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Calculus of Variations

Normalized ground states for the fractional Schrödinger–Poisson system with critical nonlinearities

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Abstract

In this paper we study the existence and properties of ground states for the fractional Schrödinger–Poisson system with combined power nonlinearities

$$\begin{cases}(-\Delta)^s u - \phi|u|^{2_s^*-3}u = \lambda u + \mu|u|^{q-2}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3,\\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3,\end{cases}$$

having prescribed mass

$$\int_{\mathbb{R}^3} |u|^2 dx = a^2$$

and doubly critical growth, where $s \in (0, 1)$, $\mu > 0$ is a parameter, $2 < q < 2_s^*$, $2_s^* := \frac{6}{3-2s}$ is the fractional critical Sobolev exponent and $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. For a L^2-subcritical, L^2-critical and L^2-supercritical perturbation $\mu|u|^{q-2}u$, respectively, we prove several existence, and non-existence results. Furthermore, the qualitative behavior of the ground states as $\mu \to 0^+$ is also studied. Our results complement and improve the existing ones in several directions, and this study seems to be the first contribution regarding existence of normalized ground states for the fractional Sobolev critical Schrödinger–Poisson system with a critical nonlocal term.

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1 Introduction and main results

In this paper we study existence and properties of ground states with prescribed mass for the nonlinear fractional Schrödinger–Poisson system with combined power nonlinearities

$$\begin{cases} i\dfrac{\partial \Psi}{\partial t} = (-\Delta)^s\Psi - \phi|\Psi|^{2_s^*-3}\Psi - \mu|\Psi|^{q-2}\Psi - |\Psi|^{2_s^*-2}\Psi, & x \in \mathbb{R}^3, \\ (-\Delta)^s\phi = |\Psi|^{2_s^*-1}, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\Psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$, $\mu > 0$, $2 < q < 2_s^*$. We look for standing wave solutions to (1.1), namely to solutions of the form $(\Psi(t, x) = e^{-i\lambda t}u(x), \phi(x))$, $\lambda \in \mathbb{R}$. Then the function $(u(x), \phi(x))$ satisfies the equation

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu|u|^{q-2}u + \phi|u|^{2_s^*-3}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s\phi = |u|^{2_s^*-1}. & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

Here $(-\Delta)^s$ is a nonlocal operator defined by

$$(-\Delta)^s u(x) = C_s \text{ P.V.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2s}} dy, \quad x \in \mathbb{R}^3,$$

and P.V. stands for the Cauchy principal value on the integral, and C_s is a suitable normalization constant. It is well-known that, the first equation in (1.2) was used by Laskin (see [26, 27]) to extend the Feynman path integral, from Brownian-like to Lévy-like quantum mechanical paths. This class of fractional Schrödinger equations with a repulsive nonlocal Coulombic potential can be approximated by the Hartree–Fock equations to describe a quantum mechanical system of many particles; see, for example, [17, 18, 32, 34]. It also appeared in many different areas, such as financial mathematics, optimization, minimal surfaces, phase transitions, conservation laws, stratified materials, crystal dislocation and water waves, we refer to [2, 11] for more applied backgrounds on the fractional Laplacian.

We note that, when the second Poisson equation of the fractional Schrödinger–Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u + K(x)\phi u = f(x, u), & x \in \mathbb{R}^3, \\ (-\Delta)^t\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

is subcritical growth, (1.3) has been studied extensively and there are many results available in the literature. In [44], Zhang et al. studied the existence and the asymptotical behaviors of positive solutions to system (1.3) for the first time by using a perturbation approach. Ji [25] showed that (1.3) has a sign-changing ground state solution by means of a quantitative deformation lemma and the constraint variational method. Teng [41] studied the existence of a ground state solution to (1.3) when $K(x) = 1$ and $f(x, u) = \mu|u|^{q-1}u + |u|^{2_s^*-2}u$, $q \in (1, 2_s^* - 1)$ by using global compactness Lemma, the monotonicity trick, Pohozaev–Nehari manifold, and arguments of Brezis–Nirenberg type. Yang et al. [43] considered (1.3), and proved the existence of infinitely many solutions (u, λ) with u having prescribed L^2-norm. In [35], combing with the Ljusternik–Schnirelmann category theory and the Nehari manifold method, Murcia and Siciliano investigated the multiplicity of semiclassical state of the fractional Schrödinger–Poisson system

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + V(x)u + K(x)\phi u = f(u), & x \in \mathbb{R}^N, \\ \varepsilon^{\theta}(-\Delta)^{\alpha/2}\phi = \gamma_\alpha u^2, & x \in \mathbb{R}^N. \end{cases} \tag{1.4}$$

concentrating on the minima of $V(x)$ for $\varepsilon > 0$ small.

When the second equation of (1.3) is of critical growth, relatively speaking, there are only few papers in the existing literature. In [19], He studied the fractional Schrödinger–Poisson system with a critical nonlocal term

$$\begin{cases}(-\Delta)^s u + V(x)u - K(x)\phi|u|^{2_s^*-3}u = f(x,u), & x \in \mathbb{R}^3,\\ (-\Delta)^s \phi = K(x)|u|^{2_s^*-1}, & x \in \mathbb{R}^3,\end{cases} \tag{1.5}$$

and proved the existence of a mountain pass solution for (1.5) with $f(x,u) = |u|^{2_s^*-2}u + h(u)$, and h being subcritical growth, by using the concentration-compactness principle and mountain pass theorem. Dou and He [13] investigated (1.5) with $f(x,u) = a(x)f(u)$, and the potentials V and a may be vanishing at infinity, the authors obtained the existence of a positive ground state solution by employing the concentration-compactness principle, the mountain pass theorem and approximation method. In [37], Qu and He considered the semiclassical state of fractional Schödinger–Poisson system with double critical exponents

$$\begin{cases}\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) + \phi|u|^{2_s^*-3}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3,\\ \varepsilon^{2s}(-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3,\end{cases} \tag{1.6}$$

and they established the existence, multiplicity and concentration of positive solutions by the Ljusternik–Schnirelmann theory. In [15], Feng proved the existence of nonnegative solutions of (1.6) with $f(u) \equiv 0$, $\varepsilon = 1$, by using concentration-compactness principle, the mountain pass theorem and approximation method.

After a bibliography review, the existing results for the fractional Schrödinger–Poisson system with a nonlocal critical term, are mainly obtained without any constrained conditions for the L^2-norm, and a natural question that arises is whether or not we can obtain the existence of solutions for the fractional Schrödinger–Poisson system with a nonlocal critical term, having a desired L^2-norm $\int_{\mathbb{R}^3} |u|^2 dx = a^2$ for some prescribed $a > 0$. The main purpose of this paper is to focuss our attention on this issue and try to establish some existence results on normalized solutions. Concretely speaking, we shall study the following fractional Schrödinger–Poisson system with doubly critical growth

$$\begin{cases}(-\Delta)^s u - \phi|u|^{2_s^*-3}u = \lambda u + \mu|u|^{q-2}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3,\\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3,\end{cases} \tag{1.7}$$

with the prescribed L^2-norm

$$\int_{\mathbb{R}^3} |u|^2 dx = a^2, \tag{1.8}$$

where $\mu > 0$ is a parameter and $\mu|u|^{q-2}u$ is a local perturbation with $q \in (2, 2_s^*)$.

It is easily seen that the fractional Schrödinger–Poisson system (1.7) can be transformed into a single fractional Schrödinger equation with a nonlocal critical term. Briefly, by the Lax-Milgram theorem, for any fixed $u \in H^s(\mathbb{R}^3)$, Poisson equation $(-\Delta)^s\phi = |u|^{2_s^*-1}$ has a unique weak solution $\phi_u \in D^{s,2}(\mathbb{R}^3)$ and ϕ_u can be expressed as (e.g. [19])

$$\phi_u(x) = C_s \int_{\mathbb{R}^3} \frac{|u(y)|^{2_s^*-1}}{|x-y|^{3-2s}} dy, \tag{1.9}$$

where $C_s = \frac{\Gamma(\frac{3-2s}{2})}{2^{2s}\pi^{\frac{3}{2}}\Gamma(s)}$. In the sequel, we often omit the constant C_s for convenience. So, substituting (1.9) into the first equation of (1.7), then (1.7) can be transformed into a single fractional Schrödinger equation as follows:

$$(-\Delta)^s u - \phi_u |u|^{2_s^*-3} u = \lambda u + \mu |u|^{q-2} u + |u|^{2_s^*-2} u, \quad \forall u \in H^s(\mathbb{R}^3). \tag{1.10}$$

When looking for solutions to (1.10), a possible choice is then to fix $\lambda \in \mathbb{R}$ and to search for solutions to (1.10) correspond to critical points of the action functional

$$I_{\lambda,\mu}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u|^2 - \lambda u^2 \right) dx - \frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{\mu}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx.$$

In this case, the existence and multiplicity of solutions have been studied in [13, 15, 19, 37] and the references therein. Alternatively, one can search for solutions to (1.10) having a prescribed L^2-norm, and in this case $\lambda \in \mathbb{R}$ is part of the unknown. Defining on $u \in H^s(\mathbb{R}^3)$ the energy functional

$$I_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_u(x) |u|^{2_s^*-1} dx - \frac{\mu}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u|^{2_s^*} dx, \tag{1.11}$$

it is standard to check that I_μ is of C^1-class and that a critical point of I_μ restricted to the (mass) constraint set

$$S_a = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = a^2 \right\},$$

gives rise to a solution to (1.11) on S_a, satisfying $\|u\|^2_{L^2(\mathbb{R}^3)} = a^2$.

Definition 1.1 We say that $u_a \in S_a$ is a ground state solution to (1.11) it is a solution having minimal energy among all the solutions which belong to S_a. Namely, if

$$I_\mu(u_a) = \inf\{I_\mu(u), u \in S_a, \ (I_\mu|_{S_a})'(u) = 0\}.$$

This definition seems particularly suited in our context, since I_μ is unbounded from below on S_a, and hence global minima do not exist.

We remark that, the prescribed mass approaches that we shall follow here, have created an increasing interest in these last years, applied to various related problems. In [29], Luo and Zhang studied the following fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu |u|^{q-2} u + |u|^{p-2} u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, & u \in H^s(\mathbb{R}^N), \end{cases} \tag{1.12}$$

where $s \in (0, 1)$ and $2 < q < p < 2_s^* = \frac{2N}{N-2s}$. The authors proved some existence and nonexistence results about the normalized solutions to (1.12) with combined subcritical nonlinearities. Li and Zou [30], Zhen and Zhang [45] studied the existence and multiple normalized solution of (1.12) with $p = 2_s^*$, and extended the main results of [1], and Soave [39] to the fractional Laplacian case. For more results about the existence of normalized solutions of (1.12), we refer to [3, 10, 12, 14, 28] and the references therein. For the results on the normalized solutions for the Schrödinger equations or systems, we refer the readers to [4–7, 21–24] and the references therein.

Motivated by the aforementioned references, in this paper, we shall study the existence of normalized ground state solutions to the following fractional critical Schrödinger–Poisson system with the prescribed L^2-norm

$$\begin{cases} (-\Delta)^s u - \phi |u|^{2_s^*-3}u = \lambda u + \mu |u|^{q-2}u + |u|^{2_s^*-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3, \\ u \in H^s(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |u|^2 dx = a^2. \end{cases} \tag{1.13}$$

Problem (1.13) characteristics doubly critical growth, in the sense that the mixed nonlinearities combined a Sobolev critical term and a critical nonlocal term in view of the Hardy–Littlewood–Sobolev inequality [33]. We shall restrict our attention on the existence of normalized ground states to (1.13) for different cases of q. The present paper seems to be the first work for the existence of normalized solutions for fractional Schröding–Poisson system with doubly critical nonlinearities.

In order to state our main results, we need to fix some notations. Let $H^s(\mathbb{R}^3)$ be the Hilbert space of function in $\mathbb{R}^3$ endowed with the standard inner product and norm

$$\langle u, v \rangle := \int_{\mathbb{R}^3} ((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv) dx, \quad \|u\|^2_{H^s(\mathbb{R}^3)} = \langle u, u \rangle,$$

and $L^s(\mathbb{R}^3)$, $1 \le s \le \infty$, be the Lebesgue space endowed with the norms

$$\|u\|_s := \left(\int_{\mathbb{R}^3} |u|^s dx \right)^{\frac{1}{s}}.$$

The Sobolev spaces $D^{s,2}(\mathbb{R}^3)$ is defined by

$$D^{s,2}(\mathbb{R}^3) = \left\{ u \in L^{2_s^*}(\mathbb{R}^3) : \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy < +\infty \right\},$$

endowed with the norm

$$\|u\|^2 := \|u\|^2_{D^{s,2}(\mathbb{R}^3)} = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy.$$

According to Propositions 3.4 and 3.6 of [11], we have that,

$$\|u\|^2 = \|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} dx dy,$$

by omitting the normalization constant. Let S be the best Sobolev constant defined by

$$S := \inf_{u \in D^{s,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{(\int_{\mathbb{R}^3} |u|^{2_s^*} dx)^{\frac{2}{2_s^*}}}, \tag{1.14}$$

and the threshold value c_s^* by

$$c_s^* := \left(\frac{\sqrt{5} - 1}{2} \right)^{\frac{3-2s}{2s}} \frac{s \left(12 + (1 - \sqrt{5})(3 - 2s) \right)}{6(3 + 2s)} S^{\frac{3}{2s}}, \tag{1.15}$$

to verify the $(PS)_c$ compactness condition in the sequel.

If $q \in (2, 2_s^*]$, we also recall that the fractional Gagliardo-Nirenberg-Sobolev inequality [36]:

$$\|u\|_q^q \le C_{q,s} \|(-\Delta)^{\frac{s}{2}} u\|_2^{q\gamma_{q,s}} \|u\|_2^{q(1-\gamma_{q,s})}, \quad \forall\, u \in H^s(\mathbb{R}^3), \tag{1.16}$$

where the optimal constant $C_{q,s}$ depends on q and s, the number

$$\gamma_{q,s} := \frac{3(q-2)}{2qs}, \quad \forall q \in (2, 2_s^*],$$

and it is easy to see that

$$q\gamma_{q,s} \begin{cases} < 2, & \text{if } 2 < q < 2 + \frac{4s}{3}, \\ = 2, & \text{if } q = \overline{q} := 2 + \frac{4s}{3}, \\ > 2, & \text{if } 2 + \frac{4s}{3} < q < 2_s^*, \end{cases} \qquad \text{and that } \gamma_{2_s^*} = 1. \tag{1.17}$$

Now we summarize our main results of this paper. In the study of problem (1.13) an important role is played by the so-called L^2-critical exponent $\overline{q} := 2 + \frac{4s}{3}$. For the L^2-subcritical case: $2 < q < \bar{q} := 2 + \frac{4s}{3}$, we have the following conclusion:

Theorem 1.1 *Assume that $a, \mu > 0$ and $2 < q < \overline{q} := 2 + \frac{4s}{3}$. If there exists a constant $\widetilde{k} = \widetilde{k}(q, s) > 0$, such that*

$$\mu a^{q(1-\gamma_{q,s})} < \widetilde{k}, \tag{1.18}$$

then $I_\mu|_{S_a}$ has a ground state u which is a positive, radially symmetric function and solves problem (1.13) for some $\lambda < 0$. Moreover,

$$m_{a,\mu} := \inf_{u \in S_a} I_\mu(u) < 0 \tag{1.19}$$

and u is an interior local minimizer of $I_\mu(u)$ on the set $A_k = \{u \in S_a : \|u\| < k\}$, for suitable k small enough, and any other ground state solution of I_μ on S_a is a local minimizer of I_μ on A_k.

In the L^2-critical case: $q = \overline{q} := 2 + \frac{4s}{3}$, the change of the geometry of $I_\mu|_{S_a}$ leads to the change of the number of critical points of I_μ. The existence of ground states can be formulated as the following theorem.

Theorem 1.2 *Assume that $a, \mu > 0$ and $2 < q = \overline{q} := 2 + \frac{4s}{3}$. If*

$$\mu a^{\overline{q}(1-\gamma_{\overline{q},s})} < \bar{q}(2C_{\bar{q},s})^{-1}, \tag{1.20}$$

then $I_\mu|_{S_a}$ has a ground state $\widetilde{u}$ which is a positive, radially symmetric function and solves problem (1.13) for some $\widetilde{\lambda} < 0$. Moreover, $0 < m_{a,\mu} < c_s^*$ and $\widetilde{u}$ is a Mountain Pass type solution, where c_s^* is given in (1.15).

In the L^2-supercritical case: $2 + \frac{4s}{3} < q < 2_s^*$, we can obtain the existence of a Mountain Pass type ground state as follows.

Theorem 1.3 *Assume that $a, \mu > 0$ and $2 + \frac{4s}{3} < q < 2_s^*$. If one of the following conditions is satisfied:*

(i) $0 < s < \frac{3}{4}$ *and* $\mu a^{q(1-\gamma_{q,s})} < \frac{1}{\gamma_{q,s}} \left(\frac{\sqrt{5}-1}{2}\right)^{-\frac{q\gamma_{q,s}-2}{2_s^*-2}} S^{\frac{3(2_s^*-q)}{2s(2_s^*-2)}}$,

(ii) $\frac{3}{4} \leq s < 1$,

then $I_\mu|_{S_a}$ has a ground state $\widetilde{u}$ which is a positive, radially symmetric function and solves problem (1.13) for some $\widetilde{\lambda} < 0$. Moreover, $0 < m_{a,\mu} < c_s^*$ and $\widetilde{u}$ is a Mountain Pass type solution.

Remark 1.1. Assumption (1.18) has explicit estimates for $\widetilde{k}(q, s)$ in terms of Gagliardo–Nirenberg and Sobolev constants according to the fact that q is L^2-subcritical, L^2-critical, or L^2-supercritical. In the case $2 + \frac{4s}{3} < q < 2_s^*$, it is remarkable that we can prove that $\widetilde{k}(q, s) = +\infty$, so that any $a, \mu > 0$ are admissible; while in the case $2 < q \leq 2 + \frac{4s}{3}$, Assumptions (1.18) and (1.20) if ($q = 2 + \frac{4s}{3}$) enters in the study of the geometry of the constrained functional $I_\mu|_{S_a}$, and used in order to ensure that the ground state level $m_{a,\mu}$ is less than c_s^*, which is an essential ingredient in our compactness argument.

The next two theorems are concerned with the behavior of the ground states found in the limit case $\mu = 0$, and from Theorems 1.1–1.3 as $\mu \to 0^+$.

Theorem 1.4 *Let $a > 0$ and $\mu = 0$. Then we have the following assertions:*

(1) *If $0 < s < \frac{3}{4}$, then I_0 on S_a has a unique positive radial ground state $U_{\varepsilon,z}$ defined in (4.6) for the unique choice of $\varepsilon > 0$ which gives $\|U_{\varepsilon,z}\|_{L^2(\mathbb{R}^3)} = a$.*
(2) *If $\frac{3}{4} \leq s < 1$, then (1.13) has no positive solutions in S_a for any $\lambda \in \mathbb{R}$.*

Theorem 1.5 *Let u_μ be the corresponding positive ground state solution obtained in Theorems 1.1–1.3 with energy level $m_{a,\mu}$. Then the following conclusions hold:*

(1) *If $2 < q < 2 + \frac{4s}{3}$, then $m_{a,\mu} \to 0$, and $\|u_\mu\| \to 0$ in $D^{s,2}(\mathbb{R}^3)$ as $\mu \to 0^+$.*
(2) *If $2 + \frac{4s}{3} \leq q < 2_s^*$, then $m_{a,\mu} \to c_s^*$ as $\mu \to 0^+$.*

Remark 1.2. (i) Theorem 1.4 reveals that the functional $I_0(u)$ has ground state energy, which is achieved by the function w_ε given in Lemma 8.2, which is a new observation for problem (1.2).

(ii) Theorems 1.1–1.5 are new results not only for the fractional Schrödinger–Poisson systems with both the nonlocal critical term and the Sobolev critical nonlinearity [19], but also for the fractional Schrödinger–Poisson systems with only the nonlocal critical term [13, 15].

Finally, we give some comments on the proof for the main results above. Since the two critical terms $|u|^{2_s^*-2}u$ and $\phi_u|u|^{2_s^*-3}u$ are all L^2-supercritical, the functional I_μ is always unbounded from below on S_a, and this makes it difficult to deal with existence of normalized solutions on the L^2- constraint. One of the main difficulties is that one has to face in such context is the analysis of the convergence of constrained Palais–Smale sequences; indeed, the critical growth term in the equation makes the bounded (PS) sequences cannot converge. Because the problem has a Sobolev critical term and a nonlocal critical convolution term, it becomes more difficult to estimate the critical value of the mountain pass, and has to consider how the interaction between the nonlocal term and the nonlinear term will affect the existence of solutions of (1.13). Another of the main difficulties is that sequences of approximated Lagrange multipliers have to be controlled, since λ is not prescribed. For addition, weak limits of the Palais–Smale sequences could leave a constraint, since the embeddings $H^s(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ and $H_{rad}^s(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ are not compact.

In order to overcome these difficulties, we employ Jeanjean's theory [20] by showing that the mountain pass geometry of $I_\mu|_{S_a}$ allows to construct a Palais–Smale sequence of functions satisfying the Pohozaev identity. This gives boundedness, which is the first step in proving strong H^s-convergence. To overcome the loss of compactness caused by the doubly critical growth, we shall employ the modified concentration-compactness principle, the mountain pass theorem and energy estimation to obtain the existence of normalized ground states of (1.13). As naturally expected, the presence of the Sobolev critical term and the critical nonlocal term in (1.13) further complicates the study of the convergence of Palais–Smale sequences. One of the most relevant aspects of our study consists in showing that, suitably combining some of the main ideas from [19, 38, 40], compactness can be restored also in the present setting.

The paper is organized as follows: in Sect. 2, we start with some preliminary results which will be frequently used to prove Theorems 1.1–1.3. In Sect. 3, we show some lemmas for L^2-subcritical perturbation case. In Sect. 4, we give some preliminaries for L^2-critical perturbation case. In Sect. 5, we present some lemmas for L^2-supercritical perturbation case. In Sect. 6, we prove Theorem 1.1. In Sect. 7, we prove Theorems 1.2–1.3. In Sect. 8, we prove Theorem 1.4. Finally, the proof of Theorem 1.5 will be given in Sect. 9.

1.1 Notation

Throughout this paper, $\|\cdot\|_q$ denotes the norm in $L^q(\mathbb{R}^3)$, $1 < q < \infty$. $B_R(y)$ denotes the ball centered at y with radius R. Capital letters $C, C_i, i = 1, 2, \ldots$ denote various positive constants whose exact values are irrelevant, and $u^{\pm} = \max\{\pm u, 0\}$.

2 Preliminaries

In this section, we present various preliminary results which are necessary in the proof of the main theorems. We first summarize some properties of the function ϕ_u given as follows.

