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Calculus of Variations

The Lavrentiev phenomenon in calculus of variations with differential forms

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Abstract

In this article we study convex non-autonomous variational problems with differential forms and corresponding function spaces. We introduce a general framework for constructing counterexamples to the Lavrentiev gap, which we apply to several models, including the double phase, borderline case of double phase potential, and variable exponent. The results for the borderline case of double phase potential provide new insights even for the scalar case, i.e., variational problems with 0-forms.

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1 Introduction

In this article we study variational problems and corresponding function spaces associated with the integral functionals of the form

$$\mathcal{F}_{\Phi,b}(\omega) := \int_{\Omega} \Phi(x, |d\omega|)\, dV + \int_{\Omega} b \wedge d\omega \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ (later we will only consider the case of a cube or ball) with $\Phi : \Omega \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a generalized Orlicz function, ω a differential k-form, $b \in L^{\Phi^*(\cdot)}(\Omega, \Lambda^{N-k-1})$, and $dV = dx^1 \dots dx^N$. For 0-forms the problem reduces to the classical problem of calculus of variations with $d\omega$ replaced by $\nabla\omega$. Further we refer to the case of 0-forms (functions) as the scalar case.

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The classical results on differential forms are collected, for example, in the books by Cartan [17], Spivak [54], Arnold [7], Flanders [33], Abraham et al. [6]. Iwaniec and Lutoborski [39], Iwaniec and Martin [40], Scott [49], Iwaniec et al. [41], Schwarz [48], Mitrea et al. [44] and Troyanov [57] studied Sobolev spaces of differential forms, Gaffney inequalities, and related problems of Hodge theory. More recent results in the framework of Calculus of Variations could be found in books by Csato et al. [18] and Agarwal et al. [1]. Recent contributions in the direction of the transportation of closed differential forms were obtained by Dacorogna and Gangbo [25, 26]; the optimal constant in Gaffney inequality was studied by Csato et al. [19].

We study calculus of variations for the non-autonomous models with general "nonstandard" growth and differential forms. To our knowledge no regularity results are known for such classes. The focus of the present paper is on the conditions separating the case with the energy gap from the regular case (density of smooth functions) for the integrands with nonstandard growth, in particular, for the variable exponent and double phase models.

In the present paper we study variational problems for the integral functional (1.1) with convex integrands $\Phi(x,t)$ that satisfy general "nonstandard" growth conditions of the type

$$-c_0 + c_1|t|^{p_-} \le \Phi(x,t) \le c_2|t|^{p_+} + c_0, \tag{1.2}$$

where $1 < p_- \le p_+ < \infty$, $c_0 \ge 0$, $c_1, c_2 > 0$.

The class of "non-standard" integrands satisfying (1.2) includes for example the $p(x)$-integrand

$$\Phi(x,t) = t^{p(x)}, \quad 1 < p_- \le p(x) \le p_+ < \infty, \quad x \in \Omega, \tag{1.3}$$

studied for the scalar case in many papers and several books, see [20, 27, 42, 60–62]. For the variable exponent model the Hölder regularity of solutions, a Harnack type inequality for non-negative solutions, and boundary regularity results were obtained by Alkhutov [3, 4] and Alkhutov and Krasheninnikova [2] under some suitable assumptions on the variable exponent of the log-Hölder type. Gradient regularity for Hölder exponent was obtained by Coscia and Mingione [24] and for the log-Hölder exponents by Acerbi and Mingione [5].

Another classical example of non-standard growth conditions is given by double-phase variational problems which correspond to the functional (1.1) with

$$\Phi(x,t) = \varphi(t) + a(x)\psi(t), \quad a \ge 0, \tag{1.4}$$

where φ and ψ are Orlicz functions with different growths rates at infinity. Two notable examples are the "standard" double phase model with

$$\varphi(t) = t^p, \quad \psi(t) = t^q, \quad 1 < p < q < \infty, \tag{1.5}$$

and the "borderline" double phase model

$$\varphi(t) = t^p \log^{-\beta}(e+t), \quad \psi(t) = t^p \log^{\alpha}(e+t). \tag{1.6}$$

Colombo and Mingione [23] obtained Hölder regularity results for double-phase potential model $\Phi(x,t) = t^p + a(x)t^q$ if $q \le p(d+\alpha)/d$ and $a \in C^\alpha(\Omega)$. Moreover, bounded minimizers are automatically $W^{1,q}(\Omega)$ if $a \in C^\alpha(\Omega)$ and $q \le p + \alpha$, see the paper [10] by Baroni et al. As it was shown in [11] those results in the scalar case are sharp in terms of the counterexamples on the Lavrentiev gap.

The special case of the model (1.4) with $\varphi(t) = t^p$ and $\psi(t) = t^p \log(e+t)$ was studied by Baroni et al. [9]. In particular they obtained the C^γ_{loc} regularity result for the minimizers provided that the weight $a(x)$ is log-Hölder continuous (with some γ) and more strong result (any $\gamma \in (0, 1)$) for the case of vanishing log-Hölder continuous weight. Skrypnik and Voitovych recently proved continuity and Harnack inequality for solutions of a general class of elliptic and parabolic equations with nonstandard growth conditions, see [56]. The results on generalized Sobolev–Orlicz spaces are collected in the book by Harjulehto and Hästö [37] and for anisotropic Musielak–Orlicz setting in the book by Chlebicka et al. [21]. In the general framework of problems with nonstandard growth and nonuniform ellipticity recent results are due to Mingione and Rădulescu [46] and to De Filippis and Mingione, see [28–30]. Recent contributions for such energies include new results on density of smooth functions and absence of Lavrentiev gap by Bulíček et al. [12], Koch [43], and Borowski et al. [8].

An essential feature of the nonautonomous models with nonstandard growth is the presence of the Lavrentiev gap phenomenon. The energy $\mathcal{F}_{\Phi,b}$ defines the corresponding generalized partial Sobolev–Orlicz spaces of differential forms $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ (the natural energy space for $\mathcal{F}_{\Phi,b}$) described in Sect. 2.5. The Lavrentiev gap in this case is the inequality

$$\inf \mathcal{F}_{\Phi,b}(W_c^{d,\Phi(\cdot)}(\Omega, \Lambda^k)) < \inf \mathcal{F}_{\Phi,b}(C_0^\infty(\Omega, \Lambda^k)). \tag{1.7}$$

where $W_c^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ is the set of $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ forms compactly supported in Ω. A similar phenomenon for boundary value problems can be expressed as the inequality

$$\inf \mathcal{F}_{\Phi,0}(\omega_0 + W_c^{d,\Phi(\cdot)}(\Omega, \Lambda^k)) < \inf \mathcal{F}_{\Phi,0}(\omega_0 + C_0^\infty(\Omega, \Lambda^k)) \tag{1.8}$$

for some $\omega_0 \in C^1(\overline{\Omega}, \Lambda^k)$.

A closely related problem is density of smooth functions in the natural energy space of the functional. Denote the closure of smooth forms from $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ in this space by $H^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$. If any function from the domain of $\mathcal{F}_{\Phi,b}$ can be approximated by smooth functions with energy convergence (equivalently, if $H^{d,\Phi(\cdot)}(\Omega, \Lambda^k) = W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$, which is abbreviated to $H = W$) then the Lavrentiev gap is obviously absent. In the autonomous case, when the integrand $\Phi = \Phi(t)$ is an Orlicz function independent of x, the Lavrentiev phenomenon is absent ($H = W$).

In the scalar case (for functions = 0-forms) the study of such models goes back to Zhikov [61, 62], who constructed the first examples on Lavrentiev phenomenon for variable exponent model and double phase model in dimension $N = 2$. Esposito et al. [31] generalized this example to any dimension (for the standard double phase model); Fonseca et al. [34] constructed examples of minimizers for the standard double phase model with large (fractal) sets of discontinuity. All these examples required the dimensional restriction $p^- < N < p^+$. This restriction was overcome by the authors of the present paper with Diening in [11] using fractal contact sets for scalar variable exponent, double phase and weighted model. In [16] the authors of the present paper studied the Lavrentiev gap property for the borderline double phase model (1.4) with one saddle point (that is, an example constructed as in [31, 61, 62]) with $p = N, \alpha, \beta > 0$.

In the present paper we study variational problems with differential forms. The study of the ρ-harmonic forms goes back to Uhlenbeck [58], who obtained classical results on the Hölder continuity (for the scalar equation this reads as C^α property of the gradient). These results were extended by Hamburger [36]. Beck and Stroffolini [15] considered partial regularity for general quasilinear systems for differential forms. Sil [51–53] studied convexity properties of integral functionals with forms and regularity estimates for inhomogeneous quasilinear systems with forms. Let us mention, that results by Sil for nonautonomous integrands are concerned with "standard growth" $p^- = p^+$, as opposed to the "nonstandard growth" problems treated in this paper.

In this paper we extend the approach of [11] to variational problems with differential forms and refine the construction by using the generalized Cantor sets which have an additional tweaking parameter. This allows for fine tuning of the singular set, while keeping the formal Hausdorff dimension. We construct examples of the Lavrentiev gap for the $p(x)$-integrand (1.3) and both "standard" double phase (1.4), (1.5) and "borderline" double phase (1.4), (1.6) integrands (the last results are new even for the case of scalar functions 0-forms). For the latter model the fine tuning of the Cantor set is crucial.

Now we state the main results of this paper. We work with three models: classical double phase potential, borderline double phase potential, and variable exponent. For each of these cases we construct examples for the Lavrentiev gap. However, the construction presented in this paper is not limited to these models. For instance, it can be also used to treat the weighted energy similar to [11, Section 4.3]. Let Ω be a ball in $\mathbb{R}^N$ and $k \in \{0, \dots, N-2\}$. The definitions of the functional spaces can be found in Sect. 2.5.

Theorem A *Let $p > 1$, $\alpha \geq 0$, and $q > p + \alpha \max((k+1)^{-1}, (p-1)(N-k-1)^{-1})$ Then there exists an integrand $\Phi(x,t) = t^p + a(x)t^q$ where nonnegative weight $a = a(x)$ is bounded, $a \in C^\alpha(\overline{\Omega})$ if $\alpha > 0$, such that $H^{d,\Phi(\cdot)}(\Omega, \Lambda^k) \neq W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$. Moreover, there exists $b \in L^{\Phi^*}(\Omega, \Lambda^{N-k-1})$ such that (1.7) holds and $\omega_0 \in C^\infty(\overline{\Omega}, \Lambda^k)$ such that (1.8) holds.*

Theorem B *Let $p_0 > 1$, $\alpha, \beta \in \mathbb{R}$, $\varkappa \geq 0$ such that $\alpha + \beta > p_0 + \varkappa$. Let φ and ψ be two Orlicz functions such that $\varphi(t) \sim t^{p_0} \ln^{-\beta} t$ and $\psi(t) \sim t^{p_0} \ln^{\alpha} t$ for large t. Then there exists an integrand $\Phi(x,t) = \varphi(t) + a(x)\psi(t)$ where $a = a(x)$ is a nonnegative function with the modulus of continuity $C \ln^{-\varkappa}(1/t)$ such that $H^{d,\Phi(\cdot)}(\Omega, \Lambda^k) \neq W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$. Moreover, there exists $b \in L^{\Phi^*}(\Omega, \Lambda^k)$ such that (1.7) holds*

Theorem C *Let $1 < p_- < p_+$. There exists a variable exponent $p : \Omega \to [p_-, p_+]$ with the modulus of continuity $\kappa_0 (\ln t^{-1})^{-1} \ln \ln t^{-1}$, $\varkappa_0 = \varkappa_0(p_-, p_+, N, k) > 0$, such that for $\Phi(x,t) = t^{p(x)}$ there holds $H^{d,\Phi(\cdot)}(\Omega, \Lambda^k) \neq W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$. Moreover, there exists $b \in L^{\Phi^*}(\Omega, \Lambda^{N-k-1})$ such that (1.7) holds and $\omega_0 \in C^\infty(\overline{\Omega}, \Lambda^k)$ such that (1.8) holds.*

Theorems A, B, C follow from Theorems 31, 33, 35, which are proved in Sect. 5. The weight $a = a(x)$ in Theorems A and B and the exponent $p = p(x)$ in Theorem C (as well as the forms providing the examples of non-density and competitors used to show the Lavrentiev phenomenon) are regular outside of a singular set of Cantor type which lies on a proper subspace of $\mathbb{R}^N$. The dimension of this subspace is either $k+1$ or $N-k-1$ depending on the parameters. Compare this to [11] where for the scalar case $k = 0$ the singular set was either a Cantor set $\mathfrak{C}$ on a line ("superdimensional" setup, which was used to construct the examples with variable exponent taking values greater than the space dimension N) or a Cantor set $\mathfrak{C}^{N-1}$ on a hyperplane ("subdimensional" setup, which was used to construct the examples with variable exponent taking values less than the space dimension N). For k-forms in the variable exponent setting the value of exponent separating these two cases is $N/(k+1)$ — for exponent taking values greater than $N/(k+1)$ the singular set will be of the form $\mathfrak{C}^{k+1} \times \{0\}^{N-k-1}$ and for exponent taking values less than $N/(k+1)$ the singular set is of the form $\mathfrak{C}^{N-k-1} \times \{0\}^{k+1}$.

Our setting can be called "semivectorial", or generalized Uhlenbeck structure, since the integrand is isotropic. In this respect it has substantially more rigid structure than the fully vectorial problems (say, of elasticity theory) with quasi-convex integrands. Note that in the "fully" vectorial setting the situation is more delicate, and the Lavrentiev phenomenon is possible even for "standard" growth conditions in the autonomous (but anisotropic!) case, see Ball and Mizel [13] and Foss et al. [32] in the context of non-linear elasticity.

The models with Lavrentiev phenomenon are also challenging to study numerically since the standard numerical schemes fail to converge to the W-minimiser of the problem. For the scalar case the problem could be solved using non-conforming methods, see Balci et al. [14]. The vectorial setting remains open.

Structure of the paper. In Sect. 2 we recall some basic definitions related to the theory of differential forms and Sobolev–Orlicz spaces. In Sect. 3 we study the existence of minimizers of the functional (1.1). In Sect. 4 we describe the general framework for construction of examples using the fractal Cantor barriers. In Sect. 5 we apply this general construction to different problems. We obtain the examples of Lavrentiev gap and non-density of smooth functions for the classical double phase in Sect. 5.1, for the borderline double phase model in Sect. 5.2, and for the variable exponent model in Sect. 5.3. The results for the borderline double phase model are new even in the scalar case.

2 Differential forms and Sobolev–Orlicz spaces

Here we recall some basic facts and definitions from the theory of differential forms. In general we follow definitions and notations from [18][Chapters 1.2;2.1;3.1−3.3], but the Hodge codifferential is the formal adjoint of the exterior derivative d (as in [35, 39]).

2.1 Exterior algebra

The Grassman algebra of exterior k-forms (i.e. skew-symmetric k-linear functions) over $\mathbb{R}^N$ is denoted by $\Lambda^k(\mathbb{R}^N)$, or for brevity just by Λ^k. The exterior product of $f \in \Lambda^k$ and $g \in \Lambda^l$ is an element $f \wedge g$ of Λ^{k+l} defined by

$$(f \wedge g)(\xi_1, \dots, \xi_{k+l}) = \sum \operatorname{sign}(i_1, \dots, i_k, j_1, \dots, j_l) f(\xi_{i_1}, \dots, \xi_{i_k}) g(\xi_{j_1}, \dots, \xi_{j_l})$$

where the summation is over permutations $(i_1, \dots, i_k, j_1, \dots, j_l)$ of $(1, 2, \dots, k+l)$ such that $i_1 < \dots < i_k$, $j_1 < \dots < j_l$. This operation is linear in both arguments, associative, and for $f \in \Lambda^k$ and $g \in \Lambda^l$ there holds $f \wedge g = (-1)^{kl} g \wedge f$.

Let e_j be an orthonormal basis $\{e_j\}_{j=1}^N$ in $\mathbb{R}^N$ and $\{e^j\}_{j=1}^N$ be its dual system in Λ^1, $e^j(e_l) = \delta_{jl}$. The monomials $e^{i_1} \wedge \ldots \wedge e^{i_k}$, $i_1 < i_2 < \dots < i_k$ form a basis in Λ^k. Denote $f_{i_1 \ldots i_k} = f(e_{i_1}, \dots, e_{i_k})$. Then the set of $f_{i_1 \ldots i_k}$ with $i_1 < i_2 < \dots < i_k$ gives the coordinates of f:

$$f = \sum_{1 \le i_1 < \dots < i_k \le N} f_{i_1 \ldots i_k} e^{i_1} \wedge \dots \wedge e^{i_k}.$$

The scalar product of $f, g \in \Lambda^k$ with coordinates $f_{i_1 \ldots i_k}$ and $g_{i_1 \ldots i_k}$ is given by

$$\langle f, g \rangle = \sum_{1 \le i_1 < \dots < i_k \le N} f_{i_1 \cdots i_k} g_{i_1, \ldots i_k}.$$

The scalar product does not depend on the particular choice of the orthonormal basis $\{e_j\}_{j=1}^N$. We denote $|f| = \langle f, f \rangle^{1/2}$.

The Hodge star operator $* : \Lambda^k \to \Lambda^{N-k}$ is defined by $f \wedge g = \langle *f, g \rangle e^1 \wedge \dots \wedge e^N$ for any $g \in \Lambda^{N-k}$, or equivalently by $f \wedge *g = \langle f, g \rangle e^1 \wedge \dots \wedge e^N$ for all $f, g \in \Lambda^k$. The

Hodge star operator $*$ is an isometry between Λ^k and Λ^{N-k} and for $f \in \Lambda^k$ there holds

$$*(*f) = (-1)^{k(N-k)} f, \quad *^{-1} f = (-1)^{k(N-k)} * f.$$

For any $f \in \Lambda^k$ and shuffle $j_1, \dots, j_N$ there holds $(*f)_{j_{k+1}\dots j_N} = \operatorname{sign}(j_1, \dots, j_N) f_{j_1\dots j_k}$.

The interior product (contraction) of $f \in \Lambda^k$ and $g \in \Lambda^l$ defined by

$$g \lrcorner f = (-1)^{(N-k)(k-l)} * (g \wedge (*f))$$

is the adjoint of the wedge product:

$$\langle g \wedge \alpha, f \rangle = \langle \alpha, g \lrcorner f \rangle \quad \text{for any} \quad \alpha \in \Lambda^{k-l}, \quad f \in \Lambda^k, \quad g \in \Lambda^l.$$

There holds $*(g \lrcorner f) = (-1)^{(k-l)l} g \wedge *f$ and $*(g \wedge f) = (-1)^{kl} g \lrcorner * f$.

If $l = 0$ then $g \lrcorner f = gf$. If $l > k$ then $g \lrcorner f = 0$, if $l = k$ then $g \lrcorner f = f \lrcorner g = \langle f, g \rangle$. If $l \leq k$ then

$$(g \lrcorner f)_{j_1\dots j_{k-l}} = \sum_{1 \leq i_1 < \dots < i_l \leq N} g_{i_1\dots i_l} f_{i_1\dots i_l j_1 \dots j_{k-l}}.$$

There holds $*f = f \lrcorner (e^1 \wedge \dots \wedge e^N)$, which can be taken as the definition for the Hodge dual.

For $w, v \in \Lambda^1$ there holds

$$w \lrcorner (v \wedge f) + v \wedge (w \lrcorner f) = \langle w, v \rangle f. \tag{2.1}$$

For a vector X the operator $\iota_X : \Lambda^k \to \Lambda^{k-1}$ is defined by $(\iota_X f)(\xi_1, \dots, \xi_{k-1}) = f(X, \xi_1, \dots, \xi_{k-1})$. There holds $\iota_v(f \wedge g) = (\iota_v f) \wedge g + (-1)^{\deg f} f \wedge (\iota_v g)$, and $\iota_v \iota_w = -\iota_w \iota_v$, $\iota_v \iota_v = 0$. For a vector $v \in \mathbb{R}^N$ and the 1-form $v^\flat \in \Lambda^1$ with the same coordinates there holds $v^\flat \lrcorner f = \iota_v f$.

2.2 Differential forms

A differential form is a mapping from $\Omega \subset \mathbb{R}^N$ to Λ^k. Further Ω will be a bounded contractible domain with sufficiently regular boundary. Using the canonical basis $dx^{i_1} \wedge \dots \wedge dx^{i_k}$ a k-form can be represented as

$$f = \sum_{1 \leq i_1 < \dots < i_k \leq N} f_{i_1\dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}. \tag{2.2}$$

Then $|f|^2(x) := \sum_{1 \leq i_1 < \dots < i_k \leq N} |f_{i_1\dots i_k}|^2(x)$.

