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Calculus of Variations

Sobolev inequalities in manifolds with asymptotically nonnegative curvature

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Abstract

Using the ABP-method as in a recent work by Brendle (Commun Pure Appl Math 76:2192–2218, 2022), we establish some sharp Sobolev and isoperimetric inequalities for compact domains and submanifolds in a complete Riemannian manifold with asymptotically nonnegative Ricci/sectional curvature. These inequalities generalize those given by Brendle in the case of complete Riemannian manifolds with nonnegative curvature.

Mathematics Subject Classification 35R45 · 53C21

1 Introduction

It is known that Sobolev inequalities, as an important analytic tool in geometric analysis, have close connections with isoperimetric inequalities. The classical isoperimetric inequality for a bounded domain D in $\mathbb{R}^n$ says that

$$n^n|B^n||D|^{n-1} \leq |\partial D|^n$$

where B^n denotes the unit ball in $\mathbb{R}^n$, and the equality holds if and only if D is a ball. There have been numerous works generalizing this inequality to different settings (cf. [14, 15, 33]).

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The isoperimetric inequalities on minimal surfaces or minimal submanifolds have a long history. For example, [13, 14, 22, 29, 35–37] investigated the isoperimetric inequality on minimal surfaces under various conditions, while the famous Michael-Simon Sobolev inequality for general dimensions [5, 32] implies an isoperimetric inequality for minimal submanifolds, but with a non-sharp constant. It is conjectured that any n-dimensional minimal submanifold Ω of $\mathbb{R}^N$ satisfies the classical isoperimetric inequality: $n^n|B^n||\Omega|^{n-1} \leq |\partial\Omega|^n$ with equality holds if and only if Ω is a ball in an n-plane of $\mathbb{R}^N$. Recently, S. Brendle [9], inspired by the ABP method as in [11] and [38], established a Michael-Simon-Sobolev type inequality on submanifolds of arbitrary dimension and codimension, which is sharp if the codimension is at most 2. In particular, his result implies a sharp isoperimetric inequality for minimal submanifolds in Euclidean space of codimension at most 2. Later, Brendle [10] also generalized his results in [9] to the case that the ambient space is a Riemannian manifold with nonnegative curvature. In [23], F. Johne gave a sharp Sobolev inequality for manifolds with nonnegative Bakry-Émery Ricci curvature, which generalizes Brendle's results in [10]. In [7], Balogh and Kristály proved a sharp isoperimetric inequality in metric measure spaces satisfying $\mathsf{CD}(0, N)$ condition which implies the sharp isoperimetric inequalities in [10] and [23]. Moreover, they also obtained a sharp L^p-Sobolev inequality for $p \in (1, n)$ on manifolds with nonnegative Ricci curvature and Euclidean volume growth. In a recent preprint [6], the authors also investigated sharp and rigid isoperimetric comparison theorems in $\mathsf{RCD}(K, N)$ metric measure spaces.

In this paper, we generalize Brendle's results in [10] to the case that the ambient space has asymptotically nonnegative curvature. The notion of asymptotically nonnegative curvature was first introduced by U. Abresch [1]. Some important geometric, topological and analysis problems have been investigated for this kind of manifolds (cf. [2, 3, 8, 21, 24, 25, 30, 31, 40, 41], etc). Now we recall its definition as follows. Let $\lambda : [0, +\infty) \to [0, +\infty)$ be a nonnegative and nonincreasing continuous function satisfying

$$b_0 := \int_0^{+\infty} s\lambda(s)ds < +\infty, \tag{1.1}$$

which implies

$$b_1 := \int_0^{+\infty} \lambda(s)ds < +\infty. \tag{1.2}$$

A complete noncompact Riemannian manifold (M, g) of dimension n is said to have asymptotically nonnegative Ricci curvature (resp. sectional curvature) if there is a base point $o \in M$ such that

$$\mathrm{Ric}_q(\cdot,\cdot) \geq -(n-1)\lambda(d(o,q))g \quad (resp.\ \mathrm{Sec}_q \geq -\lambda(d(o,q))), \tag{1.3}$$

where $d(o, q)$ is the distance function of M relative to o. Clearly, this notion includes the manifolds whose Ricci (resp. sectional) curvature is either nonnegative outside a compact set or asymptotically flat at infinity. In particular, if $\lambda \equiv 0$ in (1.3), then this becomes the case treated in [10].

Let $h(t)$ be the unique solution of

$$\begin{cases} h''(t) = \lambda(t)h(t), \\ h(0) = 0, h'(0) = 1. \end{cases} \tag{1.4}$$

By ODE theory, the solution $h(t)$ of (1.4) exists for all $t \in [0, +\infty)$. According to [41] (see also Theorem 2.14 in [34]), the function

$$\frac{|\{q \in M : d(o,q) < r\}|}{n|B^n| \int_0^r h^{n-1}(t)dt}$$

is a non-increasing function on $[0, +\infty)$ and thus we may introduce the asymptotic volume ratio of M by

$$\theta := \lim_{r\to+\infty} \frac{|\{q \in M : d(o,q) < r\}|}{n|B^n| \int_0^r h^{n-1}(t)dt}, \tag{1.5}$$

with $\theta \leq 1$. In particular, we have $|\{q \in M : d(o,q) < r\}| \leq |B^n| e^{(n-1)b_0} r^n$.

First, by combining the method in [10] with some comparison theorems, we establish a Sobolev type inequality for a compact domain in a Riemannian manifold with asymptotically nonnegative Ricci curvature as follows.

Theorem 1.1 *Let M be a complete noncompact n-dimensional manifold of asymptotically nonnegative Ricci curvature with respect to a base point $o \in M$. Let Ω be a compact domain in M with boundary $\partial\Omega$, and let f be a positive smooth function on Ω. Then*

$$\int_{\partial\Omega} f + \int_\Omega |Df| + 2(n-1)b_1 \int_\Omega f \geq n|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\right)^{\frac{n-1}{n}} \left(\int_\Omega f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}},$$

where $r_0 = \max\{d(o,x)|x \in \Omega\}$, θ is the asymptotic volume ratio of M given by (1.5) and b_0, b_1 are defined in (1.1) and (1.2).

The following result characterizes the case of equality in Theorem 1.1:

Theorem 1.2 *Let M be a complete noncompact n-dimensional manifold of asymptotically nonnegative Ricci curvature with respect to a base point $o \in M$. Let Ω be a compact domain in M with boundary $\partial\Omega$, and let f be a positive smooth function on Ω. If*

$$\int_{\partial\Omega} f + \int_\Omega |Df| + 2(n-1)b_1 \int_\Omega f = n|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\right)^{\frac{n-1}{n}} \left(\int_\Omega f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}},$$

where $r_0 = \max\{d(o,x)|x \in \Omega\}$, θ is the asymptotic volume ratio of M given by (1.5) and b_0, b_1 are defined in (1.1) and (1.2). Then $b_0 = b_1 = 0$, M is isometric to Euclidean space, Ω is a ball, and f is constant.

