hereafter we will use the notation:

$$\langle f \rangle_r := \fint_{B_r} |f| dx$$

Lemma 5 *Let $u \in H^1(B_1)$ be a Q-minimum of the functional*

$$J_n(v) := \int |Dv|^2 + a(x)\chi_{\{v>\mu\}}dx,$$

for some $a \in L^1(B_1)$. Assume

$$\fint_{B_1} u^2 dx \leq 1.$$

If $\|a\|_{L^1(B_1)} \leq \delta_0$, for some δ_0 universal, then for small numbers $0 < \lambda \ll 1/2$ and $0 < \beta < 1$ depending only on dimension and Q, there holds

$$\sqrt{\fint_{B_\lambda} |u - \langle u \rangle_\lambda|^2 \, dx} < \lambda^\beta.$$

Proof For an $\epsilon > 0$ to be chosen a posteriori, let $\delta > 0$ be the corresponding smallness condition on $\|a\|_{L^1(B_1)}$ from Lemma 4 such that

$$\fint_{B_{1/2}} |u-h|^2 dx < \epsilon, \tag{3.3}$$

for some $h \in \Xi_R^\alpha(B_{1/2})$. For $0 < \lambda \ll 1/2$ to be set and $\mu \in \mathbb{R}$ arbitrary, we can estimate, with the aid of the triangle inequality and the Cauchy-Schwarz inequality

$$\begin{aligned}\sqrt{\fint_{B_\lambda} |u - \langle u\rangle_\lambda|^2\, dx} &\le \sqrt{\fint_{B_\lambda} |u-\mu|^2\, dx} + \left|\fint_{B_\lambda} u dx - \mu\right| \\ &\le \sqrt{\fint_{B_\lambda} |u-\mu|^2\, dx} + \fint_{B_\lambda} |u-\mu|\, dx \\ &\le 2\sqrt{\fint_{B_\lambda} |u-\mu|^2\, dx}.\end{aligned} \tag{3.4}$$

Applying (3.4) with $\mu = h(0)$, where h is the function granted in (3.3), we reach

$$\sqrt{\fint_{B_\lambda} |u - \langle u\rangle_\lambda|^2\, dx} \le 2\sqrt{\fint_{B_\lambda} |u - h(0)|^2\, dx}. \tag{3.5}$$

Next we estimate, again with the aid of the triangle inequality:

$$\begin{aligned}\sqrt{\fint_{B_\lambda} |u - h(0)|^2\, dx} &\le \sqrt{\fint_{B_\lambda} |u-h|^2\, dx} + \sqrt{\fint_{B_\lambda} |h - h(0)|^2\, dx} \\ &\le \epsilon^{1/2} + \sqrt{R}\lambda^\alpha.\end{aligned} \tag{3.6}$$

It is time to make our (universal) choices. Initially we choose and fix $0 < \beta < \alpha$. With such a choice made, select $0 < \lambda < 1/2$ such that

$$\sqrt{R}\lambda^\alpha \le \frac{1}{4}\lambda^\beta. \tag{3.7}$$

Finally, we set

$$\epsilon^{1/2} := \frac{1}{4}\lambda^\beta, \tag{3.8}$$

which determines, through Lemma 4, the smallness condition $\delta > 0$ on $\|a\|_{L^1(B_1)}$. Finally, combining (3.5) and (3.7), and (3.8) we reach:

$$\begin{aligned}\sqrt{\fint_{B_\lambda} |u - \langle u\rangle_\lambda|^2\, dx} &\le 2\left(\epsilon^{1/2} + \sqrt{R}\lambda^\alpha\right) \\ &\le \lambda^\beta,\end{aligned}$$

and the proof of the lemma is complete. □

Proof of Theorem 3 As we move forward to deliver the proof of Theorem 3, let $u \in H^1(B_1)$ be a generic Q-minimum of the original energy functional J defined in (1.1) and $x_0 \in B_{1/2}$ be a point. Define

$$u_{r,s;0}(z) := \frac{u(x_0 + \sqrt{\delta_0}\cdot z)}{s},$$

where δ_0 is the constant from Lemma 5 and

$$s = \max\left\{1, \fint_{B_{\sqrt{\delta_0}}(x_0)} u^2 dx\right\}.$$

In view of Lemma 1, this sclaed function is under the assumptions of Lemma 5. Since δ_0 is a universal constant, proving u is $C^{0,\beta}$ continuous is equivalent to showing $u_{r,s;0} \in C^{0,\beta}$ with universal control.