For two differential forms f and g of order k their scalar product in the sense of $L^2(\Omega, \Lambda^k)$ is

$$(f, g)_\Omega = \int_\Omega f \wedge *g = \int_\Omega \langle f, g \rangle \, dV, \quad dV = dx^1 \wedge \dots \wedge dx^N.$$

We shall also use this notation for a more general case when $\langle f, g \rangle \in L^1(\Omega)$.

The operation of exterior differentiation d is a unique mapping from k-forms to $(k+1)$-forms such that df coincides with the differential of f for 0-forms (functions), $d \circ d = 0$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for any $\alpha \in C^1(\Omega, \Lambda^k)$ and $\beta \in C^1(\Omega, \Lambda^l)$. For a k-form f,

$$df(\xi_1, \dots, \xi_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j-1} [f'(x)\xi_j](\xi_1, \dots, \widehat{\xi_j}, \dots, \xi_{k+1}), \quad \xi_1, \dots, \xi_{k+1} \in \mathbb{R}^N.$$

The interior derivative (Hodge codifferential) of a k-form f is

$$\delta f = (-1)^{N(k-1)+1} * d * f = (-1)^k *^{-1} d * f.$$

There holds $d^2 = 0$, $\delta^2 = 0$. On k-forms

$$* \delta = (-1)^k d * \quad \text{and} \quad * d = (-1)^{k+1} \delta * . \tag{2.3}$$

For a k-form f and l-form g there holds

$$d(f \wedge g) = df \wedge g + (-1)^k f \wedge dg, \quad \delta(f \lrcorner g) = (-1)^{k+1} df \lrcorner g + (-1)^k f \lrcorner \delta g.$$

Formally one can write $df = \nabla \wedge f$, $\delta f = -\nabla \lrcorner f$, and in coordinates, for the form (2.2), using the Einstein convention of summation over repeated indices we have

$$(df)_{i_1 \dots i_{k+1}} = (-1)^{l-1} \partial_{x_{i_l}} f_{i_1 \dots \hat{\imath}_l \dots i_{k+1}}, \quad (\delta f)_{i_1 \dots i_{k-1}} = -\partial_{x_j} f_{j i_1 \dots i_{k-1}}. \tag{2.4}$$

Let $\nu_* = (\nu_1, \dots, \nu_N)$ be the unit outer normal to Ω and introduce the 1-form $\nu = \nu_1 dx^1 + \dots + \nu_N dx^N$. For a differential k-form f the standard Gauss theorem reads as (see (2.4))

$$\int\limits_\Omega (df)_{i_1 \dots i_{k+1}} dV = \int\limits_{\partial\Omega} (\nu \wedge f)_{i_1 \dots i_{k+1}} d\sigma, \quad \int\limits_\Omega (\delta f)_{i_1 \dots i_{k-1}} dV = -\int\limits_{\partial\Omega} (\nu \lrcorner f)_{i_1 \dots i_{k-1}} d\sigma$$

for each $1 \le i_1 < \dots < i_k \le N$, where $dV = dx^1 \dots dx^N$ is the standard volume form and $d\sigma$ is the surface area element. The integration-by-parts formula is

$$\int\limits_\Omega \langle df, g\rangle \, dV - \int\limits_\Omega \langle f, \delta g\rangle \, dV = \int\limits_{\partial\Omega} \langle \nu \wedge f, g\rangle \, d\sigma = \int\limits_{\partial\Omega} \langle f, \nu \lrcorner g\rangle \, d\sigma. \tag{2.5}$$

In the sense of forms, the surface element $d\sigma$ is connected to the volume form dV by $d\sigma = \iota_\nu dV$. The orientation of $\partial\Omega$ is chosen such that the integral of $d\sigma$ over any "substantial" boundary part is positive.

The operators d and δ are adjoint on compactly supported forms. By direct computation (use (2.1) with $v = \nabla$, $w = -\nabla$), $d\delta + \delta d = -\triangle$, where the Laplace operator is applied componentwise.

A form f satisfying $df = 0$ is closed. A form f satisfying $\delta f = 0$ is coclosed. If $f = dg$ then f is exact, and if $f = \delta g$ then f is coexact. If both $df = 0$ and $\delta f = 0$ the form is called harmonic (or harmonic field).

The pullback of the form f under the mapping φ is defined by $\varphi^* f$,

$$(\varphi^* f)(x; \xi_1, \dots, \xi_k) = f(\varphi(x); \varphi'(x)\xi_1, \dots, \varphi'(x)\xi_k).$$

This operation satisfies $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$ and $\varphi^* d = d\varphi^*$. In coordinates, for the form (2.2),

$$\begin{aligned}(\varphi^* f)(x) &= \sum_{1 \le i_1 < \dots < i_k \le N} f_{i_1 \dots i_k}(\varphi(x)) d\varphi^{i_1} \wedge \ldots \wedge d\varphi^{i_k} \\ &= \sum_{1 \le j_1 < \dots < j_k \le N} (\varphi^* f)_{j_1 \dots j_k}(x) dx^{j_1} \dots dx^{j_k},\end{aligned}$$

where

$$(\varphi^* f)_{j_1 \dots j_k}(x) = \sum_{1 \le i_1 < \dots < i_k \le N} \frac{\partial(\varphi^{i_1}, \dots, \varphi^{i_k})}{\partial(x^{j_1}, \dots, x^{j_k})} f_{i_1 \dots i_k}(\varphi(x)). \tag{2.6}$$

2.3 Tangential and normal part of a form

Recall that by $\nu_* = (\nu_1, \dots, \nu_N)$ we denote the unit outer normal to $\partial\Omega$ and $\nu = \nu_1 dx^1 + \dots + \nu_N dx^N$. For a differential form ω define its tangential part

$$t\omega(\xi_1, \dots, \xi_k) = \omega(\xi_1 - \nu(\xi_1)\nu_*, \dots, \xi_k - \nu(\xi_k)\nu_*)$$

and its normal part $n\omega = \omega - t\omega$. Then

$$\begin{aligned} &tf = \nu \lrcorner (\nu \wedge f), \quad nf = \nu \wedge (\nu \lrcorner f), \quad f = tf + nf, \\ &\nu \wedge tf = \nu \wedge f, \quad \nu \lrcorner tf = 0, \quad tf = 0 \Leftrightarrow \nu \wedge f = 0, \\ &\nu \wedge nf = 0, \quad \nu \lrcorner nf = \nu \lrcorner f, \quad nf = 0 \Leftrightarrow \nu \lrcorner f = 0. \end{aligned}$$

That is, setting tf is equivalent to setting $\nu \wedge f$ and setting nf is equivalent to setting $\nu \lrcorner f$.

While integrating over $\partial\Omega$, the tangential part of a form coincides with its pullback under the inclusion $\jmath : \partial\Omega \to \overline{\Omega}$, that is $t\omega = \jmath^*\omega$, and the normal part of the form vanishes.

The decomposition of a form into the tangential and normal parts can be also done using coordinates in a "collar" neighbourhood of $\partial\Omega$, by choosing (locally) an "admissible" coordinate system (map) $\varphi : U \to V, U, V \subset \mathbb{R}^N$, such that $\partial\Omega \cap V \subset \{\varphi(y', 0) : (y', 0) \subset U\}$ and $(\varphi_{y_i}(y', 0), \varphi_{y_N}(y', 0)) = \delta_{iN}$, $1 \le i \le N$ (see [45, Chapter 7, Lemma 7.5.1]). In this coordinate system for

$$\omega = \sum_{1 \le i_1 < \dots < i_k \le N} \omega_{i_1 \dots i_k} dy^{i_1} \wedge \dots \wedge dy^{i_k}$$

we have $\omega = t\omega + n\omega$, where

$$\begin{aligned} t\omega &= \sum_{1 \le i_1 < \dots < i_k < N} \omega_{i_1 \dots i_k} dy^{i_1} \wedge \dots \wedge dy^{i_k}, \\ n\omega &= \sum_{1 \le i_1 < \dots < i_{k-1} < N} \omega_{i_1 \dots i_k} dy^{i_1} \wedge \dots \wedge dy^{i_{k-1}} \wedge dy^N. \end{aligned}$$

If ω is a function (0-form) we set $t\omega = \omega$ and $n\omega = 0$. If ω is an N-form we set $t\omega = 0$, $n\omega = \omega$. The decomposition $\omega = t\omega + n\omega$ is invariant on $\partial\Omega$ and we have

$$t* = *n, \quad n* = *t, \quad td = dt, \quad n\delta = \delta n.$$

In particular, $t\omega = 0$ on $\partial\Omega$ implies $td\omega = 0$ on $\partial\Omega$ and $n\omega = 0$ on $\partial\Omega$ implies $n\delta\omega = 0$ on $\partial\Omega$.

In terms of the Stokes theorem, integration-by-parts formula (2.5) reads as follows: by (2.3) for a k-form f and a $(k+1)$-form g there holds $d(f \wedge *g) = df \wedge *g - f \wedge *\delta g$, therefore

$$\begin{aligned} (df, g)_\Omega - (f, \delta g)_\Omega &= \int_\Omega \big(df \wedge *g - f \wedge *\delta g\big) = \int_\Omega d(f \wedge *g) \\ &= \int_{\partial\Omega} f \wedge *g = \int_{\partial\Omega} tf \wedge *ng. \end{aligned} \tag{2.7}$$

2.4 Orlicz functions setup

We say that $\phi : [0,\infty) \to [0,\infty]$ is an Orlicz function if ϕ is convex, left-continuous, $\phi(0) = 0$, $\lim_{t\to 0} t^{-1}\phi(t) = 0$ and $\lim_{t\to\infty} t^{-1}\phi(t) = \infty$. The conjugate Orlicz function ϕ^* is defined by

$$\phi^*(s) := \sup_{t\geq 0}\big(st - \phi(t)\big).$$

In particular, $st \leq \phi(t) + \phi^*(s)$.

In the following we assume that $\Phi : \Omega\times[0,\infty) \to [0,\infty]$ is a generalized Orlicz function, i.e. $\Phi(x,\cdot)$ is an Orlicz function for every $x \in \Omega$ and $\Phi(\cdot,t)$ is measurable for every $t \geq 0$. We define the conjugate function Φ^* point-wise, i.e. $\Phi^*(x,\cdot) := (\Phi(x,\cdot))^*$.

We assume that Φ satisfies the “nonstandard” growth condition

$$-c_0 + c_1|t|^{p_-} \leq \Phi(x,t) \leq c_2|t|^{p_+} + c_0, \tag{2.8}$$

where $1 < p_- \leq p_+ < \infty$, $c_0 \geq 0$, $c_1, c_2 > 0$, and the following properties:

(a) Φ satisfies the Δ_2-condition, i.e. there exists $c \geq 2$ such that for all $x \in \Omega$ and all $t \geq 0$

$$\Phi(x,2t) \leq c\,\Phi(x,t). \tag{2.9}$$

(b) Φ satisfies the ∇_2-condition, i.e. Φ^* satisfies the Δ_2-condition. As a consequence, there exist $s > 1$ and $c > 0$ such that for all $x \in \Omega$, $t \geq 0$ and $\gamma \in [0,1]$ there holds

$$\Phi(x,\gamma t) \leq c\,\gamma^s\,\Phi(x,t). \tag{2.10}$$

(c) Φ and Φ^* are proper, i.e. for every $t \geq 0$ there holds

$$\int_\Omega \Phi(x,t)\,dV < \infty \quad\text{and}\quad \int_\Omega \Phi^*(x,t)\,dV < \infty.$$

2.5 Sobolev–Orlicz spaces of differential forms

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain in $\mathbb{R}^N$. In our applications this will always be a ball or a cube.

Different functional spaces like Lebesgue spaces $L^p(\Omega)$ and Lebesgue–Orlicz spaces $L^{\varphi(\cdot)}(\Omega)$, Sobolev spaces $W^{1,p}(\Omega)$ and Sobolev–Orlicz spaces $W^{1,\Phi(\cdot)}(\Omega)$, spaces of k times continuosly differentiable functions $C^k(\Omega)$ are defined in the usual way (see, for example, [37]).

The Lebesgue–Orlicz space $L^{\Phi(\cdot)}(\Omega)$ is the set of all measurable functions in Ω with finite Luxemburg norm

$$\|f\|_{L^{\Phi(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi(x,|f|\lambda^{-1})\,dV \leq 1\right\}.$$

The Sobolev–Orlicz space $W^{1,\Phi(\cdot)}(\Omega)$ is the set of functions $f \in W^{1,1}(\Omega)$ such that $|\nabla f| \in L^{\Phi(\cdot)}(\Omega)$, endowed with the norm $\|f\|_{W^{1,\Phi(\cdot)}(\Omega)} = \|f\|_{L^1(\Omega)} + \|\nabla f\|_{L^{\Phi(\cdot)}(\Omega)}$.

For a generalized Orlicz function $\Phi(x,t)$ we define the Lebesgue–Orlicz space $L^{\Phi(\cdot)}(\Omega,\Lambda^k)$ as the space of measurable differential k-form such that $|f| \in L^{\Phi(\cdot)}(\Omega)$. The norm in this space is the norm of $|f|$ in $L^{\Phi(\cdot)}(\Omega)$. For constant $\Phi \equiv p \geq 1$ we get the standard Lebesgue space $L^p(\Omega,\Lambda^k)$.

Let $r \in \mathbb{N} \cup \{0\}$. For $E = \Omega$ or $E = \overline{\Omega}$ the space $C^r(E, \Lambda^k)$ is the space of all differential k-forms for which all partial derivatives $D^\alpha f^I$ up to the order r are continuous in E. By $C_0^\infty(\Omega, \Lambda^k)$ we denote the space of smooth k-forms with compact support in Ω.

Definition 1 (*Full Sobolev–Orlicz Space*) We say that a k-form $f \in W^{1,\Phi(\cdot)}(\Omega, \Lambda^k)$ if $f_{i_1 \cdots i_k} \in W^{1,\Phi(\cdot)}(\Omega)$ for every $1 \le i_1 < \cdots < i_k \le N$. The norm is defined componentwise:

$$\|f\|_{W^{1,\Phi(\cdot)}(\Omega,\Lambda^k)} = \sum_{1 \le i_1 < \cdots < i_k \le N} \|f_{i_1 \ldots i_k}\|_{W^{1,\Phi}(\Omega)}.$$

If Ω is of class C^2 then for $f \in W^{1,1}(\Omega, \Lambda^k)$ the boundary trace of f exists, belongs to $L^1(\partial\Omega)$, and the Stokes theorem holds (see [48], [47, Theorem 6.4]):

$$\int_{\partial\Omega} f = \int_{\Omega} df.$$

We say that $u \in L^1_{loc}(\Omega, \Lambda^k)$ has a *weak differential* $du \in L^1_{loc}(\Omega, \Lambda^{k+1})$ if for any $\xi \in C_0^\infty(\Omega, \Lambda^{k+1})$ there holds

$$\int_{\Omega} \langle u, \delta\xi \rangle \, dV = \int_{\Omega} \langle du, \xi \rangle \, dV,$$

or equivalently

$$\int_{\Omega} u \wedge d\xi = (-1)^{k+1} \int_{\Omega} du \wedge \xi$$

for any $\xi \in C_0^\infty(\Omega, \Lambda^{N-k-1})$.

We say that $u \in L^1_{loc}(\Omega, \Lambda^k)$ has a *weak codifferential* $\delta u \in L^1_{loc}(\Omega, \Lambda^{k-1})$ if for any $\xi \in C_0^\infty(\Omega, \Lambda^{k-1})$ there holds

$$\int_{\Omega} \langle u, d\xi \rangle \, dV = \int_{\Omega} \langle \delta u, \xi \rangle \, dV,$$

or equivalently

$$\int_{\Omega} u \wedge \delta\xi = (-1)^{k} \int_{\Omega} \delta u \wedge \xi$$

for any $\xi \in C_0^\infty(\Omega, \Lambda^{N-k+1})$.

Both weak differential and codifferential are unique.

Definition 2 (*Partial Sobolev–Orlicz Space*) For $0 \le k \le N-1$ we define the partial Sobolev–Orlicz space $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ as the set of forms $\omega \in L^1(\Omega, \Lambda^k)$ with weak differential $d\omega \in L^{\Phi(\cdot)}(\Omega, \Lambda^k)$, endowed with the norm

$$\|\omega\|_{W^{d,\Phi(\cdot)}(\Omega,\Lambda^k)} := \|\omega\|_{L^1(\Omega,\Lambda^k)} + \|d\omega\|_{L^{\Phi(\cdot)}(\Omega,\Lambda^{k+1})}.$$

The space $H^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ is the closure of smooth forms from $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ in this space.

For $1 \le k \le N$ we define the space $W^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ as the set of forms $\omega \in L^1(\Omega, \Lambda^k)$ with weak codifferential $\delta\omega \in L^{\Phi(\cdot)}(\Omega, \Lambda^k)$ endowed with the norm

$$\|\omega\|_{W^{\delta,\Phi(\cdot)}(\Omega,\Lambda^k)} := \|\omega\|_{L^1(\Omega,\Lambda^k)} + \|\delta\omega\|_{L^{\Phi(\cdot)}(\Omega,\Lambda^{k-1})}.$$

The space $H^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ is the closure of smooth forms from $W^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ in this space.

If Ω is a bounded C^2 domain (or a polyhedral domain), then the following Green's formulas hold [18, Theorem 3.28]. Let $0 \le k \le N-1$ and let $p > 1$. If $f \in W^{d,p}(\Omega, \Lambda^k)$, $g \in W^{1,p'}(\Omega, \Lambda^{k+1})$, then

$$\int_\Omega \langle df, g\rangle\, dV - \int_\Omega \langle \delta g, f\rangle\, dV = \int_{\partial\Omega} \langle \nu \wedge f, g\rangle\, d\sigma.$$

If $f \in W^{1,p}(\Omega, \Lambda^k)$, $g \in W^{\delta,p'}(\Omega, \Lambda^{k+1})$, then

$$\int_\Omega \langle df, g\rangle\, dV - \int_\Omega \langle \delta g, f\rangle\, dV = \int_{\partial\Omega} \langle f, \nu \lrcorner g\rangle\, d\sigma.$$

The boundary traces $\nu \wedge f$ and $\nu \lrcorner g$ in these formulas are given by bounded linear mappings from $W^{d,p}(\Omega, \Lambda^k)$ to $W^{-1/p,p}(\partial\Omega, \Lambda^{k+1})$ and from $W^{\delta,p'}(\Omega, \Lambda^{k+1})$ to $W^{-1/p',p'}(\partial\Omega, \Lambda^k)$, respectively. These mappings are generated by these very integration-by-parts formulas. If f belongs to the full Sobolev space $W^{1,p}(\Omega, \Lambda^k)$, then both tangential and normal components of its boundary trace tf and nf are from $W^{1-1/p,p}(\partial\Omega, \Lambda^k)$. An extensive treatment of the boundary trace problem for spaces of differential forms can be found in [44].

Let $W_c^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ be the set of forms from $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ with compact support in Ω.

Definition 3 (*Spaces with zero tangential component*)

For $0 \le k \le N-1$ we define the space $W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ as the set of $\omega \in W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ such that

$$\int_\Omega \langle d\omega, \beta\rangle\, dV = \int_\Omega \langle \omega, d\beta\rangle\, dV$$

for all $\beta \in C^1(\overline{\Omega}, \Lambda^{k+1})$, endowed with the norm of $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$.

The space $\tilde{W}_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ is the closure of $W_c^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ in $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$.

The space $H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ is the closure of $C_0^\infty(\Omega, \Lambda^k)$ in $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$.

Clearly, $H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k) \subset \tilde{W}_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k) \subset W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$. A smooth k-form ω belongs to $H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ if and only if its tangential component $t\omega$ is zero on $\partial\Omega$.

Let $W_c^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ be the set of forms from $W^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ with compact support in Ω.

Definition 4 (*Spaces with zero normal component*)

For $1 \le k \le N$ we define the space $W_N^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ as the set of $\omega \in W^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ such that

$$\int_\Omega \langle d\omega, \beta\rangle\, dV = \int_\Omega \langle \omega, d\beta\rangle\, dV$$

for all $\beta \in C^1(\overline{\Omega}, \Lambda^{k-1})$, endowed with the norm of $W^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$.

The space $\tilde{W}_N^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ is the closure of $W_c^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ in $W^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$.

The space $H_N^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$ is the closure of $C_0^\infty(\Omega, \Lambda^k)$ in $W^{\delta,\Phi(\cdot)}(\Omega, \Lambda^k)$.