Lemma 2.1 ([19, 37]) *The function ϕ_u has the following properties:*

(i) $\phi_u \geq 0$ *for all* $u \in H^s(\mathbb{R}^3)$*;*

(ii) $\phi_{tu} = |t|^{2_s^*-1}\phi_u$ *for all* $t > 0$ *and* $u \in H^s(\mathbb{R}^3)$*;*

(iii) *For each* $u \in H^s(\mathbb{R}^3)$*,*

$$\|\phi_u\|_{D^{s,2}(\mathbb{R}^3)} \leq S^{-1/2}\|u\|_{2_s^*}^{2_s^*-1}$$

and

$$\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \leq S^{-1}\|u\|_{2_s^*}^{2(2_s^*-1)},$$

where S is the best Sobolev constant given in (1.14)*;*

(iv) *If* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, $u_n \to u$ *a.e. on* $\mathbb{R}^3$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{s,2}(\mathbb{R}^3)$, *and* $\phi_{u_n} - \phi_{u_n-u} - \phi_u \to 0$ *in* $D^{s,2}(\mathbb{R}^3)$;

(v) *If* $u_n \to u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_{u_n} \to \phi_u$ *in* $D^{s,2}(\mathbb{R}^3)$, *and* $\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}dx \to \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1}dx$*;*

(vi) *If* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$ *and* $u_n \to u$ *a.e. on* $\mathbb{R}^3$, *then*

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}dx - \int_{\mathbb{R}^3} \phi_{u_n-u}|u_n-u|^{2_s^*-1}dx - \int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx \to 0.$$

The following Pohozaev identity can be derived from [9, 31].

Proposition 2.1 *Let* $u \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ *be a positive weak solution of problem* (1.2)*, then u satisfies the equality*

$$\begin{aligned}\frac{3-2s}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx = \frac{3\lambda}{2}\int_{\mathbb{R}^3} |u|^2 dx + \frac{3-2s}{2}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1}dx \\ + \frac{3-2s}{2}\int_{\mathbb{R}^3} |u|^{2_s^*}dx + \frac{3\mu}{q}\int_{\mathbb{R}^3} |u|^q dx.\end{aligned} \tag{2.1}$$

Lemma 2.2 *Let $u \in H^s(\mathbb{R}^3)$ be a weak solution of problem (1.13), then we have the Pohozaev manifold*

$$\mathcal{N}_{a,\mu} = \{u \in S_a : \ P_\mu(u) = 0\}, \tag{2.2}$$

where

$$P_\mu(u) = s\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - s\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - s\int_{\mathbb{R}^3} |u|^{2_s^*} dx - s\mu\gamma_{q,s}\int_{\mathbb{R}^3} |u|^q dx.$$

Proof Since u is a solution of problem (1.13), we get

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \lambda\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx + \int_{\mathbb{R}^3} |u|^{2_s^*} dx + \mu\int_{\mathbb{R}^3} |u|^q dx. \tag{2.3}$$

Combining Proposition 2.1 and (2.3), we infer that

$$s\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = s\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx + s\int_{\mathbb{R}^3} |u|^{2_s^*} dx + s\mu\gamma_{q,s}\int_{\mathbb{R}^3} |u|^q dx, \tag{2.4}$$

and the conclusion follows. □

For $u \in S_a$ and $t \in \mathbb{R}$, we set

$$(t\star u)(x) = e^{\frac{3t}{2}} u(e^t x), \quad \forall\, x \in \mathbb{R}^3, \tag{2.5}$$

then $t\star u \in S_a$. For $u \in S_a$, we define the fiber map as

$$\begin{aligned}\Psi_u^\mu(t) &:= I_\mu(t\star u)\\ &= \frac{e^{2st}}{2}\|u\|^2 - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \mu\frac{e^{q\gamma_{q,s}st}}{q}\|u\|_q^q - \frac{e^{2_s^* st}}{2_s^*}\|u\|_{2_s^*}^{2_s^*}.\end{aligned} \tag{2.6}$$

An easy computation shows that $(\Psi_u^\mu)'(t) = P_\mu(t\star u)$; moreover, we have the following conclusion.

Proposition 2.2 *Let $u \in S_a$. Then $t \in \mathbb{R}$ is a critical point for $\Psi_u^\mu(t)$ if and only if $t\star u \in \mathcal{N}_{a,\mu}$. In particular, $u \in \mathcal{N}_{a,\mu}$ if and only if 0 is a critical point of $\Psi_u^\mu(t)$.*

In this spirit, we split the manifold $\mathcal{N}_{a,\mu}$ into the disjoint union

$$\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^+ \cup \mathcal{N}_{a,\mu}^0 \cup \mathcal{N}_{a,\mu}^-,$$

where

$$\begin{aligned}\mathcal{N}_{a,\mu}^+ &:= \{u \in \mathcal{N}_{a,\mu} : (\Psi_u^\mu)''(0) > 0\}\\ &= \{u \in \mathcal{N}_{a,\mu} : 2s^2\|u\|^2 > \mu q\gamma_{q,s}^2 s^2\|u\|_q^q + 2_s^* s^2\|u\|_{2_s^*}^{2_s^*} + 2(2_s^*-1)s^2\int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx\},\\ \mathcal{N}_{a,\mu}^0 &:= \{u \in \mathcal{N}_{a,\mu} : (\Psi_u^\mu)''(0) = 0\}\\ &= \{u \in \mathcal{N}_{a,\mu} : 2s^2\|u\|^2 = \mu q\gamma_{q,s}^2 s^2\|u\|_q^q + 2_s^* s^2\|u\|_{2_s^*}^{2_s^*} + 2(2_s^*-1)s^2\int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx\},\\ \mathcal{N}_{a,\mu}^- &:= \{u \in \mathcal{N}_{a,\mu} : (\Psi_u^\mu)''(0) < 0\}\\ &= \{u \in \mathcal{N}_{a,\mu} : 2s^2\|u\|^2 < \mu q\gamma_{q,s}^2 s^2\|u\|_q^q + 2_s^* s^2\|u\|_{2_s^*}^{2_s^*} + 2(2_s^*-1)s^2\int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx\}.\end{aligned} \tag{2.7}$$

Lemma 2.3 *Let $0 < s < 1, 2 < q < 2_s^*$ and $a, \mu > 0$. Let $\{u_n\} \subset S_{a,r} = S_a \cap H_r^s(\mathbb{R}^3)$ be a Palais–Smale sequence for $I_\mu|_{S_a}$ at level $m_{a,\mu}$, where $H_r^s(\mathbb{R}^3)$ is the subspace of $H^s(\mathbb{R}^3)$ consisting of radially symmetric functions. Then $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.*

Proof The proof is divided into three cases.

Case 1: $2 < q < \bar{q} = 2 + \frac{4s}{3}$. In this case, by (1.17), we have that $q\gamma_{q,s} < 2$. Since $P_\mu(u_n) \to 0$, one has

$$\|u_n\|^2 - \mu\gamma_{q,s}\int_{\mathbb{R}^3}|u_n|^q dx - \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx - \int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx = o_n(1). \tag{2.8}$$

Combining this and (1.16), we get that

$$\begin{aligned} I_\mu(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx - \frac{\mu}{q}\|u_n\|_q^q - \frac{1}{2_s^*}\|u_n\|_{2_s^*}^{2_s^*} + o_n(1) \\ &\geq \frac{1}{2}\|u_n\|^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx - \frac{\mu}{q}\|u_n\|_q^q - \frac{1}{2_s^*}\|u_n\|_{2_s^*}^{2_s^*} + o_n(1) \\ &= \frac{s}{3}\|u_n\|^2 - \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{2_s^*}\right)\|u_n\|_q^q + o_n(1) \\ &\geq \frac{s}{3}\|u_n\|^2 - \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{2_s^*}\right)C_{q,s}\|u_n\|^{q\gamma_{q,s}}a^{q(1-\gamma_{q,s})} + o_n(1). \end{aligned}$$

Since $\{u_n\}$ is a Palais–Smale sequence for $I_\mu|_{S_a}$ at level $m_{a,\mu}$, we have that $I_\mu(u_n) \leq m_{a,\mu}+1$ for n large. Thus, we obtain that

$$\frac{s}{3}\|u_n\|^2 \leq \frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{2_s^*}\right)C_{q,s}\|u_n\|^{q\gamma_{q,s}}a^{q(1-\gamma_{q,s})} + m_{a,\mu} + 2,$$

which implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Case 2: $q = \bar{q} := 2 + \frac{4s}{3}$. In this case, by (1.17), we have $q\gamma_{q,s} = 2$, and $P_\mu(u_n) \to 0$ implies that

$$\|u_n\|^2 - \mu\gamma_{q,s}\int_{\mathbb{R}^3}|u_n|^q dx - \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx - \int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx = o_n(1). \tag{2.9}$$

Hence,

$$I_\mu(u_n) = \frac{2s}{3+2s}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx + \frac{s}{3}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx + o_n(1) \leq m_{a,\mu} + 1,$$

which implies that

$$\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx \leq C \quad \text{and} \quad \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \leq C.$$

Since $q = 2 + \frac{4s}{3} \in (2, 2_s^*)$, we have $q = 2 + \frac{4s}{3} = \tau 2 + (1-\tau)2_s^*$ for some $\tau \in (0,1)$, and by Hölder inequality, we get that

$$\int_{\mathbb{R}^3}|u_n|^q dx \leq \left(\int_{\mathbb{R}^3}|u_n|^2 dx\right)^\tau \left(\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\right)^{1-\tau} \leq C.$$

Consequently, from (2.9), we know that

$$\|u_n\|^2 = \mu\gamma_{q,s}\int_{\mathbb{R}^3}|u_n|^q dx + \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx + \int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1) \leq C,$$

which implies $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Case 3: $\bar{q} := 2 + \frac{4s}{3} < q < 2_s^*$. In this case, by (1.17), one has $q\gamma_{q,s} > 2$, and from $P_\mu(u_n) \to 0$, we obtain that

$$\|u_n\|^2 - \mu\gamma_{q,s}\int_{\mathbb{R}^3}|u_n|^q dx - \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx - \int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx = o_n(1).$$

Thus, we have that

$$\begin{aligned} I_\mu(u_n) &= \frac{\mu}{q}\left(\frac{\gamma_{q,s}q}{2} - 1\right)\int_{\mathbb{R}^3}|u_n|^q dx + \frac{s}{3}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx + \frac{2s}{3+2s}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1) \\ &\le m_{a,\mu} + 1, \end{aligned}$$

which implies that $\int_{\mathbb{R}^3}|u_n|^q dx$, $\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx$ and $\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx$ are both bounded. Hence

$$\|u_n\|^2 = \mu\gamma_{q,s}\int_{\mathbb{R}^3}|u_n|^q dx + \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx + \int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-1}dx + o_n(1) \le C,$$

which completes the proof. □

Proposition 2.3 *Assume that* $0 < s < 1, 2 < q < 2_s^*$ *and* $a, \mu > 0$. *Let* $\{u_n\} \subset S_{a,r} = S_a \cap H_r^s(\mathbb{R}^3)$ *be a Palais–Smale sequence for* $I_\mu|_{S_a}$ *at level* $m_{a,\mu}$ *with*

$$m_{a,\mu} < \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} \quad \text{and} \quad m_{a,\mu} \neq 0.$$

Suppose in addition that $P_\mu(u_n) \to 0$ *as* $n \to +\infty$. *Then one of the following alternatives holds:*

(i) *either up to a subsequence* $u_n \rightharpoonup u$ *weakly in* $H^s(\mathbb{R}^3)$ *but not strongly, with* u *being a solution of problem* (1.13) *for some* $\lambda < 0$, *and*

$$I_\mu(u) \le m_{a,\mu} - \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}};$$

(ii) *or up to a subsequence* $u_n \to u$ *strongly in* $H^s(\mathbb{R}^3)$, $I_\mu(u) = m_{a,\mu}$ *and* u *solves problem* (1.13) *for some* $\lambda < 0$.

Proof By Lemma 2.3, we know that the sequence $\{u_n\}$ is a bounded sequence of radial functions in $H^s(\mathbb{R}^3)$, and by compactness of $H_r^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, up to a subsequence, there exists $u \in H_r^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H_r^s(\mathbb{R}^3)$, $u_n \to u$ strongly in $L^q(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Since $\{u_n\}$ is a bounded Palais–Smale sequence for $I_\mu|_{S_a}$, by Lagrange multipliers rule, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\begin{aligned} &\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\varphi dx - \int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2_s^*-3}u_n\varphi dx - \mu\int_{\mathbb{R}^3}|u_n|^{q-2}u_n\varphi dx \\ &\quad - \int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_n\varphi dx = \lambda_n\int_{\mathbb{R}^3}u_n\varphi dx + o_n(1)\|\varphi\| \end{aligned} \tag{2.10}$$

as $n \to \infty$ for every $\varphi \in H^s(\mathbb{R}^3)$. Choosing $\varphi = u_n$, then from (2.10) and the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^3)$, we obtain that $\{\lambda_n\}$ is bounded in $\mathbb{R}$, and up to a subsequence, $\lambda_n \to \lambda \in \mathbb{R}$.

Moreover, combining $P_\mu(u_n) \to 0$ with $\gamma_{q,s} < 1$, we infer to

$$\begin{aligned}\lambda a^2 &= \lim_{n\to\infty} \lambda_n \int_{\mathbb{R}^3} u_n^2 dx \\ &= \lim_{n\to\infty} \left(\|u_n\|^2 - \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx - \mu \int_{\mathbb{R}^3} |u_n|^q dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \right) \\ &= \lim_{n\to\infty} \mu(\gamma_{q,s} - 1) \int_{\mathbb{R}^3} |u_n|^q dx = \mu(\gamma_{q,s} - 1) \int_{\mathbb{R}^3} |u|^q dx \le 0.\end{aligned} \tag{2.11}$$

Hence, $\lambda = 0$ if and only if $u \equiv 0$. Next, we show that $u \not\equiv 0$. Assume by contradiction that $u \equiv 0$. Since $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$, up to a subsequence, $u_n \to 0$ strongly in $L^q(\mathbb{R}^3)$, then by $P_\mu(u_n) \to 0$, we have

$$\|u_n\|^2 - \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = o_n(1). \tag{2.12}$$

Without loss of generality, we may assume

$$\ell_n = \|u_n\|^2 \to \ell, \quad a_n = \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx \to a \quad \text{and} \quad b_n = \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \to b, \tag{2.13}$$

as $n \to \infty$. Note that by Young inequality, we infer to

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \phi_{u_n} (-\Delta)^{\frac{s}{2}} |u_n| dx \\ &\le \frac{\varepsilon^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} |u_n||^2 dx + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \phi_{u_n}|^2 dx \\ &= \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx + \frac{\varepsilon^2}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx.\end{aligned}$$

Thus, passing to the limit as $n \to \infty$, it follows that $b \le \frac{1}{2\varepsilon^2} a + \frac{\varepsilon^2}{2} \ell$. Choosing $\varepsilon^2 = \frac{\sqrt{5}-1}{2}$, and by (2.12), we can infer that $a \ge \frac{3-\sqrt{5}}{2} \ell$. Consequently, by (2.12)–(2.13), we derive that

$$\begin{aligned}m_{a,\mu} &= \lim_{n\to\infty} I_\mu(u_n) \\ &= \lim_{n\to\infty} \left\{ \frac{1}{2}\|u_n\|^2 - \frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^{2_s^*-1} dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx - \frac{\mu}{q} \int_{\mathbb{R}^3} |u_n|^q dx \right\} \\ &= \frac{s}{3}\ell + \frac{s(3-2s)}{3(3+2s)} a \ge \frac{s\left[12 + (1-\sqrt{5})(3-2s)\right]}{6(3+2s)} \ell = \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*} \ell.\end{aligned} \tag{2.14}$$

From (1.14), (2.12)–(2.13) and Lemma 2.1, we have that

$$\begin{aligned}\ell_n &= a_n + b_n + o_n(1) \\ &\le S^{-1} \|u_n\|_{2_s^*}^{2(2_s^*-1)} + b_n + o_n(1) \\ &\le S^{-1} \left(S^{-\frac{1}{2}} \left[\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right]^{\frac{1}{2}} \right)^{2(2_s^*-1)} + S^{-\frac{2_s^*}{2}} \left[\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \right]^{\frac{2_s^*}{2}} + o_n(1) \\ &\le S^{-2_s^*} \ell_n^{2_s^*-1} + S^{-\frac{2_s^*}{2}} \ell_n^{\frac{2_s^*}{2}} + o_n(1).\end{aligned} \tag{2.15}$$

Taking the limit in (2.15) as $n \to \infty$, we obtain that

$$\ell \leq S^{-2_s^*} \ell^{2_s^*-1} + S^{-\frac{2_s^*}{2}} \ell^{\frac{2_s^*}{2}}.$$

Therefore, either $\ell = 0$, or $\ell \geq \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}} S^{\frac{3}{2s}}$. If $\ell = 0$, from the definition of $I_\mu(u_n)$, we get that $m_{a,\mu} = 0$, which gives a contradiction to the fact that $I_\mu(u_n) \to m_{a,\mu} \neq 0$. So, $\ell \geq \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}} S^{\frac{3}{2s}}$ and by (2.14) we obtain that

$$\begin{aligned}
m_{a,\mu} &\geq \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*}\ell \\
&\geq \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}} \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*} S^{\frac{3}{2s}} \\
&= \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}},
\end{aligned}$$

which yields a contradiction to our assumptions. Therefore, $u \not\equiv 0$, and by (2.11), we see that $\lambda < 0$.

By (2.10), and a standard argument, we infer that

$$(-\Delta)^s u - \phi_u |u|^{2_s^*-3}u - \mu|u|^{q-2}u - |u|^{2_s^*-2}u = \lambda u, \quad x \in \mathbb{R}^3. \tag{2.16}$$

Indeed, for any $\varphi \in H^s(\mathbb{R}^3)$, it follows by the definition of weak convergence that

$$\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \varphi dx \to \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx \quad \text{as } n \to \infty.$$

Using $\lambda_n \to \lambda$ as $n \to \infty$, we easily get that

$$\lambda_n \int_{\mathbb{R}^3} u_n \varphi dx \to \lambda \int_{\mathbb{R}^3} u \varphi dx \quad \text{as } n \to \infty.$$

Furthermore, since $\{|u_n|^{2_s^*-2}u_n\}$ is bounded in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$ and $|u_n(x)|^{2_s^*-2}u_n(x) \to |u(x)|^{2_s^*-2}u(x)$ a.e. in $\mathbb{R}^3$. Then, we obtain that

$$|u_n|^{2_s^*-2}u_n \rightharpoonup |u|^{2_s^*-2}u \quad \text{in } L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3),$$

which yields that

$$\int_{\mathbb{R}^3} |u_n|^{2_s^*-2}u_n \varphi dx \to \int_{\mathbb{R}^3} |u|^{2_s^*-2}u \varphi dx \quad \text{as } n \to \infty.$$

It follows from Lemma 2.1 that $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{s,2}(\mathbb{R}^3)$, which implies that $\phi_{u_n} \rightharpoonup \phi_u$ in $L^{2_s^*}(\mathbb{R}^3)$. Then, we have that

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u)|u|^{2_s^*-3}u\varphi dx \to 0 \quad \text{as } n \to \infty. \tag{2.17}$$

Since $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$ and

$$\int_{\mathbb{R}^3} |\phi_{u_n}(|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u)|^{\frac{2_s^*}{2_s^*-1}}dx$$
$$\le C\left(\|\phi_{u_n}\|_{2_s^*}^{\frac{2_s^*}{2_s^*-1}}\|u_n\|_{2_s^*}^{\frac{2_s^*(2_s^*-2)}{2_s^*-1}} + \|\phi_{u_n}\|_{2_s^*}^{\frac{2_s^*}{2_s^*-1}}\|u\|_{2_s^*}^{\frac{2_s^*(2_s^*-2)}{2_s^*-1}}\right) \le C,$$

we have $\phi_{u_n}(|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u) \rightharpoonup 0$ in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$ and thus

$$\int_{\mathbb{R}^3} \phi_{u_n}(|u_n|^{2_s^*-3}u_n - |u|^{2_s^*-3}u)\varphi dx \to 0 \quad \text{as} \quad n \to \infty,$$

which together with (2.17) implies

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-3}u_n\varphi dx \to \int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-3}u\varphi dx \quad \text{as} \quad n \to \infty. \tag{2.18}$$

By the Pohozaev identity, we have $P_\mu(u) = 0$. Now, let $v_n = u_n - u$, then $v_n \rightharpoonup 0$ in $H^s(\mathbb{R}^3)$. By the well-known Brézis–Lieb lemma [8] and Lemma 2.1, we have that

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o_n(1) \quad \text{and} \quad \|u_n\|_{2_s^*}^{2_s^*} = \|v_n\|_{2_s^*}^{2_s^*} + \|u\|_{2_s^*}^{2_s^*} + o_n(1) \tag{2.19}$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}|u_n|^{2_s^*-1}dx = \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^{2_s^*-1}dx + \int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx + o_n(1). \tag{2.20}$$

Therefore, from $P_\mu(u_n) \to 0$ and $u_n \to u$ in $L^q(\mathbb{R}^3)$, we deduce by (2.19) and (2.20) that

$$\|v_n\|^2 + \|u\|^2 = \mu\gamma_{q,s}\int_{\mathbb{R}^3}|u|^q dx + \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^{2_s^*-1}dx + \int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx$$
$$+\|v_n\|_{2_s^*}^{2_s^*} + \|u\|_{2_s^*}^{2_s^*} + o_n(1).$$

Combining this with $P_\mu(u) = 0$, we conclude that

$$\|v_n\|^2 = \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^{2_s^*-1}dx + \|v_n\|_{2_s^*}^{2_s^*} + o_n(1). \tag{2.21}$$

Without loss of generality, we may assume

$$\|v_n\|^2 \to l, \quad \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^{2_s^*-1}dx \to \widetilde{a} \quad \text{and} \quad \int_{\mathbb{R}^3} |v_n|^{2_s^*}dx \to \widetilde{b}, \quad \text{as} \quad n \to \infty.$$

By Young inequality, we have that

$$\begin{aligned}\int_{\mathbb{R}^3} |v_n|^{2_s^*}dx &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}\phi_{v_n}(-\Delta)^{\frac{s}{2}}|v_n|dx \\ &\le \frac{\tau^2}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}|v_n||^2dx + \frac{1}{2\tau^2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}\phi_{v_n}|^2dx \\ &= \frac{1}{2\tau^2}\int_{\mathbb{R}^3} \phi_{v_n}|v_n|^{2_s^*-1}dx + \frac{\tau^2}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}v_n|^2dx,\end{aligned} \tag{2.22}$$

passing to the limit as $n \to \infty$, it follows that $\widetilde{b} \le \frac{1}{2\tau^2}\widetilde{a} + \frac{\tau^2}{2} l$. Taking $\tau^2 = \frac{\sqrt{5}-1}{2}$ and using (2.21), we can deduce that

$$\widetilde{a} \ge \frac{3-\sqrt{5}}{2} l. \tag{2.23}$$

Therefore, (1.14), (2.21) and Lemma 2.1 imply that

$$\begin{aligned}
\|v_n\|^2 &= \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^{2_s^*-1} dx + \|v_n\|_{2_s^*}^{2_s^*} + o_n(1) \le S^{-1} \|v_n\|_{2_s^*}^{2(2_s^*-1)} + \|v_n\|_{2_s^*}^{2_s^*} + o_n(1) \\
&\le S^{-1} \left(S^{-\frac{1}{2}} \left[\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \right]^{\frac{1}{2}} \right)^{2(2_s^*-1)} + S^{-\frac{2_s^*}{2}} \left[\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \right]^{\frac{2_s^*}{2}} + o_n(1) \\
&\le S^{-2_s^*} \|v_n\|^{2(2_s^*-1)} + S^{-\frac{2_s^*}{2}} \|v_n\|^{2_s^*} + o_n(1).
\end{aligned} \tag{2.24}$$

Passing the limit in (2.24) as $n \to \infty$, we obtain that

$$l \le S^{-2_s^*} l^{2_s^*-1} + S^{-\frac{2_s^*}{2}} l^{\frac{2_s^*}{2}}.$$

Thus, we have that

$$l = 0 \quad \text{or} \quad l \ge \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{2}{2_s^*-2}} S^{\frac{3}{2s}}.$$

Case 1: $l \ge \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{2}{2_s^*-2}} S^{\frac{3}{2s}}$. By (2.21)–(2.24), we have that

$$\begin{aligned}
m_{a,\mu} &= \lim_{n\to\infty} I_\mu(u_n) \\
&= \lim_{n\to\infty} \left(I_\mu(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^{2_s^*-1} dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |v_n|^{2_s^*} dx \right) \\
&= I_\mu(u) + \frac{s}{3} l + \frac{s(3-2s)}{3(3+2s)} \widetilde{a} \\
&\ge I_\mu(u) + \frac{s\left[12 + (1-\sqrt{5})(3-2s)\right]}{6(3+2s)} l \\
&= I_\mu(u) + \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*} l \\
&\ge I_\mu(u) + \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}}.
\end{aligned}$$

Thus, the conclusion (i) holds.