Taking $f = 1$ in Theorem 1.1, we obtain a sharp isoperimetric inequality:

Corollary 1.3 *Let $M, \Omega, r_0, \theta, b_0, b_1$ be as in Theorem 1.1. Then*

$$|\partial\Omega| \geq \left(n|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\right)^{\frac{n-1}{n}} - 2(n-1)b_1 |\Omega|^{\frac{1}{n}}\right) |\Omega|^{\frac{n-1}{n}}.$$

Furthermore, the equality holds if and only if M is isometric to Euclidean space and Ω is a ball.

Remark 1.4 If M has nonnegative Ricci curvature, then $b_0 = b_1 = 0$ and Corollary 1.3 becomes

$$|\partial\Omega| \geq n|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}},$$

which was first given by V. Agostiniani, M. Fogagnolo, and L. Mazziari [4] in dimension 3 and obtained by S. Brendle [10] for any dimension, see also [18] for related results in $3 \leq n \leq 7$.

Similarly, we may establish a Sobolev type inequality for a compact submanifold (possibly with boundary) in a Riemannian manifold with asymptotically nonnegative sectional curvature as follows.

Theorem 1.5 *Let M be a complete noncompact $(n+p)$-dimensional manifold of asymptotically nonnegative sectional curvature with respect to a base point $o \in M$. Let Σ be a compact n-dimensional submanifold of M (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ. If $p \geq 2$, then*

$$\int_{\partial\Sigma} f + \int_\Sigma \sqrt{|D^\Sigma f|^2 + f^2|H|^2} + 2nb_1 \int_\Sigma f \\ \geq n\Big(\frac{(n+p)|B^{n+p}|}{p|B^p|}\Big)^{\frac{1}{n}} \theta^{\frac{1}{n}} \Big(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\Big)^{\frac{n+p-1}{n}} \Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}},$$

where $r_0 = \max\{d(o,x)|x \in \Sigma\}$, H is the mean curvature vector of Σ, θ is the asymptotic volume ratio of M given by (1.5) and b_0, b_1 are defined in (1.1) and (1.2).

Note that $(n+2)|B^{n+2}| = 2|B^2||B^n|$. Hence, we obtain the following Sobolev type inequality for codimension 2:

Corollary 1.6 *Let M be a complete noncompact $(n+2)$-dimensional manifold of asymptotically nonnegative sectional curvature with respect to a base point $o \in M$. Let Σ be a compact n-dimensional submanifold of M (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ. Then*

$$\int_{\partial\Sigma} f + \int_\Sigma \sqrt{|D^\Sigma f|^2 + f^2|H|^2} + 2nb_1 \int_\Sigma f \\ \geq n|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \Big(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\Big)^{\frac{n+1}{n}} \Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}},$$

where $r_0 = \max\{d(o,x)|x \in \Sigma\}$, H is the mean curvature vector of Σ, θ is the asymptotic volume ratio of M given by (1.5) and b_0, b_1 are defined in (1.1) and (1.2).

The following result characterizes the case of equality in Corollary 1.6:

Theorem 1.7 *Let $M, \Sigma, f, r_0, H, \theta, b_0, b_1$ as in Corollary 1.6. If*

$$\int_{\partial\Sigma} f + \int_\Sigma \sqrt{|Df|^2 + f^2|H|^2} + 2nb_1 \int_\Sigma f \\ = n|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \Big(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\Big)^{\frac{n+1}{n}} \Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}.$$

Then $b_0 = b_1 = 0$ and M is isometric to Euclidean space, Σ is a flat ball, and f is constant.

Letting $f = 1$ in Corollary 1.6, we obtain a sharp isoperimetric inequality for minimal submanifolds of codimension 2 as follows.

Corollary 1.8 *Let M be a complete noncompact $(n+2)$-dimensional manifold of asymptotically nonnegative sectional curvature with respect to a base point $o \in M$. Let Σ be a compact n-dimensional mininal submanifold of M (possibly with boundary $\partial\Sigma$). Then*

$$|\partial\Sigma| \geq n\Big(|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \big(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\big)^{\frac{n+1}{n}} - 2b_1|\Sigma|^{\frac{1}{n}}\Big)|\Sigma|^{\frac{n-1}{n}},$$

where $r_0 = \max\{d(o,x)|x \in \Sigma\}$, θ is the asymptotic volume ratio of M given by (1.5) and b_0, b_1 are defined in (1.1) and (1.2). Furthermore, the equality holds if and only if M is isometric to Euclidean space and Σ is a flat ball.

It is obvious that the above inequalities are nontrivial only when $\theta > 0$. We say that a complete Riemannian manifold with asymptotically nonnegative (Ricci) curvature has maximal volume growth if $\theta > 0$. Examples of such manifolds may be found in [1, 12, 19, 26, 27], and the first case of Theorem 1.2 in [39], etc.

2 The case of domains

Let (M, g) be a complete noncompact n-dimensional Riemannian manifold of asymptotically nonnegative Ricci curvature with respect to a base point $o \in M$. Let Ω be a compact domain in M with smooth boundary $\partial\Omega$ and f be a smooth positive function on Ω. Without loss of generality, we assume hereafter that Ω is connected.

By scaling, we may assume that

$$\int_{\partial\Omega} f + \int_{\Omega} |Df| + \int_{\Omega} 2(n-1)b_1 f = n \int_{\Omega} f^{\frac{n}{n-1}}. \tag{2.1}$$

Due to (2.1) and the connectedness of Ω, we can find a solution of the following Neumann boundary problem

$$\begin{cases} \operatorname{div}(f Du) = n f^{\frac{n}{n-1}} - 2(n-1)b_1 f - |Df|, & \text{in } \Omega, \\ \langle Du, \nu\rangle = 1, & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where ν is the outward unit normal vector field along $\partial\Omega$. By standard elliptic regularity theory (see Theorem 6.31 in [20]), we know that $u \in C^{2,\gamma}$ for each $0 < \gamma < 1$.

As in [10], we set

$$U := \{x \in \Omega \setminus \partial\Omega : |Du(x)| < 1\}.$$

For any $r > 0$, let

$$A_r = \{\bar{x} \in U : ru(x) + \frac{1}{2} d(x, \exp_{\bar{x}}(rDu(\bar{x})))^2 \geq ru(\bar{x}) + \frac{1}{2} r^2 |Du(\bar{x})|^2, \ \forall x \in \Omega\}.$$

Define a transport map $\Phi_r : \Omega \to M$ for each $r > 0$ by

$$\Phi_r(x) = \exp_x(rDu(x)), \quad \forall x \in \Omega.$$

Since $\exp : TM \to M$ is smooth on any complete Riemannian manifold (see Proposition 5.7 in [28]), we known that the map Φ_r is of class $C^{1,\gamma}$, $0 < \gamma < 1$.

Lemma 2.1 *Assume that $x \in U$. Then we have*

$$\frac{1}{n}\Delta u \leq f^{\frac{1}{n-1}} - 2\Big(\frac{n-1}{n}\Big)b_1.$$

Proof Using the Cauchy-Schwarz inequality and the property that $|Du| < 1$ for $x \in U$, we get

$$-\langle Df, Du\rangle \leq |Df|.$$

In terms of (2.2), we derive that

$$\begin{aligned} f\Delta u &= n f^{\frac{n}{n-1}} - 2(n-1)b_1 f - |Df| - \langle Df, Du\rangle \\ &\leq n f^{\frac{n}{n-1}} - 2(n-1)b_1 f. \end{aligned}$$

This proves the assertion. □

The proofs of the following three lemmas are identical to those for Lemmas 2.2−2.4 in [10] without any change for the case of asymptotically nonnegative Ricci curvature. So we omit them here.