Clearly, $H_N^{\delta,\Phi(\cdot)}(\Omega,\Lambda^k) \subset \tilde{W}_N^{\delta,\Phi(\cdot)}(\Omega,\Lambda^k) \subset W_N^{\delta,\Phi(\cdot)}(\Omega,\Lambda^k)$. A smooth k-form ω belongs to $H_N^{\delta,\Phi(\cdot)}(\Omega,\Lambda^k)$ if and only if its normal component component $n\omega$ is zero on $\partial\Omega$.

The following proposition is straightforward.

Proposition 5 *All the spaces introduced in Definitions 2, 3 and 4 are Banach spaces.*

For $\Phi(\cdot) \equiv p \in [1,\infty]$ we get the classical partial Sobolev spaces $W_T^{d,p}(\Omega,\Lambda^k)$ and $W_N^{\delta,p}(\Omega,\Lambda^k)$ with vanishing tangential (correspondingly normal) component on the boundary. The spaces $W_T^{d,p}(\Omega,\Lambda^k)$ and $W_N^{\delta,p}(\Omega,\Lambda^k)$ coincide with the closures of $C_0^\infty(\Omega,\Lambda^k)$ in $W^{d,p}(\Omega,\Lambda^k)$ and $W^{\delta,p}(\Omega,\Lambda^k)$, respectively (see [39]). In this case there is no difference between between H and W spaces.

If $\overline{\Omega} \Subset \Omega'$, a form from $W^{d,\Phi(\cdot)}(\Omega,\Lambda^k)$ or $W^{\delta,\Phi(\cdot)}(\Omega,\Lambda^k)$ belongs to $W_T^{d,\Phi(\cdot)}(\Omega,\Lambda^k)$ or $W_N^{\delta,\Phi(\cdot)}(\Omega,\Lambda^k)$, respectively, iff its extension by zero to $\Omega'\setminus\Omega$ produces an element from $W^{d,\Phi(\cdot)}(\Omega',\Lambda^k)$ or $W^{\delta,\Phi(\cdot)}(\Omega',\Lambda^k)$, respectively.

3 Minimization problem for non-autonomous functionals with differential forms

3.1 Gauge fixing

Recall (for instance, [18, Theorem 6.5]) the following facts regarding the harmonic forms with vanishing tangential or normal components at the boundary. Let $\mathcal{H}_T(\Omega,\Lambda^k)$ be the set of harmonic forms from $W_T^{1,2}(\Omega,\Lambda^k)$ and $\mathcal{H}_N(\Omega,\Lambda^k)$ be the set of harmonic forms from $W_N^{1,2}(\Omega,\Lambda^k)$. The spaces $\mathcal{H}_T(\Omega,\Lambda^k)$ and $\mathcal{H}_N(\Omega,\Lambda^k)$ are finite dimensional, closed in $L^2(\Omega,\Lambda^k)$, for contractible domains $\mathcal{H}_T(\Omega,\Lambda^k) = \{0\}$ for $0 \le k \le N-1$ and $\mathcal{H}_N(\Omega,\Lambda^k) = \{0\}$ for $1 \le k \le N$. The space $\mathcal{H}_T(\Omega,\Lambda^N)$ is the span of $dx^1 \wedge \ldots \wedge dx^N$ and the space $\mathcal{H}_N(\Omega,\Lambda^0)$ is the span of 1.

We need the following result on the solvability of the Cauchy-Riemann type systems for differential forms. This result is a particular case of theorems [18, Theorem 7.2] for $p \ge 2$ and [50, Theorems 2.43] for any $p > 1$, and triviality of the set of harmonic forms with zero tangential component at the boundary. See also [48, Chapter 3, Theorem 3.2.5].

Corollary 6 *Let Ω be a bounded contractible C^3 domain in $\mathbb{R}^N$, $0 \le k \le N-1$, $p > 1$, $\omega_0 \in W^{1,p}(\Omega,\Lambda^k)$, and $\beta \in \omega_0 + W_T^{d,p}(\Omega,\Lambda^k)$. The problem*

$$d\omega = d\beta, \quad \delta\omega = 0 \quad in \quad \Omega,$$
$$\nu \wedge \omega = \nu \wedge \omega_0 \quad on \quad \partial\Omega$$

has a unique solution $\omega \in W^{1,p}(\Omega,\Lambda^k)$ with

$$\|\omega\|_{W^{1,p}(\Omega,\Lambda^k)} \le C\left(\|\omega_0\|_{W^{1,p}(\Omega,\Lambda^k)} + \|d\beta\|_{L^p(\Omega,\Lambda^{k+1})}\right).$$

with $C = C(N,p,\Omega)$.

3.2 Existence of minimizers

In this section $0 \le k \le N-1$, Ω is a bounded contractible C^3 domain in $\mathbb{R}^N$, $\Phi : \Omega \times [0,+\infty) \to [0,+\infty)$ is a generalized Orlicz function satisfying (2.8), $b \in$

$L^{\Phi^*(\cdot)}(\Omega, \Lambda^{N-k-1})$. We study the existence of solutions to the following two variational problems.

(*W-minimization*) Let $\omega_0 \in W^{1,\Phi(\cdot)}(\Omega, \Lambda^k)$ and minimize $\mathcal{F}_{\Phi,b}$ over the set $\omega_0 + W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$:

$$\mathcal{F}_{\Phi,b}(\omega) = \int_\Omega \Phi(x, |d\omega|)\, dV + \int_\Omega b \wedge d\omega \to \min, \quad \omega \in \omega_0 + W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k). \tag{3.1}$$

(*H-minimization*) Let $\omega_0 \in H^{1,\Phi(\cdot)}(\Omega, \Lambda^k)$ and minimize $\mathcal{F}_{\Phi,b}$ over the set $\omega_0 + H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$:

$$\mathcal{F}_{\Phi,b}(\omega) \to \min, \quad \omega \in \omega_0 + H_T^{1,\Phi(\cdot)}(\Omega, \Lambda^k). \tag{3.2}$$

Theorem 7 *The variational problem* (3.1) *has a minimizer* $\omega \in \omega_0 + W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ *with* $\delta\omega = 0$.

Proof Let ω_s be a minimizing sequence, clearly

$$\|d\omega_s\|_{L^{\Phi(\cdot)}(\Omega,\Lambda^{k+1})} \le c.$$

Due to the coercitivity condition (1.2) we have

$$\|d\omega_s\|_{L^{p-}(\Omega,\Lambda^{k+1})} \le c.$$

By Corollary 6 there exists $\alpha_s \in \omega_0 + W_T^{1,p-}(\Omega, \Lambda^k)$ satisfying $d\alpha_s = d\omega_s$ and $\delta\alpha_s = 0$ in Ω such that

$$\|\alpha_s\|_{W^{1,p-}(\Omega,\Lambda^k)} \le c.$$

Clearly $\alpha_s \in W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ and

$$\|\alpha_s\|_{W^{d,\Phi(\cdot)}(\Omega,\Lambda^k)} \le c.$$

The sequence α_s is bounded in the space $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k) \cap W^{1,p-}(\Omega, \Lambda^k)$ endowed with the norm which is the sum of norms in $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ and $W^{1,p-}(\Omega, \Lambda^k)$. Its dual space is separable, therefore there exists

$$\alpha \in \omega_0 + W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k) \cap W^{1,p-}(\Omega, \Lambda^k)$$

such that $\delta\alpha = 0$ and up to the subsequence,

$$\begin{aligned} \alpha_s \to \alpha \quad &\text{in} \quad L^{p-}(\Omega, \Lambda^k), \\ d\alpha_s \rightharpoonup d\alpha \quad &\text{in} \quad L^{\Phi(\cdot)}(\Omega, \Lambda^{k+1}). \end{aligned}$$

From the convexity of $\Phi(x, \cdot)$ and Mazur's lemma, we have the lower semicontinuity:

$$\liminf_{s\to\infty} \int_\Omega \Phi(x, |d\alpha_s|)\, dV \ge \int_\Omega \Phi(x, |d\alpha|)\, dV.$$

Since in the linear part we have convergence, the proof is complete. □

Theorem 8 *Let* ω_0 *in* $H^{1,\Phi(\cdot)}(\Omega, \Lambda^k(\mathbb{R}^N))$. *Then the problem* (3.2) *has a minimizer* $\omega \in \omega_0 + H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ *with* $\delta\omega = 0$.

Proof We keep the notation from the proof of Theorem 7. Let $\omega_s = \omega_0+\gamma_s, \gamma_s \in C_0^\infty(\Omega, \Lambda^k)$, be a minimizing sequence, clearly

$$\|d\omega_s\|_{L^{\Phi(\cdot)}(\Omega,\Lambda^{k+1})},\ \|d\gamma_s\|_{L^{\Phi(\cdot)}(\Omega,\Lambda^{k+1})} \le c.$$

Due to the coercitivity condition (1.2) we have

$$\|d\omega_s\|_{L^{p-}(\Omega,\Lambda^{k+1})},\ \|d\gamma_s\|_{L^{p-}(\Omega,\Lambda^{k+1})} \le c.$$

By Corollary 6 there exists $\alpha_s \in \omega_0 + W_T^{1,p-}(\Omega, \Lambda^k)$ satisfying $d\alpha_s = d\omega_s$ and $\delta\alpha_s = 0$ in Ω such that

$$\|\alpha_s\|_{W^{1,p-}(\Omega,\Lambda^k)} \le c.$$

Writing $\alpha_s = \omega_0 + \beta_s$, one gets $\beta_s \in W_T^{1,p-}(\Omega, \Lambda^k)$ satisfying $d\beta_s = d\gamma_s$, $\delta\beta_s = -\delta\omega_0$. Extend β_s to $\mathbb{R}^N \setminus \Omega$ by zero.

Clearly $\alpha_s \in W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ and

$$\|\alpha_s\|_{W^{d,\Phi(\cdot)}(\Omega,\Lambda^k)},\ \|\beta_s\|_{W^{d,\Phi(\cdot)}(\Omega,\Lambda^k)} \le c.$$

Let $\varphi : \Omega \times (0, 1] \to \mathbb{R}^N$ be a C^2 mapping such that $\varphi(x, 1) = x$ for all $x \in \Omega$, and set $\varphi_t(x) = \varphi(x, t)$. Let $\varphi_t^{-1}\Omega \Subset \Omega$ for every $t \in (0, 1]$. If Ω is a ball centered at the origin, one takes $\varphi_t(x) = x/t$. Consider the pullbacks $\varphi_t^*\beta_s$. These forms have compact support in Ω, with $d\varphi_t^*\beta_s$ uniformly converging to $d\beta_s = d\gamma_s \in C_0^\infty(\Omega, \Lambda^{k+1})$ and $\delta\varphi_t^*\beta_s$ converging to $-\delta\omega_0$ in $L^{p-}_{\mathrm{loc}}(\Omega, \Lambda^{k-1})$ as $t \to 1-0$. Moreover, $\varphi_t^*\beta_s$ converges to β_s in $W^{1,p-}_{\mathrm{loc}}(\Omega, \Lambda^{k-1})$ as $t \to 1-0$. This is easily seen using (2.6).

Mollifications $(\varphi_t^*\beta_s)_\varepsilon(x) = \int \chi_\varepsilon(x-y)\varphi_t^*\beta_s(y)\,dy$, where $\chi_\varepsilon(x) = \varepsilon^{-d}\chi(x/\varepsilon)$, $\chi \in C_0^\infty(\{|x| < 1\})$ with $\int \chi\,dx = 1$, converge to $\varphi_t^*\beta_s$ in $L^{p-}(\Omega, \Lambda^k)$, $d(\varphi_t^*\beta_s)_\varepsilon$ converges uniformly to $d\varphi_t^*\beta_s$, $(\varphi_t^*\beta_s)_\varepsilon \to \varphi_t^*\beta_s$ in $W^{1,p-}_{\mathrm{loc}}(\Omega, \Lambda^k)$, and $\delta(\varphi_t^*\beta_s)_\varepsilon \to \delta\varphi_t^*\beta_s$ in $L^{p-}_{\mathrm{loc}}(\Omega, \Lambda^{k-1})$ as $\varepsilon \to 0$. Clearly, $(\varphi_t^*\beta_s)_\varepsilon \in C_0^\infty(\Omega, \Lambda^k)$ for sufficiently small ε.

Therefore, keeping the same notation for β_s while replacing it by $(\varphi_t^*\beta_s)_\varepsilon$ for appropriate t and ε, we can assume that the new minimizing sequence has the form $\alpha_s = \omega_0 + \beta_s$, where $\beta_s \in C_0^\infty(\Omega, \Lambda^k)$, β_s is uniformly bounded in $W^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ and in $W^{1,p-}(\Omega', \Lambda^k)$ for all $\Omega' \Subset \Omega$. Moreover $\delta(\omega_0 + \beta_s) \to 0$ for $s \to \infty$ in $L^{p-}_{\mathrm{loc}}(\Omega, \Lambda^k)$.

Therefore there exists

$$\beta \in W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k) \cap W^{1,p-}_{\mathrm{loc}}(\Omega, \Lambda^k)$$

such that $\delta(\omega_0 + \beta) = 0$ and up to the subsequence,

$$\begin{aligned} \beta_s \rightharpoonup \beta \quad &\text{in} \quad W^{1,p-}_{\mathrm{loc}}(\Omega, \Lambda^k), \quad \beta_s \to \beta \quad \text{in} \quad L^{p-}_{\mathrm{loc}}(\Omega, \Lambda^k),\\ d\beta_s \rightharpoonup d\beta \quad &\text{in} \quad L^{\Phi(\cdot)}(\Omega, \Lambda^{k+1}). \end{aligned}$$

Due to the convexity of $\Phi(x, \cdot)$ and Mazur's lemma, we have $\beta \in H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^k)$ and for $\alpha = \omega_0 + \beta$ there holds

$$\liminf_{s\to\infty} \int_\Omega \Phi(x, |d\alpha_s|)\,dV \ge \int_\Omega \Phi(x, |d\alpha|)\,dV.$$

Since in the linear part we have convergence, the proof is complete.

□

Remark If one drops the Coulomb gauge condition $\delta\omega = 0$ in Theorem 8 then its proof can be somewhat simplified. Instead of α_s take $\tilde{\alpha}_s \in W_T^{1,p-}(\Omega, \Lambda^k)$ satisfying $d\tilde{\alpha}_s = d\gamma_s$, $\delta\tilde{\alpha}_s = 0$. Extend $\tilde{\alpha}_s$ by zero to $\mathbb{R}^N \setminus \overline{\Omega}$. For sufficiently small ε the mollifications of the pullbacks $(\varphi_t^* \tilde{\alpha}_s)_\varepsilon$ are smooth forms with compact support in Ω, $(\varphi_t^* \tilde{\alpha}_s)_\varepsilon \to \tilde{\alpha}_s$ in $W^{1,p-}_{\mathrm{loc}}(\Omega)$, and $(d\varphi_t^* \tilde{\alpha}_s)_\varepsilon$ uniformly converge to $d\gamma_s$ as $t \to 1-0$ and $\varepsilon \to 0$. Then taking appropriate t and ε we pass to the minimizing sequence $\omega_0 + (\varphi_t^* \tilde{\alpha}_s)_\varepsilon$. The rest goes as above.

Alternatively one could use [48, Chapter 3, Theorem 3.3.3] to establish the existence of $\hat{\alpha}_s \in W_0^{1,p-}(\Omega, \Lambda^k)$ with $d\hat{\alpha}_s = d\gamma_s$ and $\|\hat{\alpha}_s\|_{W^{1,p-}(\Omega)} \le c\|d\gamma_s\|_{L^{p-}(\Omega,\Lambda^{k+1})}$. Then the above argument repeats, moreover the "loc" subscript can be dropped since the extension of a $W_0^{1,p-}(\Omega)$ function by zero produces a function from $W^{1,p-}(D)$ for any bounded domain $D \subset \mathbb{R}^N$. Or use the Bogovskii type operator constructed in the paper by Costabel and McIntosh [22].

4 Lavrentiev gap and non-density

In this section we design the general framework for the construction of the examples on Lavrentiev gap. We introduce the set of assumptions for the examples in the Sect. 4.1 and show how to obtain non-density of smooth functions and the special type of the non-uniqueness of the minimisers under these assumptions. In Sect. 4.3 we introduce basic forms which will be building blocks of our examples. These building blocks correspond to the one saddle-point geometry of the classical checkerboard Zhikov example and are then used in Sects. 4.4 and 4.5 to construct more advanced examples using fractal Cantor barriers. The results are summarised in the Sect. 4.6

4.1 Separating pairs of forms and separating functionals

Here we present some "conditional" statements. We shall use two assumptions. Let Ω be a domain in $\mathbb{R}^N$ with sufficiently regular boundary, $k \in \{1, \dots, N-1\}$, and $\mathfrak{S} \subset \Omega$ be a closed set of zero Lebesgue N-measure. Our argument will be based upon defining a suitable set $\mathfrak{S}$ and $(k-1)$-form u and $(N-k-1)$-form A, which are smooth in $\Omega \setminus \mathfrak{S}$ and give a "counterexample" to the Stokes theorem. The regularity of $\partial\Omega$ is assumed to be such that the classical Stokes theorem holds. Further Ω will be either cube of ball in $\mathbb{R}^N$.

Let $\Phi : \Omega \times [0, +\infty) \to [0, +\infty)$ be a generalized Orlicz function.

Definition 9 We say that a pair of $(k-1)$-form and $(N-k-1)$-form (u, A) defined in Ω is (Φ, k)- separating if there exists a closed set $\mathfrak{S} \subset \Omega$ of zero Lebesgue N-measure such that

(i) u and A are regular outside $\mathfrak{S}$;
(ii) $u \in W^{d,1}(\Omega, \Lambda^{k-1})$ and $A \in W^{d,1}(\Omega, \Lambda^{N-k-1})$;
(iii) $\int\limits_{\partial\Omega} A \wedge du = 1$;
(iv) $|du| \cdot |dA| = 0$ in $\Omega \setminus \mathfrak{S}$;
(v) $\int\limits_{\Omega} \Phi(x, |du|)\, dV < \infty$ and $\int\limits_{\Omega} \Phi^*(x, |dA|)\, dV < \infty$.

When invoking a pair of (Φ, k)-separating forms we assume that the set $\mathfrak{S}$ comes from this definition and when necessary denote it by $\mathfrak{S}(u, A)$.