Case 2: $\ell = 0$. In this case, we can prove that $u_n \to u$ strongly in $H^s(\mathbb{R}^3)$. In fact, $\|v_n\| = \|u_n - u\| \to 0$ implies that $u_n \to u$ strongly in $D^{s,2}(\mathbb{R}^3)$ and hence in $L^{2_s^*}(\mathbb{R}^3)$ by the Sobolev inequality. We also obtain $\int_{\mathbb{R}^3} \phi_{v_n} |v_n|^{2_s^*-1} dx \to 0$ by Lemma 2.1. Next, we show that $u_n \to u$ strongly in $L^2(\mathbb{R}^3)$. If we test (2.10) with $\varphi = u_n - u$, test (2.16) with $u_n - u$, and subtract, we have that

$$\begin{aligned}&\|u_n-u\|^2-\int_{\mathbb{R}^3}(\lambda_n u_n-\lambda u)(u_n-u)dx\\&=\mu\int_{\mathbb{R}^3}(|u_n|^{q-2}u_n-|u|^{q-2}u)(u_n-u)dx+\int_{\mathbb{R}^3}(|u_n|^{2_s^*-2}u_n-|u|^{2_s^*-2}u)(u_n-u)dx\\&\quad+\int_{\mathbb{R}^3}\left[\phi_{u_n}|u_n|^{2_s^*-3}u_n-\phi_u|u|^{2_s^*-3}u\right](u_n-u)dx+o_n(1).\end{aligned}$$

Now the first, the third, and the fourth integrals tends to 0 by convergence of u_n to u in $D^{s,2}(\mathbb{R}^3)$, $L^q(\mathbb{R}^3)$ and $L^{2_s^*}(\mathbb{R}^3)$, while for the fifth integral, we have by Hölder inequality,

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\left[\phi_{u_n}|u_n|^{2_s^*-3}u_n-\phi_u|u|^{2_s^*-3}u\right](u_n-u)dx\right|\\&\le\left(\int_{\mathbb{R}^3}\left|\phi_{u_n}|u_n|^{2_s^*-3}u_n-\phi_u|u|^{2_s^*-3}u\right|^{\frac{2_s^*}{2_s^*-1}}dx\right)^{\frac{2_s^*-1}{2_s^*}}\left(\int_{\mathbb{R}^3}|u_n-u|^{2_s^*}dx\right)^{\frac{1}{2_s^*}}\\&\le CS^{-\frac{1}{2}}\|u_n-u\|_{D^{s,2}}\to 0\end{aligned}$$

as $n\to\infty$. As a consequence

$$0=\lim_{n\to\infty}\int_{\mathbb{R}^3}(\lambda_n u_n-\lambda u)(u_n-u)dx=\lim_{n\to\infty}\lambda\int_{\mathbb{R}^3}(u_n-u)^2dx,$$

which implies that $u_n\to u$ strongly in $L^2(\mathbb{R}^3)$ by $\lambda<0$. Thus, the conclusion (ii) holds, and the proof is completed. □

We end this section by stating the following variant Proposition 2.3.

Proposition 2.4 *Assume that* $0<s<1, 2<q<2_s^*$ *and* $a,\mu>0$. *Let* $\{u_n\}\subset S_{a,r}$ *be a Palais–Smale sequence for* $I_\mu|_{S_a}$ *at level* $m_{a,\mu}$, *with*

$$m_{a,\mu}<\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}\frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)}S^{\frac{3}{2s}}\quad\text{and}\quad m_{a,\mu}\ne 0.$$

Assume in addition that $P_\mu(u_n)\to 0$ *as* $n\to+\infty$, *and that there exists* $\{v_n\}\subset S_a$ *and* v_n *is radially symmetric for every* n *satisfying* $\|u_n-v_n\|\to 0$ *as* $n\to\infty$. *Then one of the alternatives (i) and (ii) in Proposition 2.3 holds.*

The proof is similar to the previous one: as in Lemma 2.3, we show that $\{u_n\}$ is bounded. Then also $\{v_n\}$ is bounded, and since each v_n is radial, we deduce that, up to a subsequence, $v_n\rightharpoonup u$ weakly in $H^s(\mathbb{R}^3)$, $v_n\to u$ strongly in $L^q(\mathbb{R}^3)$, and a.e. on $\mathbb{R}^3$. Since $\|u_n-v_n\|\to 0$, the same convergence is inherited by $\{u_n\}$, and we can proceed as in the proof of Proposition 2.3.

3 L^2-subcritical perturbation

In the L^2-subcritical case $2 < q < \bar{q} := 2 + \frac{4s}{3}$, we have $0 < q\gamma_{q,s} < 2$. To begin our argument, we first introduce the following positive constants

$$K := \frac{-(2_s^* - 1)(2_s^* - q\gamma_{q,s})}{2_s^*(2(2_s^* - 1) - q\gamma_{q,s})} + \frac{\sqrt{(2_s^* - 1)^2(2_s^* - q\gamma_{q,s})^2 + (2_s^* - 1)(2_s^*)^2(2(2_s^* - 1) - q\gamma_{q,s})(2 - q\gamma_{q,s})}}{2_s^*(2(2_s^* - 1) - q\gamma_{q,s})}, \tag{3.1}$$

$$K_1 := \frac{q\big(2_s^*(2_s^* - 1) - 2_s^* K^2 - 2(2_s^* - 1)K\big)}{22_s^*(2_s^* - 1)C_{q,s}} K^{\frac{2-q\gamma_{q,s}}{2_s^*-2}} S^{\frac{2_s^*(2-q\gamma_{q,s})}{2(2_s^*-2)}}, \tag{3.2}$$

and

$$K_2 := \frac{(2_s^* - 2)S^{\frac{2_s^*(2-q\gamma_{q,s})}{2(2_s^*-2)}}}{\gamma_{q,s}C_{q,s}(2_s^* - q\gamma_{q,s})} \times \left(\frac{-(2_s^* - q\gamma_{q,s}) + \sqrt{(2_s^* - q\gamma_{q,s})^2 + 4\big(2(2_s^* - 1) - q\gamma_{q,s}\big)(2 - q\gamma_{q,s})}}{2\big(2(2_s^* - 1) - q\gamma_{q,s}\big)}\right)^{\frac{2-q\gamma_{q,s}}{2_s^*-2}}. \tag{3.3}$$

We consider the constrained functional $I_\mu|_{S_a}$. For every $u \in S_a$, by (1.14), the fractional Gagliardo-Nirenberg-Sobolev inequality (1.16) and Lemma 2.1, we have that

$$I_\mu(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^* - 1)}S^{-2_s^*}\|u\|^{2(2_s^*-1)} - \frac{\mu}{q}C_{q,s}a^{q(1-\gamma_{q,s})}\|u\|^{q\gamma_{q,s}} - \frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}. \tag{3.4}$$

To better understand the geometry of the functional $I_\mu(u)$, we consider the function $h : \mathbb{R}^+ \to \mathbb{R}$,

$$h(t) = \frac{1}{2}t^2 - \frac{1}{2(2_s^* - 1)}S^{-2_s^*}t^{2(2_s^*-1)} - \frac{\mu}{q}C_{q,s}a^{q(1-\gamma_{q,s})}t^{q\gamma_{q,s}} - \frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}t^{2_s^*}. \tag{3.5}$$

From $\mu > 0$ and $q\gamma_{q,s} < 2$, we have that $h(0^+) = 0^-$ and $h(+\infty) = -\infty$.

Lemma 3.1 *Assume that the inequality $\mu a^{q(1-\gamma_{q,s})} < K_1$ holds, then the function h has a local strict minimum at negative level, a global maximum at positive level, and no other critical points, and there exist R_0 and R_1 both depending on a and μ, such that $h(R_0) = 0 = h(R_1)$ and $h(t) \geq 0$ if and only if $t \in (R_0, R_1)$.*

Proof For $t > 0$, we have $h(t) > 0$ if and only if

$$\varphi(t) > \frac{\mu}{q}C_{q,s}a^{q(1-\gamma_{q,s})}, \quad \text{with } \varphi(t) = \frac{1}{2}t^{2-q\gamma_{q,s}} - \frac{1}{2(2_s^* - 1)}S^{-2_s^*}t^{2(2_s^*-1)-q\gamma_{q,s}} - \frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}t^{2_s^*-q\gamma_{q,s}}.$$

In view of

$$\varphi'(t) = \frac{2 - q\gamma_{q,s}}{2} t^{1-q\gamma_{q,s}} - \frac{2(2_s^* - 1) - q\gamma_{q,s}}{2(2_s^* - 1)} S^{-2_s^*} t^{22_s^* - 3 - q\gamma_{q,s}} - \frac{2_s^* - q\gamma_{q,s}}{2_s^*} S^{-\frac{2_s^*}{2}} t^{2_s^* - 1 - q\gamma_{q,s}},$$

it is not difficult to check that $\varphi(t)$ has a unique critical point at

$$\bar{t} = K^{\frac{1}{2_s^*-2}} S^{\frac{2_s^*}{2(2_s^*-2)}},$$

and $\varphi(t)$ is increasing on $(0, \bar{t})$ and decreasing on $(\bar{t}, +\infty)$. Moreover, the maximum level is

$$\varphi(\bar{t}) = \frac{2_s^*(2_s^* - 1) - 2_s^* K^2 - 2(2_s^* - 1)K}{22_s^*(2_s^* - 1)} K^{\frac{2-q\gamma_{q,s}}{2_s^*-2}} S^{\frac{2_s^*(2-q\gamma_{q,s})}{2(2_s^*-2)}}.$$

Thus, h is positive on an open interval (R_0, R_1) if and only if $\varphi(\bar{t}) > \frac{\mu}{q} C_{q,s} a^{q(1-\gamma_{q,s})}$, that is $\mu a^{q(1-\gamma_{q,s})} < K_1$ holds. In view of $h(0^+) = 0^-$, $h(+\infty) = -\infty$ and h is positive on an open interval (R_0, R_1), it is immediate to see that h has a global maximum at positive level in (R_0, R_1), and has a local minimum point at negative level in $(0, R_0)$. Note that

$$h'(t) = t^{q\gamma_{q,s}-1} \left[t^{2-q\gamma_{q,s}} - S^{-2_s^*} t^{2(2_s^*-1)-q\gamma_{q,s}} - \mu\gamma_{q,s} C_{q,s} a^{q(1-\gamma_{q,s})} - S^{-\frac{2_s^*}{2}} t^{2_s^* - q\gamma_{q,s}} \right] = 0$$

if and only if

$$\psi(t) = \mu\gamma_{q,s} C_{q,s} a^{q(1-\gamma_{q,s})} \quad \text{with} \quad \psi(t) = t^{2-q\gamma_{q,s}} - S^{-2_s^*} t^{2(2_s^*-1)-q\gamma_{q,s}} - S^{-\frac{2_s^*}{2}} t^{2_s^* - q\gamma_{q,s}}.$$

Obviously, $\psi(t)$ has only one critical point, which is a strict maximum. Therefore, the above equation has at most two solutions. Consequently, if $\max_{t>0} \psi(t) \leq \mu\gamma_{q,s} C_{q,s} a^{q(1-\gamma_{q,s})}$, then we have a contradiction to the fact that h is positive on the open interval (R_0, R_1). Thus, $\max_{t>0} \psi(t) > \mu\gamma_{q,s} C_{q,s} a^{q(1-\gamma_{q,s})}$, which implies that h only has a local strict minimum at negative level and a global strict maximum at positive level and no other critical points. □

Lemma 3.2 *Assume that $\mu a^{q(1-\gamma_{q,s})} < K_2$, then $\mathcal{N}_{a,\mu}^0 = \emptyset$ and $\mathcal{N}_{a,\mu}$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^3)$.*

Proof We argue by contradiction that, there exists $u \in \mathcal{N}_{a,\mu}^0$. Then, $P_\mu(u) = 0$ with $(\Psi_u^\mu)''(0) = 0$, imply that

$$\|u\|^2 = \mu\gamma_{q,s} \|u\|_q^q + \|u\|_{2_s^*}^{2_s^*} + \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \tag{3.6}$$

and

$$2\|u\|^2 = \mu q \gamma_{q,s}^2 \|u\|_q^q + 2_s^* \|u\|_{2_s^*}^{2_s^*} + 2(2_s^* - 1) \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx. \tag{3.7}$$

Therefore, from (1.14), (3.6), (3.7) and Lemma 2.1 we have that

$$\mu\gamma_{q,s}(2 - q\gamma_{q,s}) \|u\|_q^q = (2_s^* - 2) \|u\|_{2_s^*}^{2_s^*} + 2(2_s^* - 2) \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx,$$

$$\begin{aligned}\|u\|^2 &= \frac{2_s^*-q\gamma_{q,s}}{2-q\gamma_{q,s}}\|u\|_{2_s^*}^{2_s^*} + \frac{2(2_s^*-1)-q\gamma_{q,s}}{2-q\gamma_{q,s}}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx \\ &\leq \frac{2_s^*-q\gamma_{q,s}}{2-q\gamma_{q,s}}S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*} + \frac{2(2_s^*-1)-q\gamma_{q,s}}{2-q\gamma_{q,s}}S^{-2_s^*}\|u\|^{2(2_s^*-1)}\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\|u\|^2 &\leq \mu\gamma_{q,s}\|u\|_q^q + \|u\|_{2_s^*}^{2_s^*} + 2\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx \leq \mu\gamma_{q,s}\frac{2_s^*-q\gamma_{q,s}}{2_s^*-2}\|u\|_q^q \\ &\leq \mu\gamma_{q,s}\frac{2_s^*-q\gamma_{q,s}}{2_s^*-2}C_{q,s}\|u\|^{q\gamma_{q,s}}a^{q(1-\gamma_{q,s})}.\end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9), we infer that

$$\begin{aligned}&\mu\gamma_{q,s}\frac{2_s^*-q\gamma_{q,s}}{2_s^*-2}C_{q,s}a^{q(1-\gamma_{q,s})} \geq S^{\frac{2_s^*(2-q\gamma_{q,s})}{2(2_s^*-2)}} \\ &\times\left(\frac{-(2_s^*-q\gamma_{q,s})+\sqrt{(2_s^*-q\gamma_{q,s})^2+4\big(2(2_s^*-1)-q\gamma_{q,s}\big)(2-q\gamma_{q,s})}}{2\big(2(2_s^*-1)-q\gamma_{q,s}\big)}\right)^{\frac{2-q\gamma_{q,s}}{2_s^*-2}},\end{aligned}$$

that is

$$\begin{aligned}&\mu a^{q(1-\gamma_{q,s})} \geq \frac{(2_s^*-2)S^{\frac{2_s^*(2-q\gamma_{q,s})}{2(2_s^*-2)}}}{\gamma_{q,s}C_{q,s}(2_s^*-q\gamma_{q,s})} \\ &\times\left(\frac{-(2_s^*-q\gamma_{q,s})+\sqrt{(2_s^*-q\gamma_{q,s})^2+4\big(2(2_s^*-1)-q\gamma_{q,s}\big)(2-q\gamma_{q,s})}}{2\big(2(2_s^*-1)-q\gamma_{q,s}\big)}\right)^{\frac{2-q\gamma_{q,s}}{2_s^*-2}} := K_2,\end{aligned} \tag{3.10}$$

which leads to a contradiction to our assumption, and so, $\mathcal{N}_{a,\mu}^0=\emptyset$.

Next, we can check that $\mathcal{N}_{a,\mu}$ is a smooth manifold of codimension 2 on $H^s(\mathbb{R}^3)$. To see this, we note that $\mathcal{N}_{a,\mu}=\{u\in H^s(\mathbb{R}^3): P_\mu(u)=0, G(u)=0\}$, for $G(u)=\int_{\mathbb{R}^3}u^2dx-a^2$, with P_μ and G being of class C^1 in $H^s(\mathbb{R}^3)$. Thus, it suffices to check that the differential $(dG(u), dP_\mu(u)):\ H^s(\mathbb{R}^3)\to\mathbb{R}^2$ is surjective, for every $u\in\mathcal{N}_{a,\mu}$. To this end, we prove that for every $u\in\mathcal{N}_{a,\mu}$, there exists $\varphi\in T_uS_a$ such that $dP_\mu(u)[\varphi]\neq 0$. Once that the existence of φ is established, the system

$$\begin{cases}dG(u)[\alpha u+\beta\varphi]=x\\ dP_\mu(u)[\alpha u+\beta\varphi]=y\end{cases} \iff \begin{cases}\alpha a^2=x\\ dP_\mu(u)[\alpha u+\beta\varphi]=y,\end{cases}$$

is solvable with respect to α,β for every $(x,y)\in\mathbb{R}^2$, and hence the surjectivity is proved.

Now, suppose by contradiction that for $u\in\mathcal{N}_{a,\mu}$ such a tangent vector φ does not exist, i.e. $dP_\mu(u)[\varphi]=0$ for every $\varphi\in T_uS_a$. Then u is a constrained critical point for the functional I_u on S_a, and hence by the Lagrange multipliers rule, there exists a $\lambda\in\mathbb{R}$ such that

$$2s(-\Delta)^s u=\lambda u+\mu sq\gamma_{q,s}|u|^{q-2}u+2_s^*s|u|^{2_s^*-2}u+2s(2_s^*-1)\phi_u|u|^{2_s^*-3}u \quad \text{in } \mathbb{R}^3.$$

However, by the Pohozaev identity for the last equation, we have

$$2s^2\|u\|^2=\mu q\gamma_{q,s}^2s^2\|u\|_q^q+2_s^*s^2\|u\|_{2_s^*}^{2_s^*}+2(2_s^*-1)s^2\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx,$$

that is $u \in \mathcal{N}^0_{a,\mu}$, a contradiction. So, for each $u \in \mathcal{N}_{a,\mu}$ there exists $\varphi \in T_u S_a$ such that $dP_\mu(u)[\varphi] \neq 0$, and we can easily solve for α, β. The function $dP_\mu(u) : S_a \to \mathbb{R}$ is surjective for each $u \in \mathcal{N}_{a,\mu}$ is proved. Hence $\mathcal{N}^0_{a,\mu}$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^3)$. Thus, $u \in \mathcal{N}_{a,\mu}$ is a natural constraint. □

The manifold $\mathcal{N}_{a,\mu}$ is then divided into its two components $\mathcal{N}^+_{a,\mu}$ and $\mathcal{N}^-_{a,\mu}$, having disjoint closure.

Lemma 3.3 *Let $u \in S_a$, then the function $\Psi^\mu_u(t)$ has exactly two critical points $\alpha_u < t_u \in \mathbb{R}$ and two zero points $c_u < d_u \in \mathbb{R}$, with $\alpha_u < c_u < t_u < d_u$. Furthermore,*

(i) *$\alpha_u \star u \in \mathcal{N}^+_{a,\mu}$, $t_u \star u \in \mathcal{N}^-_{a,\mu}$ and if $t\star u \in \mathcal{N}_{a,\mu}$, then either $t = \alpha_u$ or $t = t_u$;*
(ii) *$\|t\star u\| \leq R_0$ for each $t \leq c_u$, and*

$$I_\mu(\alpha_u\star u) = \min\{I_\mu(t\star u) : t \in \mathbb{R} \quad and \quad \|t\star u\| < R_0\} < 0;$$

(iii) *$I_\mu(t_u\star u) = \max\{I_\mu(t\star u) : t \in \mathbb{R}\} > 0$ and $\Psi^\mu_u(t)$ is strictly decreasing and concave on $(t_u, +\infty)$. Especially, if $t_u < 0$, then $P_\mu(u) < 0$;*
(iv) *The maps: $u \mapsto \alpha_u \in \mathbb{R}$ and $u \mapsto t_u \in \mathbb{R}$, $\forall u \in S_a$, are of class C^1.*

Proof Let $u \in S_a$, then by Proposition 2.2., we have $t\star u \in \mathcal{N}_{a,\mu}$ if and only if $(\Psi^\mu_u)'(t) = 0$. Firstly, we show that $\Psi^\mu_u(t)$ has at least two critical points. By (3.4), we have

$$\Psi^\mu_u(t) = I_\mu(t\star u) \geq h(\|t\star u\|) = h(e^{st}\|u\|),$$

which implies that the C^2 function $\Psi^\mu_u(t)$ is positive on $\left(s^{-1}\ln(R_0\|u\|^{-1}), s^{-1}\ln(R_1\|u\|^{-1})\right)$, $\Psi^\mu_u(-\infty) = 0^-$ and $\Psi^\mu_u(+\infty) = -\infty$. It follows that $\Psi^\mu_u(t)$ has a local minimum point α_u at a negative level in $\left(0, s^{-1}\ln(R_0\|u\|^{-1})\right)$ and has a global maximum point t_u at a positive level in $\left(s^{-1}\ln(R_0\|u\|^{-1}), s^{-1}\ln(R_1\|u\|^{-1})\right)$. Next, we prove that $\Psi^\mu_u(t)$ has no other critical points. Indeed, as $(\Psi^\mu_u)'(t) = 0$, we infer to

$$g(t) = s\mu\gamma_{q,s}\int_{\mathbb{R}^3}|u|^q dx,$$

with

$$g(t) = se^{(2-q\gamma_{q,s})st}\|u\|^2 - se^{(2(2^*_s-1)-q\gamma_{q,s})st}\int_{\mathbb{R}^3}\phi_u|u|^{2^*_s-1}dx - se^{(2^*_s-q\gamma_{q,s})st}\|u\|^{2^*_s}_{2^*_s}.$$

It follows that $g(t)$ has a unique maximum point, hence the above equation has at most two solutions.

