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Calculus of Variations

Fractional higher order thin film equation with linear mobility: gradient flow approach

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Abstract
We prove existence of weak solutions of a fractional thin film type equation with linear mobility in any space dimension and for any order of the equation. The proof is based on a gradient flow technique in the space of Borel probability measures endowed with the Wasserstein distance.

Mathematics Subject Classification 35A01 · 35R11 · 35G25 · 35K46 · 49K20 · 35B09

1 Introduction

In this paper we prove existence of non-negative solutions of the following evolution problem:

$$\begin{cases} \partial_t u - \operatorname{div}(u\nabla(\mathscr{L}_s u)) = 0 & \text{in } (0,+\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where the operator $\mathscr{L}_s$ is the s-fractional Laplacian on $\mathbb{R}^d$ and $s \in (0,+\infty)$. Since the order of the operator $\mathscr{L}_s$ is $2s$, the order of the equation in (1.1) is formally $2+2s$. We assume that the initial datum $u_0 \in L^1(\mathbb{R}^d)$ satisfies $u_0 \geq 0$, $\int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x = 1$ and $\int_{\mathbb{R}^d} |x|^2 u_0(x)\,\mathrm{d}x < +\infty$.

The linear operator $\mathscr{L}_s$, also denoted by $(-\Delta)^s$, can be defined using the Fourier transform by

$$\widehat{\mathscr{L}_s u}(\xi) := |\xi|^{2s}\hat{u}(\xi), \tag{1.2}$$

where the Fourier transform of $v \in L^1(\mathbb{R}^d)$ is defined by $\hat{v}(\xi) := \int_{\mathbb{R}^d} v(x)e^{-ix\cdot\xi}\,\mathrm{d}x$. Recalling the link between the Fourier transform and the differentiation, it is immediate to check that for $s = 1$, $\mathscr{L}_1 = -\Delta$ is the classical Laplacian, and for $s = 2$, $\mathscr{L}_2 = (-\Delta)^2$ is the classical bi-Laplacian. The operator $\mathscr{L}_s$ is called "fractional Laplacian" usually for $s \in (0,+\infty)\setminus\mathbb{N}$. In this paper we use the same terminology also in the case $s \in \mathbb{N}$.

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The aim of this paper is to prove an existence result for problem (1.1) using the gradient flow interpretation in the space of Probability measures endowed with the Wasserstein distance. This technique has two advantages. The former is to give automatically the conservation of the initial mass and the non-negativity of the solutions: these properties, in general, are not simple to prove for equations of high order. The latter is that the technique permits to treat simultaneously all the orders of the equation of this type.

The equation in (1.1), for $s \in (0, +\infty) \setminus \mathbb{N}$ is a fractional version of a Thin Film type Equation, namely

$$\partial_t u - \operatorname{div}(u\nabla(-\Delta u)) = 0. \tag{1.3}$$

In general, Thin Film equations are forth order equations of the type

$$\partial_t u - \operatorname{div}(u^n\nabla(-\Delta u)) = 0, \tag{1.4}$$

where $n \in (0, +\infty)$, or more generally,

$$\partial_t u - \operatorname{div}(m(u)\nabla(-\Delta u)) = 0, \tag{1.5}$$

where the non-negative function m is the so-called mobility function. These equations arise in the lubrication approximation theory for a viscous thin film of fluid, driven by surface tension, which spreads on a solid, where u represents the height of the fluid (see for instance the survey [35]). See [22] and [21], in the case $n = 1$, for a mathematical description of the lubrication approximation for a viscous fluid in a Hele-Shaw cell. See also the references in [6] for other models where equations of type (1.5) appear.