The essential property of (Φ, k)-separating forms is that $A \wedge du$ "contradicts" the Stokes theorem. Indeed, disregarding the singular set Σ we would arrive at

$$0 = \int_{\Omega} dA \wedge du = \int_{\Omega} d(A \wedge du) = \int_{\partial\Omega} A \wedge du = 1.$$

Definition 10 Let u and A be a pair of (Φ, k)-separating forms and $\eta \in C_0^\infty(\Omega)$ with $\eta = 1$ in a neighbourhood of $\mathfrak{S}$. Set

$$u^\circ = \eta u, \qquad u^\partial = (1-\eta)u,$$
$$A^\circ = \eta A, \qquad A^\partial = (1-\eta)A.$$

On $W^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$ we define the functionals $\mathcal{S}$, $\mathcal{S}^\circ$, and $\mathcal{S}^\partial$ by

$$\mathcal{S}(w) := \int_{\Omega} dA \wedge dw, \quad \mathcal{S}^\circ(w) := \int_{\Omega} dA^\circ \wedge dw, \quad \mathcal{S}^\partial(w) := \int_{\Omega} dA^\partial \wedge dw.$$

Proposition 11 (Separating functional) *The following holds*

(a) $\mathcal{S}, \mathcal{S}^\circ, \mathcal{S}^\partial$ *define linear functionals on* $W^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$.
(b) *For all* $w \in H^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$ *we have* $S^\circ(w) = 0$.
(c) *For the functions* u, u^∂, u° *it holds:*

$$\mathcal{S}(u) = 0, \quad \mathcal{S}(u^\partial) = 1, \quad \mathcal{S}(u^\circ) = -1,$$
$$\mathcal{S}^\partial(u) = 1, \quad \mathcal{S}^\partial(u^\partial) = 1, \quad \mathcal{S}^\partial(u^\circ) = 0,$$
$$\mathcal{S}^\circ(u) = -1, \quad \mathcal{S}^\circ(u^\partial) = 0, \quad \mathcal{S}^\circ(u^\circ) = -1.$$

Proof The first claim follows from $dA \in L^{\Phi^*}(\Omega, \Lambda^{N-k})$. Due to the Stokes theorem and by approximation for all $\omega \in H^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$ it holds

$$\int_{\Omega} dA^\circ \wedge d\omega = \int_{\partial\Omega} A^\circ \wedge d\omega = 0.$$

Now $du \wedge dA = 0$ almost everywhere, therefore $\mathcal{S}(u) = 0$. Since $u^\partial \in C^\infty(\overline{\Omega}, \Lambda^{k-1})$ and $A^\partial \in C^\infty(\overline{\Omega}, \Lambda^{N-k-1}(\mathbb{R}^N))$, we can use the Stokes theorem and the third property of (Φ, k)-separating pair to obtain

$$\mathcal{S}^\partial(u^\partial) = \int_{\Omega} dA^\partial \wedge du^\partial = \int_{\partial\Omega} A^\partial \wedge du^\partial = \int_{\partial\Omega} A \wedge du = 1.$$

Since $A^\circ \wedge du^\partial$ belongs to $C_0^\infty(\Omega, \Lambda^{N-1})$, and $d(A^\circ \wedge du^\partial) = dA^\circ \wedge du^\partial$, again by the Stokes theorem we get

$$\mathcal{S}^\circ(u^\partial) = \int_{\Omega} dA^\circ \wedge du^\partial = 0.$$

Analogously, we obtain $\mathcal{S}^\partial(u^\circ) = 0$. Now,

$$\mathcal{S}^\circ(u^\circ) = \mathcal{S}(u) - \mathcal{S}^\partial(u^\partial) - \mathcal{S}^\circ(u^\partial) - \mathcal{S}^\partial(u^\circ) = 0 - 1 - 0 - 0 = -1.$$

This proves the claim. □

Corollary 12 *If there exists a pair of (Φ, k)-separating forms then*

$$H^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1}) \neq W^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1}).$$

Proof By Proposition 11, $\mathcal{S}^\circ = 0$ on $H^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$. On the other hand, $u \in W^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$ and $\mathcal{S}^\circ(u) = -1$. □

Corollary 13 *If there exists a pair of (Φ, k)-separating forms then*

$$H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1}) \neq \tilde{W}_T^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1}).$$

Proof For any $\varphi \in C_0^\infty(\Omega, \Lambda^{k-1})$ by the Stokes theorem we have

$$\mathcal{S}(\varphi) = \int_\Omega dA \wedge d\varphi = \int_{\partial\Omega} A \wedge d\varphi = 0.$$

On the other hand, $u^\circ \in \tilde{W}_T^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$ and by Proposition 11 we have $\mathcal{S}(u^\circ) = -1$. □

Theorem 14 (Lavrentiev gap) *If there exists a pair (u, A) of (Φ, k)-separating forms then for $b = dA^\circ$ the functional*

$$\mathcal{F}_{\Phi,b}(w) = \int_\Omega \Phi(x, |dw|)\, dV + \mathcal{S}^\circ(w) = \int_\Omega \Phi(x, |dw|)\, dV + \int_\Omega b \wedge dw$$

satisfies

$$\inf \mathcal{F}_{\Phi,b}(W_c^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})) < \inf \mathcal{F}_{\Phi,b}(C_0^\infty(\Omega, \Lambda^{k-1}))$$

and as a corollary

$$\inf \mathcal{F}_{\Phi,b}(W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})) < \inf \mathcal{F}_{\Phi,b}(H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})). \tag{4.1}$$

Proof By Proposition 11 and nonnegativity of Φ, $\mathcal{F}_{\Phi,b}(w) \geq 0$ for all $w \in H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$. On the other hand, for $t > 0$, using Propositions 11 and (2.10), we have

$$\mathcal{F}_{\Phi,b}(tu^\circ) = \int_\Omega \Phi(x, t|du^\circ|)\, dV - t \leq ct^s - t$$

with some $s > 1$. This implies $\mathcal{F}_{\Phi,b}(tu^\circ) < 0$ for sufficiently small t. □

4.2 Separating pairs and BVPs

Closely related to the Lavrentiev phenomenon is a special type of nonuniqueness for boundary value problems. In the simplest form this can be expressed as (1.8) and for minimization problems this reads as

$$\begin{aligned} w_t \neq h_t \quad &\text{and} \quad \mathcal{F}_{\Phi,0}(w_t) < \mathcal{F}_{\Phi,0}(h_t) \quad \text{where} \\ w_t &= \arg\min \mathcal{F}_{\Phi,0}\big(\omega_0 + W_T^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})\big), \\ h_t &= \arg\min \mathcal{F}_{\Phi,0}\big(\omega_0 + H_T^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})\big) \end{aligned} \tag{4.2}$$

for some boundary data $\omega_0 \in H^{1,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$.

First, we repeat a certain result from [11]. Let

$$\mathcal{F}(\omega)=\mathcal{F}_{\Phi,0}(\omega)=\int\limits_{\Omega}\Phi(x,|d\omega|)\,dV,\quad \mathcal{F}^{*}(g)=\mathcal{F}_{\Phi^{*},0}(\omega)=\int\limits_{\Omega}\Phi^{*}(x,|g(x)|)\,dV.$$

Let (u, A) be a (Φ, k)-separating pair. Denote $b = dA$.

Assumption 15 There exist $s, t > 0$ such that $\mathcal{F}(tu) + \mathcal{F}^{*}(sb) < ts$.

Theorem 16 (H-harmonic $\neq$ W-harmonic)
Under Assumption 15, for $\omega_0 = tu^{\partial}$ there holds (1.8) and (4.2).

Proof Set $b = dA$. We have $tu = tu^{\partial} + tu^{\circ} \in tu^{\partial} + W_c^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$. Thus,

$$\mathcal{F}(w_t) \leq \mathcal{F}(tu). \tag{4.3}$$

By the properties of the Hodge dual and the Young inequality,

$$sb\wedge dh_t = s(*b, dh_t) \leq s|*b|\cdot|dh_t| = s|b|\cdot|dh_t| \leq \Phi(x,|dh_t|)+\Phi^{*}(x,s|b|).$$

Hence

$$\mathcal{F}(h_t)=\int_{\Omega}\Phi(x,|dh_t|)\,dx \geq s\int_{\Omega} b\wedge dh_t - \int_{\Omega}\Phi^{*}(x,s|b|)\,dx = s\,\mathcal{S}(h_t)-\mathcal{F}^{*}(sb).$$

See [38] for estimates of exterior product submultiplication constant.

Since $h_t - tu^{\partial} \in H_T^{d,\Phi(\cdot)}(\Omega)$, we have $\mathcal{S}(h_t - tu^{\partial}) = 0$ by Proposition 11. This and $\mathcal{S}(u^{\partial}) = 1$ by the same Proposition imply

$$\mathcal{F}(h_t) = s\,\mathcal{S}(tu^{\partial}) - \mathcal{F}^{*}(sb) = ts - \mathcal{F}^{*}(sb). \tag{4.4}$$

Combining (4.3) and (4.4) we get

$$\mathcal{F}(h_t) - \mathcal{F}(w_t) \geq ts - \mathcal{F}(tu) - \mathcal{F}^{*}(sb)$$

for all $t, s > 0$. By Assumption 15 the right hand-side of last inequality is positive, and thus $\mathcal{F}(h_t) > \mathcal{F}(w_t)$. This proves the claim. □

4.3 Basic forms

In this section we introduce differential forms which will be building blocks of our examples. We do necessary calculations in the cubic setting, where the boundary orientation is straightforward.

Let $k \in \{1, \dots, N-1\}$. Define two groups of variables $\bar{x} = (x_1, \dots, x_k)$ and $\hat{x} = (x_{k+1}, \dots, x_N)$. Let $\Gamma_l(x)$, $x \in \mathbb{R}^l$, denote the fundamental solution of the Laplace equation in $\mathbb{R}^l$ with pole at the origin:

$$\Gamma_l(x)=\begin{cases}\frac{1}{2}|x|, & l=1,\\ -\frac{1}{2\pi}\ln\frac{1}{|x|}, & l=2,\\ -\frac{1}{(l-2)\sigma_l}|x|^{2-l}, & l>2.\end{cases}$$

Here and below σ_l denotes the surface area ($(l-1)$-volume) of the unit sphere in $\mathbb{R}^l$, and $|x|$ denotes the standard Euclidian norm of x.

Let $\theta : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that

$$\theta(t) = 1 \quad \text{for} \quad t \geq \frac{1}{2}, \quad \theta(t) = 0 \quad \text{for} \quad t \leq \frac{1}{4}, \quad |\theta'| \leq 4.$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that

$$\eta(t) = t \quad \text{for} \quad t \leq \frac{1}{4}, \quad \eta(t) = \frac{1}{2} \quad \text{for} \quad t \geq \frac{3}{4}, \quad \eta''(t) \leq 0.$$

Our basic forms are

$$u = \theta\left(\sqrt{N}\frac{|\hat{x}|}{\eta(|\bar{x}|)}\right) *_{\hat{x}} d\Gamma_{N-k}(\hat{x}), \tag{4.5}$$

$$A = \theta\left(\sqrt{N}\frac{|\bar{x}|}{\eta(|\hat{x}|)}\right) *_{\bar{x}} d\Gamma_{k}(\bar{x}), \tag{4.6}$$

Here $*_{\hat{x}}$ and $*_{\bar{x}}$ are applied only within respective variables, that is

$$*_{\hat{x}} d\Gamma_{N-k}(\hat{x}) = \frac{1}{\sigma_{N-k}} \sum_{j=k+1}^{N} (-1)^{j-k-1} \frac{x_j}{|\hat{x}|^{N-k}} dx_{k+1} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_N,$$

$$*_{\bar{x}} d\Gamma_k(\bar{x}) = \frac{1}{\sigma_k} \sum_{j=1}^{k} (-1)^{j-1} \frac{x_j}{|\bar{x}|^k} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_k.$$

Further for $(N-k-1)$-form u from (4.5) and $(k-1)$-form A from (4.6)we use the notation $u = \mathcal{P}_1(k, N-k, 0, 0)$ and $A = \mathcal{P}_2(k, N-k, 0, 0)$. Also, in this case we denote $\mathfrak{C} = \{0\}^k \subset \mathbb{R}^k$, $\mathfrak{S} = \{0\}^k \times \{0\}^{N-k} \subset \mathbb{R}^N$, and this pair of forms is denoted by $u_{\mathfrak{S}}, A_{\mathfrak{S}}$.

The following facts are straightforward.

Proposition 17 *Both* $*_{\hat{x}} d\Gamma_{N-k}(\hat{x})$ *and* $*_{\bar{x}} d\Gamma_k(\bar{x})$ *are harmonic*

$$d(*_{\hat{x}} d\Gamma_{N-k}(\hat{x})) = 0, \quad \delta(*_{\hat{x}} d\Gamma_{N-k}(\hat{x})) = 0,$$
$$d(*_{\bar{x}} d\Gamma_{k}(\bar{x})) = 0, \quad \delta(*_{\bar{x}} d\Gamma_{k}(\bar{x})) = 0$$

outside $\hat{x} = 0$ *and* $\bar{x} = 0$ *correspondingly. For cubes* $(-\varepsilon, \varepsilon)^{N-k} \subset \mathbb{R}^{N-k}$ *and* $(-\varepsilon, \varepsilon)^k \subset \mathbb{R}^k$, $\varepsilon > 0$, *there holds*

$$\int_{\partial(-\varepsilon,\varepsilon)^{N-k}} *_{\hat{x}} d\Gamma_{N-k}(\hat{x}) = 1, \quad \int_{\partial(-\varepsilon,\varepsilon)^{k}} *_{\bar{x}} d\Gamma_{k}(\bar{x}) = 1,$$

where the natural induced orientations of the boundary are assumed.

Proposition 18 *For the forms u and A given by* (4.5) *and* (4.6)

(a) *There holds*

$$\{u \neq 0\} \subset \{|\hat{x}| > \eta(|\bar{x}|)/(4\sqrt{N})\}, \quad \{A \neq 0\} \subset \{|\bar{x}| > \eta(|\hat{x}|)/(4\sqrt{N})\}.$$

(b) *The forms u and A are smooth outside the origin,*

$$\begin{aligned} &|\nabla u| \lesssim |\hat{x}|^{k-N}, \quad |\nabla A| \lesssim |\bar{x}|^{-k}, \\ &\{|\nabla u| \neq 0\} \subset \{|\hat{x}| > \eta(|\bar{x}|)/(4\sqrt{N})\}, \\ &\{|\nabla A| \neq 0\} \subset \{|\bar{x}| > \eta(|\hat{x}|)/(4\sqrt{N})\}. \end{aligned} \tag{4.7}$$

For any bounded domain $\Omega \subset \mathbb{R}^N$ *there holds* $u \in W^{1,1}(\Omega, \Lambda^{N-k-1})$, $A \in W^{1,1}(\Omega, \Lambda^{k-1})$, *and*

$$du = d\theta\left(\sqrt{N}\frac{|\hat{x}|}{\eta(|\bar{x}|)}\right) \wedge *_{\hat{x}} d\Gamma_{N-k}(\hat{x}), \quad dA = d\theta\left(\sqrt{N}\frac{|\bar{x}|}{\eta(|\hat{x}|)}\right) \wedge *_{\bar{x}} d\Gamma_k(\bar{x}).$$

$$\begin{aligned} &|du| \lesssim |\hat{x}|^{k-N}, \quad |dA| \lesssim |\bar{x}|^{-k}, \\ &\{|du| \neq 0\} \subset \{\eta(|\bar{x}|)/(2\sqrt{N}) > |\hat{x}| > \eta(|\bar{x}|)/(4\sqrt{N})\}, \\ &\{|dA| \neq 0\} \subset \{\eta(|\hat{x}|)/(2\sqrt{N}) > |\bar{x}| > \eta(|\hat{x}|)/(4\sqrt{N})\}. \end{aligned} \tag{4.8}$$

(c) *There holds* $|du| \cdot |dA| = 0$ *in* $\mathbb{R}^N \backslash \{0\}$.

(d) *For a nonnegative function* $F = F(\cdot, \cdot)$ *with nonnegative arguments, satisfying* $\triangle_2$*–condition in the second argument and* $F(\cdot, 0) = 0$,

$$\int\limits_{[-1,1]^N} F(|\hat{x}|, |du|)\, dV \lesssim \int\limits_0^{\sqrt{N}} F\left(t, t^{k-N}\right) t^{N-1}\, dt. \tag{4.9}$$

(e) *For a nonnegative function* $G = G(\cdot, \cdot)$ *with nonnegative arguments, satisfying* $\triangle_2$*–condition in the second variable and* $G(\cdot, 0) = 0$

$$\int\limits_{[-1,1]^N} G(|\hat{x}|, |dA|)\, dV \lesssim \int\limits_0^{\sqrt{N}} G\big(t, t^{-k}\big)\, t^{N-1}\, dt. \tag{4.10}$$

Proof The first two statements follow from the definition of u and A. Assume that $\Omega \subset \{|x| < R\}$. Using polar coordinates and estimates (4.7), we evaluate

$$\int\limits_\Omega |\nabla u|\, dV \lesssim \int\limits_0^R t^{k-N} t^{N-k-1} t^k\, dt = \int\limits_0^R t^{k-1}\, dt < \infty,$$

$$\int\limits_\Omega |\nabla A|\, dV \lesssim \int\limits_0^R t^{-k} t^{k-1} t^{N-k}\, dt = \int\limits_0^R t^{N-k-1}\, dt < \infty.$$

Thus the coefficients of the forms u and A belong to the Sobolev space $W^{1,1}(\Omega)$. Since the coefficients of the exterior derivative are linear combinations of derivatives of form coefficients, this implies $u \in W^{d,1}(\Omega, \Lambda^{N-k-1})$, $A \in W^{d,1}(\Omega, \Lambda^{k-1})$ and their exterior derivatives are as above together with estimates (4.8).

To prove that $|du| \cdot |dA| = 0$ in $\mathbb{R}^N \backslash \{0\}$, we note that $|dA| \neq 0$ implies $|\bar{x}| < |\hat{x}|/(2\sqrt{N})$ and $du \neq 0$ implies $|\hat{x}| < |\bar{x}|/(2\sqrt{N})$ (recall that $\eta(t) \leq t$).

The last two statements immediately follows from the above estimates for $|\nabla u|$ and $|\nabla A|$ and using polar coordinates. □

Let $Q = [-1, 1]^d$. For $x = (x_1, \ldots, x_l) \in \mathbb{R}^l$ the norm $|x|_\infty = \max\{|x_1|, \ldots, |x|_l\}$, while the standard Euclidian norm is denoted by $|x| = \sqrt{x_1^2 + \cdots + x_l^2}$. Recall that for $x \in \mathbb{R}^l$ there holds $|x|_\infty \leq |x| \leq \sqrt{l}|x|_\infty$.

Proposition 19 *For the form* A *given by* (4.6) *on* $\partial Q \cap \{|\hat{x}|_\infty < 1\}$ *there holds* $dA = 0$. *Thus*

$$\begin{aligned} &\{dA \neq 0\} \cap \partial Q \subset \{|\hat{x}|_\infty = 1\}, \quad u = *_{\hat{x}} d\Gamma_{N-k}(\hat{x}) \quad \text{on} \quad \{dA \neq 0\} \cap \partial Q, \\ &dA = d\theta(2\sqrt{N}|\bar{x}|) \wedge *_{\bar{x}} d\Gamma_k(\bar{x}) \quad \text{on} \quad \{dA \neq 0\} \cap \partial Q. \end{aligned} \tag{4.11}$$

Proof Note that

$$\{dA \neq 0\} \subset \{|\bar{x}| < \eta(|\hat{x}|)/(2\sqrt{N})\} \subset \{|\bar{x}| < |\hat{x}|/(2\sqrt{N})\} \subset \{|\hat{x}|_\infty > 2|\bar{x}|_\infty\}.$$

Then for $x \in \{dA \neq 0\} \cap \{|\bar{x}|_\infty = 1\}$ there holds $|\hat{x}|_\infty > 2$, which implies the first claim. Thus,

$$\{dA \neq 0\} \cap \partial Q \subset \{|\hat{x}|_\infty = 1\} \cap \{1/(8\sqrt{N}) \leq |\bar{x}| \leq 1/(4\sqrt{N})\}.$$

On the set $\{dA \neq 0\} \cap \partial Q$ we have $|\hat{x}|_\infty = 1$, $|\hat{x}| \geq 1$ and $|\bar{x}| \leq 1/4$, $\eta(|\bar{x}|) \leq 1/4$, so $\theta(\sqrt{N}|\hat{x}|/\eta(|\bar{x}|)) = 1$, $\eta(|\hat{x}|) = 1$, $d\eta(|\hat{x}|) = 0$, and we get (4.11) by Definitions (4.5) and (4.6) of u and A. □

The following statement is central in our considerations. Let $\partial[-1,1]^N$ be the boundary of the cube $Q = [-1,1]^N$ with the natural induced orientation.

Lemma 20 *For the forms u and A given by* (4.5) *and* (4.6) *there holds*

$$\int_{\partial[-1,1]^N} u \wedge dA = (-1)^{k(N-k)}, \qquad \int_{\partial[-1,1]^N} A \wedge du = 1. \tag{4.12}$$

Proof Below we use the notation of integration on cubic chains see [54, Chapter 4]. Let

$$Q^l : [-1,1]^l \to \mathbb{R}^l, \quad Q^l(x) = x,$$

be the standard l-cube and ∂Q^l its boundary with the natural induced orientation.

Denote the boundary faces of Q^N as

$$I_j^{\pm} : [-1,1]^{N-1} \to \mathbb{R}^N,$$
$$I_j^{\pm}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) = (x_1, \ldots, x_{j-1}, \pm 1, x_{j+1}, \ldots, x_N).$$

Then $\partial Q^N = \sum_{j=1}^N (-1)^{j-1}(I_j^{(+)} - I_j^{(-)})$.

By $Q^{N-k}[\bar{x}]$ we denote the $(N-k)$-dimensional cubes with centers at $(\bar{x}, 0)$,

$$Q^{N-k}[\bar{x}] : [-1,1]^{N-k} \to \mathbb{R}^N, \quad Q^{N-k}[\bar{x}](x_{k+1}, \ldots, x_N) = (\bar{x}, x_{k+1}, \ldots, x_N).$$

The faces $\tilde{I}_j^{(\pm)}[\bar{x}]$ of these cubes are

$$\tilde{I}_j^{(\pm)}[\bar{x}] : [-1,1]^{N-k-1} \to \mathbb{R}^N, \quad \tilde{I}_j^{(\pm)}[\bar{x}](x_{k+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$$
$$= (\bar{x}, x_{k+1} \ldots, x_{j-1}, \pm 1, x_{j+1}, \ldots, x_N),$$

and the boundary of $Q^{N-k}[\bar{x}]$ is $\partial Q^{N-k}[\bar{x}] = \sum_{j=k+1}^N (-1)^{j-k-1}(\tilde{I}_j^{(+)}[\bar{x}] - \tilde{I}_j^{(-)}[\bar{x}])$.

