Calc. Var. (2024) 63:86
https://doi.org/10.1007/s00526-024-02686-9

Calculus of Variations

Manifolds for which Huber's Theorem holds

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Received: 10 June 2022 / Accepted: 1 March 2024 / Published online: 4 April 2024

Abstract
Extensions of Huber's Theorem to higher dimensions with $L^{\frac{n}{2}}$ bounded scalar curvature have been extensively studied over the years. In this paper, we delve into the properties of conformal metrics on a punctured ball with $\|R\|_{L^{\frac{n}{2}}} < +\infty$, aiming to identify necessary geometric constraints for Huber's theorem to be applicable. Unexpectedly, such metrics are more rigid than we initially anticipated. For instance, we found that the volume density at infinity is precisely one, and the blow-down of the metric is $\mathbb{R}^n$. Specifically, in four dimensions, we derive the L^2-integrability of the Ricci curvature, which directly leads to the conclusion that the Pfaffian 4-form is integrable and adheres to a Gauss-Bonnet-Chern formula. Additionally, we demonstrate that a Gauss-Bonnet-Chern formula, previously verified by Lu and Wang under the assumption that the second fundamental form belongs to L^4, remains valid for $R \in L^2$. Consequently, on an orientable 4-dimensional manifold conformal to a domain in a closed manifold, Huber's Theorem holds when $R \in L^2$, if and only if the negative part of the Pfaffian 4-form is integrable.

Mathematics Subject Classification 53C21 · 58J05

1 Introduction

In the famous paper [17], Huber proved a remarkable result concerning the structures of complete surfaces: every complete surface with the integrable negative part of the Gauss curvature is conformally equivalent to a compact surface with a finite number of points removed. Regrettably, this result does not extend straightforwardly to higher dimensions. For instance, the manifold $\mathbb{T}^2 \times \mathbb{R}$ is flat but not conformal to any closed manifold with

Communicated by A. Mondino.

The first author is partially supported by NSFC, Grant No.12141103.

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finite points removed. Therefore, a variety of generalizations of Huber's Theorem have been established under certain supplementary curvature and other geometric assumptions, as seen in [8, 9, 13, 16, 21], and related references.

In this paper, our focus will be on a complete manifold that conforms to a domain of a closed manifold with $\int_M |R(g)|^{\frac{n}{2}} dV_g < +\infty$. There are very few results available regarding Huber's Theorem in this particular direction. The only known sufficient and necessary condition is the combination of Theorem 2.1 in [9] with Theorem 1.2 in [2], which can be summarized as follows:

Theorem 1.1 *(Carron-Herzlich, Aldana-Carron-Tapie) Let Ω be a domain of (M, g_0), a compact Riemannian manifold of dimension $n > 2$. Assume Ω is endowed with a complete Riemannian metric g which is conformal to g_0. Then $M \setminus \Omega$ is a finite set if and only if $\mathrm{vol}(B_r^g(x_0), g) = O(r^n)$ for some point x_0 in Ω.*

The primary objective of this paper is to identify more geometric constraints for Huber's Theorem. We will investigate the geometric characteristics of a metric g defined on the punctured n-dimensional closed ball $\overline{B}\setminus 0$, which conforms to a smooth metric g_0 (defined on $\overline{B}$) with finite $\|R(g)\|_{L^{\frac{n}{2}}}$. Contrary to our expectations, such a metric exhibits a considerably higher degree of rigidity than previously anticipated. We demonstrate that the volume density of g at infinity equals 1, and the manifold blows down to an n-dimensional space. Specifically, we can state the following:

Theorem 1.2 *Let g_0 be a smooth metric defined on the closed unit n-ball $\overline{B}$, with $n \geq 3$. Let $g = u^{\frac{4}{n-2}} g_0$ be a conformal metric on $\overline{B} \setminus \{0\}$. Assuming $\|R(g)\|_{L^{\frac{n}{2}}(B,g)} < +\infty$ and $\mathrm{vol}(\overline{B} \setminus \{0\}, g) = \infty$. Then, as $r \to +\infty$, the volume ratio*

$$\frac{\mathrm{vol}(B_r^g(x), g)}{V_n r^n} \to 1,$$

and $(\overline{B} \setminus \{0\}, \frac{g}{r^2}, x)$ converges to $(\mathbb{R}^n, 0)$ in the Gromov-Hausdorff distance, where x is a fixed point in B and V_n represents the volume of the unit Euclidean ball. Additionally, $G^{-1}u \in W^{2,p}(B_{\frac{1}{2}})$ for any $p < \frac{n}{2}$, where G is the Green function defined by

$$-\Delta_{g_0} G = \delta_0, \quad G|_{\partial B} = 0.$$

Remark 1.3 Based on Proposition 2.8 and Corollary 3.3, the assumption that $\mathrm{vol}(\overline{B} \setminus 0, g) = \infty$ is equivalent to the completeness of $(\overline{B} \setminus 0, g)$, when $\|R(g)\|_{L^{\frac{n}{2}}(B,g)} < +\infty$ is satisfied.

The theorem above includes lots of unexpected pieces of information. Firstly, it implies that when (M, g) is conformal to a domain of a closed manifold, Huber's result holds if and only if the volume density at infinity equals the number of ends. In addition, each end of such a manifold is asymptotically euclidean, which is a useful property for verifying whether a manifold has a finite point conformal compactification. For instance, by setting $g = u^{\frac{4}{n-2}} g_{\mathrm{euc}}$ on $M = \mathbb{R}^n \setminus \mathbb{R}^{n-k}$, where $u = \left(\sum_{i=1}^k (x^i)^2\right)^{\frac{2-k}{2}}$, we find $R(g) = 0$, and (M, g) remains noncompact for $n > k > \frac{n}{2}+1$. Nonetheless, (M, g) does not satisfy Huber's Theorem, since its blow-down is not $\mathbb{R}^n$. This contrasts with cases where the total Q-curvature is finite [7, 12, 24]. For instance, when g_0 is the Euclidean metric and $u = r^\alpha$, the Q-curvature of g is 0, yet the volume density at infinity can vary widely. Secondly, it appears that when $\|R\|_{L^{\frac{n}{2}}} < +\infty$, conforming to a domain of a closed manifold is a quite strong assumption. For example, if we further assume $Ric(g) \geq 0$, such a manifold must be $\mathbb{R}^n$ (see Corollary 3.6).

A plausible intuitive explanation for these unusual phenomena is as follows: Firstly, we can find sequences $r_k \to 0$ and c_k such that $c_k r_k^{\frac{n-2}{2}} u(r_k x)$ converges weakly in $W^{2,p}(\mathbb{R}^n \setminus 0)$ to a positive function u' for any $p < \frac{n}{2}$. The function u' is harmonic since the limit metric $g_\infty = u'^{\frac{4}{n-2}} g_{euc}$ is scalar flat. Moreover, $(\mathbb{R}^n \setminus 0, g_\infty)$ should be extendable to a cone since it can be seen as a blow-down of $(B\setminus 0, g)$. However, a positive harmonic function on $\mathbb{R}^n \setminus 0$ must be in the form of $a + br^{2-n}$. When a and b are both non-zero, $(\mathbb{R}^n \setminus 0, (a + br^{2-n})^{\frac{4}{n-2}} g_{euc})$ becomes a complete manifold with 2 ends, and is not a cone. When $b = 0$, $(\mathbb{R}^n \setminus 0, a^{\frac{4}{n-2}} g_{euc})$ is not complete near 0. Therefore, we conclude that $u' = b|x|^{2-n}$ with $b > 0$, which implies that g_∞ is a flat metric defined on $\mathbb{R}^n \setminus 0$.

Theorem 1.2 has a number of interesting corollaries. First, we examine a conformal map from $(\overline{B} \setminus \{0\}, g_0)$ into $\mathbb{R}^{n+k}$. We show that if the second fundamental form A is in L^n and the image is noncompact, then the mapping near the origin closely resembles $\frac{x}{|x|^2}$, and the intrinsic distance is asymptotically equivalent to the distance in $\mathbb{R}^{n+k}$:

Theorem 1.4 *Let $(\overline{B}, g_0)$ be as in Theorem 1.2. Let $F : (\overline{B} \setminus \{0\}, g_0) \to \mathbb{R}^{n+k}$ be a conformal immersion with finite $\|A\|_{L^n}$. Suppose the volume is infinite. Then after changing the coordinates of $\mathbb{R}^{n+k}$, for any $r_k \to 0$ and $x_0 \in B$, there exists $\lambda_k \in \mathbb{R}$ and $y_0 \in \mathbb{R}^{n+k}$, such that a subsequence of*

$$\lambda_k(F(r_k x) - F(r_k x_0)) + y_0 \tag{1.1}$$

converges weakly in $W^{3,p}_{loc}(\mathbb{R}^n \setminus \{0\})$ to $F_\infty(x) = (\frac{x}{|x|^2}, 0)$ for any $p < \frac{n}{2}$. Consequently,

$$\lim_{x\to 0} \frac{|F(x) - F(x_0)|}{d_{g_F}(x_0, x)} = 1, \tag{1.2}$$

where g_F is the induced metric.

Note that, if we use the coordinates change: $x \to \frac{x}{|x|^2}$, the limit of $\lambda_k(F(r_k x) - F(r_k x_0)) + y_0$ is simply the identity map of $\mathbb{R}^n$ under the new coordinates. Therefore, Theorem 1.4 can be viewed as a higher-dimensional extension of a result by S. Müller and V. Šverák [22, Corollary 4.2.5], except that F does not have branches in our case.

Next, we will prove some Gauss-Bonnet-Chern formulas in 4-dimensional cases. Since the asymptotic behavior of $(\overline{B} \setminus \{0\}, g)$ at infinity is clear and simple, we can get the exact values of the error terms.

First, we discuss the formula for Q-curvature. For the Q-curvature, we use the definition in [10]. Since $R(g) \in L^2$ is not strong enough to ensure the integrability of Q-curvature (see Example 6.3), our first formula is stated as follows:

Theorem 1.5 *Let (M_0, g_0) be a compact 4-dimensional orientable manifold without boundary and let (M, g) be conformally equivalent to (M_0, g_0) with a finite number of points removed. We assume (M, g) is complete and $R(g) \in L^2(M, g)$. Then there exist domains $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \cdots$, such that*

$$\bigcup_{k=1}^{\infty} \Omega_k = M,$$

and

$$\lim_{k\to+\infty} \int_{\Omega_k} Q(g) dV_g = \int_{M_0} Q(g_0) dV_{g_0} - 8\pi^2 m,$$

where m is the number of the ends.

Under the assumptions in the above theorem, it is evident that the integrability of Q^- (or Q^+) implies the integrability of Q. When Q is integrable, the theorem above can be viewed as an intrinsic version of Theorem 1.5 in [20], where $|A|_{L^4} < +\infty$ was assumed. It may seem a bit unusual at first glance that such a formula is solely concerned with intrinsic properties.

As an application of the above theorem, we obtain the following:

Corollary 1.6 *Let (M, g) be as in Theorem 1.5. Then $Riem(g) \in L^2$, where $Riem(g)$ is the curvature tensor.*

We denote by $Pf(g)$ the standard Pfaffian of the Riemannian metric g. For a closed 4-dimensional manifold (M_0, g_0), the Chern-Gauss-Bonnet formula can be expressed as

$$\int_{M_0} Pf(g_0) = 4\pi^2 \chi(M_0).$$

where $\chi(M_0)$ is the Euler characteristic of M_0. It is well-known that

$$Pf(g) = \left(\frac{1}{8}|W(g)|^2 + \frac{1}{12}R^2(g) - \frac{1}{4}|Ric(g)|^2\right) dV_g \tag{1.3}$$

where W is the Weyl tensor. Then the integrability of the Pfaffian form is deduced from L^2-integrability of Ricci curvature and scalar curvature, along with the conformal invariance of the Weyl tensor. Furthermore, we obtain the following result:

Theorem 1.7 *Let (M, g) and m be as in Theorem 1.5. Then the Pfaffian of the curvature is integrable, and*

$$\int_M Pf(g) = 4\pi^2 \chi(M_0) - 8m\pi^2,$$

or equivalently

$$\int_M Pf(g) = 4\pi^2 \chi(M) - 4m\pi^2.$$

We set $Pf(g) = \Phi dV_g$, where dV_g is the volume form of g, and define $\Phi^- dV_g$ to be the negative part of $Pf(g)$. From the equation (1.3), we deduce that $Ric(g) \in L^2(M, g)$ whenever Φ^- is integrable. Together with Theorem 1.4 in [14], we can establish the following

Theorem 1.8 *Let (M, g_0) be a 4-dimensional oriented compact Riemannian manifold without boundary and let Ω be a domain of M. Assume Ω is endowed with a complete Riemannian metric g which is conformal to g_0 with $R(g) \in L^2(\Omega, g)$. Then $M \setminus \Omega$ is a finite set if and only if the negative part of $Pf(g)$ is integrable.*

This paper is organized as follows. Section 2 reviews some regularity results of the scalar curvature equation and establishes the 3-circle Theorem. In Sect. 3 we establish the asymptotic behaviors of the metric at infinity. Then, we prove Theorem 1.4 and Theorem 1.5, 1.7, 1.8. in Sects. 4 and 5 respectively. In the last section, we provide several examples of complete metric on the 4-dimensional punctured ball with $R \in L^2$.