Lemma 2.2 *The set*

$$\{q \in M : d(x, q) < r,\ \forall x \in \Omega\}$$

is contained in $\Phi_r(A_r)$.

Lemma 2.3 *Assume that* $\bar{x} \in A_r$*, and let* $\bar{\gamma}(t) := \exp_{\bar{x}}(t Du(\bar{x}))$ *for all* $t \in [0, r]$. *If* Z *is a smooth vector field along* $\bar{\gamma}$ *satisfying* $Z(r) = 0$*, then*

$$(D^2 u)(Z(0), Z(0)) + \int_0^r \left(|D_t Z(t)|^2 - R(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t))\right) dt \geq 0.$$

Lemma 2.4 *Assume that* $\bar{x} \in A_r$*, and let* $\bar{\gamma}(t) := \exp_{\bar{x}}(t Du(\bar{x}))$ *for all* $t \in [0, r]$. *Moreover, let* $\{e_1, \ldots, e_n\}$ *be an orthonormal basis of* $T_{\bar{x}} M$. *Suppose that* W *is a Jacobi field along* $\bar{\gamma}$ *satisfying*

$$\langle D_t W(0), e_j \rangle = (D^2 u)(W(0), e_j), \quad 1 \leq j \leq n.$$

If $W(\tau) = 0$ *for some* $\tau \in (0, r)$*, then* W *vanishes identically.*

Now, we give two comparison results for later use. The proofs of the following two lemmas are inspired by the proofs of Lemma 2.1 and Corollary 2.2 in [34].

Lemma 2.5 *Let* G *be a continuous function on* $[0, +\infty)$ *and let* $\phi, \psi \in C^2([0, +\infty))$ *be solutions of the following problems*

$$\begin{cases} \phi'' \leq G\phi, & t \in (0, +\infty), \\ \phi(0) = 1, \phi'(0) = b, \end{cases} \qquad \begin{cases} \psi'' \geq G\psi, & t \in (0, +\infty), \\ \psi(0) = 1, \psi'(0) = \tilde{b}, \end{cases}$$

where $b, \tilde{b}$ *are constants and* $\tilde{b} \geq b$. *If* $\phi(t) > 0$ *for* $t \in (0, T)$*, then* $\psi(t) > 0$ *in* $(0, T)$ *and*

$$\frac{\phi'}{\phi} \leq \frac{\psi'}{\psi} \quad \text{and} \quad \psi \geq \phi \quad \text{on } (0, T).$$

Proof Set $\beta = \sup\{t : \psi(t) > 0 \text{ in } (0, t)\}$ and $\tau = \min\{\beta, T\}$, so that ϕ and ψ are both positive in $(0, \tau)$. The function $\psi'\phi - \psi\phi'$ is continuous on $[0, +\infty)$, nonnegative at $t = 0$, and satisfies

$$(\psi'\phi - \psi\phi')' = \psi''\phi - \psi\phi'' \geq G(t)\psi\phi - G(t)\psi\phi = 0,$$

in $(0, \tau)$. Thus $\psi'\phi - \psi\phi' \geq 0$ on $[0, \tau)$, which implies

$$\frac{\psi'}{\psi} \geq \frac{\phi'}{\phi} \quad \text{in } [0, \tau). \tag{2.3}$$

Integrating (2.3) between 0 and t $(0 < t < \tau)$ yields

$$\phi(t) \leq \psi(t), \quad \text{in } [0, \tau).$$

Since $\phi > 0$ in $[0, \tau)$ by assumption, this forces $\tau = T$. □

Lemma 2.6 *Let G be a nonnegative continuous function on $[0, +\infty)$ satisfying $\int_0^{+\infty} G\,dt < +\infty$. Let $h_1, h_2 \in C^2([0, +\infty))$ be solutions of the following problems*

$$\begin{cases} h_1'' = Gh_1, & t \in (0, +\infty), \\ h_1(0) = 0, h_1'(0) = 1, \end{cases} \qquad \begin{cases} h_2'' = Gh_2, & t \in (0, +\infty), \\ h_2(0) = 1, h_2'(0) = 0. \end{cases} \tag{2.4}$$

Then we have

$$\lim_{t\to\infty} \frac{h_2}{h_1} = \lim_{t\to\infty} \frac{h_2'}{h_1'} \leq \int_0^{+\infty} G\,dt < \infty.$$

Proof From (2.4), we derive

$$(h_2h_1' - h_1h_2')'(t) \equiv 0,$$

and thus

$$(h_2h_1' - h_1h_2')(t) \equiv 1 \tag{2.5}$$

in view of the initial values for h_1 and h_2. By derivation, one can find

$$\left(\frac{h_2}{h_1}\right)' = \frac{h_2'h_1 - h_1'h_2}{h_1^2} = \frac{-1}{h_1^2} < 0,$$

which implies that $\lim_{t\to+\infty} \frac{h_2(t)}{h_1(t)}$ exists. It is easy to show that

$$0 \leq \left(\frac{h_2'}{h_1'}\right)' = \frac{G(h_2h_1' - h_1h_2')}{(h_1')^2} \leq \frac{G}{(1 + \int_0^t sG(s)ds)^2} \leq G,$$

so we get

$$\frac{h_2'(t)}{h_1'(t)} \leq \int_0^{+\infty} G\,dt.$$

By Lemma 2.13 in [34], we have $h_1(t) \geq t$. Consequently, using (2.5) and $h_1' = 1 + \int_0^t Gh_1 ds$, we obtain

$$\frac{h_2}{h_1} = \frac{h_2'}{h_1'} + \frac{1}{h_1h_1'} \leq \int_0^{+\infty} G\,dt + \frac{1}{t}, \quad t \in (0, \infty). \tag{2.6}$$

Letting $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{h_2}{h_1} = \lim_{t\to\infty} \frac{h_2'}{h_1'} \leq \int_0^{+\infty} G\,dt.$$

□

The next result is useful to study the growth of various balls on M when their radii approach to infinity.

Lemma 2.7 *Let h be the solution of (1.4). Then*

$$\lim_{t\to+\infty} \frac{h(t-C)}{h(t)} = 1 \text{ and } \lim_{t\to+\infty} \frac{h(tC)}{h(t)} = C,$$

where C is any positive constant.

Proof From Lemma 2.13 in [34], we know $t \leq h(t) \leq e^{b_0} t$, and thus

$$h'(t) = 1 + \int_0^t \lambda h \, dt \leq 1 + b_0 e^{b_0}. \tag{2.7}$$

Clearly (2.7) means that h' is nondecreasing and bounded from above. Consequently we have

$$\lim_{t \to +\infty} \frac{h(t-C)}{h(t)} = \lim_{t \to +\infty} \frac{h'(t-C)}{h'(t)} = 1$$

and

$$\lim_{t \to +\infty} \frac{h(tC)}{h(t)} = \lim_{t \to +\infty} \frac{Ch'(tC)}{h'(t)} = C.$$

□

We are now turning to the proof of Theorem 1.1.