By (4.11) we have

$$\int_{\partial Q^N} u \wedge dA = \int_{\partial Q^N} *_{\hat{x}} d\Gamma_{N-k}(\hat{x}) \wedge dA$$
$$= \sum_{j=k+1}^N (-1)^{j-1}\Big(\int_{I_j^{(+)}} - \int_{I_j^{(-)}}\Big) *_{\hat{x}} d\Gamma_{N-k}(\hat{x}) \wedge d\theta\big(2\sqrt{N}|\bar{x}|\big) \wedge *_{\bar{x}} d\Gamma_k(\bar{x})$$
$$= (-1)^{(N-k)k} \sum_{j=k+1}^N (-1)^{j-k-1}\Big(\int_{I_j^{(+)}} - \int_{I_j^{(-)}}\Big) d\theta\big(2\sqrt{N}|\bar{x}|\big) \wedge *_{\bar{x}} d\Gamma_k(\bar{x}) \wedge *_{\hat{x}} d\Gamma_{N-k}(\hat{x})$$

$$= (-1)^{(N-k)k} \int\limits_{Q^k} d(\theta(2\sqrt{N}|\bar{x}|) *_{\bar{x}} d\Gamma_k(\bar{x})) \sum_{j=k+1}^{N} (-1)^{j-k-1} \Bigg(\int\limits_{\tilde{I}_j^{(+)}[\bar{x}]} - \int\limits_{\tilde{I}_j^{(-)}[\bar{x}]} \Bigg) *_{\hat{x}} d\Gamma_{N-k}(\hat{x})$$

$$= (-1)^{(N-k)k} \int\limits_{Q^k} d(\theta(2\sqrt{N}|\bar{x}|) *_{\bar{x}} d\Gamma_k(\bar{x})) = (-1)^{(N-k)k} \int\limits_{\partial Q^k} \theta(2\sqrt{N}|\bar{x}|) *_{\bar{x}} d\Gamma_k(\bar{x})$$

$$= (-1)^{(N-k)k} \int\limits_{\partial Q^k} *_{\bar{x}} d\Gamma_k(\bar{x}) = (-1)^{(N-k)k}.$$

Thus we get the first relation in (4.12). Since

$$A \wedge du - (-1)^{k(N-k)} u \wedge dA = d\omega, \quad \omega = (-1)^{(k-1)(N-k)} u \wedge A,$$

we have

$$\int\limits_{\partial[-1,1]^N} \Big(A \wedge du - (-1)^{k(N-k)} u \wedge dA \Big) = 0.$$

This yields the second relation in (4.12). The proof of Lemma 20 is complete. □

To summarize the results of this section, we have shown that the pair of forms u and A given by (4.5) and (4.6) is $(\Phi, N-k)$-separating in $\Omega = [-1, 1]^N$ provided that the integral (4.9) converges for $F \geq \Phi$ and the integral (4.10) converges for $G \geq \Phi^*$.

4.4 Generalized Cantor sets and their properties

In this section we construct (generalized) Cantor sets.

Let l_j, $j = 0, 1, 2, \ldots$ be a decreasing sequence of positive numbers starting from $l_0 = 1$:

$$1 = l_0 > l_1 > l_2 > \cdots$$

such that $l_{j-1} > 2l_j$ for all $j \in \mathbb{N}$. We start from $I_{0,1} = [-1/2, 1/2]$. On each m-th step we we remove the open middle third of length $l_j - 2l_{j+1}$ from the interval $I_{m,j}$, $j = 1, \ldots, 2^m$ to obtain the next generation set of closed intervals $I_{m+1,j}$, $j = 1, \ldots, 2^{m+1}$. The union of the closed intervals $I_{m,j} = [a_{m,j}, b_{m,j}]$, $j = 1, \ldots, 2^m$ of length l_m from the same generation forms the pre-Cantor set $C_m = \bigcup_{j=1}^{2^m} I_{m,j}$. The Cantor set $\mathfrak{C} = \cap_{m=0}^{\infty} C_m$ is the intersection of all pre-Cantor sets C_m.

On each mth step we define the pre-Cantor measure as $\mu_m = |C_m|^{-1} \mathbb{1}_{C_m}$, where $|C_m| = 2^m l_m$ is the standard Lebesgue measure of C_m, and the weak limit of the measures μ_m is the Cantor measure corresponding to $\mathfrak{C}$.

We require further that

$$l_{m-1} - 2l_m > l_m - 2l_{m+1} \Leftrightarrow l_{m+1} > \frac{3l_m - l_{m-1}}{2},$$

at least for all sufficiently large m.

If the sequence l_j satisfies the conditions above only for sufficiently large $j \geq j_0$, then we modify it by taking the sequence $\tilde{l}_j = l_{j+j_0}(l_{j_0})^{-1}$, $j = 0, 1, 2, \ldots$

For $k \in \mathbb{N}$ by $\mathfrak{C}^k$ and μ^k we denote the Cartesian powers of k copies of $\mathfrak{C}$ and its corresponding Cantor measure, respectively.

Definition (*Generalized Cantor sets*). Let $l_j = \lambda^j j^\gamma$, $\lambda \in (0, 1/2)$, $\gamma \in \mathbb{R}$. We denote the corresponding Cantor set by $\mathfrak{C}_{\lambda,\gamma}$, the Cartesian product of its k copies is $\mathfrak{C}^k_{\lambda,\gamma}$. For $\mathfrak{C}^k_{\lambda,\gamma}$ we denote $\mathfrak{D} = -k \ln 2/\ln\lambda$, so that $\lambda^{\mathfrak{D}} = 2^{-k}$. We denote the Cantor measure corresponding to $\mathfrak{C}_{\lambda,\gamma}$ by $\mu_{\lambda,\gamma}$ and its k-th Cartesian power by $\mu^k_{\lambda,\gamma}$.

Definition (*Meager Cantor sets*). Let $l_j = \exp(-2^{j/\gamma})$, $\gamma > 0$. Denote the corresponding Cantor set by $\mathfrak{C}_{0,\gamma}$, and its Cartesian products by $\mathfrak{C}^k_{0,\gamma}$. For these sets we denote $\mathfrak{D} = 0$. We denote the corresponding Cantor measures by $\mu^k_{0,\gamma}$.

Denote $\mathfrak{C}_t = \{\mathrm{dist}(\bar{x}, \mathfrak{C}) < t\}$, where $\mathfrak{C}$ is one of $\mathfrak{C}^k_{\lambda,\gamma}$ or $\mathfrak{C}^k_{0,\gamma}$ defined above. Denote by $d_\infty(\bar{x}, \mathfrak{C})$ the distance from $\bar{x}$ to $\mathfrak{C}$ in the maximum norm and let $\mathfrak{C}_{*,t} = \{d_\infty(\bar{x}, \mathfrak{C}) < t\}$. It is clear that $\mathfrak{C}_t \subset \mathfrak{C}_{*,t}$. Let $|F|_k$ denote the standard Lebesgue k-measure of $F \subset \mathbb{R}^k$. In the following lemma $B^{\bar{x}}_t$ is the open ball in $\mathbb{R}^k$ with center at $\bar{x}$ and radius t.

Lemma 21 *We have*

$$|(\mathfrak{C}^k_{\lambda,\gamma})_t|_k \lesssim t^{k-\mathfrak{D}}(\ln t^{-1})^{\gamma\mathfrak{D}}, \quad \mu_{\lambda,\gamma}(B^{\bar{x}}_t) \lesssim t^{\mathfrak{D}}(\ln t^{-1})^{-\gamma\mathfrak{D}}, \tag{4.13}$$

and

$$|(\mathfrak{C}^k_{0,\gamma})_t|_k \lesssim t^k(\ln t^{-1})^{\gamma k}, \quad \mu_{0,\gamma}(B^{\bar{x}}_t) \lesssim (\ln t^{-1})^{-\gamma k}. \tag{4.14}$$

Proof Let l_j be the sequence of interval lengthes defining the corresponding Cantor set. Let $t \in (l_j/2 - l_{j+1}, l_{j-1}/2 - l_j)$. The set $\mathfrak{C}_{*,t}$ consists of 2^{kj} identical cubes of the form

$$|\bar{x} - \bar{x}_{j,s}|_\infty < \frac{l_j}{2} + t.$$

So

$$|\mathfrak{C}_{*,t}|_k \le 2^{k(j-1)}(l_j + 2t)^k.$$

First consider the case $\mathfrak{C} = \mathfrak{C}^k_{\lambda,\gamma}$ with $\lambda > 0$. Then $l_j + 2t \le l_{j-1} - l_j \le ct$ with some constant c independent of j, so

$$|\mathfrak{C}_t|_k \le |\mathfrak{C}_{*,t}|_k \lesssim 2^{kj}t^k.$$

Recalling that $\mathfrak{D} = -k\ln 2/\ln\lambda$ and $\lambda^{\mathfrak{D}} = 2^{-k}$, we get

$$2^{kj} = \lambda^{-j\mathfrak{D}} = l_j^{-\mathfrak{D}} j^{\gamma\mathfrak{D}} \approx l_j^{-\mathfrak{D}}(\ln(1/l_j))^{\gamma\mathfrak{D}} \approx t^{-\mathfrak{D}}(\ln t^{-1})^{\gamma\mathfrak{D}}.$$

Thus we arrive at the first inequality in (4.13).

Now consider the case $\mathfrak{C} = \mathfrak{C}^k_{0,\gamma}$ (ultrathin Cantor sets). Then we get

$$2^{kj} = \left(\ln\frac{1}{l_j}\right)^{\gamma k} \approx \left(\ln\frac{1}{t}\right)^{\gamma k}$$

and this yields the second inequality from (4.13).

Now let us estimate $\mu(B^{\bar{x}}_r)$. Any interval of length $2t$ with $t \in (l_j/2 - l_{j+1}, l_{j-1}/2 - l_j)$ can intersect at most one interval forming the j-th iteration of the pre-Cantor set. Since $B^{\bar{x}}_t$ lies within a cube with edge $2t$, then $\mu^k_{\lambda,\gamma}(B^{\bar{x}}_t) \le 2^{-jk}$. Using the above estimates for 2^{jk} we arrive at (4.14). The proof of Lemma 21 is complete.

□

4.5 From one singular point to fractal sets

Let $k \in \{1, \dots, N\}$, $\lambda \in (0, 1/2)$ and $\gamma \in \mathbb{R}$, or $\lambda = 0$ and $\gamma > 0$ be given. Let Ω be the ball of radius $\sqrt{N}$ in $\mathbb{R}^N$ centered at the origin.

Now let $\mathfrak{C} = \mathfrak{C}^k_{\lambda,\gamma}$ be the generalized Cantor set with the given parameters and $\mu = \mu^k_{\lambda,\gamma}$ be the Cantor measure corresponding to $\mathfrak{C}^k_{\lambda,\gamma}$. Our construction will be based on the singular (or fractal contact/ barrier) set $\mathfrak{S} = \mathfrak{C}^k_{\lambda,\gamma} \times \{0\}^{N-k}$. Recall that for generalized Canter sets $\mathfrak{C}^k_{\lambda,\gamma}$ we set $\mathfrak{D} = -k \ln 2/\ln \lambda$ (equivalently, $\lambda = 2^{-k/\mathfrak{D}}$) and for meager Cantor sets $\mathfrak{C}^k_{0,\gamma}$ we set $\mathfrak{D} = 0$.

Let $d(\bar{x}, \mathfrak{C})$ be the generalized distance, see [55, Chapter VI, §2] from $\bar{x}$ to $\mathfrak{C}$. In particular, $d(\bar{x}, \mathfrak{C}) \in C^\infty(\mathbb{R}^k \backslash \mathfrak{C})$,

$$\frac{1}{C}\mathrm{dist}(\bar{x}, \mathfrak{C}) \le d(\bar{x}, \mathfrak{C}) \le C\mathrm{dist}(\bar{x}, \mathfrak{C}), \quad |\nabla^j d(\bar{x}, \mathfrak{C})| \le C_j(\mathrm{dist}(\bar{x}, \mathfrak{C}))^{1-j}, \; j \in \mathbb{N}, \tag{4.15}$$

where $C, C_j > 1$ and $\mathrm{dist}(\bar{x}, \mathfrak{C})$ is the standard Euclidian distance from $\bar{x}$ to $\mathfrak{C}$. Without loss, we assume that $C \ge 4$.

Let $\theta : \mathbb{R} \to \mathbb{R}$ be a smooth nondecreasing function such that $\theta(t) = 1$ for $t \ge 1/2$, $\theta(t) = 0$ for $t \le 1/4$, $|\theta'| \le 4$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth nondecreasing concave function such that $\eta(t) = t$ for $t \le 1/4$ and $\eta(t) = 1/2$ for $t \ge 3/4$.

For $\mathfrak{S} = \mathfrak{C} \times \{0\}^{N-k}$, $\mathfrak{C} = \mathfrak{C}^k_{\lambda,\gamma}$, we define the $(N-k-1)$-form $u_{\mathfrak{S}}$, the $(k-1)$-form $A_{\mathfrak{S}}$, and the function $\rho_{\mathfrak{S}}$ by

$$u_{\mathfrak{S}} = \theta\left(\sqrt{N}C\frac{|\hat{x}|}{\eta(d(\bar{x}, \mathfrak{C}))}\right) *_{\hat{x}} d\Gamma_{N-k}(\hat{x}), \tag{4.16}$$

$$A_{\mathfrak{S}}(\bar{x}, \hat{x}) = \int A_\diamond(\bar{x} - \bar{y}, \hat{x}) d\mu(\bar{y}), \quad A_\diamond = \theta\left(\sqrt{N}\frac{|\bar{x}|}{\eta(|\hat{x}|)}\right) *_{\bar{x}} d\Gamma_k(\bar{x}), \tag{4.17}$$

$$\rho_{\mathfrak{S}} = \theta\left(C\frac{|\hat{x}|}{3\eta(d(\bar{x}, \mathfrak{C}))}\right). \tag{4.18}$$

Here the constant C is from (4.15). The integral is understood as integrating the coefficients of the form.In Ω there holds $|\nabla^j \rho_{\mathfrak{S}}(x)| \le C(j)|\hat{x}|^{-j}$, $j \in \mathbb{N}$.

Further the $(N-k-1)$-form $u_{\mathfrak{S}}$ defined by (4.16) and $(k-1)$-form $A_{\mathfrak{S}}$ defined by (4.17) corresponding to the space dimension N and the Cantor set $\mathfrak{C}_{\lambda,\gamma}$ will be denoted by $\mathcal{P}_1(k, N-k, \mathfrak{D}, \gamma)$ and $\mathcal{P}_2(k, N-k, \mathfrak{D}, \gamma)$. That is, $u_{\mathfrak{S}} = \mathcal{P}_1(k, N-k, \mathfrak{D}, \gamma)$ and $A_{\mathfrak{S}} = \mathcal{P}_2(k, N-k, \mathfrak{D}, \gamma)$. The function $\rho_{\mathfrak{S}}$ defined in (4.18) will be also denoted by $\mathcal{P}_0(k, N-k, \mathfrak{D}, \gamma)$.

Lemma 22 *There holds $u \in W^{1,1}(\Omega, \Lambda^{N-k-1}) \cap C^\infty(\overline{\Omega}\backslash\mathfrak{S}, \Lambda^{N-k-1})$ and*

$$\begin{aligned} &|\nabla u|(\bar{x}, \hat{x}) \lesssim \mathbb{1}_{\{|\hat{x}| > \mathrm{dist}\,(\bar{x},\mathfrak{C})/(8C^2\sqrt{N})\}}|\hat{x}|^{k-N}, \\ &|du|(\bar{x}, \hat{x}), |\delta u|(\bar{x}, \hat{x}) \lesssim \mathbb{1}_{\{\mathrm{dist}\,(\bar{x},\mathfrak{C})/(8C^2\sqrt{N}) < |\hat{x}| < \mathrm{dist}\,(\bar{x},\mathfrak{C})/(2\sqrt{N})\}}|\hat{x}|^{k-N}. \end{aligned} \tag{4.19}$$

For a nonnegative function $F = F(\cdot, \cdot)$ with nonnegative arguments, satisfying $\triangle_2$-condition in the second argument and $F(\cdot, 0) = 0$,

$$\int_\Omega F(|\hat{x}|, |\nabla u|)\, dV \lesssim \int_0^{\sqrt{N}} F\left(t, t^{k-N}\right) t^{N-k-1} |\mathfrak{C}_t|_k\, dt. \tag{4.20}$$

Proof Clearly, u is smooth in $\Omega \setminus \mathfrak{S}$, and in particular its coefficients have ACL property. Immediately from the definition of u, (4.15) and Proposition 17 we obtain (4.19). Then using polar coordinates in $\mathbb{R}^{N-k}$ we obtain (4.20). In particular, for $F(s,\tau) = \tau$ by Lemma 21 we get $|\nabla u| \in L^1(\Omega)$. By [63, Theorem 2.1.4] we conclude that the coefficients of u are in $W^{1,1}(\Omega)$. ◻

Lemma 23 *There holds $A \in W^{d,1}(\Omega, \Lambda^{k-1}) \cap C^\infty(\overline{\Omega}\setminus\mathfrak{S}, \Lambda^{k-1})$,*

$$dA(\bar{x},\hat{x}) = \int b_\diamond(\bar{x}-\bar{y},\hat{x})d\mu(\bar{y}), \quad \textit{where} \quad b_\diamond = dA_\diamond,$$

and

$$|dA|(\bar{x},\hat{x}) \lesssim |\hat{x}|^{-k}\mu(B^{\bar{x}}_{|\hat{x}|})\mathbb{1}_{\{\mathrm{dist}\,(\bar{x},\mathfrak{C})<|\hat{x}|/(2\sqrt{N})\}}(\bar{x},\hat{x}). \tag{4.21}$$

For a nonnegative function $G = G(\cdot,\cdot)$ with nonnegative arguments, satisfying $\triangle_2$–condition in the second variable and $G(\cdot,0) = 0$,

$$\int_\Omega G(|\hat{x}|,|dA|)\,dV \lesssim \int_0^{\sqrt{N}} G\big(t, t^{-k}\sup_{\bar{x}}\mu(B^{\bar{x}}_t\big)|\mathfrak{C}_t|_k\, t^{N-k-1}\,dt. \tag{4.22}$$

Proof Clearly, A is smooth outside the contact set $\mathfrak{S}$. Denote

$$b(\bar{x},\hat{x}) = \int b_\diamond(\bar{x}-\bar{y},\hat{x})d\mu(\bar{y}).$$

Then

$$|b|(\bar{x},\hat{x}) \le \int |b_\diamond|(\bar{x}-\bar{y},\hat{x})d\mu(\bar{y}) \lesssim \int |\hat{x}|^{-k}\mathbb{1}_{\{|\hat{x}|/(8\sqrt{N})<|\bar{x}|<|\hat{x}|/(2\sqrt{N})\}}(\bar{x}-\bar{y},\hat{x})d\mu(\bar{y})$$
$$\lesssim |\hat{x}|^{-k}\int \mathbb{1}_{\{|\bar{x}|<|\hat{x}|/(2\sqrt{N})\}}(\bar{x}-\bar{y},\hat{x})d\mu(\bar{y}) \le |\hat{x}|^{-k}\mu(B^{\bar{x}}_{|\hat{x}|/(2\sqrt{N})})\mathbb{1}_{\{\mathrm{dist}\,(\bar{x},\mathfrak{C})<|\hat{x}|/(2\sqrt{N})\}}(\bar{x},\hat{x}).$$

Using polar coordinates in $\mathbb{R}^{N-k}$ we evaluate

$$\int_\Omega G(|\hat{x}|,|b|)\,dV \lesssim \int_\Omega G\big(|\hat{x}|,|\hat{x}|^{-k}\mu(B^{\bar{x}}_{|\hat{x}|})\big)\mathbb{1}_{\{\mathrm{dist}\,(\bar{x},\mathfrak{C})<|\hat{x}|\}}(\hat{x},\bar{x})d\bar{x}d\hat{x}$$
$$\lesssim \int_{\{|\hat{x}|\le\sqrt{N}\}} G\big(|\hat{x}|,|\hat{x}|^{-k}\sup_{\bar{x}}\mu(B^{\bar{x}}_{|\hat{x}|})\big)|\{\mathrm{dist}\,(\bar{x},\mathfrak{C}) < |\hat{x}|\}|_k\,d\hat{x}$$
$$= \int_0^{\sqrt{N}} G\big(t, t^{-k}\sup_{\bar{x}}\mu(B^{\bar{x}}_t\big)|\mathfrak{C}_t|_k\, t^{N-k-1}\,dt.$$

In particular, using $G(s,t) = t$ and Lemma 21 we get $b \in L^1(\Omega)$.