2 Preliminaries

First, we introduce some notations that will be used throughout the remainder of the paper. We always assume $n \geq 3$ and denote by $(B, x^1, x^2, \cdots, x^n)$ the n-dimensional unit ball, and by B_r the n-dimensional ball of radius r centered at 0 in $\mathbb{R}^n$. We assume g_0 is a smooth metric defined on $\overline{B}$. For simplicity, we always assume $x^1, \cdots, x^n$ are normal coordinates of g_0 at 0, then we have

$$d_{g_0}(0,x) = |x|, \quad and \quad |g_{0,ij} - \delta_{ij}| \le c|x|^2. \tag{2.1}$$

2.1 Regularity

In this section, we let v be a weak solution of

$$-div(a^{ij}v_j) = fv, \tag{2.2}$$

where

$$0 < \lambda_1 \le a^{ij}, \quad \|a^{ij}\|_{C^0(B_2)} + \|\nabla a^{ij}\|_{C^0(B_2)} < \lambda_2. \tag{2.3}$$

We have the following:

Lemma 2.1 *Suppose that $v \in W^{1,2}(B_2)$ is positive and satisfies (2.2) and (2.3). We assume*

$$\int_{B_2} |f|^{\frac{n}{2}} \le \Lambda.$$

Then

$$r^{2-n}\int_{B_r(x)} |\nabla \log v|^2 < C, \quad \forall B_r(x) \subset B.$$

Lemma 2.2 *Suppose $v \in W^{1,2}(B_2)$ is positive and satisfies (2.2) and (2.3). Then for any $q \in (0, \frac{n}{2})$, there exists $\epsilon_0 = \epsilon_0(q, \lambda_1, \lambda_2) > 0$, such that if*

$$\int_{B_2} |f|^{\frac{n}{2}} < \epsilon_0,$$

then

$$\|\nabla \log v\|_{W^{1,q}(B)} \le C(\lambda_1, \lambda_2, \epsilon_0),$$

and

$$e^{-\frac{1}{|B|}\int_B \log v}\|v\|_{W^{2,q}(B)} + e^{\frac{1}{|B|}\int_B \log v}\|v^{-1}\|_{W^{2,q}(B)} \le C(\lambda_1, \lambda_2, \epsilon_0).$$

For the proofs of the above two lemmas, one can refer to [15].

Corollary 2.3 *Suppose $v \in W^{1,2}(B_2)$ is positive and satisfies (2.2) and (2.3). Assume*

$$\int_B v^2 < \Lambda.$$

Then for any $q < \frac{n}{2}$ and $p > 2$, there exists C_1 and C_2, such that

$$\|v\|_{W^{2,q}(B)} < C_1, \quad and \quad \int_B v^p \le \|v\|^2_{L^2(B)} + C_2\|v\|_{L^2(B)}.$$

Proof Note that for any $E \subset B$,

$$\int_E (\log v)^+ = \int_{E\cap\{v>1\}} \log v \le \int_E v^2 < C. \tag{2.4}$$

Utilizing (2.2), we obtain that $\|v\|_{W^{2,q}(B)} < C$.

Select $q < \frac{n}{2}$, such that $W^{2,q}(B)$ can be embedded into $L^{2p}(B)$. Since

$$|\{v \ge 1\} \cap B| \le \int_B v^2,$$

we have

$$\begin{aligned}\int_B v^p &\le \int_{\{v\le 1\}\cap B} v^2 + \left(\int_{\{v\ge 1\}\cap B} v^{2p}\right)^{\frac{1}{2}} |\{v \ge 1\} \cap B|^{\frac{1}{2}} \\ &\le \int_B v^2 + \left(\int_B v^{2p}\right)^{\frac{1}{2}} \left(\int_B v^2\right)^{\frac{1}{2}} \le \int_B v^2 + C\|v\|^p_{W^{2,q}(B)} \left(\int_B v^2\right)^{\frac{1}{2}}.\end{aligned}$$

□

2.2 Convergence of distance functions

The distance between two points x and y on a manifold (M, g) is defined as the infimum of the lengths of piecewise smooth curves joining them. We will use the following proposition:

Proposition 2.4 *Let $a_k \to 0^+$ and $g_k = g_{k,ij}dx^i \otimes dx^j$ be a smooth metric defined on $B_{\frac{1}{a_k}} \setminus B_{a_k}$. Assume g_k and g_k^{-1} converge to g_{euc} in $W^{1,p}_{loc}(\mathbb{R}^n \setminus \{0\})$ for any $p \in (n-1, n)$. Then after passing to a subsequence, d_{g_k} converges to $d_{g_{euc}}$ in $C^0((B_r \backslash B_{\frac{1}{r}}) \times (B_r \backslash B_{\frac{1}{r}}))$ for any $r > 1$.*

Proof The arguments in [15, Section 3] use properties of complete metrics, so they can not be applied here directly. For this reason, we let $t > 1$ and take nonnegative $\phi_t \in C^\infty(\mathbb{R})$, which satisfies: 1). ϕ_t is 1 on $[\frac{1}{t}, t]$ and 0 on $(-\infty, \frac{1}{2t}] \cup [2t, +\infty)$; 2). $|\phi'| < 2t$. Define

$$\hat{g}_{k,t} = \phi_t(|x|)g_k + (1 - \phi_t(|x|))g_{euc}.$$

Obviously, $\hat{g}_{k,t}$ is complete on $\mathbb{R}^n$.

We have

$$\begin{aligned}\int_{B_{2t}\backslash B_t} |\nabla(\hat{g}_{k,t} - g_{euc})|^p dx &\le Ct^p \int_{B_{2t}\backslash B_t} |g_k - g_{euc}|^p dx + C\int_{B_{2t}\backslash B_t} |\nabla(g_k - g_{euc})|^p dx \\ &\le C(t)\|g_k - g_{euc}\|^p_{W^{1,p}(B_{2t}\backslash B_t)}.\end{aligned}$$

A similar estimate can be obtained on $B_{\frac{1}{t}} \backslash B_{\frac{1}{2t}}$ using the same argument.

Note that

$$det(\phi_t g_k + (1 - \phi_t)g_{euc}) \ge \prod_{i=1}^n (\phi_t a_i + (1 - \phi_t)),$$

where $a_1, \cdots, a_n$ are eigenvalues of g_k. Then

$$det(\phi_t g_k + (1 - \phi_t)g_{euc}) \ge \frac{1}{2^n} \min\{det(g_k), 1\},$$

which implies that

$$\frac{1}{det(\hat{g}_{k,t})} \leq C(1+\frac{1}{det(g_k)}) = C(1+det(g_k^{-1})).$$

Since g_k^{-1} converges in $W^{1,p}_{loc}(\mathbb{R}^n\backslash\{0\})$ for any $p\in(n-1,n)$, (g_k^{-1}) is bounded in $L^q(B_r\backslash B_{\frac{1}{r}})$ for any q. Then $1/det(\hat{g}_{k,t})$ is bounded in $L^q(B_r)$ for any q. Recall that the inverse of a matrix is just the adjugate matrix divided by the determinant. Then $\hat{g}_{k,t}^{-1}$ is bounded in $W^{1,p}(B_r)$.

Note that $\hat{g}_{k,t}=g_{ecu}$ on B_{2t}^c. It is not difficult to check that for any fixed r, there exists r', such that any geodesic between two points $x, y\in B_r$ must lie in $B_{r'}$. Then, using the arguments in [15, Section 3], a subsequence of $d_{\hat{g}_{k,t}}$ converges to $d_{g_{euc}}$ in $C^0(B_r\times B_r)$ for any r. Thus, after passing to a subsequence, we can find $t_k\to+\infty$, such that $d_{\hat{g}_{k,t_k}}$ converges to $d_{g_{euc}}$ in $C^0(B_r\times B_r)$ for any r. For simplicity, we set $\tilde{g}_k=\hat{g}_{k,t_k}$ and assume $d_{\tilde{g}_k}$ converges in $C^0(B_r\times B_r)$ for any r.

Now, we start to prove that d_{g_k} converges to $d_{g_{euc}}$ in $C^0((B_r\backslash B_{\frac{1}{r}})\times(B_r\backslash B_{\frac{1}{r}}))$.

Let $\lambda_k(x)$ be the lowest eigenvalue of g_k^{ij}. Since $|\nabla_x^{g_k}d_{g_k}(x,y)|\leq 1$ for a.e. x, we have

$$|\nabla_x d_{g_k}(x,y)|^2 \leq \frac{1}{\lambda_k(x)} \leq C\sum_{ij}|g_{k,ij}(x)|,$$

which implies that d_{g_k} is bounded $W^{1,q}((B_r\backslash B_{\frac{1}{r}})\times(B_r\backslash B_{\frac{1}{r}}))$ for any r and $q>0$. Then, we may assume d_{g_k} converges to a function d in $C^0((B_r\backslash B_{\frac{1}{r}})\times(B_r\backslash B_{\frac{1}{r}}))$ for any r. By the trace embedding theorem, for any $x, y\in B_r\setminus B_{\frac{1}{r}}$, we have

$$d(x,y)\leq d_{g_{euc}}(x,y).$$

Next, we show $d(x,y)\geq d_{g_{euc}}(x,y)$. Let γ_k be a curve from x to y in $\mathbb{R}^n\setminus\{0\}$, such that

$$L_{g_k}(\gamma_k)\leq d_{g_k}(x,y)+\frac{1}{k}.$$

Let $\lambda>d(x,y)+r+1$. We claim that $\gamma_k\subset B_\lambda$ when k is sufficiently large. Suppose that $\gamma_k\cap\partial B_\lambda\neq\emptyset$. It is easy to check that

$$d_{g_k}(\partial B_\lambda,\partial B_r)\leq L_{g_k}(\gamma_k)\leq d_{g_k}(x,y)+\frac{1}{k}\to d(x,y).$$

However, $d_{g_k}(\partial B_\lambda,\partial B_r)=d_{\tilde{g}_k}(\partial B_\lambda,\partial B_r)$ when k is sufficiently large, and

$$\lim_{k\to+\infty} d_{\tilde{g}_k}(\partial B_\lambda,\partial B_r)=d_{g_{euc}}(\partial B_\lambda,\partial B_r)=\lambda-r,$$

which leads to a contradiction.

The rest of the proof can be divided into 2 cases. Case 1, we assume $\gamma_k\cap\partial B_{\frac{1}{t_k}}=\emptyset$. In this case,

$$L_{g_k}(\gamma_k)=L_{\tilde{g}_k}(\gamma_k)\geq d_{\tilde{g}_k}(x,y).$$

Case 2, we may assume x_k and y_k to be the first and the last point in $\gamma_k\cap\partial B_{\frac{1}{t_k}}$ respectively. Then

$$\begin{aligned} L_{g_k}(\gamma_k) &\geq L_{g_k}(\gamma_k|_{[x,x_k]})+L_{g_k}(\gamma_k|_{[y_k,y]}) \\ &= L_{\tilde{g}_k}(\gamma_k|_{[x,x_k]})+L_{\tilde{g}_k}(\gamma_k|_{[y_k,y]}) \end{aligned}$$

$$\geq d_{\tilde{g}_k}(x, x_k) + d_{\tilde{g}_k}(y_k, y)$$
$$\geq d_{\tilde{g}_k}(x, y) - d_{\tilde{g}_k}(x_k, y_k).$$

Thus, for both cases, we have

$$d(x, y) = \lim_{k\to+\infty} d_{g_k}(x, y) \geq \lim_{k\to+\infty} d_{\tilde{g}_k}(x, y) = d_{g_{euc}}(x, y).$$

□

2.3 Three circles theorem

In this section, we present the Three Circles Theorem. It is convenient to state and prove this theorem on pipes. We let $Q = [0, 3L] \times S^{n-1}$, and

$$Q_i = [(i-1)L, iL] \times S^{n-1}, \quad i = 1, 2, 3.$$

Set $g_Q = dt^2 + g_{S^{n-1}}$ and $dV_Q = dV_{g_Q}$.