The first existence result for solutions of Eq. (1.4) was obtained by Bernis and Friedman [6] in dimension $d = 1$ and $s \in \mathbb{N}$ in bounded interval with homogeneous Neumann type boundary conditions. Precisely, the result is stated for $n > 1$ but the proof should work with modifications also for $n = 1$. Further existence results in dimension 1 were obtained by Beretta, Bertsch, Dal Passo [4] and Bertozzi, Pugh [7] (periodic conditions). In higher dimension existence results was proven by Grün [24], Dal Passo, Garcke and Grün [12] and Bertsch, Dal Passo, Garcke and Grün [8] in bounded domains with Neumann boundary conditions.

In the case $s = 1/2$ and $d = 1$, a first existence result for the Neumann problem in a bounded interval was obtained by Imbert and Mellet [25] (in this case the model describes the propagation of a fracture in an elastic material under the pressure of a fluid filling the fracture). The existence result of [25] was extended to the case $s \in (0, 1)$ in the same setting by Tahrini [39].

In the case $s \in (0, 1)$ and any dimension d, a general existence result for the Cauchy problem (1.1) was obtained by Segatti and Vàzquez [38]. In Section 7 of the same paper the authors raise the problem of the proof of existence of weak solutions for (1.1) using a gradient flow technique. In the previous literature, a proof of existence of solutions for the Cauchy problem using gradient flow technique, in the case $s = 1$ and arbitrary dimension d, was given by Matthes, Mc Cann and Savaré in [34]. In the present paper we give a positive answer to the problem raised in [38].

In dimension $d = 1$ and $s = 1$, the gradient flow structure of the problem related to Eq. (1.3) was highlighted by Otto in [37] for a different notion of solution with respect to the solutions considered in the present paper. More precisely, the paper [37] is devoted to the existence of the so-called "prescribed contact angle solutions": the problem is seen as a free boundary problem, where the equation is satisfied in the positivity set $P := \{(t, x) \in (0, +\infty) \times \mathbb{R} : u(t, x) > 0\}$ and the contact angle at the boundary points $\partial P(t) = \partial\{(x \in$

$\mathbb{R} : u(t,x) > 0\}$ is prescribed and strictly positive, precisely $\pi/4$ at the points $x \in \partial P(t)$. In particular, when the measure of the set $\{(x \in \mathbb{R} : u(t,x) = 0\}$ is positive, these solutions cannot satisfy $u(t,\cdot) \in C^1(\mathbb{R})$ for $t \in (0,+\infty)$. The solutions obtained in the present paper are more regular and corresponds (still in the case $d = 1$ and $s = 1$) to the so called "zero contact angle" solutions because the obtained solutions satisfy $u(t,\cdot) \in H^2(\mathbb{R})$ for a.e. $t \in (0,+\infty)$ and in particular $u(t,\cdot) \in C^1(\mathbb{R})$ for a.e. $t \in (0,+\infty)$. Notice that in [37] the energy functional has an additional term, with respect to our energy functional, which is responsible of the fixed contact angle property of the solutions. The gradient flow structure is also used in [22], still in the case of fixed contact angle solutions in the study of lubrication approximation of the Hele-Shaw flow as Gamma convergence of the corresponding energy functionals. Again in the study of lubrication approximation, in [21] the authors obtain existence of zero contact angle solutions in one dimension.

In the case of higher order, that is when $s > 1$, general existence theorems for the Cauchy problem (1.1), to the best of the author knowledge, are not available in the literature. Only the particular case of dimension $d = 1$ and $s \in \mathbb{N}$ is contained in [6] for the Neumann problem (see also [11] and [17] for the case $d = 1$ and $s = 2$). The other objective of the paper is to give a first existence result for this type of equations of higher order.

In the rest of the introduction we describe the technique and we illustrate the main result of the paper.