For any $\varphi \in C_0^\infty(\Omega,\Lambda^k)$ using $A_\diamond \in W^{1,1}(\Omega,\Lambda^k)$ we have

$$(A,\delta\varphi) = \int_\Omega \left(\int A_\diamond(\bar{x}-\bar{y},\hat{x})d\mu(\bar{y})\right) \wedge *\delta\varphi$$
$$= \int d\mu(y)\int_\Omega A_\diamond(\bar{x}-\bar{y},\hat{x}) \wedge *\delta\varphi = \int d\mu(y)\int_\Omega b_\diamond(\bar{x}-\bar{y},\hat{x}) \wedge *\varphi$$

$$= \int_\Omega b \wedge *\varphi = (b, \varphi).$$

By definition, this implies $dA = b$, and as a consequence (4.21) and (4.22). The proof of Lemma 23 is complete. □

Proposition 24 *There holds* $|du| \cdot |dA| = 0$ *a.e. in* Ω.

Proof By Lemmas 22 and 23,

$$\{|du| > 0\} \subset \{|\hat{x}| \le \mathrm{dist}\,(\bar{x}, \mathfrak{C})/(2\sqrt{N})\}, \quad \{|dA| > 0\} \subset \{|\hat{x}| \ge 2\sqrt{N}\,\mathrm{dist}\,(\bar{x}, \mathfrak{C})\}.$$

Thus $|du| \cdot |dA| = 0$ in $\overline{\Omega}\backslash\mathfrak{S}$. The claim follows since $\mathfrak{S}$ has N-dimensional Lebesgue measure zero. □

Proposition 25 *The function* $\rho_{\mathfrak{S}} \in C^\infty(\mathbb{R}^N\backslash\mathfrak{S})$, $0 \le \rho_{\mathfrak{S}} \le 1$, $\rho_{\mathfrak{S}} = 1$ *on the support of* dA *and* $\rho_{\mathfrak{S}} = 0$ *on the support of* du.

Proof The first two properties are immediate from the definition of $\rho_{\mathfrak{S}}$. From the definition of u,

$$\mathrm{supp}\, du \subset \{|\hat{x}| \le \eta(d(\bar{x}, \mathfrak{C}))/(\sqrt{N}C)\}.$$

On this set, $\rho_{\mathfrak{S}} = 0$ (recall that $\theta(t) = 0$ if $t \le 1/4$). On the other hand,

$$\mathrm{supp}\, dA \subset \{|\hat{x}| \ge 2\sqrt{N}\mathrm{dist}(\bar{x}, \mathfrak{C})\} \subset \{|\hat{x}| \ge 2\sqrt{N}C^{-1}\eta(d(\bar{x}, \mathfrak{C}))\}.$$

On this set, $\rho_{\mathfrak{S}} = 1$ (recall that $\theta(t) = 1$ if $t \ge 1/2$). □

Proposition 26 *On the boundary of* $Q = [-1, 1]^N$ *there holds*

(a) $u \wedge dA = 0$ *on* $\partial[-1, 1]^N \cap \{|\hat{x}|_\infty < 1\}$;
(b) *On* $\partial Q \cap \{|\hat{x}|_\infty = 1\}$ *there holds*

$$u = *_{\hat{x}} d\Gamma_{N-k}, \quad dA = \int d(\theta(2\sqrt{N}|\bar{x} - \bar{y}|)) \wedge *_{\bar{x}} d\Gamma_k(\bar{x} - \bar{y})\, d\mu(\bar{y}).$$

Proof By construction,

$$\{|dA| \ne 0\} \subset \{\mathrm{dist}(\bar{x}, \mathfrak{C}) < |\hat{x}|/(2\sqrt{N})\} \subset \{\mathrm{dist}(\bar{x}, \mathfrak{C}) < |\hat{x}|_\infty/2\}.$$

If $|\bar{x}|_\infty = 1$, then $\mathrm{dist}(\bar{x}, \mathfrak{C}) \ge 1/2$ (recall that $\mathfrak{C} \subset [-1/2, 1/2]^k$), so $dA(\bar{x}, \hat{x}) \ne 0$ implies $|\hat{x}|_\infty > 1$. Thus

$$\{|dA| \ne 0\} \cap \partial Q \subset \{|\hat{x}|_\infty = 1\} \cap \partial Q.$$

Then in the definition of u for the argument of θ for $|\hat{x}|_\infty \le 1$, $|\bar{x}|_\infty = 1$ we have

$$\sqrt{N}C\frac{|\hat{x}|}{\eta(d(\bar{x}, \mathfrak{C}))} \ge \sqrt{N}\frac{|\hat{x}|}{\mathrm{dist}\,(\bar{x}, \mathfrak{C})} \ge 2.$$

This implies $\theta\left(\sqrt{N}C\frac{|\hat{x}|}{\eta(d(\bar{x}, \mathfrak{C}))}\right) = 1$, and therefore $u = *_{\hat{x}} d\Gamma_{N-k}$ on $\partial[-1, 1]^N \cap \{|\hat{x}|_\infty = 1\}$.

The formula for dA follows then from $\eta(|\hat{x}|) = 1/2$ for x in a neighbourhood of $[-1, 1]^N \cap \{|\hat{x}|_\infty = 1\}$ and smoothness of the integrand in the definition of A for $|\hat{x}| > 0$. □

Lemma 27 *For the forms u and A given by* (4.16) *and* (4.17) *there holds*

$$\int_{\partial[-1,1]^N} u \wedge dA = (-1)^{k(N-k)}, \qquad \int_{\partial[-1,1]^N} A \wedge du = 1. \tag{4.23}$$

Proof Using Proposition 26 and the notation of integration on cubic chains similar to Lemma 20 we obtain

$$\begin{aligned}
&\int_{\partial Q^N} u \wedge dA = \sum_{j=k+1}^{N} (-1)^{j-1} \left(\int_{I_j^+} - \int_{I_j^-} \right) *_{\hat{x}} d\Gamma_{N-k}(\hat{x}) \wedge \\
&\int d(\theta(2\sqrt{N}|\bar{x} - \bar{y}|)) *_{\bar{x}} d\Gamma_k(\bar{x} - \bar{y}))d\mu(\bar{y}) \\
&= (-1)^{k(N-k)} \sum_{j=k+1}^{N} (-1)^{j-k-1} \left(\int_{I_j^+} - \int_{I_j^-} \right) \\
&\quad \times \left(d \int \theta(2\sqrt{N}|\bar{x} - \bar{y}|) *_{\bar{x}} d\Gamma_k(\bar{x} - \bar{y}) d\mu(\bar{y}) \right) \wedge *_{\hat{x}} d\Gamma_{N-k}(\hat{x}) \\
&= (-1)^{k(N-k)} \int \left[\int_{Q^k} d(\theta(2\sqrt{N}|\bar{x} - \bar{y}|) *_{\bar{x}} d\Gamma_k(\bar{x} - \bar{y})) \right. \\
&\left. \sum_{j=k+1}^{N} (-1)^{j-k-1} \left(\int_{\tilde{I}_j^+(\bar{x})} - \int_{\tilde{I}_j^-(\bar{x})} \right) *_{\hat{x}} d\Gamma_{N-k}(\hat{x}) \right] d\mu(\bar{y}) \\
&= (-1)^{k(N-k)} \int d\mu(\bar{y}) = (-1)^{k(N-k)}.
\end{aligned}$$

Here we used that

$$\begin{aligned}
&\int_{Q^k} d(\theta(2\sqrt{N}|\bar{x} - \bar{y}|) *_{\bar{x}} d\Gamma_k(\bar{x} - \bar{y})) \sum_{j=k+1}^{N} (-1)^{j-k-1} \left(\int_{\tilde{I}_j^+(\bar{x})} - \int_{\tilde{I}_j^-(\bar{x})} \right) *_{\hat{x}} d\Gamma_{N-k}(\hat{x}) \\
&= \int_{Q^k} d\big(\theta(2\sqrt{N}|\bar{x} - \bar{y}|) *_{\bar{x}} d\Gamma_k(\bar{x} - \bar{y})\big) = \int_{\partial Q^k} \theta(2\sqrt{N}|\bar{x} - \bar{y}|) *_{\bar{x}} d\Gamma_k(\bar{x} - \bar{y}) \\
&= \int_{\partial Q^k} *_{\bar{x}} d\Gamma_k(\bar{x} - \bar{y}) = 1
\end{aligned}$$

for all $y \in [-1/2, 1/2]^k \subset \mathbb{R}^k$.

To calculate the second integral we use the same argument as in Lemma 20: the form $A \wedge du - (-1)^{k(N-k)} u \wedge dA$ is exact, therefore its integral over $\partial[-1, 1]^N$ is zero. □

4.6 Work-tool

Here we gather the results of Sect. 4.5, namely of Lemmas 22, 23, 27 and Propostion 24. Recall that a pair of $(k-1)$-form u and $(N-k-1)$-form A is (Φ, k)-separating if u and A are regular outside a closed set $\mathfrak{S} \subset \Omega$ of zero Lebesgue N-measure, $u \in W^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$, $A \in W^{d,\Phi^*(\cdot)}(\Omega, \Lambda^{N-k-1})$, $|du| \cdot |dA| = 0$ a.e. in Ω, and $\int_\Omega A \wedge du = 1$. The form of the following statement represents the duality between u and A. The function $\Phi = \Phi(x,t)$ is a generalized Orlicz function, as in Sect. 2.4. The following lemma gives the general work-tool to construct a (Φ, k)-separating pair. Let $\Omega = [-1,1]^N$ or $\Omega = \{|x| < \sqrt{N}\}$.

Lemma 28 *(**i**) Let $u = \mathcal{P}_1(N-k, k, \mathfrak{D}, \gamma)$ and $A = \mathcal{P}_2(N-k, k, \mathfrak{D}, \gamma)$. Let Φ be such that $\Phi(x,t) \le F_1(|\hat{x}|, t)$ on the support of du and $\Phi(x,t) \ge F_2(|\hat{x}|, t)$ on the support of dA. If*

$$\mathcal{I}_1 := \int_0^{\sqrt{N}} F_1(t, t^{-k}) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt < \infty,$$
$$\mathcal{I}_2 := \int_0^{\sqrt{N}} F_2^*\left(t, t^{k-N} \sup_{\bar{x}} \mu\left(B_t^{\bar{x}}\right)\right) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt < \infty, \tag{4.24}$$

then the pair (u, A) is (Φ, k)-separating.
*(**ii**) Let $u = \mathcal{P}_2(k, N-k, \mathfrak{D}, \gamma)$ and $A = (-1)^{k(N-k)} \mathcal{P}_1(k, N-k, \mathfrak{D}, \gamma)$. Let Φ be such that $\Phi(x,t) \le F_1(|\hat{x}|, t)$ on the support of du and $\Phi(x,t) \ge F_2(|\hat{x}|, t)$ on the support of dA. If*

$$\mathcal{I}_1 := \int_0^{\sqrt{N}} F_1\left(t, t^{-k} \sup_{\bar{x}} \mu\left(B_t^{\bar{x}}\right)\right) |\mathfrak{C}_t|_k\, t^{N-k-1}\, dt < \infty,$$
$$\mathcal{I}_2 := \int_0^{\sqrt{N}} F_2^*\big(t, t^{k-N}\big) |\mathfrak{C}_t|_k t^{N-k-1}\, dt < \infty, \tag{4.25}$$

then the pair (u, A) is (Φ, k)-separating.
Moreover, in both cases there holds

$$\int_\Omega \Phi(x, |du|)\, dV \le C(N,k)\mathcal{I}_1, \quad \int_\Omega \Phi^*(x, |dA|)\, dV \le C(N,k)\mathcal{I}_2. \tag{4.26}$$

Proof The forms u and A are regular outside $\mathfrak{S}$ by construction. By Lemmas 22,23 we have $u \in W^{d,1}(\Omega, \Lambda^{k-1})$ and $A \in W^{d,1}(\Omega, \Lambda^{N-k-1})$. By Proposition 24, $|du| \cdot |dA| = 0$ outside $\mathfrak{S}$. Since $d(A \wedge du) = dA \wedge du = 0$ outside $\mathfrak{S}$, Lemma 27 implies

$$\int_{\partial\Omega} A \wedge du = \int_{\partial[-1,1]^N} A \wedge du = 1.$$

From estimates (4.20) and (4.22) by the assumptions of the lemma we get (4.26), which completes the proof of the lemma. □

In view of Sect. 4, to construct an example for the Lavrentiev gap, it is sufficient to check the conditions of Lemma 28. In the following section we do this for the “standard” and “borderline” double phase models and for the variable exponent.

4.7 Example setups

Two cases of Lemma 28 and different choices of the fractal contact set give us several variants of example setup. Further $p_0 > 1$ will be the threshold parameter. Depending on the value of the threshold parameter p_0, we design 5 different setups:

(a) critical or one saddle point setup corresponds to the classical Zhikov checkerboard example [61] ($N = 2, k = 1$) and its development by [31] ($N > 1, k = 1$);
(b) supercritical setup corresponds to the case $p_0 > N/k$, in the scalar settting ($k = 1$) of [11] this corresponds to the superdimensional case $p_0 > N$ with singular set on a line;
(c) subcritical case corresponds to the case $1 < p_0 < N/k$, in the scalar settting ($k = 1$) of [11] this corresponds to the subdimensional case $1 < p_0 < N$ with singular set on a hyperplane;
(d) right limiting critical case corresponds to the situation when $p_0 = N/k + 0$ (that is, for the critical value $p_0 = N/k$ we use the supercritical construction);
(e) left limiting critical case corresponds to the situation when $p_0 = N/k - 0$ (that is, for the critical value $p_0 = N/k$ we use the subcritical construction).

Each of these setups includes the fractal set $\mathfrak{C}$ (see Sect. 4.4, in the "critical" case it is just one point), the barrier fractal set $\mathfrak{S}$, the pair of the forms u and A, and the function $\tilde{\rho}$ which separates the supports of du and dA: it is equal to 0 on the support of du and 1 on the support of dA. The construction of the forms u, A and the function $\tilde{\rho}$ is described in 4.3 (for one singular point case) and in 4.5 (for the rest of cases).

One can easily verify (this is done in Sect. 5.1, take there $\alpha = 0$) that $du \in L^p(\Omega, \Lambda^k)$ for any $p < p_0$ and $dA \in L^{q'}(\Omega, \Lambda^{N-k})$ for any $q > p_0$ which explains why we call this parameter "threshold". The function $\tilde{\rho}$ is then used to construct the function Φ for which the pair (u, A) is (Φ, k)-separating.

The second free parameter of the construction — the shrinking fractal parameter γ — plays an important role later in refining our examples to the limiting case and in treating the borderline double phase and the log-log-Hölder exponents.

Setup 1 *(Critical or one saddle point)* Let $p_0 = N/k$ and set

$$\mathfrak{C} = \{0\}^{N-k}, \quad \mathfrak{S} = \{0\}^N, \qquad \tilde{\rho} = \rho_{\mathfrak{S}} = \mathcal{P}_0(N-k, k, 0, 0),$$
$$u = u_{\mathfrak{S}} = \mathcal{P}_1(N-k, k, 0, 0), \qquad A = A_{\mathfrak{S}} = \mathcal{P}_2(N-k, k, 0, 0).$$

Setup 2 *(Supercritical)* Let $p_0 > N/k$. Define $\mathfrak{D} = (p_0 k - N)/(p_0 - 1)$ from $p_0 = (N - \mathfrak{D})/(k - \mathfrak{D})$ and set $\lambda = 2^{-k/\mathfrak{D}}$,

$$\mathfrak{C} = \mathfrak{C}^k_{\lambda,\gamma}, \quad \mathfrak{S} = \mathfrak{C} \times \{0\}^{N-k}, \qquad \tilde{\rho} = 1 - \rho_{\mathfrak{S}} = 1 - \mathcal{P}_0(k, N-k, \mathfrak{D}, \gamma),$$
$$u = A_{\mathfrak{S}} = \mathcal{P}_2(k, N-k, \mathfrak{D}, \gamma), \quad A = (-1)^{k(N-k)} u_{\mathfrak{S}} = (-1)^{k(N-k)} \mathcal{P}_1(k, N-k, \mathfrak{D}, \gamma).$$

Setup 3 *(Subcritical)* Let $1 < p_0 < N/k$. Define $\mathfrak{D} = N - p_0 k$ from $p_0 = (N - \mathfrak{D})/k$ and set $\lambda = 2^{-(N-k)/\mathfrak{D}}$,

$$\mathfrak{C} = \mathfrak{C}^{N-k}_{\lambda,\gamma}, \quad \mathfrak{S} = \mathfrak{C} \times \{0\}^k, \qquad \tilde{\rho} = \rho_{\mathfrak{S}} = \mathcal{P}_0(N-k, k, \mathfrak{D}, \gamma),$$
$$u = u_{\mathfrak{S}} = \mathcal{P}_1(N-k, k, \mathfrak{D}, \gamma), \qquad A = A_{\mathfrak{S}} = \mathcal{P}_2(N-k, k, \mathfrak{D}, \gamma).$$

Setup 4 *(Right limiting (critical+0))* Let $p_0 = N/k$ and set

$$\mathfrak{C} = \mathfrak{C}^k_{0,\gamma}, \quad \mathfrak{S} = \mathfrak{C}^k_{0,\gamma} \times \{0\}^{N-k}, \qquad \tilde{\rho} = 1 - \rho_{\mathfrak{S}} = 1 - \mathcal{P}_0(k, N-k, 0, \gamma),$$
$$u = A_{\mathfrak{S}} = \mathcal{P}_2(k, N-k, 0, \gamma), \qquad A = (-1)^{k(N-k)} u_{\mathfrak{S}} = (-1)^{k(N-k)} \mathcal{P}_1(k, N-k, 0, \gamma).$$

Setup 5 *(Left limiting (critical-0))* Let $p_0 = N/k$ and set

$$\mathfrak{C} = \mathfrak{C}^{N-k}_{0,\gamma}, \quad \mathfrak{S} = \mathfrak{C} \times \{0\}^k, \qquad \tilde{\rho} = \rho_{\mathfrak{S}} = \mathcal{P}_0(N-k, k, 0, \gamma),$$
$$u = u_{\mathfrak{S}} = \mathcal{P}_1(N-k, k, 0, \gamma), \qquad A = A_{\mathfrak{S}} = \mathcal{P}_2(N-k, k, 0, \gamma).$$

Setup 1, Setup 3, Setup 5 correspond to Lemma 28 (**ii**). Setup 2 and Setup 4 correspond to Lemma 28 (**i**).

5 Applications

In this section we show the presence of the Lavrentiev gap for the following models

(a) double phase;
(b) borderline double phase;
(c) variable exponent.

To this end we use the framework defined in Sect. 4 and the Cantor set-based construction from Sect. 4.5. That is, we have to show that the pair of forms u and A build as in Sect. 4.5 is (Φ, k)-separating and satisfies the conditions of Assumption 15 (the latter one for the Dirichlet problem) for the generalized Orlicz functions

(a) $\Phi(x,t) = t^p + a(x)t^q$;
(b) $\Phi(x,t) = t^p \log^{-\beta}(e+t) + a(x)t^p \log^{\alpha}(e+t)$;
(c) $\Phi(x,t) = t^{p(x)}$.

Further in this section $\Omega = \{|x| < \sqrt{N}\} \subset \mathbb{R}^N$, $k \in \{1, \ldots, N-1\}$, $\mathfrak{C} = \mathfrak{C}^l_{\lambda,\gamma}$ is a generalized Cantor set as in Sect. 4.4, and $\mathfrak{S} = \mathfrak{C} \times \{0\}^{N-l}$ is the singular contact set, where $l = k$ or $l = N-k$. As above, by $\mathfrak{C}_t$ we denote the t-neighbourhood of the set $\mathfrak{C}$.

Recall that the parameter λ of the fractal set $\mathfrak{C}^m_{\lambda,\gamma}$ is connected to its "fractal dimension" $\mathfrak{D}$ by $\mathfrak{D} = -m \ln 2/\ln \lambda$ if $\mathfrak{D} > 0$ and $\lambda = 0$ if $\mathfrak{D} = 0$, and the forms u and A defined in (4.16) and (4.17), based on the contact set $\mathfrak{C}^m_{\lambda,\gamma}$ (or (4.5) and (4.6) for $\mathfrak{D} = \gamma = 0$) are denoted by $\mathcal{P}_1(m, N-m, \mathfrak{D}, \gamma)$ and $\mathcal{P}_2(m, N-m, \mathfrak{D}, \gamma)$. That is, the forms $\mathcal{P}_j(m, N-m, \mathfrak{D}, \gamma)$, $j = 1, 2$, together with the function $\mathcal{P}_0(m, N-m, \mathfrak{D}, \gamma)$ are constructed using the singular set $\mathfrak{C}^m_{\lambda,\gamma} \times \{0\}^{N-m}$ with $\lambda = 2^{-m/\mathfrak{D}}$ if $\mathfrak{D} > 0$ and $\lambda = 0$ if $\mathfrak{D} = 0$.