We first state this theorem for the case of $g = g_Q$ and $R = (n-1)(n-2)$:

Lemma 2.5 *Let $u \neq 0$ solve the following equation on Q:*

$$-\Delta u + \frac{(n-2)^2}{4} u = 0.$$

Then there exists L_0, such that for any $L > L_0$, we have

1) $\int_{Q_1} u^2 dV_Q \leq e^{-L} \int_{Q_2} u^2 dV_Q$ implies $\int_{Q_2} u^2 dV_Q < e^{-L} \int_{Q_3} u^2 dV_Q$;
2) $\int_{Q_3} u^2 dV_Q \leq e^{-L} \int_{Q_2} u^2 dV_Q$ implies $\int_{Q_2} u^2 dV_Q < e^{-L} \int_{Q_1} u^2 dV_Q$;
3) either $\int_{Q_2} u^2 dV_Q < e^{-L} \int_{Q_1} u^2 dV_Q$ or $\int_{Q_2} u^2 dV_Q < e^{-L} \int_{Q_3} u^2 dV_Q$.

For the proof, one can refer to [6, 19]. Next, we discuss the general case.

Theorem 2.6 *Let g_0 be a metric over Q and $u \in W^{2,p}$ which solves the equation*

$$-\Delta_{g_0} u + c(n)R(g_0)u = fu.$$

Then for any $L > L_0$, there exist ϵ_0', τ, such that if

$$\|g_0 - g_Q\|_{C^2(Q)} < \tau, \quad \int_Q |f|^{\frac{n}{2}} dV_{g_0} < \epsilon_0', \tag{2.5}$$

then

1) $\int_{Q_1} u^2 dV_{g_0} \leq e^{-L} \int_{Q_2} u^2 dV_{g_0}$ implies $\int_{Q_2} u^2 dV_{g_0} \leq e^{-L} \int_{Q_3} u^2 dV_{g_0}$;
2) $\int_{Q_3} u^2 dV_{g_0} \leq e^{-L} \displaystyle\int_{Q_2} u^2 dV_{g_0}$ implies $\int_{Q_2} u^2 dV_{g_0} \leq e^{-L} \int_{Q_1} u^2 dV_{g_0}$;
3) either $\int_{Q_2} u^2 dV_{g_0} \leq e^{-L} \int_{Q_1} u^2 dV_{g_0}$ or $\int_{Q_2} u^2 dV_{g_0} \leq e^{-L} \int_{Q_3} u^2 dV_{g_0}$.

Proof If the statement in 1) is false for an $L > L_0$, we can find g_k, u_k and f_k, s.t.

$$g_k \to g_Q \text{ in } C^2(Q), \quad \int_Q |f_k|^{\frac{n}{2}} dV_{g_k} \to 0,$$
$$-\Delta_{g_k} u_k + c(n)R(g_k)u_k = f_k u_k,$$

and

$$\int_{Q_1} u_k^2 dV_{g_k} \leq e^{-L} \int_{Q_2} u_k^2 dV_{g_k}, \quad \int_{Q_2} u_k^2 dV_{g_k} > e^{-L} \int_{Q_3} u_k^2 dV_{g_k}.$$

Let

$$v_k = \frac{u_k}{\|u_k\|_{L^2(Q_2, g_k)}}.$$

We have

$$\int_{Q_1} v_k^2 dV_{g_k} \leq e^{-L} \int_{Q_2} v_k^2 dV_{g_k}, \quad \int_{Q_3} v_k^2 dV_{g_k} < e^{L} \int_{Q_2} v_k^2 dV_{g_k},$$

and

$$\int_{Q_2} v_k^2 dV_{g_k} = 1.$$

Thus

$$\int_{Q} v_k^2 dV_{g_k} \leq C.$$

v_k satisfies

$$-\Delta_{g_k} v_k + c(n) R(g_k) v_k = f_k v_k.$$

By Corollary 2.3 and Sobolev embedding theorem, v_k converges to a function v in $W^{1,2}_{loc}$, where v satisfies:

$$-\Delta v + \frac{(n-2)^2}{4} v = 0, \quad and \quad \int_{Q_2} |v|^2 dV_Q = 1.$$

Thus $v \neq 0$.

Moreover,

$$\int_{[\epsilon, L] \times S^{n-1}} v^2 dV_Q = \lim_{k \to +\infty} \int_{[\epsilon, L] \times S^{n-1}} v_k^2 dV_{g_k} \leq e^{-L} \lim_{k \to +\infty} \int_{Q_2} v_k^2 dV_{g_k},$$

letting $\epsilon \to 0$ gives

$$\int_{Q_1} v^2 dV_Q \leq e^{-L} \int_{Q_2} v^2 dV_Q.$$

Similarly, there holds

$$\int_{Q_3} v^2 dV_Q \leq e^{L} \int_{Q_2} v^2 dV_Q,$$

which contradicts Lemma 2.5. Hence, the statements in (1) are proved. Using the same arguments, we can easily carry out the proof of (2) and (3). □

Theorem 2.7 *Let $g = u^{\frac{4}{n-4}} g_0$ be a smooth metric defined on $\overline{B} \setminus \{0\}$ with*

$$\int_B |R(g)|^{\frac{n}{2}} dV_g < +\infty, \quad \mathrm{vol}(\overline{B} \setminus \{0\}, g) = +\infty.$$

Then for any $\vartheta > e^{L_0}$ *there exists* r_0*, such that for any* $r < r_0$*, there holds*

$$\int_{B_r\setminus B_{r\vartheta^{-1}}} \frac{u^2}{|x|^2} dV_{g_0} \le \frac{1}{\vartheta} \int_{B_{r\vartheta^{-1}}\setminus B_{r\vartheta^{-2}}} \frac{u^2}{|x|^2} dV_{g_0}. \tag{2.6}$$

Moreover, we have

$$\lim_{k\to+\infty} \int_{B_{\vartheta^{-k}r}\setminus B_{\vartheta^{-k-1}r}} \frac{u^2}{|x|^2} dV_{g_0} = +\infty \quad and \quad \lim_{k\to+\infty} \int_{B_{\vartheta^{-k}r}\setminus B_{\vartheta^{-k-1}r}} u^{\frac{2n}{n-2}} dV_{g_0} = +\infty. \tag{2.7}$$

Proof Put

$$\phi(t,\theta) = (e^{-t},\theta),$$

and

$$g'(t,\theta) = \phi^*(g) = v^{\frac{4}{n-2}} \hat{g}(t,\theta),$$

where $\hat{g}(t,\theta) = e^{2t}\phi^*(g_0)$, which converges to $dt^2 + g_{\mathbb{S}^{n-1}}$ as $t \to +\infty$. Then

$$\begin{aligned} v^{\frac{4}{n-2}}(t,\theta) =& u^{\frac{4}{n-2}}(e^{-t},\theta)e^{-2t}, \\ & -\Delta_{\hat{g}} v + c(n)R(\hat{g})v = c(n)R(g')v^{\frac{n+2}{n-2}} := fv, \\ & \int_{S^1\times[a,b]} |f|^{\frac{n}{2}} dV_{\hat{g}} = c\int_{B_{e^{-a}}\setminus B_{e^{-b}}} |R(g)|^{\frac{n}{2}} dV_g, \end{aligned}$$

and

$$\int_{B_r\setminus B_{r/\vartheta}} \frac{u^2}{|x|^2} dV_{g_0} = \int_{S^{n-1}\times[-\log r, -\log r + \log\vartheta]} v^2 dV_{\hat{g}}.$$

Without loss of generality, we assume

$$\|\hat{g} - g_Q\|_{C^2(S^{n-1}\times[0,+\infty))} < \tau, \quad \int_{S^{n-1}\times[0,+\infty)} |R|^{\frac{n}{2}} dV_g < \epsilon_0'. \tag{2.8}$$

Suppose (2.6) is not true, i.e., we can find $r_k \to 0$, such that

$$\int_{B_{r_k\vartheta^2}\setminus B_{r_k\vartheta}} \frac{u^2}{|x|^2} dV_{g_0} > \frac{1}{\vartheta} \int_{B_{r_k\vartheta}\setminus B_{r_k}} \frac{u^2}{|x|^2} dV_{g_0}. \tag{2.9}$$

We set

$$\Omega_{k,m} = S^{n-1} \times [-\log r_k - (m_k - m + 1)\log\vartheta, -\log r_k - (m_k - m)\log\vartheta],$$

where $m = 1, \cdots, m_k = \left[\frac{-\log r_k}{\log\vartheta}\right]$. Then (2.9) is equivalent to

$$\int_{\Omega_{k,m_k}} v^2 dV_{\hat{g}} \le \frac{1}{\vartheta} \int_{\Omega_{k,m_k-1}} v^2 dV_{\hat{g}}.$$

By Theorem 2.6, we get

$$\int_{\Omega_{k,m_k-1}} v^2 dV_{\hat{g}} \le \frac{1}{\vartheta} \int_{\Omega_{k,m_k-2}} v^2 dV_{\hat{g}}.$$

Step by step, we get

$$\int_{\Omega_{k,m}} v^2 dV_{\hat{g}} \le \vartheta^{-(m-1)} \int_{\Omega_{k,1}} v^2 dV_{\hat{g}} \le \vartheta^{-(m-1)} \int_{S^{n-1}\times[0,2\log\vartheta]} v^2 dV_{\hat{g}} \le C\vartheta^{-m}.$$

By Corollary 2.3,

$$\int_{\Omega_{k,m}} v^{\frac{2n}{n-2}} dV_{\hat{g}} \le C(\vartheta^{-m} + \vartheta^{-\frac{m}{2}}),$$

hence

$$\begin{aligned}\int_{B\setminus B_{r_k}} u^{\frac{2n}{n-2}} dV_{g_0} &\le C \int_{B\setminus B_{r_k}} u^{\frac{2n}{n-2}} dx \le C \int_{B\setminus B_{\vartheta^{-1}}} u^{\frac{2n}{n-2}} dx + C \sum_{m=1}^{m_k} \int_{B_{r_k\vartheta^m}\setminus B_{r_k\vartheta^{m-1}}} u^{\frac{2n}{n-2}} dx \\ &\le \sum_m \frac{C}{\vartheta^{\frac{m}{2}}} < C(\vartheta),\end{aligned}$$

where $C(\vartheta)$ is independent of k. Letting $k \to \infty$, we get a contradiction.

Thus, we get (2.6), which implies from Theorem 2.6 that

$$\int_{B_{r\vartheta^{-m}}\setminus B_{r\vartheta^{-m-1}}} \frac{u^2}{|x|^2} dV_{g_0} \ge C\vartheta^m.$$

Since

$$\int_{B_{r\vartheta^{-m}}\setminus B_{r\vartheta^{-m-1}}} \frac{u^2}{|x|^2} dV_{g_0} \le C(\log\vartheta)^{\frac{2}{n}} \left(\int_{B_{r\vartheta^{-m}}\setminus B_{r\vartheta^{-m-1}}} u^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}},$$

we get

$$\lim_{k\to+\infty} \int_{B_{r\vartheta^{-m}}\setminus B_{r\vartheta^{-m-1}}} u^{\frac{2n}{n-2}} dV_{g_0} = \infty.$$

□

Proposition 2.8 *Let $g = u^{\frac{4}{n-4}} g_0$ be a smooth metric defined on $\overline{B} \setminus \{0\}$ with*

$$\int_B |R(g)|^{\frac{n}{2}} dV_g < +\infty, \quad \mathrm{vol}(\overline{B}\setminus\{0\}, g) < +\infty.$$

Then $(\overline{B}\setminus\{0\}, g)$ is bounded.