The gradient flow setting and the main result

We denote by $\mathscr{P}_2(\mathbb{R}^d)$ the space of Borel probability measures on $\mathbb{R}^d$ with finite second moment. The space $\mathscr{P}_2(\mathbb{R}^d)$, endowed with the 2-Wasserstein distance W, is a complete and separable metric space (see Sect. 2.1 for the definition of W and its properties). For $u \in \mathscr{P}_2(\mathbb{R}^d)$ we define the energy functional

$$\mathscr{F}_s(u) := \frac{1}{2}\|u\|^2_{\dot{H}^s(\mathbb{R}^d)}$$

where $\|u\|_{\dot{H}^s(\mathbb{R}^d)}$ is the seminorm of the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ (see Sect. 2.2) defined as follows

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2 \,\mathrm{d}\xi.$$

We prove that a solution of the Cauchy problem (1.1) can be obtained using the so-called minimizing movement approximation scheme (in the terminology introduced by De Giorgi [14]), applied to the functional $\mathscr{F}_s$ in the metric space $(\mathscr{P}_2(\mathbb{R}^d), W)$. A general theory of minimizing movements in metric spaces and its applications to the space $(\mathscr{P}_2(\mathbb{R}^d), W)$ is contained in the book of Ambrosio-Gigli-Savaré [1]. The gradient flow approach in $(\mathscr{P}_2(\mathbb{R}^d), W)$ with this approximation scheme was first highlighted by Jordan-Kinderlehrer-Otto in the seminal paper [27] for the Fokker-Planck equation. The first study of a fourth order equation using this technique was carried out by Gianazza-Savaré-Toscani [23]. The gradient flow approach to a problem involving fractional laplacian operators is given in [30].

Let us illustrate the strategy in our case: given $u_0 \in \dot{H}^s(\mathbb{R}^d) \cap \mathscr{P}_2(\mathbb{R}^d)$ we introduce the following time discretization scheme: we consider a time step $\tau > 0$, we set $u^0_\tau := u_0$ and we recursively define

$$u^k_\tau \in \mathrm{Argmin}_{u\in\mathscr{P}_2(\mathbb{R}^d)}\left\{\mathscr{F}_s(u) + \frac{1}{2\tau}W^2(u,u^{k-1}_\tau)\right\}, \qquad \text{for } k = 1,2,\ldots. \tag{1.6}$$

The existence and uniqueness of solution for the minimization problem in (1.6) will be established in Sect. 3.2. If $\{u_\tau^k\}_{k\in\mathbb{N}} \subset \mathscr{P}_2(\mathbb{R}^d)$ is a sequence defined by (1.6), we introduce the piecewise constant interpolation

$$u_\tau(t) := u_\tau^k, \qquad \text{if } t \in ((k-1)\tau, k\tau], \quad k = 1, 2, \ldots, \qquad u_\tau(0) := u_\tau^0 = u_0, \quad (1.7)$$

We refer to u_τ as discrete solution. The family of piecewise constant curves $\{u_\tau : \tau > 0\}$ admits a limit curve, in a suitable sense, as $\tau \to 0$ and a limit curve is a weak solution of the equation in (1.1).

We state the results in the following Theorem 1.1.

Before state the theorem we point out that the space $AC^2([0, +\infty); (\mathscr{P}_2(\mathbb{R}^d), W))$ is defined in Sect. 3.2. Moreover, we denote by $[a] := \max\{n \in \mathbb{Z} : n \le a\}$, the integer part of the real number a.

Theorem 1.1 *Let $d \ge 1$, $s > 0$ and $u_0 \in \dot{H}^s(\mathbb{R}^d) \cap \mathscr{P}_2(\mathbb{R}^d)$. Then the following assertions hold:*

i) **Existence and uniqueness of discrete solutions.** *For any $\tau > 0$, there exists a unique sequence $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ satisfying (1.6). In particular the discrete solution $u_\tau : [0, +\infty) \to \mathscr{P}_2(\mathbb{R}^d)$ in (1.7) is uniquely defined.*

ii) **Convergence and regularity.** *For any vanishing sequence τ_n there exists a (non relabeled) subsequence τ_n and a curve $u \in AC^2([0, +\infty); (\mathscr{P}_2(\mathbb{R}^d), W))$ such that:*