Before passing on to the examples we make the following observation.

Lemma 29 *Let ρ be a function on Ω such that $\rho \le C$ and $|\nabla \rho(x)| \le C|\hat{x}|^{-1}$ with $C > 1$. Then the function $a_0(|\hat{x}|)\rho(x)$ has the modulus of continuity $5Ca_0(\cdot)$.*

Proof For $x = (\bar{x}, \hat{x})$ and $y = (\bar{y}, \hat{y})$ we evaluate

$$\begin{aligned} r(x,y) := |a_0(|\hat{x}|)\rho(x) - a_0(\hat{y})\rho(y)| &\le |a_0(|\hat{x}|) - a_0(|\hat{y}|)|\rho(x) + a_0(|\hat{y}|)|\rho(x) - \rho(y)| \\ &\le a_0(|\hat{x} - \hat{y}|) + a_0(|\hat{y}|)|\rho(x) - \rho(y)|. \end{aligned}$$

If $|x-y| \ge |\hat{y}|/2$ we evaluate $a_0(|\hat{y}|) \le 2a_0(|x-y|)$ using the concavity of a_0, and $|\rho(x) - \rho(y)| \le 2C$, therefore $r(x,y) \le 5Ca_0(|x-y|)$.

If $|x-y| \le |\hat{y}|/2$ then $|\hat{y}|/2 \le |\hat{x}| \le 3|\hat{y}|/2$, therefore $|\rho(x) - \rho(y)| \le 2C|\hat{y}|^{-1}|x-y|$. Now,

$$a_0(|\hat{y}|)|\rho(x) - \rho(y)| \le 2C\frac{a_0(|\hat{y}|)}{|\hat{y}|} a_0(|x-y|)\frac{|x-y|}{a_0(|x-y|)} \le 2Ca_0(|x-y|)$$

since the concavity of a_0 implies that $a(s)s^{-1} \le a(t)t^{-1}$ for $s \ge t$. Therefore, for $|x - y| \le |\hat{y}|/2$ we get $r(x, y) \le (2C + 1)a_0(|x - y|)$.

Thus in both cases we have $r(x, y) \le 5Ca_0(|x - y|)$.

5.1 Standard double phase model

Let $1 < p < q < +\infty$ and $\alpha \ge 0$,

$$\varphi(t) = t^p, \quad \psi(t) = t^q. \tag{5.1}$$

Denote

$$\begin{aligned} a_0(t) &= t^\alpha, \quad a(x) = \tilde{\rho}(x)a_0(|\hat{x}|) = \tilde{\rho}(x)|\hat{x}|^\alpha, \quad \Phi(x, t) = \varphi(t) + a(x)\psi(t) \\ &= t^p + a(x)t^q. \end{aligned} \tag{5.2}$$

where $\tilde{\rho}$ is a nonnegative function which will be described in Lemma 30 (it comes from the Setup used in Lemma 30, along with the pair of forms (u, A)).

Lemma 30 (a) *Let $p_0 = N/k$ and $p < N/k < q - \alpha k^{-1}$. Use one saddle point Setup 1.*

(b) *Let $p_0 > N/k$ and $p \le p_0 \le q - \alpha\frac{p_0-1}{N-k}$. Take $\gamma > (p_0 k - N)^{-1}$ if $p = p_0$, $\gamma < \frac{1-p_0}{p_0 k - N}$ if $q = p_0 + \alpha\frac{p_0-1}{p_0 k - N}$, and any γ otherwise. Use supercritical Setup 2.*

(c) *Let $1 < p_0 < N/k$ and $p \le p_0 \le q - \alpha k^{-1}$. Take $\gamma < (p_0 k - N)^{-1}$ if $p = p_0$, $\gamma > (q-1)/(N - p_0 k)$ if $q = p_0 + \alpha k^{-1}$, and any γ otherwise. Use subcritical Setup 3.*

(d) *Let $p_0 = N/k$ and $p \le p_0 < q - \alpha k^{-1}$. Take $\gamma > (N-k)^{-1}$ if $p = p_0$, and any $\gamma > 0$ otherwise. Use right limiting critical Setup 4.*

(e) *Let $p_0 = N/k$ and $p < p_0 \le q - \alpha k^{-1}$. Take $\gamma > (q-1)(N-k)^{-1}$ if $q = p_0 + \alpha k^{-1}$ and any $\gamma > 0$ otherwise. Use left limiting critical Setup 5.*

Then for Φ given by (5.2), the pair of forms u and A is a (Φ, k)-separating pair.

Proof We use Lemma 28 with

$$F_1(s, \tau) = \varphi(\tau) = \tau^p, \quad F_2(s, \tau) = a_0(s)\psi(\tau) = s^\alpha \tau^q$$

and the estimates provided by Lemma 21. Clearly,

$$F_2^*(s, \tau) = a_0(s)\psi^*\left(\frac{\tau}{a_0(s)}\right) = c_q s^\alpha (\tau s^{-\alpha})^{q'}.$$

We treat the five cases according to Definition 30.

(a) Case $p_0 = N/k$, $p < N/k < q - \alpha k^{-1}$. We estimate

$$\int_\Omega \varphi(|du|)\, dV \lesssim \int_0^{\sqrt{N}} t^{-pk+N-1}\, dt < \infty$$

provided that $p < N/k$. Also,

$$\int_\Omega a_0(|\hat{x}|)\psi^*(|dA|/a_0(t))\, dV \lesssim \int_0^{\sqrt{N}} t^{q'(k-N-\alpha)+\alpha+N-1}\, dt < \infty$$

provided that $q > \frac{N+\alpha}{k}$.

(b) Case $p_0 > N/k$. In this case

$$p_0 = \frac{N-\mathfrak{D}}{k-\mathfrak{D}}, \quad \mathfrak{D} = \frac{p_0 k - N}{p_0 - 1}, \quad k - \mathfrak{D} = \frac{N-k}{p_0-1}.$$

We use case (**ii**) of Lemma 28. For the first integral in (4.25), we get

$$\int_0^{\sqrt{N}} \varphi\left(\frac{\sup_{\bar{x}} \mu(B_t^{\bar{x}})}{t^k}\right) |\mathfrak{C}_t|_k \, t^{N-k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{p(\mathfrak{D}-k)} (\ln t^{-1})^{-p\gamma\mathfrak{D}} t^{k-\mathfrak{D}} (\ln t^{-1})^{\gamma\mathfrak{D}} t^{N-k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{(p_0-p)(k-\mathfrak{D})} (\ln t^{-1})^{\gamma\mathfrak{D}(1-p)} \frac{dt}{t} < \infty.$$

For the second integral in (4.25) we have

$$\int_0^{\sqrt{N}} a_0(t)\psi^*\left(\frac{t^{k-N}}{a_0(t)}\right) |\mathfrak{C}_t|_k t^{N-k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{q'(k-N-\alpha)+\alpha} t^{k-\mathfrak{D}} (\ln t^{-1})^{\gamma\mathfrak{D}} t^{N-k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{q'(k-N-\alpha)+N-\mathfrak{D}+\alpha} (\ln t^{-1})^{\gamma\mathfrak{D}} \frac{dt}{t} < \infty.$$

Here one notes that

$$q'(k-N-\alpha)+N-\mathfrak{D}+\alpha > 0 \Leftrightarrow q' < \frac{N-\mathfrak{D}+\alpha}{N-k+\alpha} \Leftrightarrow q > \frac{N-\mathfrak{D}+\alpha}{k-\mathfrak{D}} = p_0 + \frac{\alpha}{k-\mathfrak{D}}.$$

Then by Lemma 28 (**ii**) the pair of forms (u, A) is (Φ, k)-separating.

(c) Case $p_0 < N/k$. In this case $\mathfrak{D} = N - p_0 k$. We use case (**i**) of Lemma 28. For the first integral in (4.24), we have

$$\int_0^{\sqrt{N}} \varphi(t^{-k}) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{-pk} t^{N-k-\mathfrak{D}} (\ln t^{-1})^{\gamma\mathfrak{D}} t^{k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{(p_0-p)k} (\ln t^{-1})^{\gamma\mathfrak{D}} \frac{dt}{t} < \infty.$$

For the second integral in (4.24) we get

$$\int_0^{\sqrt{N}} a_0(t)\psi^*\left(\frac{\sup_{\bar{x}} \mu(B_t^{\bar{x}})}{t^{N-k} a_0(t)}\right) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt$$

$$\lesssim \int_0^{\sqrt{N}} t^{q'(\mathfrak{D}+k-N-\alpha)+\alpha} (\ln t^{-1})^{-q'\gamma\mathfrak{D}} t^{N-k-\mathfrak{D}} (\ln t^{-1})^{\gamma\mathfrak{D}} t^{k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{q'(\mathfrak{D}+k-N-\alpha)+N-\mathfrak{D}+\alpha} (\ln t^{-1})^{\gamma\mathfrak{D}/(1-q)} \frac{dt}{t} < \infty.$$

Here one notes that

$$q'(\mathfrak{D}+k-N-\alpha)+N-\mathfrak{D}+\alpha>0 \Leftrightarrow q'<\tfrac{N-\mathfrak{D}+\alpha}{N+\alpha-\mathfrak{D}-k} \Leftrightarrow q > \tfrac{N+\alpha-\mathfrak{D}}{k} = p_0+\tfrac{\alpha}{k}.$$

By Lemma 28 i), the pair (u, A) is (Φ, k)-separating.

(d) Case $p_0 = N/k + 0$. We use case (**ii**) of Lemma 28. For the first integral in (4.25), we get

$$\int_0^{\sqrt{N}} \varphi\left(\frac{\sup_{\bar{x}} \mu(B_t^{\bar{x}})}{t^k}\right) |\mathfrak{C}_t|_k\, t^{N-k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{-pk}(\ln t^{-1})^{-p\gamma k} t^k (\ln t^{-1})^{\gamma k} t^{N-k-1}\, dt$$
$$= c\int_0^{\sqrt{N}} t^{(p_0-p)k}(\ln t^{-1})^{\gamma k(1-p)}\, \frac{dt}{t} < \infty.$$

For the second integral in (4.25) we have

$$\int_0^{\sqrt{N}} a_0(t)\psi^*\left(\frac{t^{k-N}}{a_0(t)}\right) |\mathfrak{C}_t|_k t^{N-k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{q'(k-N-\alpha)+\alpha} t^k (\ln t^{-1})^{\gamma k} t^{N-k-1}\, dt$$
$$= c\int_0^{\sqrt{N}} t^{q'(k-N-\alpha)+N+\alpha}(\ln t^{-1})^{\gamma k}\, \frac{dt}{t} < \infty.$$

Here one notes that

$$q'(k-N-\alpha)+N+\alpha>0 \Leftrightarrow q' < \frac{N+\alpha}{N+\alpha-k} \Leftrightarrow q > \frac{N+\alpha}{k} = p_0 + \frac{\alpha}{k}.$$

By Lemma 28 (**ii**), the pair (u, A) is (Φ, k)-separating.

(e) Case $p_0 = N/k - 0$. We use case (**i**) of Lemma 28 For the first integral in (4.24), we have

$$\int_0^{\sqrt{N}} \varphi(t^{-k})|\mathfrak{C}_t|_{N-k} t^{k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{-pk} t^{N-k}(\ln t^{-1})^{\gamma(N-k)} t^{k-1}\, dt$$
$$= c\int_0^{\sqrt{N}} t^{(p_0-p)k}(\ln t^{-1})^{\gamma(N-k)}\, \frac{dt}{t} < \infty.$$

For the second integral in (4.25), we get

$$\int_0^{\sqrt{N}} a_0(t)\psi^*\left(\frac{\sup_{\bar{x}} \mu(B_t^{\bar{x}})}{t^{N-k} a_0(t)}\right) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt$$
$$\lesssim \int_0^{\sqrt{N}} t^{(k-N-\alpha)q'+\alpha}(\ln t^{-1})^{-q'\gamma} t^{N-k}(\ln t^{-1})^{\gamma(N-k)} t^{k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{q'(k-N-\alpha)+N+\alpha} (\ln t^{-1})^{\gamma(N-k)/(1-q)} \frac{dt}{t} < \infty.$$

By Lemma 28 (**i**), the pair of forms (u, A) is (Φ, k)-separating.

□

Theorem 31 *Let $p < N/k$ and $q > p + \alpha k^{-1}$. Then there exists $p_0 \in (1, N/k)$ such that $p < p_0 < q - \alpha k^{-1}$ (one can also take $p = p_0$ and choose $\gamma < (p_0 k - N)^{-1}$) and therefore a (Φ, k)-separating pair of forms (u, A) for Φ defined by (5.2), and $\tilde{\rho}$ from Lemma 30.*

Let $q > p + \alpha(p-1)/(N-k)$ and $q > \frac{N+\alpha}{k}$. Then there exists $p_0 > N/k$ satisfying $p < p_0 < q - \alpha(p_0 - 1)/(N-k)$ (one can also take $p = p_0$ and choose $\gamma > (p_0 k - N)^{-1}$) and therefore a (Φ, k)-separating pair of forms (u, A) for Φ defined by (5.2), and $\tilde{\rho}$ from Lemma 30.

In these cases $H^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1}) \neq W^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$. Let $\eta \in C_0^\infty(\Omega)$ be such that $\eta = 1$ in a neighbourhood of $\mathfrak{S} = \mathfrak{S}(u, A)$, $A^\circ = \eta A$, and $b = dA^\circ$. Then for the functional $\mathcal{F}_{\Phi,b}$ there is Lavrentiev gap (4.1). For sufficiently large $t > 0$ and $\omega_0 = tu^\partial \in C^\infty(\overline{\Omega}, \Lambda^{k-1})$ there holds (1.8) and (4.2).

Proof We have only to check Assumption 15. Indeed, since $\tilde{\rho} = 0$ on the support of du and $\tilde{\rho} = 1$ on the support of $b = dA$,

$$\mathcal{F}_{\Phi,0}(tu) = t^p \mathcal{F}_{\Phi,0}(u), \quad \mathcal{F}^*_{\Phi,0}(sb) \leq s^{q'} \mathcal{F}^*_{\Phi,0}(b).$$

Take $s = t^{p/q'}$ Then for sufficiently large t there holds

$$\mathcal{F}_{\Phi,0}(tu) + \mathcal{F}^*_{\Phi,0}(sb) \leq t^p (\mathcal{F}_{\Phi,0}(u) + \mathcal{F}^*_{\Phi,0}(b)) < ts = t^{1+\frac{p}{q'}}$$

since $p < 1 + \frac{p}{q'}$ if $p < q$. □

Note that here $\Phi(x, t) = t^p + a(x)t^q$ where $a \in C^\alpha(\overline{\Omega})$ (by Lemma 29). This proves Theorem A.

5.2 Borderline double phase

Let $p_0 > 1$, $\alpha, \beta \in \mathbb{R}$, $\varkappa \geq 0$ such that

$$\alpha + \beta > p_0 + \varkappa. \tag{5.3}$$

Let φ and ψ be two Orlicz functions such that

$$\varphi \lesssim \psi, \quad \varphi(t) \lesssim t^{p_0} \ln^{-\beta}(e+t), \quad \psi^*(t) \lesssim t^{p_0'} \ln^{\alpha/(1-p_0)}(e+t) \tag{5.4}$$

for large t. Denote

$$a_0(t) = \ln^{-\varkappa}(1/t), \quad a(x) = \tilde{\rho}(x) a_0(|\hat{x}|), \quad \Phi(x,t) = \varphi(t) + a(x)\psi(t), \tag{5.5}$$

where $\tilde{\rho}$ is a nonnegative function to be defined later (it is generated by the corresponding Setup in Lemma 32 together with the forms u and A).

Lemma 32 (a) *Let $p_0 = N/k$ and assume that $\beta > 1$ and $\alpha + 1 > \kappa + p_0$. Use one saddle point Setup 1.*

(b) *If $p_0 > N/k$, define $\mathfrak{D}$ from $p_0 = (N-\mathfrak{D})/(k-\mathfrak{D})$ and take γ satisfying $(1-\beta)/(p_0-1) < \gamma\mathfrak{D} < (\alpha-\varkappa-p_0+1)/(p_0-1)$. Use supercritical Setup 2.*

(c) *If $1 < p_0 < N/k$, define $\mathfrak{D}$ from $p_0 = (N-\mathfrak{D})/k$ and take γ satisfying $p+\varkappa-\alpha-1 < \gamma\mathfrak{D} < \beta-1$. Use subcritical Setup 3.*

(d) *If $p_0 = N/k$ and additionally $\alpha > p_0-1+\varkappa$, take $\gamma > 0$ satisfying $(1-\beta)/(p_0-1) < \gamma k < (\alpha-\varkappa-p_0+1)/(p_0-1)$. Use right limiting critical Setup 4.*

(e) *If $p_0 = N/k$ and additionally $\beta > 1$, take $\gamma > 0$ satisfying $p_0+\varkappa-\alpha-1 < \gamma > 0(N-k) < \beta-1$. Use left limiting critical Setup 5.*

Then for Φ given by (5.5), the pair of forms u and A is a (Φ, k)-separating pair.

Proof To shorten notation we write here p instead of p_0. We use Lemma 28 with

$$F_1(s,\tau) = \tau^p \ln^{-\beta}(e+\tau), \quad F_2(s,\tau) = a_0(s)\psi(\tau)$$

and the estimates provided by Lemma 21. We have

$$F_2^*(s,\tau) = a_0(s)\psi^*\left(\frac{\tau}{a_0(s)}\right).$$

We treat the five cases according to Definition 32.

(a) Case $p = N/k$. We estimate

$$\int_\Omega \Phi(x,|du|)\,dV = \int_\Omega \varphi(|du|)\,dV \lesssim \int_0^{\sqrt{N}} t^{-pk+N-1}\ln^{-\beta}(e+t^{-1})\,dt$$

$$= \int_0^{\sqrt{N}} \ln^{-\beta}(e+t^{-1})\,\frac{dt}{t} < \infty,$$

$$\int_\Omega \Phi^*(x,|dA|)\,dV \le \int_\Omega a_0(|\hat{x}|)\psi^*(|dA|/a_0(t))\,dV \lesssim \int_0^{\sqrt{N}} \ln^{(\kappa-\alpha)/(p-1)}(e+t^{-1})\frac{dt}{t} < \infty.$$

Therefore the pair of forms (u, A) is (Φ, k)-separating.

(b) Case $p > N/k$. We use case (**ii**) of Lemma 28. For the first integral in (4.25), we get

$$\begin{aligned}
&\int_0^{\sqrt{N}} \varphi\left(\frac{\sup_{\bar{x}}\mu(B_t^{\bar{x}})}{t^k}\right)|\mathfrak{C}_t|_k\, t^{N-k-1}\,dt < \infty\\
&\lesssim \int_0^{\sqrt{N}} t^{p(\mathfrak{D}-k)}(\ln t^{-1})^{-p\gamma\mathfrak{D}-\beta}t^{k-\mathfrak{D}}(\ln t^{-1})^{\gamma\mathfrak{D}}t^{N-k-1}\,dt\\
&= c\int_0^{\sqrt{N}} t^{-1}(\ln t^{-1})^{\gamma\mathfrak{D}(1-p)-\beta}\,dt < \infty.
\end{aligned}$$

For the second integral in (4.25), using $p' = (N-\mathfrak{D})/(N-k)$ we have

$$\int_0^{\sqrt{N}} a_0(t)\psi^*\left(\frac{t^{k-N}}{a_0(t)}\right)|\mathfrak{C}_t|_k t^{N-k-1}\,dt$$

$$\lesssim \int_0^{\sqrt{N}} t^{p'(k-N)} (\ln t^{-1})^{\varkappa p' - \alpha/(p-1)} t^{k-\mathfrak{D}} (\ln t^{-1})^{\gamma \mathfrak{D} - \varkappa} t^{N-k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{-1} (\ln t^{-1})^{\gamma \mathfrak{D} - (\alpha - \varkappa)/(p-1)}\, dt < \infty.$$

Then by Lemma 28 (**ii**) the pair of forms (u, A) is (Φ, k)-separating.
(c) Case $p < N/k$. We use case (**i**) of Lemma 28 for the first integral in (4.24), we have

$$\int_0^{\sqrt{N}} \varphi(t^{-k}) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{-pk} (\ln t^{-1})^{-\beta} t^{N-k-\mathfrak{D}} (\ln t^{-1})^{\gamma \mathfrak{D}} t^{k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{-1} (\ln t^{-1})^{\gamma \mathfrak{D} - \beta}\, dt < \infty.$$

For the second integral in (4.24), using $p' = (N - \mathfrak{D})/(N - k - \mathfrak{D})$ we get

$$\int_0^{\sqrt{N}} a_0(t) \psi^* \left(\frac{\sup_{\bar{x}} \mu(B_t^{\bar{x}})}{t^{N-k} a_0(t)} \right) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt$$

$$\lesssim \int_0^{\sqrt{N}} t^{p'(\mathfrak{D}+k-N)} (\ln t^{-1})^{p'\varkappa - p'\gamma \mathfrak{D} + \alpha/(1-p)} t^{N-k-\mathfrak{D}} (\ln t^{-1})^{\gamma \mathfrak{D} - \varkappa} t^{k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{-1} (\ln t^{-1})^{(\gamma \mathfrak{D} + \alpha - \varkappa)/(1-p)}\, dt < \infty.$$

By Lemma 28 (**i**), the pair (u, A) is (Φ, k)-separating.