Proof We need to show that there exist r and C, such that for any x sufficiently close to 0, we can find $x' \in \bar{B}_1 \setminus B_r$, such have $d_g(x, x') < C$.

Let g', $\hat{g}$, v be as in the proof of Theorem 2.7 and assume (2.8) holds. Set

$$\Omega_m = S^{n-1} \times [-\log r_0 + (m-1)\log\vartheta, -\log r_0 + m\log\vartheta],$$

where $-\log r_0 \in [0, \log\vartheta)$ such that $x = (-\log r_0 + m_0 \log\vartheta, \theta)$ for some $m_0 \in \mathbb{Z}^+$ and $\theta \in S^{n-1}$.

By Theorem 2.6, if there exists m, such that $\int_{\Omega_m} v^2 dV_{\hat{g}} \le \vartheta^{-1} \int_{\Omega_{m+1}} v^2 dV_{\hat{g}}$, then

$$\int_{\Omega_{m+m'}} v^2 dV_{\hat{g}} \ge C\vartheta^{m'} \to +\infty,$$

which is impossible, since

$$\int_{\Omega_{m+m'}} v^2 dV_{\hat{g}} \le C\left(\int_{\Omega_{m+m'}} v^{\frac{2n}{n-2}} dV_{\hat{g}}\right)^{\frac{n-2}{n}} \le C(\mathrm{vol}(B\setminus\{0\}, g))^{\frac{n-2}{n}} < +\infty.$$

Then

$$\int_{\Omega_{m+1}} v^2 dV_{\hat{g}} \le \vartheta^{-1}\int_{\Omega_m} v^2 dV_{\hat{g}},$$

which implies that

$$\int_{\Omega_m} v^2 dV_{\hat{g}} < C\vartheta^{-m}.$$

By Corollary 2.3, $\|v\|_{L^p(\Omega_m)} < C(p)\vartheta^{-m/(2p)}$ for any p, hence, for any $q < \frac{n}{2}$, $\|\Delta_{\hat{g}} v\|_{L^q(\Omega_m)} < C(q)\vartheta^{-a(q)m}$ for some $a(q) > 0$. By the standard elliptic estimate, we get $\|v\|_{W^{2,q}(\Omega_m)} < C\vartheta^{-a(q)m}$. It follows from the Sobolev inequality that for any $q' \in (n-1, n)$, the inequality $\|v^{\frac{2n}{n-2}}\|_{W^{1,q'}(\Omega_m)} < C(q')\vartheta^{-a(q')m}$ holds for some positive constants $C(q')$ and $a(q')$.

For convenience, we set

$$t_m = -\log r_0 + m\log\vartheta, \quad x_m = (t_m, \theta).$$

By the classical trace embedding theorem (cf. [1, Theorem 4.12]), we have

$$\begin{aligned} d_{\hat{g}}(x_m, x_{m+1}) &\le C\int_0^{\log\vartheta} v^{\frac{2}{n-2}}(t+t_m,\theta)dt \le C(\int_0^{\log\vartheta} v^{\frac{2n}{n-2}}(t+t_m,\theta))^{\frac{1}{n}} \\ &\le C\|v^{\frac{2n}{n-2}}\|^{\frac{1}{n}}_{W^{1,q'}(\Omega_m)} \le C\vartheta^{-\frac{a(q')m}{n}}. \end{aligned}$$

Then

$$d(x, x_0) < \sum_{m=1}^{m_0} d(x_m, x_{m-1}) < C.$$

□

3 Asymptotic properties

In this section, we always assume that (2.1) holds and $g = u^{\frac{4}{n-4}} g_0$ denotes a smooth metric on $\overline{B}\setminus\{0\}$ with

$$\mathrm{vol}(B\setminus\{0\}, g) = \infty.$$

First of all, we prove the following lemma:

Lemma 3.1 *Let $r_k \to 0$. After passing to a subsequence, we can find $c_k > 0$, such that $c_k r_k^{\frac{n-2}{2}} u(r_k x)$ converges to $|x|^{2-n}$ weakly in $W^{2,p}_{loc}(\mathbb{R}^n\setminus\{0\})$ for any $p \in [1, \frac{n}{2})$. Moreover, d_{g_k} converges to $d_{|x|^{-4}g_{euc}}$ in $C^0((B_r\backslash B_{\frac{1}{r}}) \times (B_r\backslash B_{\frac{1}{r}}))$ for any $r > 1$, where $g_{k,ij} = r_k^2(c_k u(r_k x))^{\frac{4}{n-2}} g_{0,ij}(r_k x)$.*

Proof Define

$$u_k(x) = r_k^{\frac{n-2}{2}} u(r_k x) c_k.$$

Choose c_k such that $\int_{\partial B_1} \log u_k d\mathbb{S}^{n-1} = 0$. It is easy to check that $-\Delta_{g_k} u_k = f_k u_k$, where

$$f_k = -c(n) r_k^2 R_{g_0}(r_k x) + c(n) R(r_k x)(r_k^{\frac{n-2}{2}} u(r_k x))^{\frac{4}{n-2}},$$

and for sufficiently large k,

$$\|f_k\|_{L^{\frac{n}{2}}(B_r \setminus B_{\frac{1}{r}})} \leq C r_k^2 + C \left(\int_{B_{rr_k} \setminus B_{\frac{r_k}{r}}} |R|^{\frac{n}{2}} u^{\frac{2n}{n-2}} dx \right)^{\frac{2}{n}} \leq \min\{\epsilon_0, \epsilon_0'\}.$$

By Lemma 2.1 and the Poincaré inequality (c.f. [3, Theorem 5.4.3]), $\log u_k$ is bounded in $W^{1,2}(B_r \setminus B_{\frac{1}{r}})$. Then u_k is bounded in $W^{2,p}(B_r \setminus B_{\frac{1}{r}})$ by using Lemma 2.2. Thus u_k converges weakly to a positive harmonic function u' locally on $\mathbb{R}^n \setminus \{0\}$ with $\int_{\partial B_1} \log u' d\mathbb{S}^{n-1} = 0$. According Corollary 3.14 in [5], u' is written as

$$u' = a + b|x|^{2-n},$$

where a and b are nonnegative constants. Applying Theorem 2.7, we get

$$\int_{B_{r_k r} \setminus B_{r_k r \vartheta^{-1}}} \frac{u^2}{|x|^2} dV_{g_0} \leq \frac{1}{\vartheta} \int_{B_{r_k r \vartheta^{-1}} \setminus B_{r_k r \vartheta^{-2}}} \frac{u^2}{|x|^2} dV_{g_0},$$

which implies that

$$\int_{B_r \setminus B_{r\vartheta^{-1}}} \frac{u_k^2}{|x|^2} dV_{g_{0,k}} \leq \frac{1}{\vartheta} \int_{B_{r\vartheta^{-1}} \setminus B_{r\vartheta^{-2}}} \frac{u_k^2}{|x|^2} dV_{g_{0,k}},$$

where $g_{0,k} = g_{0,ij}(r_k x) dx^i \otimes dx^j$. Taking the limit, we obtain

$$\int_{B_r \setminus B_{r\vartheta^{-1}}} \frac{u'^2}{|x|^2} dx \leq \frac{1}{\vartheta} \int_{B_{r\vartheta^{-1}} \setminus B_{r\vartheta^{-2}}} \frac{u'^2}{|x|^2} dx.$$

Letting r be sufficiently large, we get $a = 0$. Since $\int_{\partial B_1} \log u' = 0$, $b = 1$.

By changing coordinates: $x \to \frac{x}{|x|^2}$, we see that $u'^{\frac{4}{n-2}} g_{euc}$ is just g_{euc} in the new coordinates. The convergence of d_{g_k} follows from Proposition 2.4 directly. □

In the preceding lemma, we did not express c_k in terms of r_k, which limits our understanding of the behavior of $u(r_k x)$. Nonetheless, the lemma is sufficiently strong to derive the following decay properties:

Corollary 3.2 *For any $\tau \in (0, 1)$, there exists δ such that for any $r < \delta$, the following hold:*

$$(1-\tau)\frac{1}{2^n} \leq \frac{\mathrm{vol}(B_{2r} \setminus B_r, g)}{\mathrm{vol}(B_r \setminus B_{r/2}, g)} \leq \frac{1}{2^n}(1+\tau); \tag{3.1}$$

$$(1-\tau)(2^n - 1) \leq \frac{\mathrm{vol}(B_{2r} \setminus B_{r,g})}{V_n(d_g(2rx_0, rx_0))^n} \leq (2^n - 1)(1+\tau), \quad \forall x_0 \in \partial B; \tag{3.2}$$

$$\frac{\int_{B_r \setminus B_{r/2}} |x|^\beta |\nabla_{g_0} u|^\alpha dV_{g_0}}{\int_{B_{2r} \setminus B_r} |x|^\beta |\nabla_{g_0} u|^\alpha dV_{g_0}} < 2^{(n-1)\alpha - n - \beta}(1+\tau), \quad \forall \alpha \in [1, n), \quad \beta \in \mathbb{R}; \tag{3.3}$$

$$\frac{\int_{B_r\setminus B_{r/2}}|x|^\beta u^\alpha dV_{g_0}}{\int_{B_{2r}\setminus B_r}|x|^\beta u^\alpha dV_{g_0}} < 2^{(n-2)\alpha-n-\beta}(1+\tau), \quad \forall\alpha\in[1,+\infty), \quad \beta\in\mathbb{R}; \tag{3.4}$$

$$\frac{1}{2^{n-2}}(1-\tau) \le \frac{\int_{\partial B_r}|\nabla_{g_0}\log u|dS_{g_0}}{\int_{\partial B_{2r}}|\nabla_{g_0}\log u|dS_{g_0}} \le \frac{1}{2^{n-2}}(1+\tau); \tag{3.5}$$

$$2^{-1}(1-\tau) \le \frac{d_g(2rx_0, rx_0)}{d_g(rx_0, rx_0/2)} \le 2^{-1}(1+\tau), \quad \forall x_0\in\partial B; \tag{3.6}$$

$$(1-\tau) \le \frac{d_g(rx_0, \partial B_{2r})}{d_g(rx_0, 2rx_0)} \le (1+\tau), \quad \forall x_0\in\partial B; \tag{3.7}$$

$$2^{-1}(1-\tau) \le \frac{diam(\partial B_{2r})}{diam(\partial B_r)} \le 2^{-1}(1+\tau); \tag{3.8}$$

$$2^{-1}(1-\tau) \le \frac{d_g(\partial B_{2r}, \partial B_r)}{d_g(\partial B_r, \partial B_{r/2})} \le 2^{-1}(1+\tau); \tag{3.9}$$

$$4(1-\tau) \le \frac{diam(B_{2r}\setminus B_r)}{d_g(\partial B_{2r}, \partial B_r)} \le 4(1+\tau). \tag{3.10}$$

Proof Let $u_k = c_k r_k^{\frac{n-2}{2}} u(r_k x)$ be as in the proof of Lemma 3.1, which converges to $u' = |x|^{2-n}$ weakly in $W^{2,p}_{loc}(\mathbb{R}^n\setminus\{0\})$ for any $p < \frac{n}{2}$. Then we may assume ∇u_k converges in $L^q_{loc}(\mathbb{R}^n\setminus\{0\})$ for any $q < n$, and u_k converges in $L^q_{loc}(\mathbb{R}^n\setminus\{0\})$ for any $q > 0$. By the trace inequality, we can also assume $\log u_k$ converges in $L^1(\partial B_t)$.

Now, we prove the right-hand side inequality of (3.1): assume it is not valid, then there exists $r_k\to 0$, such that

$$\frac{\mathrm{vol}(B_{2r_k}\setminus B_{r_k}, g)}{\mathrm{vol}(B_{r_k}\setminus B_{r_k/2}, g)} > \frac{1}{2^n}(1+\tau),$$

which means that

$$\frac{\mathrm{vol}(B_2\setminus B_1, g_k)}{\mathrm{vol}(B_1\setminus B_{1/2}, g_k)} > \frac{1}{2^n}(1+\tau),$$

where $g_{k,ij} = u_k^{\frac{4}{n-2}} g_{0,ij}(r_k x)$. Letting $k\to+\infty$, we get

$$\frac{\mathrm{vol}(B_2\setminus B_1, g_\infty)}{\mathrm{vol}(B_1\setminus B_{1/2}, g_\infty)} \ge \frac{1}{2^n}(1+\tau),$$

where $g_\infty = |x|^{2-n}g_{euc}$. A contradiction.