From $u \in S_a$ and Proposition 2.2, we have $\alpha_u\star u, t_u\star u \in \mathcal{N}_{a,\mu}$, and $t\star u \in \mathcal{N}_{a,\mu}$ implying $t \in \{\alpha_u, t_u\}$. Since α_u is a local minimum point of $\Psi^\mu_u(t)$, we see that $(\Psi^\mu_{\alpha_u\star u})''(0) = (\Psi^\mu_u)''(\alpha_u) \geq 0$. As $\mathcal{N}^0_{a,\mu} = \emptyset$, we get $(\Psi^\mu_{\alpha_u\star u})''(0) = (\Psi^\mu_u)''(\alpha_u) > 0$, which implies that $\alpha_u\star u \in \mathcal{N}^+_{a,\mu}$. Similarly, we have that $t_u\star u \in \mathcal{N}^-_{a,\mu}$.

By the monotonicity and recalling the behavior at infinity of $\Psi^\mu_u(t)$, we see that $\Psi^\mu_u(t)$ has exactly two zero points $c_u < d_u$ with $\alpha_u < c_u < t_u < d_u$, and $\Psi^\mu_u(t)$ has exactly two inflection points. Particularly, $\Psi^\mu_u(t)$ is concave on $(t_u, +\infty)$, and hence, if $t_u < 0$, then $P_\mu(u) = (\Psi^\mu_u)'(0) < 0$.

Finally, we show that $u \mapsto \alpha_u \in \mathbb{R}$ and $u \mapsto t_u \in \mathbb{R}, \forall u \in S_a$, are of class C^1. Indeed, we can apply the implicit function theorem on the C^1 function $\Phi(t, u) := (\Psi^\mu_u)'(t)$. We use that $\Phi(\alpha_u, u) = (\Psi^\mu_u)'(\alpha_u) = 0$, that $\partial_t\Phi(\alpha_u, u) = (\Psi^\mu_u)''(\alpha_u) < 0$, and the fact that $\mathcal{N}^0_{a,\mu} = \emptyset$ implies that it is not possible to pass with continuity from $\mathcal{N}^+_{a,\mu}$ to $\mathcal{N}^-_{a,\mu}$. Thus, we know that $u \mapsto \alpha_u \in \mathbb{R}$, $\forall u \in S_a$, is of class C^1. Analogously, we can show that $u \mapsto t_u \in \mathbb{R}$, $\forall u \in S_a$, is of class C^1. □

For $k > 0$, we define

$$A_k = \{u \in S_a : \|u\| < k\} \quad \text{and} \quad m_{a,\mu} = \inf_{u \in A_{R_0}} I_\mu(u).$$

Then, we can conclude the following conclusion from Lemma 3.3.

Corollary 3.1 *There holds that the set $\mathcal{N}_{a,\mu}^+ \subset A_{R_0} = \{u \in S_a : \|u\| < R_0\}$ and*

$$\sup_{u \in \mathcal{N}_{a,\mu}^+} I_\mu(u) \leq 0 \leq \inf_{u \in \mathcal{N}_{a,\mu}^-} I_\mu(u).$$

Lemma 3.4 *The level $m_{a,\mu} \in (-\infty, 0)$, and verifies*

$$m_{a,\mu} = \inf_{\mathcal{N}_{a,\mu}} I_\mu = \inf_{\mathcal{N}_{a,\mu}^+} I_\mu \quad \text{and} \quad m_{a,\mu} < \inf_{\overline{A_{R_0}} \setminus A_{R_0 - r}} I_\mu$$

for $r > 0$ sufficiently small.

Proof For each $u \in A_{R_0}$, we have that

$$I_\mu(u) \geq h(\|u\|) \geq \min_{t \in [0, R_0]} h(t) > -\infty.$$

Hence, $m_{a,\mu} > -\infty$. Moreover, for each $u \in S_a$, we have $\|t \star u\| < R_0$ and $I_\mu(t \star u) < 0$ for $t \ll -1$, and so $m_{a,\mu} < 0$.

From $\mathcal{N}_{a,\mu}^+ \subset A_{R_0}$, we have $m_{a,\mu} \leq \inf_{\mathcal{N}_{a,\mu}^+} I_\mu$. On the other hand, if $u \in A_{R_0}$, by Lemma 3.3, we see that $\alpha_u \star u \in \mathcal{N}_{a,\mu}^+ \subset A_{R_0}$ and

$$I_\mu(\alpha_u \star u) = \min\{I_\mu(t \star u) : t \in \mathbb{R} \quad \text{and} \quad \|t \star u\| < R_0\} \leq I_\mu(u),$$

which implies that $\inf_{\mathcal{N}_{a,\mu}^+} I_\mu \leq m_{a,\mu}$. Since $I_\mu > 0$ on $\mathcal{N}_{a,\mu}^-$ by Corollary 3.1, we infer to $\inf_{\mathcal{N}_{a,\mu}^+} I_\mu = \inf_{\mathcal{N}_{a,\mu}} I_\mu$.

Finally, by the continuity of h there exists $r > 0$ such that $h(t) \geq \frac{m_{a,\mu}}{2}$ if $t \in [R_0 - r, R_0]$. Thus, for any $u \in S_a$ with $R_0 - r \leq \|u\| \leq R_0$, we have that

$$I_\mu(u) \geq h(\|u\|) \geq \frac{m_{a,\mu}}{2} > m_{a,\mu},$$

and this completes the proof. □

4 L^2-critical perturbation

In this section, we deal with the L^2-critical case $q = \bar{q} := 2 + \frac{4s}{3}$ and a, μ satisfy the inequality

$$\mu a^{\frac{4s}{3}} < \bar{q}(2C_{\bar{q},s})^{-1}. \tag{4.1}$$

Recalling the decomposition of

$$\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^+ \cup \mathcal{N}_{a,\mu}^0 \cup \mathcal{N}_{a,\mu}^-,$$

we have the following assertion.

Lemma 4.1 *$\mathcal{N}_{a,\mu}^0 = \emptyset$ and $\mathcal{N}_{a,\mu}$ is a smooth manifold of codimension 2 in $H^s(\mathbb{R}^3)$.*

Proof We argue by contradiction that, there exists $u \in \mathcal{N}^0_{a,\mu}$. Then, by $P_\mu(u) = 0$ and $(\Psi^\mu_u)''(0) = 0$, we have that

$$\|u\|^2 = \mu\gamma_{q,s}\|u\|^q_q + \|u\|^{2^*_s}_{2^*_s} + \int_{\mathbb{R}^3} \phi_u |u|^{2^*_s-1} dx, \tag{4.2}$$

and

$$2\|u\|^2 = \mu q\gamma^2_{q,s}\|u\|^q_q + 2^*_s\|u\|^{2^*_s}_{2^*_s} + 2(2^*_s - 1)\int_{\mathbb{R}^3} \phi_u |u|^{2^*_s-1} dx. \tag{4.3}$$

Thus, from (4.2) and (4.3), we infer to $\|u\|^{2^*_s}_{2^*_s} + 2\int_{\mathbb{R}^3} \phi_u|u|^{2^*_s-1}dx = 0$, which is not possible since $u \in S_a$, here we used the fact $\overline{q}\gamma_{\overline{q},s} = 2$. The rest of the proof is similar to that of Lemma 3.2, and so the details are omitted. □

Lemma 4.2 *Under the condition* (4.1)*, then for each $u \in S_a$, there exists a unique $t_u \in \mathbb{R}$ such that $t_u \star u \in \mathcal{N}_{a,\mu}$, where t_u is the unique critical point of the function of Ψ^μ_u and is a strict maximum point at positive level. Moreover,*

- (i) $\mathcal{N}_{a,\mu} = \mathcal{N}^-_{a,\mu}$;
- (ii) $\Psi^\mu_u(t)$ *is strict decreasing and concave on* $(t_u, +\infty)$ *and* $t_u < 0$ *implies that* $P_\mu(u) < 0$;
- (iii) *The map* $u \in S_a \mapsto t_u \in \mathbb{R}$ *is of* C^1;
- (iv) *If* $P_\mu(u) < 0$*, then* $t_u < 0$.

Proof Note that $\overline{q}\gamma_{\overline{q},s} = 2$, we get that

$$\begin{aligned}\Psi^\mu_u(t) &= I_\mu(t\star u)\\ &= \left(\frac{1}{2}\|u\|^2 - \frac{\mu}{\bar{q}}\|u\|^{\bar{q}}_{\bar{q}}\right)e^{2st} - \frac{e^{2(2^*_s-1)st}}{2(2^*_s-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2^*_s-1}dx - \frac{e^{2^*_s st}}{2^*_s}\|u\|^{2^*_s}_{2^*_s},\end{aligned} \tag{4.4}$$

where

$$\frac{1}{2}\|u\|^2 - \frac{\mu}{\bar{q}}\|u\|^{\bar{q}}_{\bar{q}} \geq \left(\frac{1}{2} - \frac{\mu}{\bar{q}}C_{\bar{q},s}a^{\frac{4s}{3}}\right)\|u\|^2 > 0,$$

by the condition (4.1) and the fractional Gagliardo-Nirenberg-Sobolev inequality (1.16). From (4.4), we know that Ψ^μ_u has a unique critical point t_u, which is a strict maximum point at positive level. Moreover, if $u \in \mathcal{N}_{a,\mu}$, then $t_u = 0$, and is a maximum point such that $(\Psi^\mu_u)''(0) \leq 0$. By virtue of $\mathcal{N}^0_{a,\mu} = \emptyset$, we have $(\Psi^\mu_u)''(0) < 0$. Thus, $\mathcal{N}_{a,\mu} = \mathcal{N}^-_{a,\mu}$. The smoothness of the map $u \in S_a \mapsto t_u \in \mathbb{R}$ can be deduced by applying the implicit function theorem as in Lemma 3.3. Finally, since $(\Psi^\mu_u)'(t) < 0$ if and only if $t > t_u$, we get $P_\mu(u) = (\Psi^\mu_u)'(0) < 0$ if and only if $t_u < 0$. □

Lemma 4.3 *Under the condition* (4.1)*, then $m_{a,\mu} = \inf_{\mathcal{N}_{a,\mu}} I_\mu > 0$.*

Proof Let $u \in \mathcal{N}_{a,\mu}$, then $P_\mu(u) = 0$, and by (1.14), the fractional Gagliardo–Nirenberg–Sobolev inequality (1.16) and Lemma 2.1, we derive that

$$\begin{aligned}\|u\|^2 &= \mu\frac{2}{\bar{q}}\|u\|^{\bar{q}}_{\bar{q}} + \|u\|^{2^*_s}_{2^*_s} + \int_{\mathbb{R}^3}\phi_u|u|^{2^*_s-1}dx\\ &\leq \mu\frac{2}{\bar{q}}C_{\bar{q},s}a^{\frac{4s}{3}}\|u\|^2 + S^{-\frac{2^*_s}{2}}\|u\|^{2^*_s} + S^{-2^*_s}\|u\|^{2(2^*_s-1)}.\end{aligned}$$

Combining this with (4.1), we have

$$S^{\frac{-2_s^*}{2}}\|u\|^{2_s^*}+S^{-2_s^*}\|u\|^{2(2_s^*-1)}\geq\left(1-\mu\frac{2}{\bar{q}}C_{\bar{q},s}a^{\frac{4s}{3}}\right)\|u\|^2\Rightarrow\|u\|>0,\tag{4.5}$$

which implies $u\not\equiv 0$. Moreover, by $P_\mu(u)=0$, we infer to

$$I_\mu(u)=\frac{2_s^*-2}{22_s^*}\|u\|_{2_s^*}^{2_s^*}+\frac{2_s^*-2}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx>0,$$

and hence,

$$m_{a,\mu}=\inf_{\mathcal{N}_{a,\mu}}I_\mu>0.$$

□

Lemma 4.4 *There exists $k>0$ sufficiently small, such that $0<\sup_{\overline{A}_k}I_\mu<m_{a,\mu}$. Moreover,*

$$u\in\overline{A}_k\Rightarrow I_\mu(u)>0\quad\textit{and}\quad P_\mu(u)>0,$$

where $A_k=\{u\in S_a:\ \|u\|<k\}$.

Proof By (1.14), the fractional Gagliardo–Nirenberg–Sobolev inequality (1.16) and Lemma 2.1, we have that

$$I_\mu(u)\geq\left(\frac{1}{2}-\frac{\mu}{\bar{q}}C_{\bar{q},s}a^{\frac{4s}{3}}\right)\|u\|^2-\frac{1}{2(2_s^*-1)}S^{-2_s^*}\|u\|^{2(2_s^*-1)}-\frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}>0,$$

and then

$$\begin{aligned}P_\mu(u)&=s\|u\|^2-s\mu\gamma_{\bar{q},s}\|u\|_{\bar{q}}^{\bar{q}}-s\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx-s\|u\|_{2_s^*}^{2_s^*}\\&\geq s\left(1-\frac{2\mu}{\bar{q}}C_{\bar{q},s}a^{\frac{4s}{3}}\right)\|u\|^2-sS^{-2_s^*}\|u\|^{2(2_s^*-1)}-sS^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}>0,\end{aligned}$$

having assumed $u\in\overline{A}_k$ with k small enough. By Lemma 4.3, we see that $m_{a,\mu}>0$, thus if k is small enough, we also have that

$$I_\mu(u)\leq\frac{1}{2}\|u\|^2<m_{a,\mu}.$$

□

In what follows, we shall use Proposition 2.3 to recover compactness. To this aim, we need an estimate from above for the value $m_{r,a,\mu}:=\inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}}I_\mu$, where $S_{r,a}$ is the subset of the radial functions in S_a.

We recall that the minimizer for S in (1.14) is given by the function

$$U_{\varepsilon,z}(x):=C\left(\frac{\varepsilon}{\varepsilon^2+|x-z|^2}\right)^{\frac{3-2s}{2}},\tag{4.6}$$

where $\varepsilon>0$, $C>0$ and $z\in\mathbb{R}^3$ is any fixed point (e.g. [38]).

Lemma 4.5 ([19]) *Let $A,B,C>0$ and define $g:[0,\infty)\to\mathbb{R}$ by*

$$g(t)=\frac{A}{2}t^2-\frac{B}{2(2_s^*-1)}t^{2(2_s^*-1)}-\frac{C}{2_s^*}t^{2_s^*}.$$

Then

$$\sup_{t\geq 0} g(t)=\left(\frac{\sqrt{C^2+4AB}-C}{2B}\right)^{\frac{2}{2_s^*-2}}\frac{(2_s^*-2)(22_s^*AB+C^2-C\sqrt{C^2+4AB})}{4(2_s^*-1)2_s^*B}.$$

Lemma 4.6 *Assume that condition (4.1) holds, then*

$$m_{r,a,\mu}<\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}\frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)}S^{\frac{3}{2s}}.$$

Proof Let $\eta(x)\in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\eta\in[0,1]$, $\eta\equiv 1$ on $B_1(0)$ and $\eta\equiv 0$ on $\mathbb{R}^3\backslash B_2(0)$. We define

$$u_\varepsilon(x)=\eta(x)U_\varepsilon(x),\quad v_\varepsilon=a\frac{u_\varepsilon}{\|u_\varepsilon\|_2},\tag{4.7}$$

where $U_\varepsilon(x)$ is given in (4.6) by taking $z=0$, the origin point. By [38], we can derive the following estimations:

$$\|u_\varepsilon\|^2=\iint_{\mathbb{R}^6}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{3+2s}}dxdy\leq S^{\frac{3}{2s}}+O(\varepsilon^{3-2s}),\tag{4.8}$$

$$\int_{\mathbb{R}^3}u_\varepsilon^2dx=\begin{cases}C\varepsilon^{2s}+O(\varepsilon^{3-2s}), & \text{if } 0<s<\frac{3}{4};\\ C\varepsilon^{2s}\log(\frac{1}{\varepsilon}), & \text{if } s=\frac{3}{4};\\ C\varepsilon^{3-2s}+O(\varepsilon^{2s}), & \text{if } \frac{3}{4}<s<1,\end{cases}\tag{4.9}$$

$$\int_{\mathbb{R}^3}|u_\varepsilon|^{2_s^*}dx=S^{\frac{3}{2s}}+O(\varepsilon^3)\tag{4.10}$$

and

$$\int_{\mathbb{R}^3}|u_\varepsilon|^qdx=C\varepsilon^{3-\frac{3-2s}{2}q}+O(\varepsilon^{\frac{3-2s}{2}q})=O(\varepsilon^{3-\frac{3-2s}{2}q}),\ \ 2+\frac{4s}{3}\leq q<2_s^*.\tag{4.11}$$

Since $v_\varepsilon\in C_0^\infty(\mathbb{R}^3)$, $v_\varepsilon\in S_{r,a}$ and Lemma 4.2, we know that

$$m_{r,a,\mu}=\inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}}I_\mu\leq I_\mu(t_{v_\varepsilon}\star v_\varepsilon)=\max_{t\in\mathbb{R}}I_\mu(t\star v_\varepsilon).$$

Next, we focus on an upper estimate of

$$I_\mu(t_{v_\varepsilon}\star v_\varepsilon)=\max_{t\in\mathbb{R}}I_\mu(t\star v_\varepsilon),$$

and split the argument into three steps:
Step 1. Consider the case $\mu=0$ and estimate

$$\max_{t\in\mathbb{R}}\Psi_{v_\varepsilon}^0(t)=I_0(t\star v_\varepsilon).$$

In view of (2.6), we have that

$$\Psi_{v_\varepsilon}^0(t)=I_0(t\star v_\varepsilon)=\frac{e^{2st}}{2}\|v_\varepsilon\|^2-\frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx-\frac{e^{2_s^*st}}{2_s^*}\|v_\varepsilon\|_{2_s^*}^{2_s^*}.\tag{4.12}$$

A direct computation shows that for each $v_\varepsilon \in S_a$ the function $\Psi^0_{v_\varepsilon}(t)$ has a unique critical point $t_{v_\varepsilon,0}$, which is a strict maximum point given by

$$e^{st_{v_\varepsilon,0}} = \left(\frac{-\|v_\varepsilon\|^{2^*_s}_{2^*_s} + \sqrt{\|v_\varepsilon\|^{22^*_s}_{2^*_s} + 4\|v_\varepsilon\|^2 \int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx}}{2\int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx} \right)^{\frac{1}{2^*_s-2}}. \tag{4.13}$$

Applying (4.8)–(4.10) and the definition of v_ε, we have that

$$\begin{aligned}
\Psi^0_{v_\varepsilon}(t_{v_\varepsilon,0}) &= \frac{1}{2}\|v_\varepsilon\|^2 \left(\frac{-\|v_\varepsilon\|^{2^*_s}_{2^*_s} + \sqrt{\|v_\varepsilon\|^{22^*_s}_{2^*_s} + 4\|v_\varepsilon\|^2 \int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx}}{2\int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx} \right)^{\frac{2}{2^*_s-2}} \\
&- \frac{1}{2(2^*_s-1)} \int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx \left(\frac{-\|v_\varepsilon\|^{2^*_s}_{2^*_s} + \sqrt{\|v_\varepsilon\|^{22^*_s}_{2^*_s} + 4\|v_\varepsilon\|^2 \int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx}}{2\int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx} \right)^{\frac{2(2^*_s-1)}{2^*_s-2}} \\
&- \frac{1}{2^*_s}\|v_\varepsilon\|^{2^*_s}_{2^*_s} \left(\frac{-\|v_\varepsilon\|^{2^*_s}_{2^*_s} + \sqrt{\|v_\varepsilon\|^{22^*_s}_{2^*_s} + 4\|v_\varepsilon\|^2 \int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx}}{2\int_{\mathbb{R}^3} \phi_{v_\varepsilon}|v_\varepsilon|^{2^*_s-1}dx} \right)^{\frac{2^*_s}{2^*_s-2}} \\
&= \frac{1}{2}\|u_\varepsilon\|^2 \left(\frac{-\|u_\varepsilon\|^{2^*_s}_{2^*_s} + \sqrt{\|u_\varepsilon\|^{22^*_s}_{2^*_s} + 4\|u_\varepsilon\|^2 \int_{\mathbb{R}^3} \phi_{u_\varepsilon}|u_\varepsilon|^{2^*_s-1}dx}}{2\int_{\mathbb{R}^3} \phi_{u_\varepsilon}|u_\varepsilon|^{2^*_s-1}dx} \right)^{\frac{2}{2^*_s-2}} \\
&- \frac{1}{2(2^*_s-1)} \int_{\mathbb{R}^3} \phi_{u_\varepsilon}|u_\varepsilon|^{2^*_s-1}dx \left(\frac{-\|u_\varepsilon\|^{2^*_s}_{2^*_s} + \sqrt{\|u_\varepsilon\|^{22^*_s}_{2^*_s} + 4\|u_\varepsilon\|^2 \int_{\mathbb{R}^3} \phi_{u_\varepsilon}|u_\varepsilon|^{2^*_s-1}dx}}{2\int_{\mathbb{R}^3} \phi_{u_\varepsilon}|u_\varepsilon|^{2^*_s-1}dx} \right)^{\frac{2(2^*_s-1)}{2^*_s-2}} \\
&- \frac{1}{2^*_s}\|u_\varepsilon\|^{2^*_s}_{2^*_s} \left(\frac{-\|u_\varepsilon\|^{2^*_s}_{2^*_s} + \sqrt{\|u_\varepsilon\|^{22^*_s}_{2^*_s} + 4\|u_\varepsilon\|^2 \int_{\mathbb{R}^3} \phi_{u_\varepsilon}|u_\varepsilon|^{2^*_s-1}dx}}{2\int_{\mathbb{R}^3} \phi_{u_\varepsilon}|u_\varepsilon|^{2^*_s-1}dx} \right)^{\frac{2^*_s}{2^*_s-2}} \\
&= \sup_{t\geq 0} \Psi^0_{u_\varepsilon}(t) = \Psi^0_{u_\varepsilon}(t_{u_\varepsilon,0}),
\end{aligned} \tag{4.14}$$

where $\Psi^0_{u_\varepsilon}(t)$ and $e^{t_{u_\varepsilon,0}}$ are given in (4.12) and (4.13), respectively, replacing v_ε by u_ε.