(d) Case $p = N/k, \alpha > p - 1$. We use case (**ii**) of Lemma 28. For the first integral in (4.25), we get

$$\int_0^{\sqrt{N}} \varphi \left(\frac{\sup_{\bar{x}} \mu(B_t^{\bar{x}})}{t^k} \right) |\mathfrak{C}_t|_k\, t^{N-k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{-pk} (\ln t^{-1})^{-p\gamma k - \beta} t^k (\ln t^{-1})^{\gamma k} t^{N-k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{-1} (\ln t^{-1})^{\gamma k(1-p) - \beta}\, dt < \infty.$$

For the second integral in (4.25), using $p' = N/(N-k)$ we have

$$\int_0^{\sqrt{N}} a_0(t) \psi^* \left(\frac{t^{k-N}}{a_0(t)} \right) |\mathfrak{C}_t|_k t^{N-k-1}\, dt$$

$$\lesssim \int_0^{\sqrt{N}} t^{p'(k-N)} (\ln t^{-1})^{p'\varkappa - \alpha/(p-1)} t^k (\ln t^{-1})^{\gamma k - \varkappa} t^{N-k-1}\, dt$$

$$= c \int_0^{\sqrt{N}} t^{-1} (\ln t^{-1})^{\gamma k - (\alpha - \varkappa)/(p-1)}\, dt < \infty.$$

By Lemma 28 (**ii**), the pair (u, A) is (Φ, k)-separating.
(e) Case $p = N/k$, $\beta > 1$. We use case (**i**) of Lemma 28 For the first integral in (4.24), we have

$$\int_0^{\sqrt{N}} \varphi(t^{-k}) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt \lesssim \int_0^{\sqrt{N}} t^{-pk} (\ln t^{-1})^{-\beta} t^{N-k} (\ln t^{-1})^{\gamma(N-k)} t^{k-1}\, dt$$
$$= c \int_0^{\sqrt{N}} t^{-1} (\ln t^{-1})^{\gamma(N-k)-\beta}\, dt < \infty.$$

For the second integral in (4.25), we get

$$\int_0^{\sqrt{N}} a_0(t) \psi^* \left(\frac{\sup_{\bar{x}} \mu(B_t^{\bar{x}})}{t^{N-k} a_0(t)} \right) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt$$
$$\lesssim \int_0^{\sqrt{N}} t^{-(N-k)p'} (\ln t^{-1})^{p'\varkappa - p'\gamma(N-k) + \alpha/(1-p)} t^{N-k} (\ln t^{-1})^{\gamma(N-k)-\varkappa} t^{k-1}\, dt$$
$$= c \int_0^{\sqrt{N}} t^{-1} (\ln t^{-1})^{(\gamma(N-k)+\alpha-\varkappa)/(1-p)}\, dt < \infty.$$

By Lemma 28 (**i**), the pair of forms (u, A) is (Φ, k)-separating. □

Theorem 33 *Under condition (5.3), for any $k = 1, \dots, N-1$ and any $p > 1$ there exists $\tilde{p}$ and a (Φ, k)-separating pair of forms (u, A) for Φ defined by (5.4) and (5.5). Therefore in these cases $H^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1}) \neq W^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$. Let $\eta \in C_0^\infty(\Omega)$ be such that $\eta = 1$ in a neighbourhood of $\mathfrak{S} = \mathfrak{S}(u, A)$, $A^\circ = \eta A$, and $b = dA^\circ$. For the functional $\mathcal{F}_{\Phi,b}$ there is Lavrentiev gap (4.1).*

Note that here we have Φ given by $\Phi(x,t) = \varphi(t) + a(x)\psi(t)$, with $a \in C^{\omega(\cdot)}(\overline{\Omega})$, $\omega(t) \le C \ln^{-\kappa}(1/t)$ for some $C > 1$ (see Lemma 29). This proves Theorem B.

5.3 Variable exponent model

A classical example of an integrand from the class (1.2) is the variable exponent model

$$\Phi(x,t) = t^{p(x)}, \tag{5.6}$$

where $p:\ \Omega \to [p_-, p_+]$ is a variable exponent. Let $p_0 \in (p_-, p_+)$,

$$\sigma(t) = \kappa \frac{\ln\ln \frac{1}{t}}{\ln \frac{1}{t}}, \tag{5.7}$$

with $\kappa > 0$ and $\tilde{\rho}$ be a function to be defined later (see Lemma 34). Let $\xi \in C^\infty(\mathbb{R})$ be a positive nondecreasing function such that $\xi(t) = t$ if $t \in [(p_- + p_0)/2, (p_+ + p_0)/2]$, $\xi(t) = \xi(p_-) = (3p_- + p_0)/4$ if $t \le p_-$, $\xi(t) = \xi(p_+) = (3p_+ + p_0)/4$ if $t \ge p_+$. Set

$$p(x) = \xi\big(p_0 + \sigma(|\hat{x}|)(2\tilde{\rho} - 1)\big), \tag{5.8}$$

and let Φ be defined by (5.6).

Recall that due to the well-know result from [59] if the exponent p has the modulus of continuity (5.7) with sufficiently small κ then smooth functions are dense in corresponding Sobolev–Orlicz space and the Lavrentiev phenomenon is absent. On the other hand, the example with one saddle point provided in [59] ($k = 0$, $N = 2$, $p_- < 2 < p_+$) shows that for sufficiently large κ the Lavrentiev gap occurs, while for the scalar case ($k = 0$) the smallness of κ gives $H = W$. We construct examples of the Lavrentiev phenomenon for $p(x)$-integrand in arbitrary dimension and for any $1 < p_- < p_+ < \infty$.

Lemma 34 (a) *Let $p_0 = N/k$ and $\kappa > k^{-2}\max(k, N-k)$. Use one saddle point Setup 1.*
(b) *Let $p_0 > N/k$,*

$$\kappa > \frac{p_0(p_0-1)}{2(N-k)}, \quad \text{and} \quad 1 - \kappa\frac{N-k}{p_0-1} < \gamma(kp_0 - N) < \kappa\frac{N-k}{p_0-1} - (p_0-1). \tag{5.9}$$

Use supercritical Setup 2.
(c) *Let $1 < p_0 < N/k$,*

$$\kappa > \frac{p_0}{2k} \quad \text{and} \quad p_0 - 1 - \kappa k < \gamma(N - p_0k) < \kappa k - 1. \tag{5.10}$$

Use subcritical Setup 3.
(d) *Let $p_0 = N/k$,*

$$\kappa > \max\left(\frac{N}{2k^2}, \frac{N-k}{k^2}\right) \quad \text{and} \quad \max\left(\frac{k-\kappa k^2}{N-k}, 0\right) < \gamma k < \frac{k - N + \kappa k^2}{N-k}. \tag{5.11}$$

Use right limiting critical Setup 4.
(e) *Let $p_0 = N/k$,*

$$\kappa > \max\left(\frac{N}{2k^2}, \frac{k}{k^2}\right), \quad \text{and} \quad \max\left(\frac{N-k-\kappa k^2}{k}, 0\right) < \gamma k < \kappa k - 1. \tag{5.12}$$

Use left limiting critical Setup 5.
Then for Φ given by (5.6) *and* (5.8)*, the pair of forms u and A is a (Φ, k)-separating pair.*

Proof We use Lemma 28 with $F_1(s,\tau) = \tau^{p_0 - \sigma(s)}$ and $F_2(s,\tau) = \tau^{p_0+\sigma(s)}$. Clearly, $F_2^*(s,\tau) \le c(p_-, p_+)\tau^{(p_0+\sigma(s))'}$. Note that $t^{\sigma(t)} = (\ln t^{-1})^{-\kappa}$.

(a) Case $p_0 = N/k$. We evaluate

$$\int_\Omega \Phi(x, |du|)\, dV \lesssim \int_\Omega |du|^{p_0 - \sigma(|\hat{x}|)}\, dV \lesssim \int_0^{\sqrt{N}} t^{N-1-k(p_0-\sigma(t))}\, dt$$

$$\lesssim \int_0^{\sqrt{N}} (\ln t^{-1})^{-k\kappa} t^{-1}\, dt < \infty$$

provided that $k\kappa > 1$. Also

$$\int_\Omega \Phi^*(x, |dA|)\, dV \lesssim \int_\Omega |dA|^{(p_0+\sigma(\hat{x}))'}\, dV$$

$$\lesssim \int_0^{\sqrt{N}} t^{(k-N)(p_0+\sigma(t))/(p_0+\sigma(t)-1)} t^{N-1}\, dt = \int_0^{\sqrt{N}} (\ln t^{-1})^{r(t)} t^{-1}\, dt,$$

where $r(t) = -\kappa k^2/(N-k+k\sigma(t))$. Since $\lim_{t\to+0} r(t) < -1$, the last integral converges.

(b) Case $p_0 > N/k$. We have $\mathfrak{D} = \frac{p_0 k - N}{p_0 - 1}$, $p_0 = \frac{N-\mathfrak{D}}{k-\mathfrak{D}}$, and the conditions (5.9) on κ and γ can be rewritten as

$$\kappa > \frac{N-\mathfrak{D}}{2(k-\mathfrak{D})^2} \quad \text{and} \quad \frac{k-\mathfrak{D}}{N-k} - \kappa\frac{(k-\mathfrak{D})^2}{N-k} < \gamma\mathfrak{D} < \kappa\frac{(k-\mathfrak{D})^2}{N-k} - 1. \tag{5.13}$$

We use case (**ii**) of Lemma 28. For the first integral in (4.25), we get

$$\int_0^{\sqrt{N}} F_1\big(t, t^{-k} \sup_{\bar{x}} \mu(B_t^{\bar{x}})\big) |\mathfrak{C}_t|_k\, t^{N-k-1} dt$$

$$= \int_0^{\sqrt{N}} (t^{-k} \sup_{\bar{x}} \mu(B_t^{\bar{x}}))^{p_0-\sigma(t)} |\mathfrak{C}_t|_k\, t^{N-k-1}\, dt$$

$$\lesssim \int_0^{\sqrt{N}} (t^{-k} t^{\mathfrak{D}} (\ln(t^{-1})^{-\gamma\mathfrak{D}}))^{\frac{N-\mathfrak{D}}{k-\mathfrak{D}}-\sigma(t)} t^{k-\mathfrak{D}} (\ln(t^{-1}))^{\gamma\mathfrak{D}} t^{N-k} \frac{dt}{t}$$

$$= \int_0^{\sqrt{N}} t^{(k-\mathfrak{D})\sigma(t)} (\ln(t^{-1}))^{(\frac{k-N}{k-\mathfrak{D}}+\sigma(t))\gamma\mathfrak{D}} \frac{dt}{t}$$

$$= \int_0^{\sqrt{N}} (\ln(t^{-1}))^{r(t)} \frac{dt}{t}, \qquad r(t) = \left(\frac{k-N}{k-\mathfrak{D}} + \sigma(t)\right)\gamma\mathfrak{D} - \kappa(k-\mathfrak{D}).$$

Since (5.13) implies that $\lim_{t\to+0} r(t) < -1$, the last integral converges.

For the second integral in (4.25), we get

$$\int_0^{\sqrt{N}} F_2^*\big(t, t^{k-N}\big) |\mathfrak{C}_t|_k t^{N-k-1}\, dt$$

$$= \int_0^{\sqrt{N}} (t^{k-N})^{(\frac{N-\mathfrak{D}}{k-\mathfrak{D}}+\sigma(t))'} t^{k-\mathfrak{D}} (\ln(t^{-1}))^{\gamma\mathfrak{D}} t^{N-k-1}\, dt$$

$$= \int_0^{\sqrt{N}} (\ln t^{-1})^{r(t)} \frac{dt}{t}, \quad r(t) = \gamma \mathfrak{D} - \kappa \frac{(k-\mathfrak{D})^2}{N-k+(k-\mathfrak{D})\sigma(t)}.$$

Since (5.13) implies that $\lim_{t\to+0} r(t) < -1$, the last integral converges.

By Lemma 28 (**ii**) the pair (u, A) is (Φ, k)-separating.

(c) Case $p_0 < N/k$. We have $\mathfrak{D} = N - p_0 k$ and the conditions (5.10) on κ and γ can be rewritten as

$$\kappa > \frac{N-\mathfrak{D}}{2k^2} \quad \text{and} \quad \frac{N-\mathfrak{D}-k}{k} - \kappa k < \gamma \mathfrak{D} < \kappa k - 1. \tag{5.14}$$

We use case (**i**) of Lemma 28 for the first integral in (4.24), we have

$$\int_0^{\sqrt{N}} F_1(t, t^{-k}) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt = \int_0^{\sqrt{N}} (t^{-k})^{p_0-\sigma(t)} t^{N-k-\mathfrak{D}} (\ln t^{-1})^{\gamma\mathfrak{D}} t^{k-1}\, dt$$

$$= \int_0^{\sqrt{N}} (\ln t^{-1})^{\gamma\mathfrak{D}-\kappa k} \frac{dt}{t} < \infty$$

since (5.14) implies $\gamma\mathfrak{D} - \kappa k < -1$.

For the second integral in (4.24), using $p' = (N-\mathfrak{D})/(N-k-\mathfrak{D})$ we get

$$\int_0^{\sqrt{N}} F_2^*\left(t, t^{k-N} \sup_{\bar{x}} \mu\left(B_t^{\bar{x}}\right)\right) |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt$$

$$= \int_0^{\sqrt{N}} \left(t^{k-N} \sup_{\bar{x}} \mu\left(B_t^{\bar{x}}\right)\right)^{(p_0+\sigma(t))'} |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt$$

$$\leq \int_0^{\sqrt{N}} (t^{k-N} t^{\mathfrak{D}} (\ln t^{-1})^{-\gamma\mathfrak{D}})^{(p_0+\sigma(t))'} t^{N-k-\mathfrak{D}} (\ln t^{-1})^{\gamma\mathfrak{D}} t^{k-1}\, dt = \int_0^{\sqrt{N}} (\ln t^{-1})^{r(t)} \frac{dt}{t},$$

where $r(t) = \dfrac{-\kappa k^2 - k\gamma\mathfrak{D}}{N-\mathfrak{D}-k+k\sigma(t)}$.

Since (5.14) implies that $\lim_{t\to+0} r(t) < -1$, the last integral converges.

By Lemma 28 (**i**), the pair (u, A) is a (Φ, k)-separating.

(d) Case $p_0 = N/k + 0$. For the first integral in (4.25), we get

$$\int_0^{\sqrt{N}} F_1\left(t, t^{-k} \sup_{\bar{x}} \mu\left(B_t^{\bar{x}}\right)\right) |\mathfrak{C}_t|_k\, t^{N-k-1} dt$$

$$= \int_0^{\sqrt{N}} \left(t^{-k} \sup_{\bar{x}} \mu\left(B_t^{\bar{x}}\right)\right)^{p_0-\sigma(t)} |\mathfrak{C}_t|_k\, t^{N-k-1}\, dt$$

$$\lesssim \int_0^{\sqrt{N}} (t^{-k} (\ln(t^{-1})^{-\gamma k}))^{\frac{N}{k}-\sigma(t)} t^k (\ln(t^{-1}))^{\gamma k} t^{N-k} \frac{dt}{t}$$

$$= \int_0^{\sqrt{N}} (\ln(t^{-1}))^{\gamma(k-N+k\sigma(t))-\kappa k} \frac{dt}{t} < \infty$$

since (5.11) implies $\gamma(k-N) - \kappa k < -1$.

For the second integral in (4.25) we have

$$\int_0^{\sqrt{N}} F_2^*\big(t, t^{k-N}\big)|\mathfrak{C}_t|_k t^{N-k-1}\, dt = \int_0^{\sqrt{N}} (t^{k-N})^{(\frac{N}{k}+\sigma(t))'} t^k (\ln(t^{-1}))^{\gamma k} t^{N-k-1}\, dt$$

$$= \int_0^{\sqrt{N}} (\ln t^{-1})^{r(t)} \frac{dt}{t},$$

$$r(t) = \gamma k - \frac{\kappa k^2}{N-k+k\sigma(t)}.$$

Since (5.11) implies that $\lim_{t\to+0} r(t) < -1$, the last integral converges.

(e) Case $p_0 = N/k - 0$. We use case (**i**) of Lemma 28 for the first integral in (4.24), we have

$$\int_0^{\sqrt{N}} F_1(t, t^{-k})|\mathfrak{C}_t|_{N-k} t^{k-1}\, dt = \int_0^{\sqrt{N}} (t^{-k})^{p_0-\sigma(t)} t^{N-k-\mathfrak{D}} (\ln t^{-1})^{\gamma k} t^{k-1}\, dt$$

$$= \int_0^{\sqrt{N}} (\ln t^{-1})^{\gamma k - \kappa k} \frac{dt}{t} < \infty$$

since (5.12) implies $\gamma k - \kappa k < -1$.

For the second integral in (4.24), using $p_0' = N/(N-k)$ we get

$$\int_0^{\sqrt{N}} F_2^*\left(t, t^{k-N} \sup_{\bar{x}} \mu\left(B_t^{\bar{x}}\right)\right) |\mathfrak{C}_t|_{N-k} t^{k-1} dt$$

$$= \int_0^{\sqrt{N}} \left(t^{k-N} \sup_{\bar{x}} \mu\left(B_t^{\bar{x}}\right)\right)^{(p_0+\sigma(t))'} |\mathfrak{C}_t|_{N-k} t^{k-1}\, dt$$

$$\le \int_0^{\sqrt{N}} (t^{k-N}(\ln t^{-1})^{-\gamma k}))^{(p_0+\sigma(t))'} t^{N-k} (\ln t^{-1})^{\gamma k} t^{k-1}\, dt$$

$$= \int_0^{\sqrt{N}} (\ln t^{-1})^{r(t)} \frac{dt}{t}, \quad r(t) = \frac{-\kappa k^2 - \gamma k^2}{N-k+k\sigma(t)}.$$

Since (5.12) implies that $\lim_{t\to+0} r(t) < -1$, the last integral converges. □

Theorem 35 *Let* $1 < p^- < p^+ < \infty$. *Then there exists a variable exponent* $p : \Omega \to [p^-, p^+]$ *(defined by* (5.8) *and* (5.7)*) and* (Φ, k)*-separating pair* (u, A) *for* $\Phi(x,t) = t^{p(x)}$. *Moreover,* $p \in C^\infty(\overline{\Omega}\setminus\mathfrak{S}) \cap C(\overline{\Omega})$, *where* $\mathfrak{S} = \mathfrak{S}(u, A)$ *is a closed set of Lebesgue measure* *zero. For this Φ there holds $H^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1}) \neq W^{d,\Phi(\cdot)}(\Omega, \Lambda^{k-1})$. Let $\eta \in C_0^\infty(\Omega)$ be such that $\eta = 1$ in a neighbourhood of $\mathfrak{S} = \mathfrak{S}(u, A)$, $A^\circ = \eta A$, $b = dA^\circ$. Then for the functional $\mathcal{F}_{\Phi,b}$ there is Lavrentiev gap (4.1). For sufficiently large $t > 0$ and $\omega_0 = tu^\partial \in C^\infty(\overline{\Omega}, \Lambda^{k-1})$ there holds (1.8) and (4.2).*

Proof We have to check only the last statement (different solutions of the Dirichlet problem). By Theorem 16, it remains to show that for our (Φ, k)-separating pair (u, A) there holds

$$\mathcal{F}_{\Phi,0}(tu) + \mathcal{F}^*_{\Phi,0}(s\,dA) \leq \tfrac{1}{2}st$$

for suitable large s, t. The argument repeats that given in the proof of Theorem 32 in [11] and we omit it. □

In this construction by Lemma 29 the variable exponent $p(\cdot)$ has the modulus of continuity $C\left(\ln t^{-1}\right)^{-1} \ln\ln t^{-1}$. This proves Theorem C.

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