Proof of Theorem 1.1 For any $r > 0$ and $\bar{x} \in A_r$, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_{\bar{x}} M$. Choosing the geodesic normal coordinates $(x^1, \ldots, x^n)$ around $\bar{x}$, such that $\frac{\partial}{\partial x^i} = e_i$ at $\bar{x}$. Let $\bar{\gamma}(t) := \exp_{\bar{x}}(tDu(\bar{x}))$ for all $t \in [0, r]$. For $1 \leq i \leq n$, let $E_i(t)$ be the parallel transport of e_i along $\bar{\gamma}$. For $1 \leq i \leq n$, let $X_i(t)$ be the Jacobi field along $\bar{\gamma}$ with the initial conditions of $X_i(0) = e_i$ and

$$\langle D_t X_i(0), e_j \rangle = (D^2 u)(e_i, e_j), \quad 1 \leq j \leq n.$$

Let $P(t) = (P_{ij}(t))$ be a matrix defined by

$$P_{ij}(t) = \langle X_i(t), E_j(t) \rangle, \quad 1 \leq i, j \leq n.$$

From Lemma 2.4, we known $\det P(t) > 0, \forall t \in [0, r)$. Obviously, $|\det D\Phi_t(\bar{x})| = \det P(t) > 0$ for $t \in [0, r)$. Let $S(t) = (S_{ij}(t))$ be a matrix defined by

$$S_{ij}(t) = R(\bar{\gamma}'(t), E_i(t), \bar{\gamma}'(t), E_j(t)), \quad 1 \leq i, j \leq n,$$

where R denotes the Riemannian curvature tensor of M. By the Jacobi equation, one can obtain

$$\begin{cases} P''(t) = -P(t)S(t), & t \in [0, r], \\ P_{ij}(0) = \delta_{ij}, \, P'_{ij}(0) = (D^2 u)(e_i, e_j). \end{cases} \tag{2.8}$$

Let $Q(t) = P(t)^{-1} P'(t)$, $t \in (0, r)$. Using (2.8), a simple computation yields

$$\frac{d}{dt} Q(t) = -S(t) - Q^2(t),$$

where $Q(t)$ is symmetric. The assumption of asymptotically nonnegative Ricci curvature gives

$$\begin{aligned} \frac{d}{dt}[\mathrm{tr} Q(t)] + \frac{1}{n}[\mathrm{tr} Q(t)]^2 &\leq \frac{d}{dt}[\mathrm{tr} Q(t)] + \mathrm{tr}[Q^2(t)] \\ &= -\mathrm{tr} S(t) \\ &\leq (n-1)|Du(\bar{x})|^2 \lambda(d(o, \bar{\gamma}(t))), \end{aligned} \tag{2.9}$$

where o is the base point. Using triangle inequality and the definition of A_r, it is easy to see that

$$d(o,\bar{\gamma}(t)) \geq \big|d(o,\bar{x}) - d(\bar{x},\bar{\gamma}(t))\big| = \big|d(o,\bar{x}) - t|Du(\bar{x})|\big|. \tag{2.10}$$

Set

$$g = \frac{1}{n}\mathrm{tr}Q,$$
$$\Lambda_{\bar{x}}(t) = \frac{(n-1)}{n}|Du(\bar{x})|^2\lambda(\big|d(o,\bar{x}) - t|Du(\bar{x})|\big|).$$

Noting that λ is nonincreasing, it follows from (2.8), (2.9), (2.10) that

$$\begin{cases} g'(t) + g(t)^2 \leq \Lambda_{\bar{x}}(t), & t \in (0,r), \\ g(0) = \dfrac{1}{n}\Delta u(\bar{x}). \end{cases}$$

If we take $\phi = e^{\int_0^t g(\tau)d\tau}$, then ϕ satisfies

$$\begin{cases} \phi'' \leq \Lambda_{\bar{x}}(t)\phi, & t \in (0,r), \\ \phi(0) = 1, \phi'(0) = \dfrac{1}{n}\Delta u(\bar{x}). \end{cases} \tag{2.11}$$

Next, we denote by ψ_1, ψ_2 the solutions of the following problems

$$\begin{cases} \psi_1'' = \Lambda_{\bar{x}}(t)\psi_1, & t \in (0,r), \\ \psi_1(0) = 0, \psi_1'(0) = 1, \end{cases} \quad \begin{cases} \psi_2'' = \Lambda_{\bar{x}}(t)\psi_2, & t \in (0,r), \\ \psi_2(0) = 1, \psi_2'(0) = 0. \end{cases} \tag{2.12}$$

Similar to the proof of (2.6), it is easy to verify that

$$\frac{\psi_2}{\psi_1}(r) \leq \int_0^{+\infty} \Lambda_{\bar{x}}(t)\,dt + \frac{1}{r} \leq 2\Big(\frac{n-1}{n}\Big)b_1|Du(\bar{x})| + \frac{1}{r}.$$

Since $|Du(\bar{x})| < 1$, we obtain

$$\frac{\psi_2}{\psi_1}(r) \leq 2\Big(\frac{n-1}{n}\Big)b_1 + \frac{1}{r}. \tag{2.13}$$

Using Lemma 2.13 in [34] and (2.12), we deduce that

$$\begin{aligned} \psi_1(t) &\leq \int_0^t e^{\int_0^s \tau\Lambda_{\bar{x}}(\tau)d\tau}ds \\ &\leq te^{\int_0^\infty \tau\Lambda_{\bar{x}}(\tau)d\tau} \\ &= te^{\frac{n-1}{n}\int_0^\infty w\lambda(|d(o,\bar{x})-w|)dw} \\ &\leq te^{\frac{n-1}{n}(2r_0b_1+b_0)}, \end{aligned} \tag{2.14}$$

where $r_0 = \max\{d(o,x)|x \in \Omega\}$.

Let $\psi(t) = \psi_2(t) + \frac{1}{n}\Delta u(\bar{x})\psi_1(t)$. Using Lemma 2.5, one can get

$$\frac{1}{n}\mathrm{tr}Q(t) = \frac{\phi'}{\phi} \leq \frac{\psi'}{\psi}, \quad \forall t \in (0,r).$$

Thus,

$$\frac{d}{dt}\log\det P(t) = \mathrm{tr}Q(t) \leq n\frac{\psi'}{\psi}. \tag{2.15}$$