Now, we claim that

$$\sup_{t\geq 0} \Psi^0_{u_\varepsilon}(t) \leq \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}). \tag{4.15}$$

In fact, recalling that $(-\Delta)^s \phi_{u_\varepsilon} = |u_\varepsilon|^{2^*_s-1}$ there holds

$$\begin{aligned}
\int_{\mathbb{R}^3} |u_\varepsilon|^{2^*_s} dx &= \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \phi_{u_\varepsilon} (-\Delta)^{\frac{s}{2}} |u_\varepsilon| dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} |u_\varepsilon||^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} \phi_{u_\varepsilon}|^2 dx \\
&= \frac{1}{2} \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^{2^*_s-1} dx + \frac{1}{2} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx.
\end{aligned}$$

Then, thanks to (4.8) and (4.10) we derive that, for $\varepsilon > 0$ sufficiently small,

$$\int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^{2_s^*-1} dx \geq 2 \int_{\mathbb{R}^3} |u_\varepsilon|^{2_s^*} dx - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx = S^{\frac{3}{2s}} - O(\varepsilon^{3-2s}), \tag{4.16}$$

from which, together with Lemma 4.5 we derive to

$$\begin{aligned} \Psi_{u_\varepsilon}^0(t) &= \frac{e^{2st}}{2} \|u_\varepsilon\|^2 - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^{2_s^*-1} dx - \frac{e^{2_s^* st}}{2_s^*} \|u_\varepsilon\|_{2_s^*}^{2_s^*} \\ &\leq \frac{e^{2st}}{2} (S^{\frac{3}{2s}} + O(\varepsilon^{3-2s})) - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)} (S^{\frac{3}{2s}} - O(\varepsilon^{3-2s})) - \frac{e^{2_s^* st}}{2_s^*} (S^{\frac{3}{2s}} + O(\varepsilon^3)) \\ &\leq \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} \frac{s \left(12 + (1-\sqrt{5})(3-2s) \right)}{6(3+2s)} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \end{aligned}$$

for $\varepsilon > 0$ sufficiently small, and the claim is checked.

Step 2. Estimate on $t_{\varepsilon,\mu}$. Here $t_{\varepsilon,\mu} := t_{v_\varepsilon,\mu}$ denotes the unique maximum point of the function

$$\begin{aligned} \Psi_{v_\varepsilon}^\mu(t) = I_\mu(t \star v_\varepsilon) = \frac{e^{2st}}{2} \|v_\varepsilon\|^2 - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx - \mu \frac{e^{\bar{q}\gamma_{\bar{q},s} st}}{\bar{q}} \|v_\varepsilon\|_{\bar{q}}^{\bar{q}} \\ - \frac{e^{2_s^* st}}{2_s^*} \|v_\varepsilon\|_{2_s^*}^{2_s^*}. \end{aligned}$$

Since $(\Psi_{v_\varepsilon}^\mu)'(t) = P_\mu(t_{\varepsilon,\mu} \star v_\varepsilon) = 0$, we have

$$\|v_\varepsilon\|^2 - \frac{2\mu}{\bar{q}} \|v_\varepsilon\|_{\bar{q}}^{\bar{q}} - e^{2(2_s^*-2)st_{\varepsilon,\mu}} \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx - e^{(2_s^*-2)st_{\varepsilon,\mu}} \|v_\varepsilon\|_{2_s^*}^{2_s^*} = 0,$$

which implies that

$$\begin{aligned} e^{(2_s^*-2)st_{\varepsilon,\mu}} &= \frac{-\|v_\varepsilon\|_{2_s^*}^{2_s^*} + \sqrt{\|v_\varepsilon\|_{2_s^*}^{22_s^*} + 4 \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx (\|v_\varepsilon\|^2 - \frac{2\mu}{\bar{q}} \|v_\varepsilon\|_{\bar{q}}^{\bar{q}})}}{2 \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx} \\ &\geq \frac{-\|v_\varepsilon\|_{2_s^*}^{2_s^*} + \sqrt{\|v_\varepsilon\|_{2_s^*}^{22_s^*} + 4 \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx \left(\|v_\varepsilon\|^2 - \frac{2\mu}{\bar{q}} C_{\bar{q},s} a^{\frac{4s}{3}} \|v_\varepsilon\|^2 \right)}}{2 \int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx}. \end{aligned}$$

Step 3. Estimate on $\sup_{t\in\mathbb{R}} \Psi^{\mu}_{v_\varepsilon}$. By Step 1, Step 2 and the definition of v_ε, we obtain that

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \Psi^{\mu}_{v_\varepsilon}(t) &= \Psi^{\mu}_{v_\varepsilon}(t_{\varepsilon,\mu}) = \Psi^{0}_{v_\varepsilon}(t_{\varepsilon,\mu}) - \frac{\mu}{\bar{q}} e^{2st_{\varepsilon,\mu}} \|v_\varepsilon\|_{\bar{q}}^{\bar{q}} \\
&\le \sup_{t\in\mathbb{R}} \Psi^{0}_{v_\varepsilon}(t) - \|v_\varepsilon\|_{\bar{q}}^{\bar{q}} \\
&\quad \times \frac{\mu}{\bar{q}} \left(\frac{-\|v_\varepsilon\|_{2_s^*}^{2_s^*} + \sqrt{\|v_\varepsilon\|_{2_s^*}^{22_s^*} + 4\int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx \left(\|v_\varepsilon\|^2 - \frac{2\mu}{\bar{q}} C_{\bar{q},s} a^{\frac{4s}{3}} \|v_\varepsilon\|^2 \right)}}{2\int_{\mathbb{R}^3} \phi_{v_\varepsilon} |v_\varepsilon|^{2_s^*-1} dx} \right)^{\frac{2}{2_s^*-2}} \\
&= \sup_{t\in\mathbb{R}} \Psi^{0}_{u_\varepsilon}(t) - \frac{a^{\frac{4s}{3}} \|u_\varepsilon\|_{\bar{q}}^{\bar{q}}}{\|u_\varepsilon\|_2^{\frac{4s}{3}}} \\
&\quad \times \frac{\mu}{\bar{q}} \left(\frac{-\|u_\varepsilon\|_{2_s^*}^{2_s^*} + \sqrt{\|u_\varepsilon\|_{2_s^*}^{22_s^*} + 4\int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^{2_s^*-1} dx \left(\|u_\varepsilon\|^2 - \frac{2\mu}{\bar{q}} C_{\bar{q},s} a^{\frac{4s}{3}} \|u_\varepsilon\|^2 \right)}}{2\int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^{2_s^*-1} dx} \right)^{\frac{2}{2_s^*-2}} \\
&\le \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) - C_{a,\mu,s} \frac{\|u_\varepsilon\|_{\bar{q}}^{\bar{q}}}{\|u_\varepsilon\|_2^{\frac{4s}{3}}},
\end{aligned} \tag{4.17}$$

where $C_{a,\mu,s} > 0$ is a positive constant independent of ε, and we have used the following estimates:

$$\frac{1}{C_1} \le \|u_\varepsilon\|^2 \le C_1, \quad \frac{1}{C_2} \le \|u_\varepsilon\|_{2_s^*}^{2_s^*} \le C_2, \quad \frac{1}{C_3} \le \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^{2_s^*-1} dx \le C_3 \tag{4.18}$$

and

$$\frac{1}{C_4} \le \frac{-\|u_\varepsilon\|_{2_s^*}^{2_s^*} + \sqrt{\|u_\varepsilon\|_{2_s^*}^{22_s^*} + 4\|u_\varepsilon\|^2 \int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^{2_s^*-1} dx}}{2\int_{\mathbb{R}^3} \phi_{u_\varepsilon} |u_\varepsilon|^{2_s^*-1} dx} \le C_4, \tag{4.19}$$

which are deduced from (4.8), (4.10) and Lemma 2.1. Moreover, by (4.9) and (4.11), we see that

$$\frac{\|u_\varepsilon\|_{\bar{q}}^{\bar{q}}}{\|u_\varepsilon\|_2^{\frac{4s}{3}}} = \begin{cases} C\varepsilon^{\frac{4s^2}{3} - \frac{4s^2}{3}} = C, & \text{if } 0 < s < \frac{3}{4}; \\ C\varepsilon^{\frac{4s^2}{3} - \frac{4s^2}{3}} |\ln \varepsilon|^{-\frac{1}{2}} = C|\ln \varepsilon|^{-\frac{1}{2}}, & \text{if } s = \frac{3}{4}; \\ C\varepsilon^{\frac{4s^2}{3} - \frac{2s(3-2s)}{3}} = C\varepsilon^{\frac{2s(4s-3)}{3}}, & \text{if } \frac{3}{4} < s < 1. \end{cases}$$

In particular, any term of order $O(\varepsilon)$ is negligible with respect to this ratio for ε small, and hence coming back to (4.17) we deduce that

$$\sup_{t\in\mathbb{R}} \Psi^{\mu}_{v_\varepsilon}(t) < \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}},$$

for any ε small enough. Therefore, we have

$$m_{r,a,\mu} = \inf_{\mathcal{N}_{a,\mu} \cap S_{r,a}} I_\mu \le \max_{t\in\mathbb{R}} \Psi^{\mu}_{v_\varepsilon}(t) < \left(\frac{\sqrt{5}-1}{2} \right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}},$$

which in turn gives the thesis of the lemma. □

5 L^2-supercritical perturbation

In this section, we deal with the L^2-supercritical case $\bar{q} := 2 + \frac{4s}{3} < q < 2_s^*$. To begin our argument, we consider once again the Pohozaev manifold $\mathcal{N}_{a,\mu}$, which can be decomposed as

$$\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^+ \cup \mathcal{N}_{a,\mu}^0 \cup \mathcal{N}_{a,\mu}^-. \tag{5.1}$$

Lemma 5.1 $\mathcal{N}_{a,\mu}^0 = \emptyset$ *and* $\mathcal{N}_{a,\mu}$ *is a smooth manifold of dimension 2 in* $H^s(\mathbb{R}^3)$.

Proof Suppose on the contrary that, there exists some $u \in \mathcal{N}_{a,\mu}^0$, then

$$\|u\|^2 = \mu\gamma_{q,s}\|u\|_q^q + \|u\|_{2_s^*}^{2_s^*} + \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \tag{5.2}$$

and

$$2\|u\|^2 = \mu q\gamma_{q,s}^2\|u\|_q^q + 2_s^*\|u\|_{2_s^*}^{2_s^*} + 2(2_s^*-1)\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx, \tag{5.3}$$

which leads to

$$(2 - q\gamma_{q,s})\mu\gamma_{q,s}\|u\|_q^q = (2_s^* - 2)\|u\|_{2_s^*}^{2_s^*} + 2(2_s^*-2)\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx.$$

Since $2 - q\gamma_q < 0$ and $2_s^* - 2 > 0$, we have $u \equiv 0$, but this is not possible, thanks to $u \in S_a$. The rest of the proof is similar to that of Lemma 3.2, and we omit the details here. □

Lemma 5.2 *For each* $u \in S_a$*, there exists a unique* $t_u \in \mathbb{R}$ *such that* $t_u \star u \in \mathcal{N}_{a,\mu}$*, where* t_u *is the unique critical point of the function of* Ψ_u^μ *and is a strict maximum point at positive level. Moreover,*

(i) $\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^-$;
(ii) $\Psi_u^\mu(t)$ *is strict decreasing and concave on* $(t_u, +\infty)$*, and* $t_u < 0$ *implies that* $P_\mu(u) < 0$;
(iii) *The map* $u \in S_a \mapsto t_u \in \mathbb{R}$ *is of class* C^1;
(iv) *If* $P_\mu(u) < 0$*, then* $t_u < 0$.

Proof In view of

$$\Psi_u^\mu(t) = I_\mu(t\star u) = \frac{e^{2st}}{2}\|u\|^2 - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx - \mu\frac{e^{q\gamma_{q,s}st}}{q}\|u\|_q^q - \frac{e^{2_s^*st}}{2_s^*}\|u\|_{2_s^*}^{2_s^*},$$

and

$$(\Psi_u^\mu)'(t) = se^{2st}\|u\|^2 - se^{2(2_s^*-1)st}\int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx - \mu s\gamma_{q,s}e^{q\gamma_{q,s}st}\|u\|_q^q - se^{2_s^*st}\|u\|_{2_s^*}^{2_s^*},$$

it is easy to see that $(\Psi_u^\mu)'(t) = 0$ if and only if

$$\|u\|^2 = e^{2(2_s^*-2)st}\int_{\mathbb{R}^3} \phi_u|u|^{2_s^*-1}dx + \mu\gamma_{q,s}e^{(q\gamma_{q,s}-2)st}\|u\|_q^q + e^{(2_s^*-2)st}\|u\|_{2_s^*}^{2_s^*} \triangleq g(t). \tag{5.4}$$

Clearly, $g(t)$ is positive, continuous and monotone increasing, and $g(t) \to 0^+$ as $t \to -\infty$ and $g(t) \to +\infty$ as $t \to +\infty$. Therefore, there exists a unique point t_u such that $t_u\star u \in \mathcal{N}_{a,\mu}$, where t_u is the unique critical point of $\Psi_u^\mu(t)$ and is a strict maximum point at positive level. By maximality, we have that $(\Psi_u^\mu)''(t_u) \leq 0$, and since $\mathcal{N}_{a,\mu}^0 = \emptyset$, we conclude that $t_u \star u \in \mathcal{N}_{a,\mu}^-$, and $\mathcal{N}_{a,\mu} = \mathcal{N}_{a,\mu}^-$ since $\Psi_u^\mu(t)$ has exactly one maximum point. To show that the map $u \in S_a \mapsto t_u \in \mathbb{R}$ is of class C^1, we can apply the implicit function theorem as in Lemma 3.3. Finally, since $(\Psi_u^\mu)'(t) < 0$ if and only if $t > t_u$, so $P_\mu(u) = (\Psi_u^\mu)'(0) < 0$ if and only if $t_u < 0$. □

Lemma 5.3 $m_{a,\mu} = \inf_{\mathcal{N}_{a,\mu}} I_\mu > 0$.

Proof If $u \in \mathcal{N}_{a,\mu}$, then by (1.14), the fractional Gagliardo–Nirenberg–Sobolev inequality (1.16) and Lemma 2.1, we have that

$$\begin{aligned} \|u\|^2 &= \mu\gamma_{q,s}\|u\|_q^q + \|u\|_{2_s^*}^{2_s^*} + \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx \\ &\leq \mu\gamma_{q,s} C_{q,s} a^{q(1-\gamma_{q,s})}\|u\|^{q\gamma_{q,s}} + S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*} + S^{-2_s^*}\|u\|^{2(2_s^*-1)}. \end{aligned}$$

Dividing by $\|u\|^2$, we can deduce that

$$\mu\gamma_{q,s} C_{q,s} a^{q(1-\gamma_{q,s})}\|u\|^{q\gamma_{q,s}-2} + S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*-2} + S^{-2_s^*}\|u\|^{2(2_s^*-2)} \geq 1, \quad \forall u \in \mathcal{N}_{a,\mu},$$

which implies that $\inf_{u\in\mathcal{N}_{a,\mu}} \|u\| > 0$ and so,

$$\inf_{u\in\mathcal{N}_{a,\mu}} \left[\mu\gamma_{q,s}\|u\|_q^q + \|u\|_{2_s^*}^{2_s^*} + \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx\right] > 0. \tag{5.5}$$

Thus, from (5.5), $P_\mu(u) = 0$ and the fact $q\gamma_{q,s} > 2$, we obtain that

$$\begin{aligned} \inf_{u\in\mathcal{N}_{a,\mu}} I_\mu(u) &= \inf_{u\in\mathcal{N}_{a,\mu}} \left[\frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{\mu}{q}\|u\|_q^q - \frac{1}{2_s^*}\|u\|_{2_s^*}^{2_s^*}\right] \\ &= \inf_{u\in\mathcal{N}_{a,\mu}} \left[\frac{\mu}{q}\left(\frac{q\gamma_{q,s}}{2} - 1\right)\|u\|_q^q + \frac{2_s^*-2}{22_s^*}\|u\|_{2_s^*}^{2_s^*} + \frac{2_s^*-2}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx\right] \\ &> 0. \end{aligned}$$

□

Lemma 5.4 *There exists $k > 0$ sufficiently small, such that $0 < \sup_{\overline{A_k}} I_\mu < m_{a,\mu}$. Moreover,*

$$u \in \overline{A_k} \Rightarrow I_\mu(u) > 0 \quad \textit{and} \quad P_\mu(u) > 0,$$

where $A_k = \{u \in S_a : \|u\| < k\}$.

Proof By (1.14), the fractional Gagliardo–Nirenberg–Sobolev inequality (1.16), Lemma 2.1 and $q\gamma_{q,s} > 2$, we have that

$$I_\mu(u) \geq \frac{1}{2}\|u\|^2 - \frac{\mu}{q} C_{q,s} a^{q(1-\gamma_{q,s})}\|u\|^{q\gamma_{q,s}} - \frac{1}{2(2_s^*-1)} S^{-2_s^*}\|u\|^{2(2_s^*-1)} - \frac{1}{2_s^*} S^{-\frac{2_s^*}{2}}\|u\|^{2_s^*} > 0,$$

and

$$\begin{aligned} P_\mu(u) &= s\|u\|^2 - s\mu\gamma_{q,s}\|u\|_q^q - s\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - s\|u\|_{2_s^*}^{2_s^*} \\ &\geq s\|u\|^2 - s\mu\gamma_{q,s} C_{q,s} a^{q(1-\gamma_{q,s})}\|u\|^{q\gamma_{q,s}} - sS^{-2_s^*}\|u\|^{2(2_s^*-1)} - sS^{-\frac{2_s^*}{2}}\|u\|^{2_s^*} > 0, \end{aligned}$$

if $u \in \overline{A_k}$ with k small enough. By Lemma 5.3, we see that $m_{a,\mu} > 0$, thus if necessary replacing k with a smaller quantity, we also have that

$$I_\mu(u) \le \frac{1}{2}\|u\|^2 < m_{a,\mu}.$$

□

As in the previous section, the following estimate will play a crucial role in the proof of existence of a ground state. Let $m_{r,a,\mu} := \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu$, where $S_{r,a}$ is the subset of the radial functions in S_a.

Lemma 5.5 *If one of the following conditions is satisfied:*

(i) $0 < s < \frac{3}{4}$ *and* $\mu a^{q(1-\gamma_{q,s})} < \frac{1}{\gamma_{q,s}} \left(\frac{\sqrt{5}-1}{2}\right)^{-\frac{q\gamma_{q,s}-2}{2_s^*-2}} S^{\frac{3(2_s^*-q)}{2s(2_s^*-2)}}$;

(ii) $\frac{3}{4} \le s < 1$,

then we have $m_{r,a,\mu} < \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}}$.

Proof The structure of the proof is similar to that of Lemma 4.6, but we took advantage of the fact that $q\gamma_{q,s} > 2$ in order to make direct computations in several steps. Let us recall the definition of u_ε and v_ε given in Lemma 4.6, we know that $u_\varepsilon \in C_0^\infty(\mathbb{R}^3, [0, 1])$ and $v_\varepsilon \in S_{r,a}$. By Lemma 5.3, we have that

$$m_{r,a,\mu} = \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu \le I_\mu(t_{v_\varepsilon,\mu}\star v_\varepsilon) = \max_{t\in\mathbb{R}} I_\mu(t\star v_\varepsilon).$$

From the Step 1 of Lemma 4.6, we get that

$$\Psi^0_{v_\varepsilon}(t_{v_\varepsilon,0}) \le \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}). \tag{5.6}$$

Step 1. Estimate on $t_{\varepsilon,\mu}$. Let $t_{\varepsilon,\mu} := t_{v_\varepsilon,\mu}$ be the maximum point of

$$\begin{aligned}\Psi^\mu_{v_\varepsilon}(t) &:= I_\mu(t\star v_\varepsilon)\\ &= \frac{e^{2st}}{2}\|v_\varepsilon\|^2 - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx - \mu\frac{e^{q\gamma_{q,s}st}}{q}\|v_\varepsilon\|_q^q - \frac{e^{2_s^*st}}{2_s^*}\|v_\varepsilon\|_{2_s^*}^{2_s^*}.\end{aligned}$$

By $(\Psi^\mu_{v_\varepsilon})'(t_{\varepsilon,\mu}) = P_\mu(t_{\varepsilon,\mu}\star v_\varepsilon) = 0$, we have that

$$\begin{aligned}e^{2(2_s^*-1)st_{\varepsilon,\mu}}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx + e^{2_s^*st_{\varepsilon,\mu}}\|v_\varepsilon\|_{2_s^*}^{2_s^*} &= e^{2st_{\varepsilon,\mu}}\|v_\varepsilon\|^2 - \mu\gamma_{q,s}e^{q\gamma_{q,s}st_{\varepsilon,\mu}}\|v_\varepsilon\|_q^q\\ &\le e^{2st_{\varepsilon,\mu}}\|v_\varepsilon\|^2,\end{aligned}$$

whence it follows that

$$e^{st_{\varepsilon,\mu}} \le \left(\frac{-\|v_\varepsilon\|_{2_s^*}^{2_s^*} + \sqrt{\|v_\varepsilon\|_{2_s^*}^{22_s^*} + 4\|v_\varepsilon\|^2\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}}{2\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}\right)^{\frac{1}{2_s^*-2}}. \tag{5.7}$$

By virtue of (5.7), $P_\mu(t_{\varepsilon,\mu}\star v_\varepsilon)=0$ and the fact $q\gamma_{q,s}>2$, we infer that

$$
\begin{aligned}
&e^{2(2_s^*-2)st_{\varepsilon,\mu}}+e^{(2_s^*-2)st_{\varepsilon,\mu}}\frac{\|v_\varepsilon\|_{2_s^*}^{2_s^*}}{\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}\\
&=\frac{\|v_\varepsilon\|^2}{\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}-\mu\gamma_{q,s}e^{(q\gamma_{q,s}-2)st_{\varepsilon,\mu}}\frac{\|v_\varepsilon\|_q^q}{\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}\\
&\geq\frac{\|v_\varepsilon\|^2}{\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}-\mu\gamma_{q,s}\left(\frac{-\|v_\varepsilon\|_{2_s^*}^{2_s^*}+\sqrt{\|v_\varepsilon\|_{2_s^*}^{22_s^*}+4\|v_\varepsilon\|^2\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}}{2\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}\right)^{\frac{q\gamma_{q,s}-2}{2_s^*-2}}\\
&\quad\times\frac{\|v_\varepsilon\|_q^q}{\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^{2_s^*-1}dx}.
\end{aligned}\tag{5.8}
$$

By the definition of v_ε, we have that

$$
\begin{aligned}
&e^{2(2_s^*-2)st_{\varepsilon,\mu}}+e^{(2_s^*-2)st_{\varepsilon,\mu}}\frac{\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}\frac{\|u_\varepsilon\|_{2_s^*}^{2_s^*}}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\\
&\geq\frac{\|u_\varepsilon\|_2^{2(2_s^*-2)}}{a^{2(2_s^*-2)}}\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}-\mu\gamma_{q,s}\frac{\|u_\varepsilon\|_2^{2(2_s^*-2)+q\gamma_{q,s}-q}}{a^{2(2_s^*-2)+q\gamma_{q,s}-q}}\frac{\|u_\varepsilon\|_q^q}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\\
&\quad\times\left(\frac{-\|u_\varepsilon\|_{2_s^*}^{2_s^*}+\sqrt{\|u_\varepsilon\|_{2_s^*}^{22_s^*}+4\|u_\varepsilon\|^2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}}{2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\right)^{\frac{q\gamma_{q,s}-2}{2_s^*-2}}\qquad(5.9)\\
&=\frac{\|u_\varepsilon\|_2^{2(2_s^*-2)}}{a^{2(2_s^*-2)}}\Bigg[\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}-\mu\gamma_{q,s}\frac{a^{q(1-\gamma_{q,s})}}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}\\
&\quad\times\left(\frac{-\|u_\varepsilon\|_{2_s^*}^{2_s^*}+\sqrt{\|u_\varepsilon\|_{2_s^*}^{22_s^*}+4\|u_\varepsilon\|^2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}}{2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\right)^{\frac{q\gamma_{q,s}-2}{2_s^*-2}}\Bigg].
\end{aligned}
$$

Using the estimates in (4.8)–(4.11) and Lemma 2.1, we can take constants $C_i>0$ such that

$$
C_1\leq\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx\leq\frac{1}{C_1},\quad\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\geq C_2,\quad\frac{\|u_\varepsilon\|_2^{2_s^*-2}\|u_\varepsilon\|_{2_s^*}^{2_s^*}}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\leq C_3\tag{5.10}
$$

$$
\frac{1}{C_4}\leq\left(\frac{-\|u_\varepsilon\|_{2_s^*}^{2_s^*}+\sqrt{\|u_\varepsilon\|_{2_s^*}^{22_s^*}+4\|u_\varepsilon\|^2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}}{2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\right)^{\frac{q\gamma_{q,s}-2}{2_s^*-2}}\leq C_4,\tag{5.11}
$$

and

$$
\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}\leq\begin{cases}C_5\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})}=C_5, & \text{if } 0<s<\frac{3}{4};\\ C_5\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}}, & \text{if } s=\frac{3}{4};\\ C_5\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}}, & \text{if } \frac{3}{4}<s<1.\end{cases}\tag{5.12}
$$

Next, we show that

$$e^{st_{\varepsilon,\mu}} \geq C\frac{\|u_\varepsilon\|_2}{a}. \tag{5.13}$$

under suitable conditions.