Since the proofs of other inequalities are almost the same, we omit them. □

The inequalities (3.1)–(3.10) will be used to estimate quantities on $B_{2r}\setminus B_r$. For instance, using (3.1), we have

$$\mathrm{vol}(B_{2^k r}\setminus B_{2^{k-1}r}, g) \le \left(\frac{1+\tau}{2^n}\right)^{k-1}\mathrm{vol}(B_{2r}\setminus B_r, g),$$

which implies

$$\mathrm{vol}(B\setminus B_r, g) \le C\mathrm{vol}(B_{2r}\setminus B_r, g). \tag{3.11}$$

We provide several additional applications.

Corollary 3.3 *The manifold* $(\overline{B}\setminus\{0\}, g)$ *is complete.*

Proof Consider a sequence $\{x_k\}$ that does not contain a convergent subsequence. Then x_k converges to 0. Let $x_0 \in \partial B$ be fixed. To show completeness, it suffices to prove that $d_g(x_k, x_0) \to +\infty$.

By (3.6) and (3.7), we have $d_g(x_k, 2x_k) \to +\infty$, and

$$d_g(x_0, x_k) \geq d_g(x_k, \partial B_{2|x_k|}) \geq (1-\tau)d_g(x_k, 2x_k) \to +\infty.$$

□

Corollary 3.4 *Let r_k, c_k be as in Lemma 3.1. Let $x' \in \partial B$ and $\rho_k = d_g(x', r_k x')$. Then*

$$\lim_{k\to+\infty} c_k^{\frac{2}{n-2}} \rho_k = 1.$$

Proof Let u_k and g_k be as in Lemma 3.1. Set $g_\infty = |x|^{-4} g_{euc}$. By Lemma 3.1, for any $\sigma = 2^m$, we have

$$d_{g_k}(x', \sigma x') \to d_{g_\infty}(x', \sigma x') = 1 - 1/\sigma.$$

Thus

$$c_k^{\frac{2}{n-2}} d_g(r_k x', r_k \sigma x') = d_{g_k}(x', \sigma x') \to 1 - 1/\sigma.$$

Using (3.6), we get

$$c_k^{\frac{2}{n-2}} d_g(r_k \sigma x', x') \leq c_k^{\frac{2}{n-2}} C d_g(r_k \sigma x', 2r_k \sigma x') \to C d_{g_\infty}(\sigma x', 2\sigma x') = \frac{C}{2\sigma}.$$

The proof is completed by applying triangle inequality. □

Corollary 3.5 *We have*

$$\lim_{\rho\to+\infty} \frac{\mathrm{vol}(B_\rho^g(x_0), g)}{V_n \rho^n} = 1$$

and $(\overline{B} \setminus \{0\}, \rho^{-2} g, x_0)$ converges to $(\mathbb{R}^n, 0)$ in the Gromov-Hausdorff distance for any x_0.

Proof First, we prove the convergence of volume ratio. It suffices to prove that for any $\rho_k \to +\infty$, a subsequence of $\frac{\mathrm{vol}(B_{\rho_k}^g(x_0), g)}{V_n {\rho_k}^n}$ converges to 1.

Let $\rho_k \to +\infty$, and $x_k = a_k x_0$ for some $a_k \in \mathbb{R}^+$, such that

$$d_g(x_0, x_k) = \rho_k.$$

Put $y_k = \sigma x_k$, where $\sigma = 2^m$ is sufficiently large. We denote

$$\tau_k = d_g(x_k, y_k).$$

We will first approximate $B_{\rho_k}^g(x_0)$ with $B_{\tau_k}^g(y_k)$, and subsequently approximate $B_{\tau_k}^g(y_k)$ by its intersection with $B_{|y_k|}$, that is, $B_{\tau_k}^g(y_k) \cap B_{|y_k|}$. The reason for doing this is that after rescaling, $B_{\tau_k}^g(y_k) \cap B_{|y_k|}$ exhibits very good convergence properties.

By (3.6),

$$d_g(x_0, y_k) < 3d_g(y_k, \frac{1}{2}y_k) := \sigma_k,$$

hence

$$\tau_k - \sigma_k \le \rho_k \le \tau_k + \sigma_k. \tag{3.12}$$

Since

$$d_g(x, x_0) \le d_g(x, y_k) + d_g(x_0, y_k),$$

for any $x \in B^g_{\tau_k - 2\sigma_k}(y_k)$,

$$d_g(x, x_0) \le \tau_k - 2\sigma_k + \sigma_k \le \rho_k.$$

Similarly, for any $x \in B^g_{\rho_k}(x_0)$

$$d_g(x, y_k) \le d_g(x, x_0) + d_g(x_0, y_k) \le \rho_k + \sigma_k \le \tau_k + 2\sigma_k.$$

Then

$$B^g_{\tau_k - 2\sigma_k}(y_k) \subset B^g_{\rho_k}(x_0), \quad B^g_{\rho_k}(x_0) \subset B^g_{\tau_k + 2\sigma_k}(y_k). \tag{3.13}$$

Let $u_k = u(r_k x) r^{\frac{n-2}{2}} c_k$, where $r_k = |x_k|$ and c_k is as in Lemma 3.1. By Lemma 3.1, u_k converges to $|x|^{2-n}$, and

$$\frac{\sigma_k}{\tau_k} \to \frac{3}{\sigma - 1}. \tag{3.14}$$

Then, by choosing σ sufficiently large, for a fixed ϵ and sufficiently large k,

$$B^g_{(1-\epsilon)\tau_k}(y_k) \subset B^g_{\rho_k}(x_0) \subset B^g_{(1+\epsilon)\tau_k}(y_k). \tag{3.15}$$

Next, we estimate

$$\frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k), g)}{(\lambda\tau_k)^n}.$$

By (3.11), (3.2) and (3.6), we have

$$\begin{aligned}
\frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k) \cap B_{|y_k|}, g)}{(\lambda\tau_k)^n} &\le \frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k), g)}{(\lambda\tau_k)^n} \\
&= \frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k) \cap B_{|y_k|}, g) + \mathrm{vol}(B^g_{\lambda\tau_k}(y_k) \setminus B_{|y_k|}, g)}{(\lambda\tau_k)^n} \\
&\le \frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k) \cap B_{|y_k|}, g) + \mathrm{vol}(B \setminus B_{|y_k|}, g)}{(\lambda\tau_k)^n} \\
&\le \frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k) \cap B_{|y_k|}, g) + C\mathrm{vol}(B_{2|y_k|} \setminus B_{|y_k|}, g)}{(\lambda\tau_k)^n} \\
&\le \frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k) \cap B_{|y_k|}, g)}{(\lambda\tau_k)^n} + C\frac{d^n_g(2y_k, y_k)}{(\lambda\tau_k)^n} \\
&\le \frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k) \cap B_{|y_k|}, g)}{(\lambda\tau_k)^n} + C\frac{\sigma^n_k}{\lambda^n\tau^n_k}.
\end{aligned}$$

It is easy to check that

$$\frac{\mathrm{vol}(B^g_{\lambda\tau_k}(y_k) \cap B_{|y_k|}, g)}{(\lambda\tau_k)^n} \to \frac{\mathrm{vol}(B^{g_\infty}_{\lambda(1-\sigma^{-1})}(\sigma x_0/|x_0|) \cap B_\sigma, g_\infty)}{(\lambda(1-\sigma^{-1}))^n}.$$

To calculate $\mathrm{vol}(B^{g_\infty}_{(1-\sigma^{-1})}(\sigma x_0/|x_0|)\cap B_\sigma, g_\infty)$, we use the coordinates change

$$B\to\mathbb{R}^n\setminus B:\quad x\to z=\frac{x}{|x|^2}.$$

In the new coordinates, $g_\infty=g_{euc}$, and

$$B^{g_\infty}_{\lambda(1-\sigma^{-1})}(\sigma x_0/|x_0|)\cap B_\sigma = B_{\lambda(1-\sigma^{-1})}(\sigma^{-1}x_0/|x_0|)\setminus B_{\sigma^{-1}}.$$

Thus

$$\mathrm{vol}(B^{g_\infty}_{\lambda(1-\sigma^{-1})}(\sigma x_0/|x_0|)\cap B_\sigma, g_\infty)=V_n((\lambda(1-\sigma^{-1}))^n-O(\sigma^{-n})).$$

Then we can select σ to be sufficiently large and let $\lambda=1\pm\epsilon$, such that

$$1-\epsilon\le\frac{\mathrm{vol}(B^g_{(1+\epsilon)\tau_k}(y_k),g)}{V_n((1+\epsilon)\tau_k)^n},\quad\frac{\mathrm{vol}(B^g_{(1-\epsilon)\tau_k}(y_k),g)}{V_n((1-\epsilon)\tau_k)^n}\le(1+\epsilon)$$

when k is sufficiently large. By (3.15) and (3.12),

$$1-C\epsilon\le\frac{\mathrm{vol}(B^g_{\rho_k}(x_0))}{V_n\rho_k^n}\le 1+C\epsilon$$

when k is sufficiently large, we complete the proof of the ratio convergence.

Next, we prove the Gromov-Hausdorff convergence. It suffices to prove that a subsequence of $(\overline{B_1^{g/\rho_k^2}(x_0)}, d_{g/\rho_k^2}, x_0)$ converges to $(\overline{B},0)$.

By (3.12), (3.13), and (3.14),

$$\lim_{\sigma\to+\infty}\lim_{k\to+\infty}d_{GH}\left((\overline{B_1^{g/\rho_k^2}(x_0)}, d_{g/\rho_k^2}, x_0),(\overline{B_1^{g/\tau_k^2}(y_k)}, d_{g/\tau_k^2}, y_k)\right)=0.$$

Combining (3.9) with (3.10), we have

$$diam(B_1\setminus B_{|y_k|},g)<Cd(\partial B_{|y_k|},\partial B_{|y_k|/2},g)\le C\tau_k\frac{d(\partial B_{|y_k|},\partial B_{|y_k|/2},g)}{d(x_k,y_k)}\le\frac{C}{\sigma}\tau_k,$$

since $\frac{d(\partial B_{|y_k|},\partial B_{|y_k|/2},g)}{d(x_k,y_k)}\to\frac{1/(2\sigma)}{1-1/\sigma}$. Then

$$\lim_{\sigma\to+\infty}\lim_{k\to+\infty}d_{GH}\left((\overline{B_1^{g/\tau_k^2}(y_k)}, d_{g/\tau_k^2}, y_k),(\overline{B_1^{g/\tau_k^2}(y_k)\cap B_{|y_k|}}, d_{g/\tau_k^2}, y_k)\right)=0.$$

Note that

$$(B_1^{g/\tau_k^2}(y_k)\cap B_{|y_k|}, d_{g/\tau_k^2}, y_k)=(B_1^{g_k/(c_k^{\frac{4}{n-2}}\tau_k^2)}(y_k/r_k)\cap B_{|y_k/r_k|}, d_{g_k/(c_k^{\frac{4}{n-2}}\tau_k^2)}, y_k/r_k).$$

Since $d_{g_k}(x_k,y_k)\to 1-1/\sigma$ and $d_{g_k}=c_k^{\frac{2}{n-2}}d_g$, we have $c_k^{\frac{2}{n-2}}\tau_k\to 1-1/\sigma$, which implies that

$$\lim_{\sigma\to+\infty}\lim_{k\to+\infty}d_{GH}\left(\overline{B_1^{g/\tau_k^2}(y_k)\cap B_{|y_k|}}, d_{g/\tau_k^2}, y_k),(\overline{B_1^{g_k}(y_k/r_k)\cap B_{|y_k/r_k|}}, d_{g_k}, y_k/r_k)\right)=0.$$

However, we have

$$(\overline{B_1^{g_k}(y_k/r_k)\cap B_{|y_k/r_k|}}, d_{g_k}, y_k/r_k)\to(\overline{B_1^{g_\infty}(\sigma x_0/|x_0|)\cap B_\sigma}, d_{g_\infty}, \sigma x_0/|x_0|),$$