Consequently, (2.15) implies

$$|\det D\Phi_t(\bar{x})| = \det P(t) \leq \psi^n(t) = (\psi_2(t) + \frac{1}{n}\Delta u(\bar{x})\psi_1(t))^n$$

for all $t \in [0, r]$. This gives

$$|\det D\Phi_r(\bar{x})| \leq \Big(\frac{\psi_2(r)}{\psi_1(r)} + \frac{1}{n}\Delta u(\bar{x})\Big)^n \psi_1^n(r)$$

for any $\bar{x} \in A_r$. Note that $0 \leq \phi \leq \psi$. Using (2.13), (2.14) and Lemma 2.1, we derive that

$$\begin{aligned}|\det D\Phi_r(\bar{x})| &\leq e^{(n-1)(2r_0b_1+b_0)}\Big(2(\frac{n-1}{n})b_1 + \frac{1}{r} + \frac{1}{n}\Delta u(\bar{x})\Big)^n r^n \\ &\leq e^{(n-1)(2r_0b_1+b_0)}\Big(\frac{1}{r} + f^{\frac{1}{n-1}}(\bar{x})\Big)^n r^n\end{aligned} \tag{2.16}$$

for any $\bar{x} \in A_r$. Moreover, by (1.4), we obtain $h(t) \geq t$ and

$$\lim_{t\to\infty} h'(t) = 1 + \int_0^\infty h(s)\lambda(s)\,ds \geq 1 + \int_0^\infty s\lambda(s)\,ds = 1 + b_0. \tag{2.17}$$

Combining Lemma 2.2, (2.16) with the formula for change of variables in multiple integrals, we find that

$$\begin{aligned}&|\{q \in M : d(x,q) < r \text{ for all } x \in \Omega\}| \\ &\leq \int_{A_r} |\det D\Phi_r| \\ &\leq \int_\Omega e^{(n-1)(2r_0b_1+b_0)}(\frac{1}{r} + f^{\frac{1}{n-1}})^n r^n.\end{aligned} \tag{2.18}$$

For $r > r_0$, the triangle inequality implies that

$$B_{r-r_0}(o) \subset \{q \in M : d(x,q) < r \text{ for all } x \in \Omega\} \subset B_{r+r_0}(o). \tag{2.19}$$

From (1.5), (2.19) and Lemma 2.7, it is easy to show that

$$\begin{aligned}|B^n|\theta &= \lim_{r\to+\infty} \frac{B_{r-r_0}(o)}{n\int_0^{r-r_0} h(t)^{n-1}dt}\frac{\int_0^{r-r_0} h(t)^{n-1}dt}{\int_0^r h(t)^{n-1}dt} \\ &\leq \lim_{r\to+\infty} \frac{|\{q \in M : d(x,q) < r \text{ for all } x \in \Omega\}|}{n\int_0^r h(t)^{n-1}dt} \\ &\leq \lim_{r\to+\infty} \frac{B_{r+r_0}(o)}{n\int_0^{r+r_0} h(t)^{n-1}dt}\frac{\int_0^{r+r_0} h(t)^{n-1}dt}{\int_0^r h(t)^{n-1}dt} \\ &= |B^n|\theta.\end{aligned} \tag{2.20}$$

Dividing (2.18) by $n\int_0^r h(t)^{n-1}dt$ and sending $r \to \infty$, it follows from (2.17) and (2.20) that

$$\begin{aligned}|B^n|\theta &\leq e^{(n-1)(2r_0b_1+b_0)}\int_\Omega f^{\frac{n}{n-1}} \lim_{r\to\infty}\frac{r^n}{n\int_0^r h(t)^{n-1}dt} \\ &= e^{(n-1)(2r_0b_1+b_0)}\int_\Omega f^{\frac{n}{n-1}} \lim_{r\to\infty}\frac{1}{h'(t)^{n-1}} \\ &\leq \Big(\frac{e^{2r_0b_1+b_0}}{1+b_0}\Big)^{n-1}\int_\Omega f^{\frac{n}{n-1}}.\end{aligned}$$

Hence we obtain

$$\int_{\partial\Omega} f + \int_{\Omega} |Df| + 2(n-1)b_1 \int_{\Omega} f \geq n|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\frac{1+b_0}{e^{2r_0 b_1 + b_0}}\right)^{\frac{n-1}{n}} \left(\int_{\Omega} f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

□

Proof of Theorem 1.2 Suppose the equality of Theorem 1.1 holds. Then we have equalities in (2.13) and (2.17) which force $\lambda \equiv 0$. Thus M has nonnegative Ricci curvature. The assertion follows immediately from Theorem 1.2 in [10]. □

3 The case of submanifolds

In this section, we assume that the ambient space M is a complete noncompact $(n+p)$-dimensional Riemannian manifold of asymptotically nonnegative sectional curvature with respect to a base point $o \in M$. Let $\Sigma \subset M$ be a compact submanifold of dimension n with or without boundary, and f be a positive smooth function on Σ. Let $\bar{D}$ denote the Levi-Civita connection of M and let D^Σ denote the induced connection on Σ. The second fundamental form B of Σ is given by

$$\langle B(X,Y), V\rangle = \langle \bar{D}_X Y, V\rangle,$$

where X, Y are the tangent vector fields on Σ, V is a normal vector field along Σ. The mean curvature vector of Σ is defined by $H = \mathrm{tr} B$.

We only need to treat the case that Σ is connected. By scaling, we can assume that

$$\int_{\partial\Sigma} f + \int_{\Sigma} \sqrt{|D^\Sigma f|^2 + f^2|H|^2} + 2nb_1 \int_{\Sigma} f = n \int_{\Sigma} f^{\frac{n}{n-1}}. \tag{3.1}$$

By the connectedness of Σ and (3.1), there exists a solution of the following Neumann boundary problem

$$\begin{cases} \mathrm{div}_\Sigma(f D^\Sigma u) = nf^{\frac{n}{n-1}} - 2nb_1 f - \sqrt{|D^\Sigma f|^2 + f^2|H|^2}, & \text{in } \Sigma, \\ \langle D^\Sigma u, \nu\rangle = 1, & \text{on } \partial\Sigma, \end{cases} \tag{3.2}$$

where ν is the outward unit normal vector field of $\partial\Sigma$ with respect to Σ. Note that if $\partial\Sigma = \varnothing$, then the boundary condition in (3.2) is void. By standard elliptic regularity theory (see Theorem 6.31 in [20]), we know that $u \in C^{2,\gamma}$ for each $0 < \gamma < 1$.

As in [10], we define

$$U := \{x \in \Sigma \setminus \partial\Sigma : |D^\Sigma u(x)| < 1\},$$
$$E := \{(x,y) : x \in U, y \in T_x^\perp \Sigma, |D^\Sigma u(x)|^2 + |y|^2 < 1\}.$$

For each $r > 0$, we denote by A_r the set of all points $(\bar{x}, \bar{y}) \in E$ satisfying

$$ru(x) + \frac{1}{2} d(x, \exp_{\bar{x}}(rD^\Sigma u(\bar{x})) + r\bar{y})^2 \geq ru(\bar{x}) + \frac{1}{2} r^2 (|D^\Sigma u(\bar{x})|^2 + |\bar{y}|^2)$$

for all $x \in \Sigma$. Define the transport map $\Phi_r : T^\perp\Sigma \to M$ for each $r > 0$ by

$$\Phi_r(x,y) = \exp_x(rD^\Sigma u(x) + ry)$$

for all $x \in \Sigma$ and $y \in T_x^\perp\Sigma$. The regularity of u implies that Φ_r is of class $C^{1,\gamma}$, $0 < \gamma < 1$.