Case 1: $0 < s < \frac{3}{4}$. In this case, it holds that

$$\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})} = \varepsilon^0 = 1, \tag{5.14}$$

and from (5.10)–(5.12) we get

$$e^{2(2_s^*-2)st_{\varepsilon,\mu}}(1+C_6) \geq \frac{\|u_\varepsilon\|_2^{2(2_s^*-2)}}{a^{2(2_s^*-2)}}\left(C_2 - \mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_4}{C_1}\right),$$

and we see that inequality (5.13) holds only when $\mu\gamma_{q,s}a^{q(1-\gamma_{q,s})} \leq C_1C_2/C_4$. Thus, we have to give a more precise estimate, let us come back to (5.9) and observe that by well-known interpolation inequality, we have that

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \leq \frac{\|u_\varepsilon\|_2^{\frac{2(2_s^*-q)}{2_s^*-2}}\|u_\varepsilon\|_{2_s^*}^{\frac{2_s^*(q-2)}{2_s^*-2}}}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} = \|u_\varepsilon\|_{2_s^*}^{\frac{2_s^*(q-2)}{2_s^*-2}}. \tag{5.15}$$

Therefore, by (5.9) and (5.15), we have

$$\begin{aligned} &e^{2(2_s^*-2)st_{\varepsilon,\mu}} + e^{(2_s^*-2)st_{\varepsilon,\mu}}\frac{\|u_\varepsilon\|_2^{2_s^*-2}}{a^{2_s^*-2}}\frac{\|u_\varepsilon\|_{2_s^*}^{2_s^*}}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx} \\ &\geq \frac{\|u_\varepsilon\|_2^{2(2_s^*-2)}}{a^{2(2_s^*-2)}}\left[\frac{\|u_\varepsilon\|^2}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx} - \mu\gamma_{q,s}\frac{a^{q(1-\gamma_{q,s})}}{\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\|u_\varepsilon\|_{2_s^*}^{\frac{2_s^*(q-2)}{2_s^*-2}}\right. \\ &\left.\times\left(\frac{-\|u_\varepsilon\|_{2_s^*}^{2_s^*}+\sqrt{\|u_\varepsilon\|_{2_s^*}^{22_s^*}+4\|u_\varepsilon\|^2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}}{2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\right)^{\frac{q\gamma_{q,s}-2}{2_s^*-2}}\right]. \end{aligned} \tag{5.16}$$

From (1.14), (4.8)–(4.10), (4.16) and Lemma 2.1, we see that the right hand side of (5.16) is positive provided that

$$\begin{aligned} \mu\gamma_{q,s}a^{q(1-\gamma_{q,s})} &< \frac{\|u_\varepsilon\|^2}{\|u_\varepsilon\|_{2_s^*}^{\frac{2_s^*(q-2)}{2_s^*-2}}}\left(\frac{-\|u_\varepsilon\|_{2_s^*}^{2_s^*}+\sqrt{\|u_\varepsilon\|_{2_s^*}^{22_s^*}+4\|u_\varepsilon\|^2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}}{2\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx}\right)^{-\frac{q\gamma_{q,s}-2}{2_s^*-2}} \\ &\leq \frac{S^{\frac{3}{2s}}+O(\varepsilon^{3-2s})}{\left(S^{\frac{3}{2s}}+O(\varepsilon^3)\right)^{\frac{q-2}{2_s^*-2}}} \\ &\times\left(\frac{-(S^{\frac{3}{2s}}+O(\varepsilon^3))+\sqrt{(S^{\frac{3}{2s}}+O(\varepsilon^3))^2+4(S(S^{\frac{3}{2s}}+O(\varepsilon^3))^{\frac{2}{2_s^*}})(S^{\frac{3}{2s}}-O(\varepsilon^{3-2s}))}}{2S^{-2_s^*}(S^{\frac{3}{2s}}+O(\varepsilon^{3-2s}))^{(2_s^*-1)}}\right)^{-\frac{q\gamma_{q,s}-2}{2_s^*-2}} \\ &= \left(\frac{\sqrt{5}-1}{2}\right)^{-\frac{q\gamma_{q,s}-2}{2_s^*-2}}S^{\frac{3(2_s^*-q)}{2s(2_s^*-2)}}+O(\varepsilon^{3-2s}). \end{aligned}$$

Therefore, if $0 < s < \frac{3}{4}$ and $\mu a^{q(1-\gamma_{q,s})} < \frac{K_3}{\gamma_{q,s}}$, we have

$$e^{st_{\varepsilon,\mu}} \geq C\frac{\|u_\varepsilon\|_2}{a}.$$

Case 2: $s = \frac{3}{4}$. In this case we have $3 < q < 4$, and

$$\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}} = |\ln\varepsilon|^{\frac{q}{2}-2} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{5.17}$$

Consequently,

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \leq C_5\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}} = o_\varepsilon(1).$$

Therefore, we get

$$e^{2(2_s^*-2)st_{\varepsilon,\mu}}(1+C_6) \geq \frac{\|u_\varepsilon\|_2^{2(2_s^*-2)}}{a^{2(2_s^*-2)}}\left(C_2 - \mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_4}{C_1}o_\varepsilon(1)\right) \geq C\frac{\|u_\varepsilon\|_2^{2(2_s^*-2)}}{a^{2(2_s^*-2)}},$$

that is

$$e^{st_{\varepsilon,\mu}} \geq C\frac{\|u_\varepsilon\|_2}{a}.$$

Case 3: $\frac{3}{4} < s < 1$. By the definition of $\gamma_{q,s}$ and a direct computation, we get that

$$3 - \frac{3-2s}{2}q - \frac{(3-2s)q(1-\gamma_{q,s})}{2} = \frac{3-4s}{4s}\left(q - \frac{6}{3-2s}\right)(3-2s) > 0.$$

Thus,

$$\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

and so

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \leq C\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}} = o_\varepsilon(1).$$

Therefore, we get

$$e^{2(2_s^*-2)st_{\varepsilon,\mu}}(1+C_6) \geq \frac{\|u_\varepsilon\|_2^{2(2_s^*-2)}}{a^{2(2_s^*-2)}}\left(C_2 - \mu\gamma_{q,s}a^{q(1-\gamma_{q,s})}\frac{C_4}{C_1}o_\varepsilon(1)\right) \geq C\frac{\|u_\varepsilon\|_2^{2(2_s^*-2)}}{a^{2(2_s^*-2)}},$$

that is

$$e^{st_{\varepsilon,\mu}} \geq C\frac{\|u_\varepsilon\|_2}{a}.$$

Step 2. Estimate on $\max_{t\in\mathbb{R}} \Psi^{\mu}_{v_\varepsilon}(t)$. By Step 1 and (5.6), we have that

$$\begin{aligned}
\max_{t\in\mathbb{R}} \Psi^{\mu}_{v_\varepsilon}(t) &= \Psi^{\mu}_{v_\varepsilon}(t_{\varepsilon,\mu}) = \Psi^{0}_{v_\varepsilon}(t_{\varepsilon,\mu}) - \mu\frac{e^{q\gamma_{q,s}st_{\varepsilon,\mu}}}{q}\int_{\mathbb{R}^3}|v_\varepsilon|^q dx \\
&\le \sup_{t\in\mathbb{R}} \Psi^{0}_{v_\varepsilon}(t) - \frac{C\mu}{q}\frac{\|u_\varepsilon\|_2^{q\gamma_{q,s}}}{a^{q\gamma_{q,s}}}\frac{a^q}{\|u_\varepsilon\|_2^q}\int_{\mathbb{R}^3}|u_\varepsilon|^q dx \\
&= \sup_{t\in\mathbb{R}} \Psi^{0}_{v_\varepsilon}(t) - \frac{C\mu a^{q(1-\gamma_{q,s})}}{q}\frac{\int_{\mathbb{R}^3}|u_\varepsilon|^q dx}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \\
&\le \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}) \\
&\quad - \frac{C\mu a^{q(1-\gamma_{q,s})}}{q}\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}}.
\end{aligned} \tag{5.18}$$

Similarly as in (5.12), we have that

$$\frac{\|u_\varepsilon\|_q^q}{\|u_\varepsilon\|_2^{q(1-\gamma_{q,s})}} \ge \begin{cases} C\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})} = C, & \text{if } 0<s<\frac{3}{4}; \\ C\varepsilon^{3-\frac{3-2s}{2}q-sq(1-\gamma_{q,s})}|\ln\varepsilon|^{\frac{q(\gamma_{q,s}-1)}{2}}, & \text{if } s=\frac{3}{4}; \\ C\varepsilon^{3-\frac{3-2s}{2}q-\frac{(3-2s)q(1-\gamma_{q,s})}{2}}, & \text{if } \frac{3}{4}<s<1. \end{cases} \tag{5.19}$$

Finally, by (5.18)–(5.19), we infer to

$$m_{r,a,\mu} = \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu \le \max_{t\in\mathbb{R}} \Psi^{\mu}_{v_\varepsilon}(t) < \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}},$$

for any $\varepsilon > 0$ small enough, which is the desired result. □

6 Proof of Theorem 1.1

In this section we shall prove that for the L^2-subcritical case: $2 < q < \bar{q} := 2 + \frac{4s}{3}$, Theorem 1.1 holds, for any $a, \mu > 0$ satisfying condition (1.18), i.e.,

$$\mu a^{q(1-\gamma_{q,s})} < \widetilde{k}, \tag{6.1}$$

with $\widetilde{k} = \min\{K_1, K_2, K_3\}$, where K_1, K_2 are given in (3.2), (3.3), respectively, and

$$\begin{aligned}
K_3 := \frac{2_s^* q}{C_{q,s}(2_s^* - q\gamma_{q,s})}\left[\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}} \frac{(2_s^*-2)(22_s^*+1-\sqrt{5})}{4(2_s^*-1)2_s^*} S^{\frac{3}{2s}}\right]^{\frac{2-q\gamma_{q,s}}{2}} \\
\times\left[\frac{1}{2}\left(\frac{2_s^*}{2_s^*-2}\right)^{\frac{q\gamma_{q,s}}{2-q\gamma_{q,s}}}(q\gamma_{q,s})^{\frac{q\gamma_{q,s}}{2-q\gamma_{q,s}}}(2-q\gamma_{q,s})\right]^{-\frac{2-q\gamma_{q,s}}{2}}.
\end{aligned} \tag{6.2}$$

Let $\{v_n\}$ be a minimizing sequence for $\inf_{A_{R_0}} I_\mu$, and we may assume that $\{v_n\} \subset S_{r,a}$ is radially decreasing for every $n \in \mathbb{N}$. Otherwise, we can replace v_n with $|v_n|^*$, the Schwarz rearrangement of $|v_n|$, and we have another function in A_{R_0} with $I_\mu(|v_n|^*) \le I_\mu(|v_n|)$.

Moreover, by Lemma 3.3, for every n we may take $\alpha_{v_n}\star v_n \in \mathcal{N}^+_{a,\mu}$ such that $\|\alpha_{v_n}\star v_n\| \le R_0$ and

$$I_\mu(\alpha_{v_n}\star v_n) = \min\{I_\mu(t\star v_n) : \ t\in\mathbb{R} \ \text{ and } \ \|t\star v_n\| < R_0\} \le I_\mu(v_n).$$

In this way, we obtain a new minimizing sequence $\{w_n = \alpha_{v_n}\star v_n\}$ with $w_n \in S_{r,a}\cap\mathcal{N}^+_{a,\mu}$ radially decreasing for each n. By Lemma 3.4, we have $\|w_n\| \le R_0 - r$ for each n and hence by Ekeland's variational principle [42] in a standard way, we know that the existence of a new minimizing sequence $\{u_n\}\subset A_{R_0}$ for $m_{a,\mu}$ with the property that $\|w_n - u_n\| \to 0$ as $n\to+\infty$, which is also a Palais–Smale sequence for I_μ on S_a. Thus, from Brezis–Lieb lemma [8] and Sobolev embedding theorem, we have

$$\|u_n\|^2 = \|u_n - w_n\|^2 + \|w_n\|^2 + o_n(1) = \|w_n\|^2 + o_n(1),$$
$$\int_{\mathbb{R}^3}|u_n|^p dx = \int_{\mathbb{R}^3}|u_n - w_n|^p dx + \int_{\mathbb{R}^3}|w_n|^p dx + o_n(1) = \int_{\mathbb{R}^3}|w_n|^p dx + o_n(1),$$

for $p\in[2,2^*_s]$. Now, by $\|u_n - w_n\|\to 0$ as $n\to\infty$ and Lemma 2.1, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{u_n}|u_n|^{2^*_s-1}dx &= \int_{\mathbb{R}^3}\phi_{u_n-w_n}|u_n-w_n|^{2^*_s-1}dx + \int_{\mathbb{R}^3}\phi_{w_n}|w_n|^{2^*_s-1}dx + o_n(1)\\ &= \int_{\mathbb{R}^3}\phi_{w_n}|w_n|^{2^*_s-1}dx + o_n(1).\end{aligned}$$

Consequently, we obtain that

$$P_\mu(u_n) = P_\mu(w_n) + o_n(1) \to 0 \quad \text{as} \quad n\to+\infty.$$

Hence, one of the alternative in Proposition 2.3 occurs. We can show that the second alternative in Proposition 2.3 holds. Suppose by contradiction that, there exists a sequence $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^3)$ but not strongly, where $u\not\equiv 0$ solves problem (1.13) for some $\lambda < 0$ and

$$I_\mu(u) \le m_{a,\mu} - \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}}.$$

Since u is a solution of problem (1.13), by the Pohozaev identity $P_\mu(u) = 0$, one has

$$\|u\|^2 = \mu\gamma_{q,s}\|u\|_q^q + \|u\|_{2^*_s}^{2^*_s} + \int_{\mathbb{R}^3}\phi_u|u|^{2^*_s-1}dx.$$

Therefore, by the fractional Gagliardo–Nirenberg–Sobolev inequality (1.16), we have that

$$\begin{aligned} m_{a,\mu} &\ge I_\mu(u) + \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}}\\ &= \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + \frac{2^*_s-2}{22^*_s}\|u\|^2\\ &\quad + \frac{2^*_s-2}{22^*_s(2^*_s-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2^*_s-1}dx - \frac{\mu}{q}\left(1-\frac{q\gamma_{q,s}}{2^*_s}\right)\|u\|_q^q\\ &\ge \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + \frac{2^*_s-2}{22^*_s}\|u\|^2\\ &\quad - \frac{\mu}{q}\left(1-\frac{q\gamma_{q,s}}{2^*_s}\right)C_{q,s}a^{q(1-\gamma_{q,s})}\|u\|^{q\gamma_{q,s}}. \end{aligned} \tag{6.3}$$

Now, we show that the right side of the above inequality is positive, which shall contradicts with the fact that $m_{a,\mu} < 0$. To this aim, we define

$$h(t) := \frac{2_s^* - 2}{22_s^*} t^2 - \frac{\mu}{q} \left(1 - \frac{q\gamma_{q,s}}{2_s^*}\right) C_{q,s} a^{q(1-\gamma_{q,s})} t^{q\gamma_{q,s}}, \quad \forall\, t \geq 0.$$

Since $q\gamma_{q,s} < 2$, the function $h(t)$ has a global minimum at negative level

$$\begin{aligned} h(t_{\min}) &= \min_{t>0} h(t) \\ &= -\frac{1}{2}\left(\frac{2_s^*}{2_s^* - 2}\right)^{\frac{q\gamma_{q,s}}{2-q\gamma_{q,s}}} \left[\frac{\mu}{q}\left(1 - \frac{q\gamma_{q,s}}{2_s^*}\right) C_{q,s} a^{q(1-\gamma_{q,s})}\right]^{\frac{2}{2-q\gamma_{q,s}}} (q\gamma_{q,s})^{\frac{q\gamma_{q,s}}{2-q\gamma_{q,s}}} (2 - q\gamma_{q,s}) \\ &< 0. \end{aligned} \tag{6.4}$$

By (6.1)–(6.2), we have

$$\begin{aligned} \mu a^{q(1-\gamma_{q,s})} &< \frac{2_s^* q}{C_{q,s}(2_s^* - q\gamma_{q,s})} \left[\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}} \frac{(2_s^* - 2)(22_s^* + 1 - \sqrt{5})}{4(2_s^* - 1)2_s^*} S^{\frac{3}{2s}}\right]^{\frac{2-q\gamma_{q,s}}{2}} \\ &\quad \times \left[\frac{1}{2}\left(\frac{2_s^*}{2_s^* - 2}\right)^{\frac{q\gamma_{q,s}}{2-q\gamma_{q,s}}} (q\gamma_{q,s})^{\frac{q\gamma_{q,s}}{2-q\gamma_{q,s}}} (2 - q\gamma_{q,s})\right]^{-\frac{2-q\gamma_{q,s}}{2}} := K_3. \end{aligned} \tag{6.5}$$

Combining (6.4) and (6.5), we infer to

$$h(t_{\min}) > -\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}}.$$

Therefore, coming back to (6.3), we have that

$$\begin{aligned} m_{a,\mu} &\geq \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + h(\|u\|) \\ &\geq \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + h(t_{\min}) > 0, \end{aligned}$$

in contradiction with the fact that $m_{a,\mu} < 0$. This means that necessarily $u_n \to u$ strongly in $H^s(\mathbb{R}^3)$, $I_\mu(u) = m_{a,\mu}$ and u solves problem (1.13) for some $\lambda < 0$. It remains to show that any ground state is a local minimizer for I_μ on A_{R_0}. Using the fact that $I_\mu(u) = m_{a,\mu} < 0$, and then $u \in \mathcal{N}_{a,\mu}$, so by Lemma 3.3 we know that $u \in \mathcal{N}_{a,\mu}^+ \subset A_{R_0}$ and

$$I_\mu(u) = m_{a,\mu} = \inf_{A_{R_0}} I_\mu(u) \quad \text{with} \quad \|u\| < R_0.$$

Finally, we prove that the ground state solution is positive. Let $u^+ := \max\{u, 0\}$ and it is easy to see that all the arguments above can be repeated word by word, replacing I_μ by the functional

$$\begin{aligned} I_\mu^+(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{1}{2(2_s^* - 1)} \int_{\mathbb{R}^3} \phi_{u^+} |u^+|^{2_s^*-1} dx \\ &\quad - \frac{\mu}{q}\int_{\mathbb{R}^3} |u^+|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u^+|^{2_s^*} dx. \end{aligned} \tag{6.6}$$

Using $u^- := \min\{u, 0\}$ as a test function in (6.6), and arguing as in the proof of Proposition 3.1 [37], we have $u(x) > 0$ in $\mathbb{R}^3$. This completes the proof. □

7 Proof of Theorems 1.2 and 1.3

We first recall the following useful preliminary results, which are needed in proving Theorems 1.2 and 1.3 below.

Definition 7.1 ([16, Definition 3.1]). Let B be a closed subset of X. We shall say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with boundary B provided

(i) every set in $\mathcal{F}$ contains B;
(ii) for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times X) \cup ([0,1] \times B)$, we have that $\eta(\{1\} \times A) \in \mathcal{F}$.

Proposition 7.1 ([16, Theorem 3.2]). *Let ψ be a C^1 function on a complete connected C^1-Finsler manifold X (without boundary) and consider a homotopy-stable family $\mathcal{F}$ of compact subsets of X with a closed boundary B. Set $c = c(\psi, \mathcal{F}) = \inf_{A\in\mathcal{F}} \max_{u\in A} \psi(u)$ and suppose that*

$$\sup_{u\in B} \psi(u) < c.$$

Then, for any sequence of sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\lim_n \sup_{A_n} \psi = c$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\lim_{n\to+\infty} \psi(u_n) = c, \quad \lim_{n\to+\infty} \|d\psi(u_n)\| = 0 \quad \textit{and} \quad \lim_{n\to+\infty} dist(u_n, A_n) = 0.$$

Moreover, if $d\psi$ is uniformly continuous, then u_n can be chosen to be in A_n for each n.