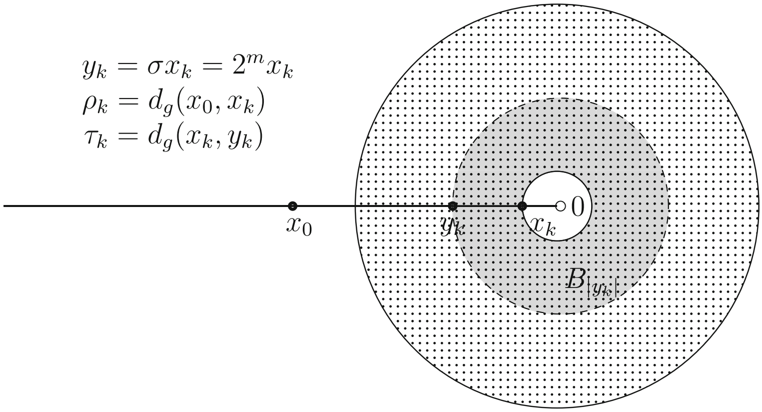

Fig. 1 $B^{g}_{\tau_k}(y_k)$ is the whole region filled with small dots. The shaded region is $B^{g}_{\tau_k}(y_k) \cap B_{|y_k|}$. We use $B^{g_k}_{\tau_k}(y_k) \cap B_{|y_k|}$ to approximate $B^{g_k}_{\rho_k}(x_0)$

and the limit is isometric to

$$(\overline{B_1(\sigma^{-1}x_0/|x_0|) \setminus B_{\sigma^{-1}}}, d_{g_{euc}}, \sigma^{-1}x_0/|x_0|).$$

Letting $\sigma \to \infty$, we complete the proof. □

Corollary 3.6 *Assume (M, g) is conformally equivalent to a domain of a compact manifold without boundary. If $\|R\|_{L^{\frac{n}{2}}} < +\infty$ and $Ric \geq 0$, and if (M, g) is complete and noncompact, then $(M, g) = \mathbb{R}^n$.*

Proof By the Bishop-Gromov Theorem, $\mathrm{vol}(B^g_r(x), g) \leq V_n r^n$. Then, by a result in [9], there exists a compact manifold (M_0, g_0) and a finite set $A \subset M_0$, such that (M, g) is conformal to $(M_0 \setminus A, g_0)$. By Corollary 3.5, A contains a single point, so the corollary follows from the Bishop-Gromov Theorem. □

Next, we derive a stronger version of Lemma 3.1 and finish the proof of Theorem 1.2:

Proposition 3.7 *$w = G^{-1}u$ is in $W^{2,p}(B)$ for any $p \in [1, \frac{n}{2})$, where G is the Green function defined by*

$$-\Delta_{g_0} G = \delta_0, \quad G|_{\partial B} = 0.$$

Proof By direct computation,

$$\Delta_{g_0} u = G\Delta_{g_0} w + 2\nabla_{g_0} G \nabla_{g_0} w = -c(n)Ru^{\frac{n+2}{n-2}} + c(n)R(g_0)u.$$

Then

$$\begin{aligned} -\Delta_{g_0} w &= c(n)G^{-1}Ru^{\frac{n+2}{n-2}} + 2\nabla_{g_0} \log G \nabla_{g_0} w - c(n)R(g_0)w \\ &= c(n)Ru^{\frac{4}{n-2}} w + 2\nabla_{g_0} \log G \nabla_{g_0} w - c(n)R(g_0)w \\ &:= f. \end{aligned}$$

It is well known (cf. [4]) that $G = r^{2-n}(1 + O(1))$ near 0 and

$$|\nabla_{g_0} \log G|(x) \leq \frac{C}{|x|}$$

when x is small.

First, we show $w \in L^q$ for any q. Indeed, applying (3.4) to $\alpha = q$, $\beta = (n-2)q$, we get

$$\int_{B_t} |w|^q \le C \sum_i \int_{B_{2^{-i}t}} |x|^{q(n-2)} |u|^q < C \sum_i ((1+\tau)2^{-n})^i < +\infty.$$

Next, we show $w \in W^{1,p}(B)$ for any $p < n$. Since

$$|\nabla_{g_0} w| \le G^{-1} |\nabla_{g_0} u| + uG^{-1} |\nabla_{g_0} \log G| \le C(|x|^{n-2} |\nabla_{g_0} u| + |x|^{n-3} u),$$

we may apply (3.4) to $\alpha = p$ and $\beta = (n-3)p$ to get

$$\int_{B_t} (|x|^{n-3} |u|)^p < +\infty$$

and (3.3) to $\alpha = p$ and $\beta = (n-2)p$ to obtain

$$\int_{B_t} |x|^{n-2} |\nabla u|^p < +\infty.$$

Then $\int_{B_t} |\nabla w|^p < +\infty$ for any $p < n$. Let $\varphi \in \mathcal{D}(B)$, and η_ϵ be a cutoff function which is 1 in $B \setminus B_{2\epsilon}$, 0 in B_ϵ, and satisfies $|\nabla \eta_\epsilon| < \frac{C}{\epsilon}$. It is easy to check that $\eta_\epsilon w$ is bounded in $W^{1,p}(B)$, hence a subsequence of $\eta_\epsilon w$ converges weakly in $W^{1,p}(B)$. Obviously, w is the limit, hence $w \in W^{1,p}(B)$.

Next, we show $f \in L^p$ for any $p < \frac{n}{2}$ and w solves the equation $-\Delta_{g_0} w = f$ weakly in B. Since

$$|f| \le C(|R| u^{\frac{4}{n-2}} w + |x|^{-1} |\nabla_{g_0} w| + w),$$

by the fact that $|R| u^{\frac{4}{n-2}} \in L^{\frac{n}{2}}$, $w \in L^q$ for any $q > 0$ and $\nabla w \in L^p$ for any $p < n$, it is easy to check that $f \in L^p$ for any $p < \frac{n}{2}$. Then

$$\int_B \nabla_{g_0} \varphi \nabla_{g_0} w dV_{g_0} = \lim_{\epsilon \to 0} \int_B \nabla_{g_0} \eta_\epsilon \varphi \nabla_{g_0} w dV_{g_0} = \lim_{\epsilon \to 0} \int_B \eta_\epsilon \varphi f = \int_B \varphi f,$$

hence w is a weak solution.

The proof can be completed without difficulty using the theory of elliptic equations. □

4 Conformally immersed submanifolds in $\mathbb{R}^{n+k}$

In this section, we consider a conformal immersion $F : (B \backslash 0, g_0) \to \mathbb{R}^{n+k}$ satisfying $|A|_{L^n} < +\infty$, where A represents the second fundamental form. We define

$$g = F^*(g_{euc}) = u^{\frac{4}{n-2}} g_0.$$

Obviously

$$\int_B |R|^{\frac{n}{2}} dV_g < +\infty.$$

For the purposes of this section, we assume $\mathrm{vol}(F(B \backslash 0)) = +\infty$. As a consequence of Corollary 3.3, the space $(\overline{B}_{\frac{1}{2}} \backslash 0, g)$ is complete. The goal of this section is to prove Theorem 1.4.

Proof of Theorem 1.4 First of all, we can find c_k, such that $c_k r_k^{\frac{n-2}{2}} u(r_k x)$ converges to $|x|^{2-n}$ weakly in $W^{2,p}_{loc}(\mathbb{R}^n \setminus \{0\})$ for any $p < \frac{n}{2}$. Set

$$F_k = c_k^{\frac{2}{n-2}}(F(r_k x) - F(r_k x_0)) + y_0,$$

where y_0 will be defined later. Since

$$\left|\frac{\partial F_k}{\partial x^i}\right|^2 = r_k^2 c_k^{\frac{4}{n-2}}(u(r_k x))^{\frac{4}{n-2}} = (c_k r_k^{\frac{n-2}{2}} u(r_k x))^{\frac{4}{n-2}},$$

$|\nabla F_k|$ is bounded in $W^{2,p}(B_r \setminus B_{\frac{1}{r}})$ by Lemma 3.1. Thus, we may assume F_k converges weakly in $W^{3,p}_{loc}(\mathbb{R}^n \setminus \{0\})$ to a map F_∞ which satisfies $F_\infty(\mathbb{R}^n \setminus \{0\}) \subset \mathbb{R}^n$ and

$$\frac{\partial F_\infty}{\partial x^i}\frac{\partial F_\infty}{\partial x^j} = |x|^{-4}\delta_{ij}.$$

For convenience, we transition to new coordinates

$$x \to y = \frac{x}{|x|^2}.$$

In these coordinates,

$$g_\infty(y) = g_{euc}(y).$$

Then F_∞ can be considered as an isometric map from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{R}^n$.

Let $\gamma(t) = ty$, where $y \in S^{n-1}$. Since $\gamma(t)$ is a geodesic in $\mathbb{R}^n \setminus \{0\}$, $f(\gamma(t))$ must be a ray of $\mathbb{R}^n$. In addition, it is easy to check that as y', y'' approach 0 $d_{g_\infty}(F_\infty(y'), F_\infty(y'')) \to 0$. Then $\lim_{y\to 0} F_\infty(y)$ exists. Select y_0 such that the limit is $\lim_{y\to 0} F_\infty(y) = 0$. Then $F_\infty(\gamma(t)) = tF_\infty(y)$.

Since F_∞ is isometric, for any $X \in T\partial B_1 = S^{n-1}$ with $|X| = 1$, it holds

$$F_{\infty,*}(X) \perp F_{\infty,*}(\frac{\partial}{\partial r}), \quad |F_{\infty,*}(X)| = 1.$$

Hence, the restriction $F_\infty|_{S^{n-1}}$ is an isometric map from S^{n-1} to itself. Since S^{n-1} is simply connected, $F_\infty|_{S^{n-1}}$ is a homeomorphism, as follows from the fact that an isometric map is a covering map. Therefore, we may assume $F_\infty(y) = y$.

In the original coordinates, this translates to

$$F_\infty(x) = \frac{x}{|x|^2}.$$

Proceeding, we consider a sequence $x_k \to 0$. Assuming $a = |x_0|$ and setting $x'_k = a\frac{x_k}{|x_k|}$, $r_k = |x_k|$ and $\sigma = 2^m$, we find that

$$d_g(\sigma x_k, x'_k) \le C d_g(\sigma x_k, 2\sigma x_k), \quad and \quad \frac{d_g(\sigma x_k, 2\sigma x_k)}{d_g(x_k, \sigma x_k)} \to \frac{1}{2(\sigma - 1)},$$

thus we can choose m, such that for large k,

$$1 - \epsilon \le \frac{d_g(x_k, x_0)}{d_g(x_k, \sigma x_k)} \le 1 + \epsilon.$$

By arguments as in the proof of Lemma 3.2, for any $\tau \in (0, 1)$,

$$\frac{1-\tau}{2} \le \frac{|F(rx) - F(rx/2)|}{|F(rx/2) - F(rx/4)|} \le \frac{1+\tau}{2}$$

for any $x \in \partial B$ and sufficiently small r. Then we can choose m, such that for sufficiently large k,

$$1 - \epsilon \le \frac{|F(x_k) - F(x_0)|}{|F(x_k) - F(\sigma x_k)|} < 1 + \epsilon.$$

Since

$$\frac{|F(x_k) - F(\sigma x_k)|}{d_g(x_k, \sigma x_k)} = \frac{|F_k(x_k/|x_k|) - F_k(\sigma x_k/|x_k|)|}{d_{g_k}(x_k/|x_k|, \sigma x_k/|x_k|)} \to 1,$$

it follows that

$$1 - C\epsilon < \frac{|F(x_k) - F(x_0)|}{d_g(x_k, x_0)} \le 1 + C\epsilon$$

when k is sufficiently large. □

5 4 Dimensional Gauss-Bonnet-Chern formulas

In this section, we assume $n = 4$ and discuss Gauss-Bonnet-Chern formulas.