Lemma 3.1 *Assume that* $(x, y) \in E$. *Then we have*

$$\frac{1}{n}\left(\Delta_\Sigma u(x) - \langle H(x), y\rangle\right) \le f^{\frac{1}{n-1}}(x) - 2b_1.$$

Proof Combining $|D^\Sigma u(x)|^2 + |y|^2 < 1$ with Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &- \langle D^\Sigma f(x), D^\Sigma u(x)\rangle - f(x)\langle H(x), y\rangle \\ &\le \sqrt{|D^\Sigma f(x)|^2 + f(x)^2|H(x)|^2}\sqrt{|D^\Sigma u(x)|^2 + |y|^2} \\ &\le \sqrt{|D^\Sigma f(x)|^2 + f(x)^2|H(x)|^2}. \end{aligned} \tag{3.3}$$

In terms of (3.2) and (3.3), one derives that

$$\begin{aligned} & f(x)\Delta_\Sigma u(x) - f(x)\langle H(x), y\rangle \\ &= nf(x)^{\frac{n}{n-1}} - 2nb_1 f - \sqrt{|D^\Sigma f(x)|^2 + f(x)^2|H(x)|^2} \\ &\quad - \langle D^\Sigma f(x), D^\Sigma u(x)\rangle - f(x)\langle H(x), y\rangle \\ &\le nf(x)^{\frac{n}{n-1}} - 2nb_1 f. \end{aligned}$$

The proof is completed. □

The following three lemmas are due to Brendle (Lemmas 4.2, 4.3, 4.5 in [10]). Their proofs are independent of the curvature condition of ambient space too.

Lemma 3.2 *For each* $0 \le \sigma < 1$, *the set*

$$\{q \in M : \sigma r < d(x, q) < r,\ \forall x \in \Sigma\}$$

is contained in the set

$$\Phi_r(\{(x, y) \in A_r : |D^\Sigma u(x)|^2 + |y|^2 > \sigma^2\}).$$

Lemma 3.3 *Assume that* $(\bar{x}, \bar{y}) \in A_r$, *and let* $\bar{\gamma}(t) := \exp_{\bar{x}}(tD^\Sigma u(\bar{x}) + t\bar{y})$ *for all* $t \in [0, r]$. *If* Z *is a smooth vector field along* $\bar{\gamma}$ *satisfying* $Z(0) \in T_{\bar{x}}\Sigma$ *and* $Z(r) = 0$, *then*

$$\begin{aligned} &((D^\Sigma)^2 u)(Z(0), Z(0)) - \langle B(Z(0), Z(0)), \bar{y}\rangle \\ &+ \int_0^r \left(|\bar{D}_t Z(t)|^2 - \bar{R}(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t))\right)dt \ge 0. \end{aligned}$$

Lemma 3.4 *Assume that* $(\bar{x}, \bar{y}) \in A_r$, *and let* $\bar{\gamma}(t) := \exp_{\bar{x}}(tD^\Sigma u(\bar{x}) + t\bar{y})$ *for all* $t \in [0, r]$. *Let* $\{e_1, \ldots, e_n\}$ *be an orthonormal basis of* $T_{\bar{x}}\Sigma$. *Suppose that* W *is a Jacobi field along* $\bar{\gamma}$ *satisfying* $W(0) \in T_{\bar{x}}\Sigma$ *and* $\langle \bar{D}_t W(0), e_j\rangle = ((D^\Sigma)^2 u)(W(0), e_j) - \langle B(W(0), e_j), \bar{y}\rangle$ *for each* $1 \le j \le n$. *If* $W(\tau) = 0$ *for some* $\tau \in (0, r)$, *then* W *vanishes identically.*

Now we begin the proof of Theorem 1.5.

Proof of Theorem 1.4 For any $r > 0$ and $(\bar{x}, \bar{y}) \in A_r$, let $\{e_i\}_{1\le i\le n}$ be any given orthonormal basis in $T_{\bar{x}}\Sigma$. Choose a normal coordinate system $(x^1, \cdots, x^n)$ on Σ around $\bar{x}$ such that $\frac{\partial}{\partial x^i} = e_i$ at $\bar{x}$ $(1 \le i \le n)$. Let $\{e_\alpha\}_{n+1\le\alpha\le n+p}$ be an orthonormal frame field of $T^\perp\Sigma$ around $\bar{x}$ such that $\left((D^\Sigma)^\perp e_\alpha\right)_{\bar{x}} = 0$ for $n + 1 \le \alpha \le n + p$, where $(D^\Sigma)^\perp$ denotes the normal connection in the normal bundle $T^\perp\Sigma$ of Σ. Any normal vector y around $\bar{x}$ can be written as $y = \sum_{\alpha=n+1}^{n+p} y^\alpha e_\alpha$, and thus $(x^1, \cdots, x^n, y^{n+1}, \cdots, y^{n+p})$ becomes a local coordinate system on the total space of the normal bundle $T^\perp\Sigma$.

Let $\bar{\gamma}(t) := \exp_{\bar{x}}(tD^\Sigma u(\bar{x}) + t\bar{y})$ for all $t \in [0, r]$. For each $1 \le A \le n+p$, we denote by $E_A(t)$ the parallel transport of $e_A(\bar{x})$ along $\bar{\gamma}$. For each $1 \le i \le n$, let X_i be the Jacobi field along $\bar{\gamma}$ with the following initial conditions

$$\begin{aligned} X_i(0) &= e_i, \\ \langle \bar{D}_t X_i(0), e_j\rangle &= ((D^\Sigma)^2 u)(e_i, e_j) - \langle B(e_i, e_j), \bar{y}\rangle, \quad 1 \le j \le n, \\ \langle \bar{D}_t X_i(0), e_\beta\rangle &= \langle B(e_i, D^\Sigma u(\bar{x})), e_\beta\rangle, \quad n+1 \le \beta \le n+p. \end{aligned} \tag{3.4}$$

For each $n+1 \le \alpha \le n+p$, let X_α be the Jacobi field along $\bar{\gamma}$ satisfying

$$X_\alpha(0) = 0, \quad \bar{D}_t X_\alpha(0) = e_\alpha. \tag{3.5}$$

Using Lemma 3.4, we known that $\{X_A(t)\}_{1 \le A \le n+p}$ are linearly independent for each $t \in (0, r)$.