In conclusion, it suffices to establish Theorem 3 for functions under the structural assumptions of Lemma 5. Under such conditions, the thesis of Lemma 5 yields:

$$\fint_{B_\lambda} |u - \langle u \rangle_\lambda|^2 \, dx \leq \lambda^{2\beta}.$$

Define $u_1 \colon B_1 \to \mathbb{R}$ as

$$u_1(x) := \frac{u(\lambda x) - \langle u \rangle_\lambda}{\lambda^\beta}. \tag{3.9}$$

Clearly,

$$\fint_{B_1} |u_1|^2 dx \leq 1.$$

In addition, Lemma 1 gives that u_1 is a Q-minimum of the energy functional

$$J_1(v) = \int |Dv|^2 + \lambda^{2(1-\beta)} \delta_0 \chi_{\{v > \langle u \rangle_\lambda\}} dx,$$

Since $\lambda^{2(1-\beta)} < 1$, u_1 is entitled to the thesis of Lemma 5; i.e.

$$\fint_{B_\lambda} |u_1 - \langle u_1 \rangle_\lambda|^2 \, dx \leq \lambda^{2\beta} \tag{3.10}$$

Direct calculation yields

$$\langle u_1 \rangle_\lambda = \frac{\langle u \rangle_{\lambda^2} - \langle u \rangle_\lambda}{\lambda^\beta}.$$

Thus, (3.10) yields

$$\fint_{B_{\lambda^2}} \left| u(x) - \langle u \rangle_{\lambda^2} \right|^2 dx \leq \lambda^{4\beta}.$$

Continuing the process recursively, we conclude

$$\fint_{B_{\lambda^k}} \left| u(x) - \langle u \rangle_{\lambda^k} \right|^2 dx \leq \lambda^{2k\beta},$$

for all $k = 1, 2, \cdots$. Hölder continuity now follows by standard considerations, which we omit. □

4 Free boundary Lipchitz regularity

In this section we focus on the one-phase case, i.e. we assume the Q-minimum is nonnegative. The main result of this section is

Theorem 4 *Let $u \in H^1(B_1)$ be a nonnegative Q-minimum of J and $z_0 \in \partial\{u > 0\} \cap B_{1/2}$ an interior free boundary point. Then for all $x \in B_{1/2}(z_0)$ there holds*

$$u(x) \leq C|x - z_0|,$$

where $C > 0$ is a constant depending only on dimension, Q, and $\|u\|_\infty$.

The proof of Theorem 4 generalizes the findings of [11] and its proof relies on the following key flatness improvement lemma:

Lemma 6 *Let $u \in H^1(B_1)$ be a nonnegative, normalized Q-minimum of*

$$J_a(v) = \int |Dv|^2 + a(x)\chi_{\{v>0\}} dx,$$

and assume 0 is a free boundary point. Given $\epsilon > 0$, there exists a $\delta > 0$, depending only on ϵ, Q, and dimension, but independent of u, such that if $|a(x)| \leq \delta$ in $B_{2/3}$, then

$$\sup_{B_{1/2}} u \leq \epsilon.$$

Proof Assume, seeking a contradiction, that the thesis of the lemma fails to hold. This means there exists a positive number $\epsilon_0 > 0$, a sequence of bounded functions a_n with $\|a_n\|_\infty = o(1)$ and a sequence of Q-minima of