Lemma 7.1 ([4, Lemma 3.6]) *For $u \in S_a$ and $t \in \mathbb{R}$ the map*

$$T_u S_a \to T_{t\star u} S_a, \quad \varphi \mapsto t\star\varphi$$

is a linear isomorphism with the inverse $\psi \mapsto (-t)\star\psi$.

Now, we are in a position to prove Theorems 1.2 and 1.3.
Case 1. L^2-critical perturbation for $q = \bar{q} = 2 + \frac{4s}{3}$. We use the strategy firstly introduced in [20] and consider the functional $\widetilde{I}_\mu : \mathbb{R} \times H^s(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\begin{aligned}\widetilde{I}_\mu(t,u) &:= I_\mu(t\star u)\\ &= \left(\frac{1}{2}\|u\|^2 - \frac{\mu}{\bar{q}}\|u\|_{\bar{q}}^{\bar{q}}\right) e^{2st} - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)} \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{e^{2_s^* st}}{2_s^*} \|u\|_{2_s^*}^{2_s^*}.\end{aligned} \tag{7.1}$$

It is easy to see that $\widetilde{I}_\mu$ is of C^1-class, and $\widetilde{I}_\mu$ is invariant under rotations applied to u, a Palais–Smale sequence for $\widetilde{I}_\mu|_{\mathbb{R}\times S_{r,a}}$ is a Palais–Smale sequence $\widetilde{I}_\mu|_{\mathbb{R}\times S_a}$. We define the minimax level

$$\sigma(a,\mu) := \inf_{\gamma\in\Gamma} \max_{(t,u)\in\gamma([0,1])} \widetilde{I}_\mu(t,u)$$

among the associated minimax class

$$\Gamma := \left\{\gamma = (\alpha, \beta) \in C([0,1], \mathbb{R} \times S_{r,a}) | \gamma(0) \in (0, \overline{A_k}), \gamma(1) \in (0, I_\mu^0)\right\}, \tag{7.2}$$

where $k>0$ be defined by Lemma 4.4 and $I_\mu^c:=\{u\in S_a: I_\mu(u)\le c\}$. Let $u\in S_{r,a}$. Since $\|t\star u\|^2\to 0^+$ as $t\to-\infty$ and $I_\mu(t\star u)\to-\infty$ as $t\to+\infty$, there exist $t_0\ll -1$ and $t_1\gg 1$ such that

$$\gamma_u:\ s\in[0,1]\mapsto \big(0,\big((1-s)t_0+st_1\big)\star u\big)\in\mathbb{R}\times S_{r,a} \tag{7.3}$$

is a path in Γ. Then $\sigma(a,\mu)$ is a real value.

Now, for any path $\gamma=(\alpha,\beta)\in\Gamma$, we consider the function

$$T_\gamma:\ t\in[0,1]\mapsto P_\mu\left(\alpha(t)\star\beta(t)\right)\in\mathbb{R}.$$

By Lemmas 4.3 and 4.4 we get $T_\gamma(0)=P_\mu(\beta(0))>0$. Note that $\Psi^\mu_{\beta(1)}(t)>0$ for every $t\in(-\infty,t_{\beta(1)})$ and $\Psi^\mu_{\beta(1)}(0)=I_\mu(\beta(1))\le 0$, we have $t_{\beta(1)}<0$. Thus, by Lemma 4.2, we have that $T_\gamma(1)=P_\mu(\beta(1))<0$. Moreover, the map $s\mapsto\alpha(s)\star\beta(s)$ is continuous from $[0,1]$ to $H^s(\mathbb{R}^3)$, and hence we deduce that there exists $s_\gamma\in(0,1)$ such that $T_\gamma(s_\gamma)=0$, i.e., $\alpha(s_\gamma)\star\beta(s_\gamma)\in\mathcal{N}_{a,\mu}$, this implies that

$$\max_{\gamma([0,1])}\widetilde{I}_\mu\ge\widetilde{I}_\mu(\gamma(s_\gamma))=I_\mu\big(\alpha(s_\gamma)\star\beta(s_\gamma)\big)\ge\inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}}I_\mu=m_{r,a,\mu}.$$

Consequently, we have $\sigma(a,\mu)\ge m_{r,a,\mu}$. On the other hand, if $u\in\mathcal{N}^-_{a,\mu}\cap S_{r,a}$, then

$$I_\mu(u)=\max_{\gamma_u([0,1])}\widetilde{I}_\mu\ge\sigma(a,\mu),$$

where γ_u defined in (7.3) is a path in Γ. Thus, we have that $m_{r,a,\mu}\ge\sigma(a,\mu)$. Combining this with Lemmas 4.3–4.4, we derive that

$$\sigma(a,\mu)=m_{r,a,\mu}>\sup_{(\overline{A}_k\cup I_\mu^0)\cap S_{r,a}}I_\mu=\sup_{((0,\overline{A}_k)\cup(0,I_\mu^0))\cap(\mathbb{R}\times S_{r,a})}\widetilde{I}_\mu.$$

According to Proposition 7.1, we know that $\{\gamma([0,1]):\gamma\in\Gamma\}$ is a homotopy stable family of compact subsets of $\mathbb{R}\times S_{r,a}$ with extended closed boundary $(0,\overline{A}_k)\cup(0,I_\mu^0)$ and the superlevel set $\{\widetilde{I}_\mu\ge\sigma(a,\mu)\}$ is a dual set for Γ. Using Proposition 7.1, we can take any minimizing sequence $\{\gamma_n=(\alpha_n,\beta_n)\}\subset\Gamma_n$ for $\sigma(a,\mu)$ with the property that $\alpha_n\equiv 0$ and $\beta_n(s)\ge 0$ a.e. in $\mathbb{R}^3$ for every $s\in[0,1]$, then there exists a Palais–Smale sequence $\{(t_n,w_n)\}\subset\mathbb{R}\times S_{r,a}$ for $\widetilde{I}_\mu|_{\mathbb{R}\times S_{r,a}}$ at level $\sigma(a,\mu)$ satisfying

$$\partial_t\widetilde{I}_\mu(t_n,w_n)\to 0\quad\text{and}\quad\|\partial_u\widetilde{I}_\mu(t_n,w_n)\|_{(T_{w_n}S_{r,a})^*}\to 0\quad\text{as }\ n\to+\infty, \tag{7.4}$$

with the property that

$$|t_n|+\operatorname{dist}_{H^s}(w_n,\beta_n([0,1]))\to 0\quad\text{as }\ n\to+\infty. \tag{7.5}$$

By the definition of $\widetilde{I}_\mu(t_n,w_n)$ in (7.1) and the first condition in (7.4), we obtain $P_\mu(t_n\star w_n)\to 0$. The second condition in (7.4) shows that for every $\phi\in T_{w_n}S_{r,a}$

$$dI_\mu(t_n\star w_n)[t_n\star\phi]=o_n(1)\|\phi\|=o_n(1)\|t_n\star\phi\|\quad\text{as }\ n\to+\infty, \tag{7.6}$$

in the last equality, we used that $|t_n|$ is bounded from (7.5).

Let then $u_n=t_n\star w_n$, by Lemmas 7.1 and (7.6), we can deduce that $\{u_n\}\subset S_{r,a}$ is a Palais–Smale sequence for $I_\mu|_{S_{r,a}}$ (thus a PS sequence for $I_\mu|_{S_a}$, since the problem is invariant under rotations) at level $\sigma(a,\mu)=m_{r,a,\mu}$ with $P_\mu(u_n)\to 0$. Hence, by Lemmas 4.3–4.5, we have that

$$m_{r,a,\mu}\in\left(0,\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}\frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)}S^{\frac{3}{2s}}\right), \tag{7.7}$$

then one of the two alternatives in Proposition 2.3 occurs.

Assume that (i) of Proposition 2.3 occurs, then there exists $u \in H^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^3)$ but not strongly, where $u \not\equiv 0$ is a solution of problem (1.13) for some $\lambda < 0$ and

$$I_\mu(u) \le m_{r,a,\mu} - \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} < 0. \tag{7.8}$$

Moreover, by Pohozaev identity $P_\mu(u) = 0$, which reads as

$$\|u\|^2 - \frac{2\mu}{\bar{q}}\|u\|_{\bar{q}}^{\bar{q}} - \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \|u\|_{2_s^*}^{2_s^*} = 0,$$

together with condition (4.1), we have that

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{\mu}{\bar{q}}\int_{\mathbb{R}^3} |u|^{\bar{q}} dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= \frac{2_s^*-2}{22_s^*}\|u\|^2 + \frac{2_s^*-2}{22_s^*(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{(2_s^*-2)\mu}{2_s^*\bar{q}}\|u\|_{\bar{q}}^{\bar{q}} \\ &\ge \frac{2_s^*-2}{22_s^*}\left(1 - \frac{2\mu}{\bar{q}} C_{\bar{q},s} a^{\frac{4s}{3}}\right)\|u\|^2 + \frac{2_s^*-2}{22_s^*(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx > 0, \end{aligned}$$

a contradiction with (7.8). This shows that necessarily the alternative (ii) of Proposition 2.3 holds, namely there exists a subsequence $u_n \to u$ strongly in $H^s(\mathbb{R}^3)$, $I_\mu(u) = m_{r,a,\mu}$ and u solves problem (1.13) for some $\lambda < 0$. Combining $\beta_n(s) \ge 0$ a.e. in $\mathbb{R}^3$ for every $s \in [0, 1]$, (7.5) and the convergence imply that $u \ge 0$, and utilizing the same argument as Sect. 6, we have that u is positive. Finally, we prove that u is a ground state solution. Since any normalized solution stays on $\mathcal{N}_{a,\mu}$ and satisfies that

$$I_\mu(u) = m_{r,a,\mu} = \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu.$$

It is sufficient to check that

$$\inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu = \inf_{\mathcal{N}_{a,\mu}} I_\mu = m_{a,\mu}.$$

Suppose by contradiction that there exists a $w \in \mathcal{N}_{a,\mu}\backslash S_{r,a}$ such that $I_\mu(w) < \inf_{\mathcal{N}_{a,\mu}\cap S_{r,a}} I_\mu$. Then we let $v := |w|^*$ be the symmetric decreasing rearrangement of w, which lies in $S_{r,a}$. By standard properties, we have that

$$\|v\|^2 \le \|w\|^2, \quad I_\mu(v) \le I_\mu(w) \quad \text{and} \quad P_\mu(v) \le P_\mu(w) = 0.$$

If $P_\mu(v) = 0$, then $P_\mu(v) = P_\mu(w) = 0$, a contradiction with the above inequalities and hence we can assume that $P_\mu(v) < 0$. In this case, by Lemma 4.2, we see that $t_v < 0$. But then we have again a contradiction in the following way:

$$\begin{aligned} I_\mu(w) \le I_\mu(t_v \star v) &= \frac{(2_s^*-2)e^{2(2_s^*-1)st_v}}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_v |v|^{2_s^*-1} dx + \frac{(2_s^*-2)e^{2_s^* st_v}}{22_s^*}\|v\|_{2_s^*}^{2_s^*} \\ &\le e^{2_s^* st_v} I_\mu(w) < I_\mu(w), \end{aligned}$$

where we use the fact that $t_v \star v, w \in \mathcal{N}_{a,\mu}$. Therefore,

$$m_{a,\mu} = m_{r,a,\mu},$$

and so u is a ground state solution.

Case 2: L^2-supercritical perturbation for $\bar{q} = 2 + \frac{4s}{3} < q < 2_s^*$. Proceeding exactly as in the case $q = \bar{q} = 2 + \frac{4s}{3}$, we can obtain a Palais–Smale sequence $\{u_n\} \subset S_{r,a}$ for $I_\mu|_{S_a}$ at level $\sigma(a, \mu) = m_{r,a,\mu}$ with $P_\mu(u_n) \to 0$. Hence, by Lemma 5.5, we have that $m_{r,a,\mu}$ satisfies (7.7), then one of the two alternatives in Proposition 2.3 occurs.

Assume that (i) of Proposition 2.3 occurs, then there exists $u \in H^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^3)$ but not strongly, where $u \not\equiv 0$ is a solution of problem (1.13) for some $\lambda < 0$ and

$$I_\mu(u) \le m_{r,a,\mu} - \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12 + (1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} < 0. \tag{7.9}$$

However, by Pohozaev identity $P_\mu(u) = 0$, we have that

$$\|u\|^2 - \mu\gamma_{q,s}\|u\|_q^q - \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \|u\|_{2_s^*}^{2_s^*} = 0,$$

and by virtue of $q\gamma_{q,s} > 2$, we get that

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{\mu}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= \frac{\mu}{q}\left(\frac{q\gamma_{q,s}}{2} - 1\right)\int_{\mathbb{R}^3} |u|^q dx + \frac{2_s^*-2}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx + \frac{2_s^*-2}{22_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx > 0, \end{aligned}$$

a contradiction with (7.9). Therefore, the alternative (ii) of Proposition 2.2 occurs. Namely, there exists a subsequence $u_n \to u$ strongly in $H^s(\mathbb{R}^3)$, $I_\mu(u) = m_{r,a,\mu}$ and u solves problem (1.13) for some $\lambda < 0$. By the convergence, u is also nonnegative, and utilizing the same argument as Sect. 6, we have that u is positive. It remains to show that u is a ground state. The rest part of the proof is similar to that of Case 1. The thesis follows. □

8 Proof of Theorem 1.4

In this section, we focus on problem (1.13) in the limit case $\mu = 0$. In this situation, the action functional of (1.13) is given by

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{1}{2_s^*}\|u\|_{2_s^*}^{2_s^*},$$

and the associated Pohozaev identity reads as

$$\mathcal{N}_{a,0} = \left\{u \in S_a : \|u\|^2 - \int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \|u\|_{2_s^*}^{2_s^*} = 0\right\} = \left\{u \in S_a : (\Psi_u^0)'(0) = 0\right\},$$

where

$$\Psi_u^0(t) := \frac{e^{2st}}{2}\|u\|^2 - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_u |u|^{2_s^*-1} dx - \frac{e^{2_s^* st}}{2_s^*}\|u\|_{2_s^*}^{2_s^*}$$

and $\mathcal{N}_{a,0}$ can be decomposed as

$$\mathcal{N}_{a,0} = \mathcal{N}_{a,0}^+ \cup \mathcal{N}_{a,0}^0 \cup \mathcal{N}_{a,0}^-.$$

Before further studying for problem (1.13), the solutions of the following equations must be clearly studied. To be specific, we consider, the Euler–Lagrange equation of I_0 expressed as

$$(-\Delta)^s u = \phi_u |u|^{2_s^*-3}u + |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^3, \tag{8.1}$$

and the equation

$$(-\Delta)^s u = |u|^{2_s^*-2}u, \quad x \in \mathbb{R}^3. \tag{8.2}$$

Lemma 8.1 *The solutions of problem (8.1) and problem (8.2) are one-to-one correspondence. Moreover, (8.1) has a positive ground state solution, unique up to translation and scaling.*

Proof Assume that $w_1(x)$ is a solution of problem (8.2), then $w_2(x) = K_{w_1} w_1(x)$, solves problem (8.1), where $K_{w_1} > 0$ satisfying

$$K_{w_1}^{2_s^*-2} = \frac{-\|w_1\|^2 + \|w_1\| \sqrt{\|w_1\|^2 + 4\int_{\mathbb{R}^3} \phi_{w_1} |w_1|^{2_s^*-1} dx}}{2\int_{\mathbb{R}^3} \phi_{w_1} |w_1|^{2_s^*-1} dx}. \tag{8.3}$$

For any solutions $w_1(x)$ and $w_2(x)$ of problem (8.2), if $K_{w_1} w_1(x) = K_{w_2} w_2(x)$ holds, then

$$w_1(x) = \frac{K_{w_2}}{K_{w_1}} w_2(x).$$

Since both $w_1(x)$ and $w_2(x)$ are the solutions of problem (8.2), we have $\frac{K_{w_2}}{K_{w_1}} = 1$ and hence $w_1(x) = w_2(x)$.

On the other hand, assume that $w_2(x)$ is a solution of problem (8.1), then $w_1(x) = T_{w_2} w_2(x)$ solves problem (8.2), where $T_{w_2} > 0$ and

$$T_{w_2}^{2_s^*-2} = \frac{\|w_2\|^2}{\|w_2\|_{2_s^*}^{2_s^*}}. \tag{8.4}$$

Combining with those, it is easy to see that $w_1(x) \xrightarrow{K_{w_1}} w_2(x) \xrightarrow{T_{w_2}} w_1(x)$.

Note that all positive ground state solutions to (8.2) are the functions $U_{\varepsilon,z}$ defined in (4.6). Then $\psi_{\varepsilon,z} = K_{U_{\varepsilon,z}} U_{\varepsilon,z}$ is a positive ground state solution of problem (8.1), where $K_{U_{\varepsilon,z}} > 0$ and

$$K_{U_{\varepsilon,z}}^{2_s^*-2} = \frac{\sqrt{5}-1}{2}. \tag{8.5}$$

To see this fact, we introduce the following "limit equation"

$$\begin{cases} (-\Delta)^s u = \phi |u|^{2_s^*-3} u, & x \in \mathbb{R}^3, \\ (-\Delta)^s \phi = |u|^{2_s^*-1}, & x \in \mathbb{R}^3. \end{cases} \tag{8.6}$$

We claim that: All positive solutions of (8.6) have the form $u(x) = \phi(x) = U_{\varepsilon,z}(x)$ for any $\varepsilon > 0$ and $z \in \mathbb{R}^3$.

Indeed, assume that (u, ϕ) is a pair of positive solution to (8.6), then we have that

$$(-\Delta)^s (u - \phi) = (\phi - u)|u|^{2_s^*-2}, \quad x \in \mathbb{R}^3. \tag{8.7}$$

Multiplying both sides of this equation by $(u - \phi)$ and integrating by part, we obtain that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} (u - \phi)|^2 dx + \int_{\mathbb{R}^3} |u - \phi|^2 |u|^{2_s^*-2} dx = 0.$$

Hence, we can conclude $u(x) = \phi(x) = U_{\varepsilon,z}(x)$ and

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} U_{\varepsilon,z}|^2 dx = \int_{\mathbb{R}^3} \phi_{U_{\varepsilon,z}} |U_{\varepsilon,z}|^{2_s^*-1} dx = \int_{\mathbb{R}^3} |U_{\varepsilon,z}|^{2_s^*} dx = S^{\frac{2_s^*}{2_s^*-2}}, \tag{8.8}$$

which implies that (8.6) is equivalent to system (8.2).

Now, for any positive solutions $w_1(x)$ and $w_2(x)$ of problem (8.1), then by (8.4),

$$u_1(x) = T_{w_1} w_1(x) \quad \text{and} \quad u_2(x) = T_{w_2} w_2(x)$$

are the positive solutions of problem (8.2). Combining this with the fact that the positive solution is of the form $U_{\varepsilon,z}(x)$, we then have that $\|u_1\| = \|u_2\|$, i.e.

$$\frac{\|w_1\|^{2_s^*}}{\|w_1\|_{2_s^*}^{2_s^*}} = \frac{\|w_2\|^{2_s^*}}{\|w_2\|_{2_s^*}^{2_s^*}} \Longleftrightarrow \left\| \frac{w_1}{\|w_1\|_{2_s^*}} \right\| = \left\| \frac{w_2}{\|w_2\|_{2_s^*}} \right\|,$$

which implies that $\|\widetilde{w}_1\| = \|\widetilde{w}_2\|$ with $\widetilde{w}_i := w_i/\|w_i\|_{2_s^*}, i = 1, 2$, in the sense of the $L^{2_s^*}$-normalized norm. Hence from the argument aforementioned, we know that any positive solution of (8.1) is a ground state solution. The uniqueness of the ground state solutions of problem (8.1) follows from (8.5) and $U_{\varepsilon,z}$. The proof is completed. □

From the above discussion, we can derive the following conclusion.

Lemma 8.2 *The free functional I_0 has least energy value*

$$\inf_{u\in\mathcal{M}} I_0(u) = \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}},$$

where $\mathcal{M} = \{u \in D^{s,2}(\mathbb{R}^3)\backslash\{0\} : I_0'(u)u = 0\}$ is the Nehari manifold of I_0. The infimum is achieved only by functions $w_\varepsilon(x) = k_{U_{\varepsilon,z}} U_{\varepsilon,z}(x)$, where $U_{\varepsilon,z}$ is given by (4.6).

Proof From the proof of Lemma 8.1, we see that critical points of I_0 correspond to the positive ground state solutions w_ε of (8.1), and

$$w_\varepsilon(x) = k_{U_{\varepsilon,z}} U_{\varepsilon,z}(x), \quad \text{with } k_{U_{\varepsilon,z}} = \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{2_s^*-2}},$$

where $U_{\varepsilon,z}(x)$ is the ground states of (8.2). Therefore, by (8.8) and a direct computation, we obtain that

$$\begin{aligned}
\inf_{u\in\mathcal{M}} I_0(u) &= I_0(w_\varepsilon) \\
&= \frac{1}{2}\|w_\varepsilon\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_{w_\varepsilon}|w_\varepsilon|^{2_s^*-1}dx - \frac{1}{2_s^*}\|w_\varepsilon\|_{2_s^*}^{2_s^*} \\
&= \frac{k_{U_{\varepsilon,z}}^2}{2}\|U_{\varepsilon,z}\|^2 - \frac{k_{U_{\varepsilon,z}}^{2(2_s^*-1)}}{2(2_s^*-1)}\int_{\mathbb{R}^3} \phi_{U_{\varepsilon,z}}|U_{\varepsilon,z}|^{2_s^*-1}dx - \frac{k_{U_{\varepsilon,z}}^{2_s^*}}{2_s^*}\|U_{\varepsilon,z}\|_{2_s^*}^{2_s^*} \\
&= \frac{k_{U_{\varepsilon,z}}^2}{2} S^{\frac{2_s^*}{2_s^*-2}} - \frac{k_{U_{\varepsilon,z}}^{2(2_s^*-1)}}{2(2_s^*-1)} S^{\frac{2_s^*}{2_s^*-2}} - \frac{k_{U_{\varepsilon,z}}^{2_s^*}}{2_s^*} S^{\frac{2_s^*}{2_s^*-2}} \\
&= \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}}.
\end{aligned}$$

□

Proof of Theorems 1.4. It is easy to see that for every $u \in S_a$, the function

$$\Psi_u^0(t) = \frac{e^{2st}}{2}\|u\|^2 - \frac{e^{2(2_s^*-1)st}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx - \frac{e^{2_s^*st}}{2_s^*}\|u\|_{2_s^*}^{2_s^*},$$

has a unique maximum point $t_{u,0}$, given by

$$e^{st_{u,0}} = \left(\frac{-\|u\|_{2_s^*}^{2_s^*} + \sqrt{\|u\|_{2_s^*}^{22_s^*} + 4\|u\|^2\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx}}{2\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx}\right)^{\frac{1}{2_s^*-2}}. \tag{8.9}$$

By the definition of $\mathcal{N}_{a,0}^+$ and $\mathcal{N}_{a,0}^0$, we can deduce that $\mathcal{N}_{a,0}^+ = \mathcal{N}_{a,0}^0 = \emptyset$. Indeed, suppose that there exists $u \in \mathcal{N}_{a,0}$ such that $u \in \mathcal{N}_{a,0}^0 \cup \mathcal{N}_{a,0}^+$, then $(\Psi_u^0)''(0) \geq 0$, that is,

$$2\|u\|^2 \geq 2_s^*\|u\|_{2_s^*}^{2_s^*} + 2(2_s^*-1)\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx.$$

By $u \in \mathcal{N}_{a,0}$, we have that

$$\|u\|^2 = \|u\|_{2_s^*}^{2_s^*} + \int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx,$$

and hence,

$$\|u\|_{2_s^*}^{2_s^*} + \int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx \leq 0,$$

which implis that $u \equiv 0$, contradicting to $u \in S_a$. Thus, $\mathcal{N}_{a,0} = \mathcal{N}_{a,0}^-$.