(1) $\forall r \in [0, s)$, $u \in C([0, +\infty); H^r(\mathbb{R}^d))$, $\lim_{t\to 0^+} \|u(t) - u_0\|_{H^s(\mathbb{R})} = 0$ and

$$u_{\tau_n}(t) \to u(t) \quad \text{strongly in } H^r(\mathbb{R}^d) \text{ as } n \to \infty, \forall t \in [0, +\infty),$$

(2) $u \in C([0, +\infty); H^s_w(\mathbb{R}^d))$, where $H^s_w(\mathbb{R}^d)$ denotes the space $H^s(\mathbb{R}^d)$ endowed with the weak topology, and

$$u_{\tau_n}(t) \to u(t) \quad \text{weakly in } H^s(\mathbb{R}^d) \text{ as } n \to \infty, \forall t \in [0, +\infty),$$

(3) $u \in L^2((0, T); H^{1+s}(\mathbb{R}^d))$ for every $T > 0$, and

$$u_{\tau_n} \to u \quad \text{strongly in } L^2((0, T); H^{1+r}(\mathbb{R}^d)) \quad \text{as } n \to \infty, \forall r \in [0, s),$$
$$u_{\tau_n} \to u \quad \text{weakly in } L^2((0, T); H^{1+s}(\mathbb{R}^d)) \quad \text{as } n \to \infty.$$

iii) **Solution of the equation.** *u satisfies the equation in (1.1) in the following weak form:*

$$\int_0^{+\infty}\int_{\mathbb{R}^d} u\, \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty} N(u(t,\cdot), \nabla\varphi(t,\cdot))\,\mathrm{d}t = 0, \quad \text{for any } \varphi \in C_c^\infty((0, +\infty) \times \mathbb{R}^d), \quad (1.8)$$

where $N : H^{1+s}(\mathbb{R}^d) \times C_c^\infty(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}$ is defined by

$$N(v, \eta) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} ((\mathscr{L}_{s-m} v)\mathscr{L}_m(\mathrm{div}(\eta\, v))\,\mathrm{d}x, & \text{if } s \in [2m, 2m+1], \\ \displaystyle\int_{\mathbb{R}^d} \nabla((\mathscr{L}_{s-m-1} v) \cdot \nabla(\mathscr{L}_m(\mathrm{div}(\eta\, v)))\,\mathrm{d}x, & \text{if } s \in (2m+1, 2m+2), \end{cases} \quad (1.9)$$

and $m := [s/2]$ is the integer part of $s/2$.

iv) **Entropy dissipation inequality.** *Denoting by $\mathscr{H}(u) := \int_{\mathbb{R}^d} u \ln u \,\mathrm{d}x$ the logarithmic entropy, the following inequality holds*

$$\mathscr{H}(u(T)) + \int_0^T \|u(t)\|^2_{\dot{H}^{s+1}(\mathbb{R}^d)}\,\mathrm{d}t \le \mathscr{H}(u_0), \qquad \forall T \in (0, +\infty). \quad (1.10)$$

We provide some comments about the statement of Theorem 1.1.

The regularity of the solutions $u \in L^2((0,T); H^{1+s}(\mathbb{R}^d))$, given in the point ii), implies that $u(t,\cdot) \in H^{1+s}(\mathbb{R}^d)$ for a.e. $t \in (0,+\infty)$. In particular, in dimension $d = 1$, if $s > \frac{1}{2}$, then $\partial_x u(t,\cdot)$ is continuous and u is a zero-contact angle solution.