Let $P(t) = (P_{AB}(t))$ and $S(t) = (S_{AB}(t))$ be the matrices given by

$$\begin{aligned} P_{AB}(t) &= \langle X_A(t), E_B(t)\rangle, \\ S_{AB}(t) &= \bar{R}(\bar{\gamma}'(t), E_A(t), \bar{\gamma}'(t), E_B(t)) \end{aligned}$$

for $1 \le A, B \le n+p$ and $t \in [0, r]$, where $\bar{R}$ denotes the Riemannian curvature tensor of M. Using the Jacobi equation and the initial conditions (3.4), (3.5), we have

$$\begin{aligned} P''(t) &= -P(t)S(t), \\ P_{AB}(0) &= \begin{bmatrix} \delta_{ij} & 0 \\ 0 & 0 \end{bmatrix}, \\ P'_{AB}(0) &= \begin{bmatrix} ((D^\Sigma)^2 u)(e_i, e_j) - \langle B(e_i, e_j), \bar{y}\rangle & \langle B(e_i, D^\Sigma u(\bar{x})), e_\beta\rangle \\ 0 & \delta_{\alpha\beta} \end{bmatrix}. \end{aligned} \tag{3.6}$$

Set $Q(t) = P(t)^{-1}P'(t)$, $t \in (0, r)$. By (3.6), a simple computation yields

$$\frac{d}{dt} Q(t) = -S(t) - Q^2(t), \tag{3.7}$$

where $Q(t)$ is symmetric. For the matrices $P(t)$, $Q(t)$, it is easy to derive their following asymptotic expansions (cf. [10])

$$\begin{aligned} P(t) &= \begin{bmatrix} \delta_{ij} + O(t) & O(t) \\ O(t) & t\delta_{\alpha\beta} + O(t^2) \end{bmatrix}, \\ Q(t) &= \begin{bmatrix} (D^\Sigma)^2 u(e_i, e_j) - \langle B(e_i, e_j), \bar{y}\rangle + O(t) & O(1) \\ O(1) & \frac{1}{t}\delta_{\alpha\beta} + O(1) \end{bmatrix} \end{aligned} \tag{3.8}$$

as $t \to 0^+$. In terms of (3.7) and the curvature assumption for M, we deduce

$$\begin{aligned} \frac{d}{dt} Q_{AA}(t) + Q_{AA}(t)^2 &\le \frac{d}{dt} Q_{AA}(t) + \sum_{B=1}^{n+p} Q_{AB}Q_{BA}(t) \\ &= -S_{AA}(t) \\ &\le (|D^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 - \langle D^\Sigma u(\bar{x}) + \bar{y}, e_A\rangle^2)\lambda(d(o, \bar{\gamma}(t))) \\ &\le (|D^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 - \langle D^\Sigma u(\bar{x}) + \bar{y}, e_A\rangle^2)\lambda(\left|d(o, \bar{x}) - t|D^\Sigma u(\bar{x}) + \bar{y}|\right|) \end{aligned} \tag{3.9}$$

for $1 \le A \le n+p$, where the last inequality follows from the following triangle inequality

$$d(o, \bar\gamma(t)) \ge \big|d(o, \bar x) - d(\bar x, \bar\gamma(t))\big| = \big|d(o, \bar x) - t|D^\Sigma u(\bar x) + \bar y|\big|.$$

For $1 \le A \le n+p$, we set

$$\Lambda_{\bar x, A}(t) = (|D^\Sigma u(\bar x)|^2 + |\bar y|^2 - \langle D^\Sigma u(\bar x) + \bar y, e_A\rangle^2)\lambda(\big|d(o, \bar x) - t|D^\Sigma u(\bar x) + \bar y|\big|).$$

Then we have

$$\begin{cases} Q_{ii}'(t) + Q_{ii}(t)^2 \le \Lambda_{\bar x, i}(t), & t \in (0, r), \\ \lim\limits_{t\to 0^+} Q_{ii}(t) = \lambda_i, \end{cases}$$

where $\lambda_i = P_{ii}'(0)$. Let ϕ_i be defined by

$$\phi_i(t) = e^{\int_0^t Q_{ii}(\tau)d\tau}.$$

Then ϕ_i satisfies

$$\begin{cases} \phi_i'' \le \Lambda_{\bar x, i}\phi_i, & t \in (0, r), \\ \phi_i(0) = 1, \phi_i'(0) = \lambda_i. \end{cases} \tag{3.10}$$

Next, we denote by $\psi_{1,i}, \psi_{2,i}$ solutions to the following problems

$$\begin{cases} \psi_{1,i}'' = \Lambda_{\bar x, i}\psi_{1,i}, & t \in (0, r), \\ \psi_{1,i}(0) = 0, \psi_{1,i}'(0) = 1, \end{cases} \quad \begin{cases} \psi_{2,i}'' = \Lambda_{\bar x, i}\psi_{2,i}, & t \in (0, r), \\ \psi_{2,i}(0) = 1, \psi_{2,i}'(0) = 0. \end{cases} \tag{3.11}$$

Similar to the proof of (2.6), (2.13) and (2.14), we obtain

$$\begin{aligned} \frac{\psi_{2,i}}{\psi_{1,i}}(r) &\le \int_0^{+\infty} \Lambda_{\bar x, i}(t)\, dt + \frac{1}{r} \\ &\le 2b_1 \frac{|D^\Sigma u(\bar x)|^2 + |\bar y|^2 - \langle D^\Sigma u(\bar x) + \bar y, e_i\rangle^2}{\sqrt{|D^\Sigma u(\bar x)|^2 + \bar y^2}} + \frac{1}{r} \\ &\le 2b_1 \sqrt{|D^\Sigma u(\bar x)|^2 + \bar y^2} + \frac{1}{r} \end{aligned} \tag{3.12}$$

and

$$\psi_{1,i}(t) \le t e^{\frac{|D^\Sigma u(\bar x)|^2 + \bar y^2 - \langle D^\Sigma u(\bar x) + \bar y, e_i\rangle^2}{|D^\Sigma u(\bar x)|^2 + \bar y^2}(2r_0 b_1 + b_0)}, \quad t \in (0, r), \tag{3.13}$$

where $r_0 = \max\{d(o, x) | x \in \Sigma\}$. Using Lemma 2.5, one can find from (3.10) and (3.11) that

$$Q_{ii}(t) = \frac{\phi_i'}{\phi_i}(t) \le \frac{\psi_{2,i}' + \lambda_i \psi_{1,i}'}{\psi_{2,i} + \lambda_i \psi_{1,i}}(t). \tag{3.14}$$

Similarly we obtain from (3.8) and (3.9) that

$$\begin{cases} Q_{\alpha\alpha}'(t) + Q_{\alpha\alpha}(t)^2 \le \Lambda_{\bar x, \alpha}(t), & t \in (0, r), \\ Q_{\alpha\alpha}(t) = \dfrac{1}{t} + O(1), & \text{as } t \to 0^+ \end{cases}$$

for $n+1 \le \alpha \le n+p$. Set $\phi_\alpha(t) = t e^{\int_0^t (Q_{\alpha\alpha}(\tau) - \frac{1}{\tau})d\tau}$. Then ϕ_α satisfies

$$\begin{cases} \phi_\alpha'' \le \Lambda_{\bar x, \alpha}\phi_\alpha, & t \in (0, r), \\ \phi_\alpha(0) = 0, \phi_\alpha'(0) = 1. \end{cases}$$

Next, we denote by $\psi_{1,\alpha}$ the unique solution to the following problem

$$\begin{cases} \psi_{1,\alpha}'' = \Lambda_{\bar{x},\alpha}\psi_{1,\alpha}, \quad t \in (0,r), \\ \psi_{1,\alpha}(0) = 0, \psi_{1,\alpha}'(0) = 1. \end{cases} \tag{3.15}$$