For our purpose, we set $dS_{g_0} = \Theta(r,\theta)dS^3$ and define $\phi = \log u$ and

$$F_1(r) = \int_{S^3} R(r,\theta)u^2(r,\theta)\Theta(r,\theta)dS^3, \quad H_1 = -\int_{S^3} R(r,\theta)\frac{\partial}{\partial r}(u^2(r,\theta)\Theta(r,\theta))dS^3$$
$$F_2(r) = \int_{S^3} \left(\Delta_{g_0}\phi(r,\theta)\right)\Theta(r,\theta)dS^3, \quad H_2 = -\int_{S^3} \left(\Delta_{g_0}\phi\right)\frac{\partial}{\partial r}\Theta(r,\theta)dS^3.$$

Let $n_{g,\partial B_r}$ be the unit normal vector of ∂B_r with respect to g. If we choose $x^1, \cdots, x^n$ to be normal coordinates of g_0, then $B_r = B_r^{g_0}(0)$, $n_{g,\partial B_r} = u^{-1}\frac{\partial}{\partial r}$,and

$$\int_{\partial B_r} n_{g,\partial B_r}(R)dS_g = \int_{S^3} u^2\frac{\partial R}{\partial r}\Theta dS^3 = F_1'(r) + H_1(r). \tag{5.1}$$

Moreover, we have

$$\int_{\partial B_r} \frac{\partial \Delta_{g_0}\phi}{\partial r}dS_{g_0} = F_2'(r) + H_2(r). \tag{5.2}$$

Equations (5.1) and (5.2) will help us to calculate $\int_{B_r} \Delta_g R dV_g$ and $\int_{B_r} \Delta_{g_0}^2 \log u dV_{g_0}$. For example, by

$$\int_{B_r} \Delta_g R dV_g = \lim_{r_k \to 0}(F_1' + H_1(r))|_{r_k}^{r},$$

if we can find a sequence $r_k \to 0$, such that the limit of $F_i'(r_k) + H_i(r_k)$ is known, then we will get the exact value of $\int_{B_r} \Delta_g R dV_g$.

The following lemma will play a vital role in the following discussions.

Lemma 5.1 *Let $f \in C^1[\frac{r_0}{4}, 2r_0]$, $h \in C^0[\frac{r_0}{4}, 2r_0]$ and b_1, b_2 are constants. Assume*

$$\left| \int_{\frac{r_0}{4}}^{\frac{r_0}{2}} f dt - \frac{3}{32} b_1 r_0^2 \right| + \left| \int_{r_0}^{2r_0} f dt - \frac{3}{2} b_1 r_0^2 \right| \le a r_0^2, \quad \int_{\frac{r_0}{4}}^{2r_0} |h - b_2| < a r_0.$$

Then there exists $\xi \in [r_0/4, 2r_0]$ such that

$$|f'(\xi) + h(\xi) - b_1 - b_2| \le 12a.$$

Proof Since we can replace f with $f - b_1 r$ and h with $h - b_2$, it suffices for our aim to prove the case when $b_1 = b_2 = 0$. By The Mean Value Theorem for Integrals, there exists $\xi_1 \in [r_0/4, r_0/2]$ and $\xi_2 \in [r_0, 2r_0]$, such that

$$\frac{r_0}{4} f(\xi_1) = \int_{\frac{r_0}{4}}^{\frac{r_0}{2}} f(t) dt, \quad r_0 f(\xi_2) = \int_{r_0}^{2r_0} f(t) dt,$$

which yields that

$$|f(\xi_1)| \le 4 a r_0, \quad |f(\xi_2)| \le a r_0.$$

Then

$$\left| \int_{\xi_1}^{\xi_2} (f' + h) \right| \le |f(\xi_1) - f(\xi_2)| + \int_{\xi_1}^{\xi_2} |h| \le 5 a r_0 + \int_{r_0/4}^{2r_0} |h| \le 6 a r_0.$$

Using the Mean Value Theorem for Integrals again, we can find $\xi \in [\xi_1, \xi_2]$, such that

$$(\xi_2 - \xi_1)|f'(\xi) + h(\xi)| \le 6 a r_0.$$

Noting that $r_0/2 < \xi_2 - \xi_1$, we complete the proof. □

Lemma 5.2 *For any sufficiently small r, we have*

$$\left| \int_r^{2r} F_1(t) dt \right| < \alpha(r) r^2, \quad \int_r^{2r} |H_1|(t) dt < \alpha(r) r,$$

and

$$\left| \int_r^{2r} F_2(t) dt + 6\omega_3 r^2 \right| < \alpha(r) r^2, \quad \int_r^{2r} |H_2 - 12\omega_3|(t) dt < \alpha(r) r,$$

where $\lim_{r \to 0} \alpha(r) = 0$, and $\omega_3 = 2\pi^2$ is the volume of the 3-dimensional sphere.

Proof We have

$$\begin{aligned} \left| \int_r^{2r} F_1(t) dt \right| &\le \int_r^{2r} \int_{S^3} |R(g)| u^2 \Theta dS^3 dt = \int_{B_{2r} \setminus B_r} |R(g)| u^2 dV_{g_0} \\ &\le \left(\int_{B_{2r} \setminus B_r} |R(g) u^2|^2 dV_{g_0} \right)^{\frac{1}{2}} \left(\int_{B_{2r} \setminus B_r} dV_{g_0} \right)^{\frac{1}{2}} \\ &\le C \|R\|_{L^2(B_{2r}, g)} r^2. \end{aligned}$$

and

$$\int_r^{2r} |H_1(t)| dt \le 2 \int_r^{2r} \int_{S^3} |R(g)| u^2 \left| \frac{\partial \log u}{\partial t} \right| \Theta dS^3 dt$$

$$+\int_r^{2r}\int_{S^3}|R(g)|u^2\left|\frac{\partial\log\Theta(t,\theta)}{\partial t}\right|(\Theta dS^3dt)$$
$$\leq C(\|R\|_{L^2(B_{2r},g)}\|\nabla\phi\|_{L^2(B_{2r}\setminus B_r,g_0)}+\int_{B_{2r}\setminus B_r}|R(g)|u^2\frac{1}{r}dV_{g_0})$$
$$\leq C\|R\|_{L^2(B_{2r},g)}\|\nabla\phi\|_{L^2(B_{2r}\setminus B_r,g_0)}+Cr\|R(g)\|_{L^2(B_{2r},g)}.$$

By Lemma 2.1,

$$\|\nabla\phi\|_{L^2(B_{2r}\setminus B_r,g_0)}\leq Cr\|Ru^2+R(g_0)\|_{L^2(B_{2r}\setminus B_r,g_0)}\leq C(\|R\|_{L^2(B_{2r}\setminus B_r,g)}r+r^2),$$

hence

$$\int_r^{2r}|H_1(t)|dt\leq C(\|R\|_{L^2(B_{2r}\setminus B_r,g)}r+r^2).$$

Next, we discuss F_2. Since

$$-\Delta_{g_0}\phi-|\nabla_{g_0}\phi|^2=c(n)Ru^2-c(n)R(g_0),$$

we obtain

$$\int_r^{2r}F_2(t)dt=\int_{B_{2r}\setminus B_r}\Delta_{g_0}\phi dV_{g_0}$$
$$=-\int_{B_{2r}\setminus B_r}|\nabla_{g_0}\phi|^2dV_{g_0}-c(4)\int_{B_{2r}\setminus B_r}(R(g)u^2-R(g_0))dV_{g_0}.$$

Note that

$$\int_{B_{2r}\setminus B_r}|R(g)u^2-R(g_0)|dV_{g_0}\leq C(\|R\|_{L^2(B_{2r}\setminus B_r,g)}r^2+r^4).$$

To get the estimate of $\int_r^{2r}F_2$, we only need to show that

$$\lim_{r\to 0}\frac{1}{r^2}\int_{B_{2r}\setminus B_r}|\nabla_{g_0}\phi|^2dV_{g_0}=6\omega_3.$$

Assume there exists $r_k\to 0$, such that

$$\lim_{k\to\infty}\frac{1}{r_k^2}\int_{B_{2r_k}\setminus B_{r_k}}|\nabla_{g_0}\phi|^2dV_{g_0}=\lambda\neq 6\omega_3.$$

Set $u_k=c_kr_ku(r_kx)$, where c_k is chosen such that $\int_{\partial B_1}\log u_k=0$. By the arguments in Section 3, $\log u_k(x)$ converges to $\log|x|^{-2}$ weakly in $W^{2,p}_{loc}(\mathbb{R}^4\setminus\{0\})$. Then, after passing to a subsequence, $\int_{B_2\setminus B_1}|\nabla_{g_0(r_kx)}\log u_k|^2$ converges to $6\omega_3$, hence

$$\lim_{k\to+\infty}\frac{1}{r_k^2}\int_{B_{2r_k}\setminus B_{r_k}}|\nabla_{g_0}\phi|^2dV_{g_0}=\lim_{k\to+\infty}\int_{B_2\setminus B_1}|\nabla_{g_0(r_kx)}\log u_k|^2dV_{g_0(r_kx)}=6\omega_3.$$

This leads to a contradiction.

Lastly, we calculate $\int_r^{2r}|H_2-12\omega_3|$:

$$\int_r^{2r}|H_2(t)-12\omega_3|dt\leq\int_{B_{2r}\setminus B_r}\left||\nabla_{g_0}\phi|^2\frac{\partial\log\Theta}{\partial r}-\frac{12}{\Theta}\right|dV_{g_0}$$
$$+C\int_{B_{2r}\setminus B_r}|R(g)u^2-R(g_0)|\frac{1}{r}dV_{g_0}.$$

The same argument as above shows that

$$\lim_{r\to 0}\frac{1}{r}\int_{B_{2r}\setminus B_r}\left||\nabla_{g_0}\phi|^2\frac{\partial\log\Theta}{\partial r}-\frac{12}{\Theta}\right|dV_{g_0}=0.$$

This completes the proof. □

We will provide several applications here. First, we calculate $\int_{B_r}\Delta_g R$:

Lemma 5.3 *There exists $r_k\to 0$ such that*

$$\int_{B_{\frac{1}{2}}\setminus B_{r_k}}\Delta_g R(g)dV_g\to\int_{\partial B_{\frac{1}{2}}}\frac{\partial R}{\partial r}dS_g.$$

Proof We have

$$\int_{B_{\frac{1}{2}}\setminus B_r}\Delta_g R(g)dV_g=\int_{\partial B_{\frac{1}{2}}}n_{g,\partial B_{\frac{1}{2}}}(R)dS_g-\int_{\partial B_r}n_{g,\partial B_r}(R)dS_g.$$

Applying Lemma 5.1 to $b_1=b_2=0$ and $f=F_1$, $h=H_1$, we deduce this lemma from (5.1). □

We recall some basic properties of Q-curvatures, cf. [10, 11]. On a 4-dimensional manifold, the Paneitz operator is defined as follows:

$$P_{g_0}\varphi=\Delta_{g_0}^2\varphi+div_{g_0}\left(\frac{2}{3}R_{g_0}\nabla_{g_0}\varphi-2Ric_{g_0}^{ij}\varphi_i\frac{\partial}{\partial x^j}\right).$$

The Q-curvature of g satisfies the following equations:

$$Q(g)=-\frac{1}{12}\Delta_g R(g)-\frac{1}{4}|Ric(g)|^2+\frac{1}{12}R^2,$$
$$P_{g_0}\phi+2Q(g_0)=2Q(g)e^{4\phi}.$$

For simplicity, we define

$$T(\phi)=\frac{1}{3}R_{g_0}\frac{\partial\phi}{\partial r}-Ric_{g_0}(\nabla_{g_0}\phi,\frac{\partial}{\partial r}).$$

Lemma 5.4 *There exists $r_k\to 0$, such that*

$$\lim_{k\to+\infty}\int_{B_{\frac{1}{2}}\setminus B_{r_k}}Q(g)dV_g=\int_{B_{\frac{1}{2}}}Q(g_0)dV_{g_0}-4\omega_3+\int_{\partial B_{\frac{1}{2}}}\left(\frac{1}{2}\frac{\partial\Delta_{g_0}\phi}{\partial r}+T(\phi)\right).$$

Proof We have

$$\begin{aligned}\int_{B_{\frac{1}{2}}\setminus B_r}Q_g dV_g&=\int_{B_{\frac{1}{2}}\setminus B_r}Q_{g_0}dV_{g_0}+\frac{1}{2}\int_{B_{\frac{1}{2}}\setminus B_r}P_{g_0}(\phi)dV_{g_0}\\&=\int_{B_{\frac{1}{2}}\setminus B_r}Q_{g_0}dV_{g_0}\\&\quad+\frac{1}{2}\int_{\partial(B_{\frac{1}{2}}\setminus B_r)}\left(\frac{\partial\Delta_{g_0}\phi}{\partial r}+\frac{2}{3}R_{g_0}\frac{\partial\phi}{\partial r}-2Ric_{g_0}(\nabla_{g_0}\phi,\frac{\partial}{\partial r})\right)dS_{g_0}.\end{aligned}$$

By (3.5),

$$\lim_{r\to 0}\int_{\partial B_r}|\nabla_{g_0}\phi|dS_{g_0}=0.$$

Then

$$\int_{\partial B_r}\left(\frac{2}{3}R_{g_0}\frac{\partial\phi}{\partial r}-2Ric_{g_0}(\nabla_{g_0}\phi,\frac{\partial}{\partial r})\right)dS_{g_0}\to 0. \tag{5.3}$$

By applying Lemma 5.1 to $f=F_2/2, h=H_2/2$ and $(b_1,b_2)=(-2\omega_3,6\omega_3)$, we deduce from (5.2) that there exists r_k, such that

$$\frac{1}{2}\int_{\partial B_{r_k}}\frac{\partial\Delta_{g_0}\phi}{\partial r}dS_{g_0}\to 4\omega_3.$$

Therefore, we complete the proof. □

Next, we discuss the relationship between $\|R\|_{L^2}$ and $\|Riem\|_{L^2}$:

Lemma 5.5 *We have*

$$\int_{B_{\frac{1}{2}}}|Riem(g)|^2dV_g<+\infty.$$

Proof It is well-known that

$$Riem(g)=W(g)+\frac{1}{2}(Ric(g)-\frac{1}{6}R(g)g)\owedge g,$$

where W is the Weyl tensor and $\owedge$ is the Kulkarni-Nomizu product. Since $|W|^2dV_g$ is conformally invariant, we only need to check $Ric(g)\in L^2$ here.