Next, we prove that $\mathcal{N}_{a,0}$ is a smooth manifold of codimension 1 on S_a. Since

$$\mathcal{N}_{a,0} = \left\{u \in S_a : \|u\|^2 - \int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx - \|u\|_{2_s^*}^{2_s^*} = 0\right\},$$

$\mathcal{N}_{a,0}$ can be defined by $P_0(u) = 0$, $G(u) = 0$, where

$$P_0(u) = \|u\|^2 - \int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx - \|u\|_{2_s^*}^{2_s^*} \quad \text{and} \quad G(u) = \int_{\mathbb{R}^3}|u|^2dx - a^2.$$

Since $P_0(u)$ and $G(u)$ are class of C^1, it suffices to check that $d(P_0(u), G(u)) : H^s(\mathbb{R}^3) \to \mathbb{R}^2$ is surjective. If this is not true, then $dP_0(u)$ must be linearly dependent from $dG(u)$, that is, there exist some $\nu \in \mathbb{R}$ such that

$$2s(-\Delta)^s u = \nu u + 2_s^* s|u|^{2_s^*-2}u + 2s(2_s^*-1)\phi_u|u|^{2_s^*-3}u \quad \text{in } \mathbb{R}^3.$$

By the Pohozaev identity and the last equation, we obtain that

$$2s^2\|u\|^2 = 2_s^*s^2\|u\|_{2_s^*}^{2_s^*} + 2(2_s^*-1)s^2\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx,$$

that is $u \in \mathcal{N}_{a,0}^0$, a contradiction. Moreover, $\mathcal{N}_{a,0}$ is a natural constraint. Indeed, if $u \in \mathcal{N}_{a,0}$ is a critical point of $I_0|_{\mathcal{N}_{a,0}}$, then by the Lagrange multipliers rule there exists $\lambda, \nu \in \mathbb{R}$ such that

$$I_0'(u)\varphi = \lambda\int_{\mathbb{R}^3}u\varphi dx + \nu P_0'(u)\varphi,$$

for every $\varphi \in H^s(\mathbb{R}^3)$. That is, u solves the following equation

$$(1-2\nu)(-\Delta)^s u = \lambda u + \big(1-2(2_s^*-1)\nu\big)\phi_u|u|^{2_s^*-3}u + (1-2_s^*\nu)|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3. \tag{8.10}$$

We have to prove that $\nu = 0$, and to this end we observe that by the Pohozaev identity

$$\begin{aligned} &\frac{(1-2\nu)(3-2s)}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx = \frac{3\lambda}{2}\int_{\mathbb{R}^3}|u|^2dx \\ &\quad + \frac{\big(1-2(2_s^*-1)\nu\big)(3-2s)}{2}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx + \frac{(1-2_s^*\nu)(3-2s)}{2}\int_{\mathbb{R}^3}|u|^{2_s^*}dx. \end{aligned} \tag{8.11}$$

Combining (8.10) and (8.11), we have that

$$(1-2\nu)\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx = \big(1-2(2_s^*-1)\nu\big)\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx + (1-2_s^*\nu)\int_{\mathbb{R}^3}|u|^{2_s^*}dx.$$

Since $u \in \mathcal{N}_{a,0}$, this implies that

$$\nu\left[2\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2dx - 2(2_s^*-1)\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx - 2_s^*\int_{\mathbb{R}^3}|u|^{2_s^*}dx\right] = 0.$$

But the term inside the bracket cannot be 0, since $u \notin \mathcal{N}_{a,0}^0$, and then necessarily $\nu = 0$. Thus, u is a critical point of $I_0|_{S_a}$. Hence, for every $u \in S_a$, there exists a unique $t_{u,0} \in \mathbb{R}$ such that $t_{u,0}\star u \in \mathcal{N}_{a,0}$ and $t_{u,0}$ is a strict maximum point of $\Psi_u^0(t)$, if $u \in \mathcal{N}_{a,0}$, we have $t_{u,0} = 0$ and

$$I_0(u) = \max_{t\in\mathbb{R}} I_0(t\star u) \geq \inf_{u\in S_a}\max_{t\in\mathbb{R}} I_0(t\star u).$$

On the other hand, if $u \in S_a$, then $t_{u,0}\star u \in \mathcal{N}_{a,0}$ and

$$\max_{t\in\mathbb{R}} I_0(t\star u) = I_0(t_{u,0}\star u) \geq \inf_{u\in\mathcal{N}_{a,0}} I_0(u).$$

Therefore, we conclude that

$$\inf_{u\in\mathcal{N}_{a,0}} I_0(u) = \inf_{u\in S_a}\max_{t\in\mathbb{R}} I_0(t\star u).$$

Now we show that the infimum of I_0 in $\mathcal{N}_{a,0}$ is not achieved. By (8.9) and (4.15), we derive that

$$\begin{aligned}
&\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} \\
&= \inf_{u\in\mathcal{M}} I_0(u) \leq \inf_{u\in\mathcal{N}_{a,0}} I_0(u) \\
&= \inf_{u\in S_a} \max_{t\in\mathbb{R}} I_0(t\star u) \\
&= \inf_{u\in\mathcal{N}_{a,0}} \left[\frac{e^{2st_{u,0}}}{2}\|u\|^2 - \frac{e^{2(2_s^*-1)st_{u,0}}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx - \frac{e^{2_s^* st_{u,0}}}{2_s^*}\|u\|_{2_s^*}^{2_s^*}\right] \\
&= \inf_{u\in H^s(\mathbb{R}^3)\setminus\{0\}} \left[\frac{e^{2st_{u,0}}}{2}\|u\|^2 - \frac{e^{2(2_s^*-1)st_{u,0}}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx - \frac{e^{2_s^* st_{u,0}}}{2_s^*}\|u\|_{2_s^*}^{2_s^*}\right] \\
&\leq \frac{e^{2st_{u_\varepsilon,0}}}{2}\|u_\varepsilon\|^2 - \frac{e^{2(2_s^*-1)st_{u_\varepsilon,0}}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{u_\varepsilon}|u_\varepsilon|^{2_s^*-1}dx - \frac{e^{2_s^* st_{u_\varepsilon,0}}}{2_s^*}\|u_\varepsilon\|_{2_s^*}^{2_s^*} \\
&= \Psi_{u_\varepsilon}^0(t_{u_\varepsilon,0}) = \sup_{t\geq 0}\Psi_{u_\varepsilon}^0(t) \\
&\leq \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}} + O(\varepsilon^{3-2s}),
\end{aligned}$$

for any $\varepsilon > 0$. By density of $H^s(\mathbb{R}^3)$ in $D^{s,2}(\mathbb{R}^3)$, Lemma 8.2, we infer that the infimum is $\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}} \frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)} S^{\frac{3}{2s}}$, and is achieved if and only if the extremal functions $U_{\varepsilon,z}$ defined in (4.6) when $0 < s < \frac{3}{4}$ and stay in $L^2(\mathbb{R}^3)$. In the case $\frac{3}{4} \leq s < 1$, we show that the infimum of I_0 on $\mathcal{N}_{a,0}$ is not achieved. Assume by contradiction that there exists a minimizer u, let $v := |u|^*$ be the symmetric decreasing rearrangement of u, which lies in $S_{r,a}$. Then, by the properties of symmetric decreasing rearrangement, we infer to

$$\|v\|^2 \leq \|u\|^2, \quad I_0(v) \leq I_0(u) \quad \text{and} \quad P_0(v) \leq P_0(u) = 0.$$

If $P_0(v) < 0$, then $t_{v,0}$ defined in (8.9) is negative. Hence, by $P_0(t_{v,0}\star v) = 0$ and $P_0(u) = 0$, we derive that

$$\begin{aligned}
I_0(u) \leq I_0(t_{v,0}\star v) &= \frac{e^{2st_{v,0}}}{2}\|v\|^2 - \frac{e^{2(2_s^*-1)st_{v,0}}}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_v|v|^{2_s^*-1}dx - \frac{e^{2_s^* st_{v,0}}}{2_s^*}\|v\|_{2_s^*}^{2_s^*} \\
&= \frac{(2_s^*-2)e^{2st_{v,0}}}{22_s^*}\|v\|^2 + \frac{(2_s^*-2)e^{2(2_s^*-1)st_{v,0}}}{2(2_s^*-1)2_s^*}\int_{\mathbb{R}^3}\phi_v|v|^{2_s^*}dx \\
&\leq \frac{(2_s^*-2)e^{2st_{v,0}}}{22_s^*}\|u\|^2 + \frac{(2_s^*-2)e^{2st_{v,0}}}{2(2_s^*-1)2_s^*}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*}dx \\
&= e^{2st_{v,0}}\left[\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|u\|^2 - \left(\frac{1}{2(2_s^*-1)}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx\right] \\
&= e^{2st_{v,0}}\left[\frac{1}{2}\|u\|^2 - \frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_u|u|^{2_s^*-1}dx - \frac{1}{2_s^*}\|u\|_{2_s^*}^{2_s^*}\right] \\
&= e^{2st_{v,0}}I_0(u) < I_0(u),
\end{aligned}$$

which is a contradiction. Thus, it is necessary that $P_0(v)=0$, that is, $v\in\mathcal{N}_{a,0}$, and v is a nonnegative radial minimizer. Since $\mathcal{N}_{a,0}$ is a natural constraint, we infer that

$$\begin{cases}(-\Delta)^s v-\phi|v|^{2_s^*-3}v=\lambda v+|v|^{2_s^*-2}v, & x\in\mathbb{R}^3,\\ (-\Delta)^s\phi=|v|^{2_s^*-1}, & x\in\mathbb{R}^3,\end{cases} \tag{8.12}$$

for some $\lambda\in\mathbb{R}$, and by the maximum principle [9], $v>0$ in $\mathbb{R}^3$. By $P_0(v)=0$, necessarily $\lambda=0$, and so v solves the equation

$$\begin{cases}(-\Delta)^s v=\phi|v|^{2_s^*-3}v+|v|^{2_s^*-1}, & x\in\mathbb{R}^3,\\ (-\Delta)^s\phi=|v|^{2_s^*-1}, & x\in\mathbb{R}^3,\end{cases} \tag{8.13}$$

Therefore, by Lemmas 8.1 and 8.2, we have that $v=k_{U_{\varepsilon,z}}U_{\varepsilon,z}$. But this is not possible, since $U_{\varepsilon,z}\notin H^s(\mathbb{R}^3)$ for $\frac{3}{4}\le s<1$. This completes the proof. □

9 Proof of Theorem 1.5

This section is devoted to prove Theorem 1.5. We begin with the following two lemmas, which are necessary to the proof.

Lemma 9.1 *Let $a>0$, $\mu\ge 0$ and $\bar{q}=2+\frac{4s}{3}\le q<2_s^*$ holds. Then*

$$\inf_{u\in\mathcal{N}_{a,\mu}} I_\mu(u)=\inf_{u\in S_a}\max_{t\in\mathbb{R}} I_\mu(t\star u).$$

Proof By virtue of $\bar{q}=2+\frac{4s}{3}\le q<2_s^*$ and $\mu\ge 0$, we know from Lemmas 4.2 and 5.2 that $\mathcal{N}_{a,0}=\mathcal{N}_{a,0}^-$. For every $u\in S_a$, there exists a unique $t_{u,\mu}\in\mathbb{R}$ such that $t_{u,\mu}\star u\in\mathcal{N}_{a,\mu}$, and that $t_{u,\mu}$ is a strict maximum point of the functional Ψ_u^μ. Thus, if $u\in\mathcal{N}_{a,\mu}$ we get $t_{u,\mu}=0$ and

$$I_\mu(u)=\max_{t\in\mathbb{R}} I_\mu(t\star u)\ge\inf_{v\in S_a}\max_{t\in\mathbb{R}} I_\mu(t\star v).$$

On the other hand, if $u\in S_a$, then $t_{u,\mu}\star u\in\mathcal{N}_{a,\mu}$ and so

$$\max_{t\in\mathbb{R}} I_\mu(t\star u)=I_\mu(t_{u,\mu}\star u)\ge\inf_{v\in\mathcal{N}_{a,\mu}} I_\mu(v),$$

which completes the proof. □

Lemma 9.2 *Let $a>0$, $\mu^*>0$ and $\bar{q}=2+\frac{4s}{3}\le q<2_s^*$ holds. Then the function $\mu:[0,\mu^*]\mapsto m_{a,\mu}\in\mathbb{R}$ is monotone and non-increasing.*

Proof Let $0\le\mu_1\le\mu_2\le\mu^*$, by Lemma 9.1, we have that

$$\begin{aligned} m_{a,\mu_2}&=\inf_{u\in S_a}\max_{t\in\mathbb{R}} I_{\mu_2}(t\star u)=\inf_{u\in S_a} I_{\mu_2}(t_{u,\mu_2}\star u)\\ &=\inf_{u\in S_a}\left[I_{\mu_1}(t_{u,\mu_2}\star u)+\frac{\mu_1-\mu_2}{q}e^{q\gamma_{q,s}st_{u,\mu_2}}\int_{\mathbb{R}^3}|u|^q\,dx\right]\\ &\le\inf_{u\in S_a}\max_{t\in\mathbb{R}} I_{\mu_1}(t\star u)=m_{a,\mu_1}, \end{aligned}$$

as desired. □

Proof of Theorems 1.5. The proof is divided into two cases.

Case 1: $2 < q < \bar{q} = 2 + \frac{4s}{3}$. We recall that u_μ is a positive ground state solution of $I_\mu(u)$ on $\{u \in S_a : \|u_\mu\| < R_0\}$, where $R_0(a, \mu)$ is defined in Lemma 3.1 such that $h(R_0) = 0$ and h is given in (3.5), and we can check that $R_0 = R_0(a, \mu) \to 0$ as $\mu \to 0^+$. Thus, $\|u_\mu\| < R_0 \to 0$ as $\mu \to 0^+$. Moreover, for every $u \in S_a$, according to (1.14), the fractional Gagliardo–Nirenberg–Sobolev inequality (1.16) and Lemma 2.1, we have that

$$\begin{aligned} 0 > m_{a,\mu} = I_\mu(u_\mu) &\geq \frac{1}{2}\|u_\mu\|^2 - \frac{1}{2(2_s^*-1)} S^{-2_s^*}\|u_\mu\|^{2(2_s^*-1)} \\ &\quad - \frac{\mu}{q} C_{q,s} a^{q(1-\gamma_{q,s})}\|u_\mu\|^{q\gamma_{q,s}} - \frac{1}{2_s^*} S^{-\frac{2_s^*}{2}}\|u_\mu\|^{2_s^*} \to 0 \end{aligned}$$

as $\mu \to 0^+$.

Case 2: $\bar{q} = 2 + \frac{4s}{3} \leq q < 2_s^*$. Let $a > 0$, $\mu^* > 0$ and in this case (1.20) holds. Firstly, we show that the family of positive radial ground states $\{u_\mu : 0 < \mu < \mu^*\}$ is bounded in $H^s(\mathbb{R}^3)$. If $q = \bar{q}$, then by $P_\mu(u_\mu) = 0$ and Lemma 9.2, we get that

$$\begin{aligned} m_{a,0} \geq m_{a,\mu} = I_\mu(u_\mu) &= \frac{2_s^*-2}{22_s^*}\|u_\mu\|^2 + \frac{2_s^*-2}{2(2_s^*-1)2_s^*}\int_{\mathbb{R}^3}\phi_{u_\mu}|u_\mu|^{2_s^*-1}dx \\ &\quad - \frac{(2_s^*-2)\mu}{2_s^*\bar{q}}\|u_\mu\|_{\bar{q}}^{\bar{q}} \\ &\geq \frac{2_s^*-2}{22_s^*}\left(1 - \frac{2\mu}{\bar{q}} C_{\bar{q},s} a^{\frac{4s}{3}}\right)\|u_\mu\|^2. \end{aligned}$$

If $\bar{q} = 2 + \frac{4s}{3} < q < 2_s^*$, in a similar way, we infer to

$$\begin{aligned} m_{a,0} &\geq m_{a,\mu} = I_\mu(u_\mu) \\ &= \frac{\mu}{q}\left(\frac{q\gamma_{q,s}}{2} - 1\right)\int_{\mathbb{R}^3}|u_\mu|^q dx + \frac{2_s^*-2}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{u_\mu}|u_\mu|^{2_s^*-1}dx + \frac{2_s^*-2}{22_s^*}\int_{\mathbb{R}^3}|u_\mu|^{2_s^*}dx. \end{aligned}$$

Hence, by $q\gamma_{q,s} > 2$ and $P_\mu(u_\mu) = 0$, we also have $\{u_\mu\}$ is bounded in $H^s(\mathbb{R}^3)$. Since in particular $\{u_\mu\}$ is bounded in $L^q(\mathbb{R}^3)$, we have that

$$\begin{aligned} \overline{\lambda}_\mu a^2 &= \|u_\mu\|^2 - \int_{\mathbb{R}^3}\phi_{u_\mu}|u_\mu|^{2_s^*-1}dx - \mu\int_{\mathbb{R}^3}|u_\mu|^q dx - \int_{\mathbb{R}^3}|u_\mu|^{2_s^*}dx \\ &= \mu(\gamma_{q,s}-1)\int_{\mathbb{R}^3}|u_\mu|^q dx \to 0 \end{aligned}$$

as $\mu \to 0^+$. Therefore, we deduce that up to a subsequence $u_\mu \rightharpoonup u$ weakly in $H^s(\mathbb{R}^3)$, in $D^{s,2}(\mathbb{R}^3)$ and in $L^{2_s^*}(\mathbb{R}^3)$; $u_\mu \to u$ strongly in $L^q(\mathbb{R}^3)$. Let $\|u_\mu\|^2 \to \ell \geq 0$. If $\ell = 0$, then $u_\mu \to 0$ strongly in $D^{s,2}(\mathbb{R}^3)$, and hence $I_\mu(u_\mu) \to 0$. But, by Lemma 9.2, we get $I_\mu(u_\mu) \geq m_{a,\mu^*} > 0$ for each $\mu \in (0, \mu^*)$, a contradiction. Hence, $\ell > 0$. By $P_\mu(u_\mu) = 0$ we have as $\mu \to 0^+$,

$$\int_{\mathbb{R}^3}\phi_{u_\mu}|u_\mu|^{2_s^*-1}dx + \|u_\mu\|_{2_s^*}^{2_s^*} = \|u_\mu\|^2 - \mu\gamma_{q,s}\|u_\mu\|_q^q \to \ell, \tag{9.1}$$

Then, we may assume that

$$\int_{\mathbb{R}^3}\phi_{u_\mu}|u_\mu|^{2_s^*-1}dx \to a \quad \text{and} \quad \|u_\mu\|_{2_s^*}^{2_s^*} \to b, \quad \text{as } \mu \to 0^+. \tag{9.2}$$

On the other hand, by Young inequality, we have that

$$\begin{aligned}\int_{\mathbb{R}^3}|u_\mu|^{2_s^*}dx &= \int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}\phi_{u_\mu}(-\Delta)^{\frac{s}{2}}|u_\mu|dx\\ &\le \frac{\theta^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}|u_\mu||^2dx+\frac{1}{2\theta^2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}\phi_{u_\mu}|^2dx\\ &= \frac{1}{2\theta^2}\int_{\mathbb{R}^3}\phi_{u_\mu}|u_\mu|^{2_s^*-1}dx+\frac{\theta^2}{2}\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_\mu|^2dx.\end{aligned}$$

Thus, passing to the limit as $\mu\to 0^+$, it follows that $b\le\frac{1}{2\theta^2}a+\frac{\theta^2}{2}\ell$. Choosing $\theta^2=\frac{\sqrt{5}-1}{2}$, and using (9.1), we derive $a\ge\frac{3-\sqrt{5}}{2}\ell$. Consequently, by (9.2) again, we get

$$\begin{aligned}m_{a,\mu} &= \lim_{\mu\to 0^+}I_\mu(u_\mu)\\ &= \lim_{\mu\to 0^+}\left\{\frac{1}{2}\|u_\mu\|^2-\frac{1}{2(2_s^*-1)}\int_{\mathbb{R}^3}\phi_{u_\mu}|u_\mu|^{2_s^*-1}dx-\frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_\mu|^{2_s^*}dx\right\}\\ &= \frac{s}{3}\ell+\frac{s(3-2s)}{3(3+2s)}a\ge\frac{s\left[12+(1-\sqrt{5})(3-2s)\right]}{6(3+2s)}\ell.\end{aligned}\tag{9.3}$$

From (1.14), (9.1) and Lemma 2.1 we have that

$$\ell=a+b+o_\mu(1)\le S^{-2_s^*}\ell^{2_s^*-1}+S^{-\frac{2_s^*}{2}}\ell^{\frac{2_s^*}{2}}+o_\mu(1).\tag{9.4}$$

Taking the limit as $\mu\to 0^+$, we obtain that $\ell\ge\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{2}{2_s^*-2}}S^{\frac{3}{2s}}$. From (9.4), we infer to

$$m_{a,\mu}\ge\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}\frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)}S^{\frac{3}{2s}}.\tag{9.5}$$

Meanwhile, we have that

$$\lim_{\mu\to 0^+}I_\mu(u_\mu)\le m_{a,0}=\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}\frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)}S^{\frac{3}{2s}}.\tag{9.6}$$

Finally, combining (9.5) with (9.6), we obtain that

$$m_{a,\mu}=\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{3-2s}{2s}}\frac{s\left(12+(1-\sqrt{5})(3-2s)\right)}{6(3+2s)}S^{\frac{3}{2s}},$$

and the conclusion follows. □

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