Similar to (2.14), we derive that

$$\psi_{1,\alpha} \le e^{\frac{|D^\Sigma u(\bar{x})|^2+\bar{y}^2-\langle D^\Sigma u(\bar{x})+\bar{y},e_\alpha\rangle^2}{|D^\Sigma u(\bar{x})|^2+\bar{y}^2}(2r_0b_1+b_0)}t, \tag{3.16}$$

for $t \in (0,r)$. By Lemma 2.1 in [34] we have

$$Q_{\alpha\alpha}(t) = \frac{\phi_\alpha'}{\phi_\alpha}(t) \le \frac{\psi_{1,\alpha}'}{\psi_{1,\alpha}}(t). \tag{3.17}$$

From (3.14) and (3.17), it follows that

$$\frac{d}{dt}\log\det P(t) = \mathrm{tr}(Q(t)) \le \sum_i \frac{\psi_{2,i}' + \lambda_i\psi_{1,i}'}{\psi_{2,i}+\lambda_i\psi_{1,i}}(t) + \sum_\alpha \frac{\psi_{1,\alpha}'}{\psi_{1,\alpha}}(t). \tag{3.18}$$

Combining (3.11), (3.15) with the asymptotic properties in (3.8), we conclude that

$$\lim_{t\to 0^+} \frac{\det P(t)}{\prod_i(\psi_{2,i}(t)+\lambda_i\psi_{1,i}(t))\prod_\alpha \psi_{1,\alpha}(t)} = 1. \tag{3.19}$$

Integrating (3.18) over $[\varepsilon, t]$ for $0 < \varepsilon < t$ and using (3.19) by letting $\varepsilon \to 0^+$, it is easy to show that

$$|\det \bar{D}\Phi_t(\bar{x},\bar{y})| = \det P(t) \le \prod_i(\psi_{2,i}(t)+\lambda_i\psi_{1,i}(t))\prod_\alpha \psi_{1,\alpha}(t).$$

Note that $0 \le \phi_i \le (\psi_{2,i}+\lambda_i\psi_{1,i})$ and $\psi_{1,i} \ge 0$ $(1 \le i \le n)$. Combining (3.13), (3.16) with arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} |\det \bar{D}\Phi_t(\bar{x},\bar{y})| &\le \Big(\frac{1}{n}\sum_i \frac{\psi_{2,i}(t)}{\psi_{1,i}(t)} + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}),\bar{y}\rangle)\Big)^n \prod_A \psi_{1,A}(t) \\ &\le \Big(\frac{1}{n}\sum_i \frac{\psi_{2,i}(t)}{\psi_{1,i}(t)} + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}),\bar{y}\rangle)\Big)^n t^{n+p}e^{(n+p-1)(2r_0b_1+b_0)} \end{aligned}$$

which yields by (3.12) that

$$\begin{aligned} |\det \bar{D}\Phi_r(\bar{x},\bar{y})| \le (2b_1\sqrt{|Du(\bar{x})|^2+\bar{y}^2} + \frac{1}{r} + \frac{1}{n}(\Delta_\Sigma u(\bar{x}) \\ - \langle H(\bar{x}),\bar{y}\rangle))^n r^{n+p}e^{(n+p-1)(2r_0b_1+b_0)} \end{aligned} \tag{3.20}$$

for all $(\bar{x},\bar{y}) \in A_r$. Noting that $\sqrt{|Du(\bar{x})|^2+\bar{y}^2} < 1$, we derive by Lemma 3.1 and (3.20) that

$$|\det \bar{D}\Phi_r(\bar{x},\bar{y})| \le (\frac{1}{r} + f^{\frac{1}{n-1}}(\bar{x}))^n r^{n+p}e^{(n+p-1)(2r_0b_1+b_0)} \tag{3.21}$$

for all $(\bar{x},\bar{y}) \in A_r$. Using Lemma 3.2 and (3.21), one may find in a similar way as the proof of Theorem 1.4 in [10] that

$$\begin{aligned} &|\{p \in M : \sigma r < d(x,p) < r, \forall x \in \Sigma\}| \\ &\le \frac{p}{2}|B^p|(1-\sigma^2)e^{(n+p-1)(2r_0b_1+b_0)}\int_\Sigma (\frac{1}{r} + f^{\frac{1}{n-1}}(\bar{x}))^n r^{n+p}, \end{aligned} \tag{3.22}$$

for all $r > 0$ and all $0 \leq \sigma < 1$. Similar to the proof of (2.20), one can obtain by using Lemma 2.7 that

$$\begin{aligned} \lim_{r\to+\infty} & \frac{|\{p \in M : \sigma r < d(x,p) < r, \forall x \in \Sigma\}|}{(n+p)\int_0^r h^{n+p-1}dt} \\ &= |B^{n+p}|\theta \lim_{r\to+\infty}(1-\sigma\frac{h^{n+p-1}(\sigma r)}{h^{n+p-1}(r)}) \\ &= |B^{n+p}|(1-\sigma^{n+p})\theta. \end{aligned} \tag{3.23}$$

Dividing (3.22) by $(n+p)\int_0^r h(t)^{n+p-1}dt$ and sending $r \to +\infty$, we deduce by using (2.17) and (3.23) that

$$\begin{aligned} &= |B^{n+p}|(1-\sigma^{n+p})\theta \\ &\leq \frac{p}{2}|B^p|(1-\sigma^2)e^{(n+p-1)(2r_0b_1+b_0)}\int_\Sigma f^{\frac{n}{n-1}} \lim_{r\to+\infty}\frac{r^{n+p}}{(n+p)\int_0^r h(t)^{n+p-1}dt} \\ &\leq \frac{p}{2}|B^p|(1-\sigma^2)\Big(\frac{e^{2r_0b_1+b_0}}{1+b_0}\Big)^{n+p-1}\int_\Sigma f^{\frac{n}{n-1}}. \end{aligned} \tag{3.24}$$

for all $0 \leq \sigma < 1$. Now, if we divide (3.24) by $1-\sigma$ and let $\sigma \to 1$, we have

$$(n+p)|B^{n+p}|\theta \leq p|B^p|\Big(\frac{e^{2r_0b_1+b_0}}{1+b_0}\Big)^{n+p-1}\int_\Sigma f^{\frac{n}{n-1}}. \tag{3.25}$$

Hence (3.1) and (3.25) imply that

$$\begin{aligned} &\int_{\partial\Sigma} f + \int_\Sigma \sqrt{|D^\Sigma f|^2 + f^2|H|^2} + 2nb_1\int_\Sigma f \\ &\geq n\Big(\frac{(n+p)|B^{n+p}|}{p|B^p|}\Big)^{\frac{1}{n}}\theta^{\frac{1}{n}}\Big(\frac{1+b_0}{e^{2r_0b_1+b_0}}\Big)^{\frac{n+p-1}{n}}\Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}. \end{aligned}$$

□

Proof of Theorem 1.6 Suppose the equality of Theorem 1.5 holds. Then we have equality in both (2.17) and (3.12) and either one forces $\lambda \equiv 0$. Thus M has nonnegative sectional curvature. The assertion follows immediately from Theorem 1.6 in [10]. □

Finally we would like to mention that we have established a Sobolev type inequality for manifolds with density and asymptotically nonnegative Bakery-Émery Ricci curvature in [16] and a logarithmic Sobolev type inequality for closed submanifolds in manifolds with asymptotically nonnegative sectional curvature in [17].

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