Recall that $Q(g)=-\frac{1}{12}\Delta_g R(g)-\frac{1}{4}|Ric(g)|^2+\frac{1}{12}R^2$, which means that

$$\begin{aligned}\int_{B_{\frac{1}{2}}\setminus B_r}|Ric(g)|^2dV_g&=\frac{1}{3}\int_{B_{\frac{1}{2}}\setminus B_r}R^2dV_g-\frac{1}{3}\int_{B_{\frac{1}{2}}\setminus B_r}\Delta_g R(g)dV_g-4\int_{B_{\frac{1}{2}}\setminus B_r}Q(g)dV_g\\&=\frac{1}{3}\int_{B_{\frac{1}{2}}\setminus B_r}R^2dV_g-4\int_{B_{\frac{1}{2}}\setminus B_r}Q(g_0)dV_{g_0}\\&\quad-\frac{1}{3}\int_{\partial(B_{\frac{1}{2}}\setminus B_r)}\frac{\partial R}{\partial r}dS_g-2\int_{\partial(B_{\frac{1}{2}}\setminus B_r)}(\frac{\partial\Delta_{g_0}\phi}{\partial r}+2T(\phi))dS_{g_0}\end{aligned}$$

Applying Lemma 5.1 to $f=\frac{1}{3}F_1+2F_2$, $h=\frac{1}{3}H_1+2H_2$, $(b_1,b_2)=(-8\omega_3,24\omega_3)$, we can find $r_k\to 0$, such that

$$\frac{1}{3}\int_{\partial B_{r_k}}\frac{\partial R}{\partial r}dS_g+2\int_{\partial B_{r_k}}(\frac{\partial\Delta_{g_0}\phi}{\partial r}+2T(\phi))\to 16\omega_3.$$

□

Lastly, we consider the formula for Pfaffian form:

Lemma 5.6 *We have*

$$\int_{B_{\frac{1}{2}}}Pf(g)=-4\omega_3+\int_{B_{\frac{1}{2}}}Pf(g_0)+\frac{1}{2}\int_{\partial B_{\frac{1}{2}}}\frac{\partial\Delta_{g_0}\phi}{\partial r}dS_{g_0}$$

$$+\frac{1}{12}\int_{\partial B_{\frac{1}{2}}} u^2\frac{\partial R}{\partial r}dS_{g_0} - \frac{1}{12}\int_{B_{\frac{1}{2}}} \Delta_{g_0}R(g_0)dV_{g_0} + \int_{\partial B_{\frac{1}{2}}} T(\phi)dS_{g_0}.$$

Proof Recall that

$$Pf(g) = \frac{1}{8}|W(g)|^2 + \frac{1}{12}R^2 - \frac{1}{4}|Ric(g)|^2,$$

where W is the Weyl tensor. Since

$$\int_B |W(g)|^2 dV_g = \int_B |W(g_0)|^2 dV_{g_0} < +\infty,$$

$Pf(g)$ is integrable. Recall that (c.f. [11])

$$Pf(g) = \frac{1}{8}|W(g)|^2 dV_g + Q(g)dV_g + \frac{1}{12}\Delta_g R(g)dV_g,$$
$$Pf(g_0) = \frac{1}{8}|W(g_0)|^2 dV_{g_0} + Q(g_0)dV_{g_0} + \frac{1}{12}\Delta_{g_0} R(g_0)dV_{g_0},$$

and

$$P_{g_0}\phi + 2Q(g_0) = 2Q(g)e^{4\phi},$$

where $g = u^2 g_0 = e^{2\phi}g_0$, we have

$$\begin{aligned}
\int_{B_{\frac{1}{2}}\setminus B_r} Pf(g) &= \int_{B_{\frac{1}{2}}\setminus B_r} Pf(g_0) + \frac{1}{2}\int_{B_{\frac{1}{2}}\setminus B_r} P_{g_0}\phi dV_{g_0} \\
&\quad + \frac{1}{12}\int_{B_{\frac{1}{2}}\setminus B_r} \Delta_g R(g)dV_g - \frac{1}{12}\int_{B_{\frac{1}{2}}\setminus B_r} \Delta_{g_0}R(g_0)dV_{g_0} \\
&= \int_{B_{\frac{1}{2}}\setminus B_r} Pf(g_0) + \frac{1}{2}\int_{\partial(B_{\frac{1}{2}}\setminus B_r)} \frac{\partial \Delta_{g_0}\phi}{\partial r}dS_{g_0} \\
&\quad + \frac{1}{12}\int_{\partial(B_{\frac{1}{2}}\setminus B_r)} u^2\frac{\partial R}{\partial r}dS_{g_0} - \frac{1}{12}\int_{B_{\frac{1}{2}}\setminus B_r} \Delta_{g_0}R(g_0)dV_{g_0} \\
&\quad + \frac{1}{2}\int_{\partial(B_{\frac{1}{2}}\setminus B_r)} \left(R_{g_0}\frac{\partial\phi}{\partial r} - 2Ric_{g_0}(\nabla_{g_0}\phi, \frac{\partial}{\partial r})\right)dS_{g_0}.
\end{aligned}$$

Apply Lemma 5.1 to $f = \frac{1}{12}F_1 + \frac{1}{2}F_2$, and $h = \frac{1}{12}H_1 + \frac{1}{2}H_2$, $(b_1, b_2) = (-2\omega_3, 6\omega_3)$, which suffices to complete the proof. □

Theorem 1.5, 1.7 and 1.6 can be deduced from Lemma 5.4, 5.6 and 5.5 easily.

6 Examples

In the last section, we provide examples of metrics on $B^4_{1/2}\setminus\{0\}$ that are conformal to g_{euc} and satisfy $\|R(g)\|_{L^2} < +\infty$. We will set

$$u = r^{-2}e^v, \quad \phi = \log u, \quad and \quad g = u^2 g_{euc},$$

where $v = v(r)$ is radial.

We have

$$|R_g|^2 dV_g = (c\frac{\Delta u}{u})^2 dx, \quad |Q_g| dV_g = |\Delta^2 \phi| dx,$$

where c is a constant. We observe that

$$\frac{\Delta u}{u} = v'' - \frac{v'}{r} + (v')^2, \quad |\Delta^2 \phi| = |\Delta^2 v| \le C(|v''''| + \frac{|v'''|}{r} + \frac{|v''|}{r^2} + \frac{|v'|}{r^3}). \tag{6.1}$$

Recall that (cf. [23, Ch. 5])

$$R_{ij}(g) = -(\log u^2)_{,ij} + \frac{1}{2}(\log u^2)_i (\log u^2)_j - \frac{1}{2}(\Delta(\log u^2) + |\nabla \log u^2|^2)(g_{euc})_{ij}.$$

Note that the Hessian tensor of $\log u^2$ in the euclidean metric is

$$Hess(\log u^2, g_{euc}) = (\log u^2)'' dr \otimes dr + r(\log u^2)' g_{S^3}.$$

It follows that

$$\begin{aligned} Ric(g) &= \left(-(\log u^2)'' + \frac{1}{2}|(\log u^2)'|^2 - \frac{1}{2}(\Delta(\log u^2) + |(\log u^2)'|^2)\right) dr \otimes dr \\ &\quad - \left(r(\log u^2)' + \frac{1}{2}(\Delta(\log u^2) + |(\log u^2)'|^2) r^2\right) g_{S^3} \\ &= -\frac{3}{2}\left((\log u^2)'' + \frac{1}{r}(\log u^2)'\right) dr \otimes dr \\ &\quad - \left(\frac{5}{2} r(\log u^2)' + \frac{r^2}{2}(\log u^2)'' + \frac{r^2}{2}|(\log u^2)'|^2\right) g_{S^3} \\ &= -3\left(v'' + \frac{1}{r}(v)'\right) dr \otimes dr - \left(-3rv' + r^2 v'' + 2|v'|^2 r^2\right) g_{S^3}, \end{aligned}$$

leading to

$$|Ric(g)|^2 \sqrt{|g|} \le C(|v''|^2 + \frac{|v'|^2}{r^2}). \tag{6.2}$$

Example 6.1 Consider $v = r^a \log r$, $g = e^{-2(2-r^a)\log r} g_{euc}$. We have

$$\frac{\Delta u}{u} = r^{2a-2}(a \log r + 1)^2 + r^{a-2}(a^2 \log r - 2a \log r + 2a - 2).$$

Then, $R(g) \in L^2$ if and only if $a > 0$. In this setting, it is easy to verified that $Ric(g) \in L^2$ and $Q(g) \in L^1$ from (6.1) and (6.2).

Example 6.2 Let $v = -a \log(-\log r)$, $g = \frac{g_{euc}}{r^4 |\log r|^{2a}}$. We find

$$\frac{\Delta u}{u} = \frac{a(1+a)}{r^2 \log^2 r} + \frac{2a}{r^2 \log r}.$$

Then, $R(g) \in L^2$ for any a. We can check that $Ric(g) \in L^2$ and $Q(g)$ is integrable.

This example extends the metric

$$g = \frac{|dz|^2}{|z|^2 |\log |z||^{2a}}.$$

constructed by Hulin-Troyanov [18] on a 2 dimensional disk, which has finite total Gauss curvature. Depending on the value of a, their metric can be either bounded or unbounded, finite area or infinite area. However, in our case, the metric is always unbounded and of infinite volume.

Note that $r^2u = e^v = |\log r|^{-a}$ does not belong to $W^{2,2}$ when $a > -\frac{1}{2}$. This indicates that the conclusion '$G^{-1}u \in W^{2,p}$ for any $p < \frac{n}{2}$' in Theorem 1.2 can not be extended to $p = \frac{n}{2}$.

Example 6.3 Consider $v = r^4 \sin\frac{1}{r}$, $g = r^{-4}e^{2r^4 \sin\frac{1}{r}}g_{euc}$. We observe that

$$v'' = O(1), \quad v'/r^2 = O(1), \quad \Delta^2 v = \frac{\sin\frac{1}{r}}{r^4} + O(\frac{1}{r^3}).$$

Consequently, the scalar curvature $R(g)$ and the Ricci curvature $Ric(g)$ are in L^2. Since

$$\int_{B_{\frac{1}{2}}} \frac{|\sin(\frac{1}{r})|}{r^4}dx = \int_0^{\frac{1}{2}} \frac{|\sin(\frac{1}{r})|}{r^4}r^3dr = \int_2^{\infty} \frac{|\sin(t)|}{t}dt > \sum_{k=2}^{\infty} \frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin(t)|dt = +\infty,$$

the Q-curvature $Q(g)$ is not integral in this case.

Acknowledgements We are grateful to the referee for helpful corrections and highly constructive suggestions which very substantially improved the exposition.

Data availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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