$$J_{a_n}(v) = \int |Dv|^2 + a_n(x)\chi_{\{v>0\}} dx,$$

in B_1, with $0 \in \{u_n > 0\}$, such that

$$\sup_{B_{1/2}} u_n \geq \epsilon_0, \tag{4.1}$$

for all $n \in \mathbb{N}$. By Theorem 3, up to a subsequence, we can assume $u_n \to u_0$ uniformly in $B_{2/3}$, to a nonnegative function u_0. In particular, it also follows that $u_0 = 0$. By the weak closeness lemma, i.e. Lemma 3, it follows that u_0 is a Q-minimum of the Dirichlet functional, and thus entitled to the conclusion of Theorem 2. Thus,

$$\sup_{B_{1/2}} u_0 \leq C u_0(0) = 0.$$

However, in view of (4.1) and the uniform convergence of u_n to u_0, for n large enough we would conclude:

$$0 < \epsilon_0 \leq \sup_{B_{1/2}} u_n \leq \frac{\epsilon_0}{2},$$

leading to a contradiction. □

Proof of Theorem 4 Let u be a Q-minimum of J. We start off with some universal choices. Take $\epsilon = 2^{-1}$ and let $\delta_\star > 0$ be the corresponding number yielded by Lemma 6. Consider the function

$$v(y) = \frac{u(z_0 + \mu_0 y)}{\|u\|_\infty},$$

defined over B_1, where $0 < \mu_0 \ll 1$ is chosen by the equation

$$\mu_0 := \sqrt{\delta_\star} \cdot \min\{\|u\|_\infty, 1\}. \tag{4.2}$$

In view of Lemma 1, v is a normalized Q-minimum of the functional

$$\tilde{J} := \int_\Omega |Du|^2 + (\|u\|_\infty^{-1}\mu_0)^2 \cdot \chi_{\{u>0\}} dx = \int_\Omega |Du|^2 + \delta_\star \cdot [\|u\|_\infty^{-1} \cdot \min\{\|u\|_\infty, 1\}]^2 \cdot \chi_{\{u>0\}} dx. \tag{4.3}$$

Applying Lemma 6 to v yields

$$\sup_{B_{1/2}} v \le 2^{-1}. \tag{4.4}$$

Next, define $v_2\colon B_1 \to [0, 1]$ as:

$$v_2(y) = 2v(\frac{y}{2}).$$

In view of (4.4), we have

$$0 \le v_2 \le 1.$$

Applying Lemma 1 to v_2, we discover v_2 is too a Q-minimum of the same Alt–Caffarelli functional as in (4.3). Thus it is entitled to the estimate displayed in (4.4). That is:

$$\sup_{B_{1/2}} v_2 \le 2^{-1}.$$

In view of the definition of v_2, the above inequality yields to v the estimate

$$\sup_{B_{1/4}} v \le 4^{-1}. \tag{4.5}$$

Repeating the argument recursively, we conclude

$$\sup_{B_{1/2^n}} v \le 2^{-n}, \tag{4.6}$$

for all natural numbers $n \in \mathbb{N}$.

Next, given a positive number $\rho > 0$, let

$$n_\rho := \lfloor - \log_2 \rho \rfloor;$$

the largest natural number which is less than or equal to $-\log_2 \rho$. We can estimate, in view of (4.6),

$$\begin{aligned} \sup_{B_\rho} v(x) &\le \sup_{B_{2^{n_\rho}}} v(x) \\ &\le 2^{-n_\rho} \\ &\le 2\rho. \end{aligned}$$

In view of the definition of v, we obtain, labeling $x = x_0 + \mu_0 y$

$$u(x) = \|u\|_\infty v(\mu_0^{-1}(x - z_0)).$$

Set $t = |x - z_0|$. We finally have

$$\sup_{B_t(z_0)} u(x) = \|u\|_\infty \sup_{B_{\mu_0^{-1} t}} v \le 2\mu_0^{-1}|x - z_0|,$$

and the Theorem is proven. □

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