For the sake of clarity, since the weak formulation (1.8) of the equation (1.1) is written in terms of the nonlinear operator (1.9), we explicit the weak formulation in the case $s \in (0,1]$, namely

$$\int_0^{+\infty}\int_{\mathbb{R}^d} u\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty}\int_{\mathbb{R}^d} (-\Delta)^s u\,\operatorname{div}(\nabla\varphi\,u)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \forall\,\varphi \in C_c^\infty((0,+\infty)\times\mathbb{R}^d).$$

This formulation and the regularity obtained for u coincide with the one obtained in [38] without gradient flow technique. In the case $s = 1$, the weak formulation of the equation can be rewritten as

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}^d} u\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^{+\infty}\int_{\mathbb{R}^d} (\Delta u\ \Delta\varphi\,u + \Delta u\ \nabla\varphi\cdot\nabla u)\,\mathrm{d}x\,\mathrm{d}t = 0,\\ &\qquad \forall\,\varphi \in C_c^\infty((0,+\infty)\times\mathbb{R}^d).\end{aligned} \tag{1.11}$$

Also in this case the results of Theorem 1.1 are very similar to the existence results obtained in [34], which makes use of the gradient flow technique.

We give some comments on the strategy of the proof of Theorem 1.1.

The existence of both the discrete solutions and a limit curve is consequence of standard arguments in the general theory of minimizing movements. This fact is illustrated in Sect. 3. The minimizers in (1.6), and consequently $u_\tau(t)$ for any $t \geq 0$, belong to $H^s(\mathbb{R}^d)$ by construction. This regularity and a bound on the H^s norms allow to obtain the convergence in points $ii)$(1)-(2). In order to obtain the improved regularity in point $ii)$(3) we make variations of the minimizers in (1.6) along the heat flow, which is the Wasserstein gradient flow of the classical logarithmic entropy $\mathscr{H}$, using the flow interchange technique stated in [34]. This variation provides the regularity $H^{1+s}(\mathbb{R}^d)$ of the minimizers of the scheme and a control of the $H^{1+s}(\mathbb{R}^d)$ seminorm in terms of the entropy (see Lemma 4.4). As a consequence of Lemma 4.4 we obtain a discrete version of the Entropy dissipation inequality (1.10), which is (4.13), and a uniform bound on the $L^2((0,T); \dot{H}^{1+s}(\mathbb{R}^d))$ norm of the discrete solutions (see Corollary 4.5). Using these properties we obtain the convergence in point $ii)$(3) (see Lemma 4.6). Using the regularity $H^{1+s}(\mathbb{R}^d)$, we can make variations of the minimizers of the scheme along the flow generated by a smooth vector field, obtaining a discrete weak formulation of the equation in terms of the nonlinear operator (1.9) (see Sect.4.4 and (4.40)). Finally, using the convergence of point $ii)$ we can pass to the limit in the discrete formulation of the equation obtaining point $iii)$ (see Theorem 4.9).

The case $d = 1$ and $s = 1$ We conclude the introduction analysing the case $d = 1$ and $s = 1$, showing that the solutions given by Theorem 1.1 enjoy some additional properties.

Theorem 1.2 *Let $d = 1$, $s = 1$, $u_0 \in \dot{H}^1(\mathbb{R}) \cap \mathscr{P}_2(\mathbb{R})$ and u a solution given by Theorem 1.1. Then*

$$u \in C^{1/8,1/2}([0,+\infty)\times\mathbb{R}) \cap L^\infty([0,+\infty)\times\mathbb{R}), \tag{1.12}$$

$$u(t,\cdot) \in H^2(\mathbb{R}) \cap C^{1,1/2}(\mathbb{R}), \qquad \textit{for a.e. } t \in (0,+\infty), \tag{1.13}$$

$$\textit{for a.e. } t \in (0,+\infty) \quad \partial^2_{xx}u(t,\cdot) \quad \textit{is differentiable a.e. on the set } \{x \in \mathbb{R} : u(t,x) > 0\}, \tag{1.14}$$

$$u^{1/2}\,\partial^3_{xxx}u \in L^2((0,+\infty)\times\mathbb{R}), \tag{1.15}$$

where we used the convention $u\,\partial^3_{xxx}u = 0$ *on the set* $\{x \in \mathbb{R} : u(t,x) = 0\}$*, and the weak formulation of the equation can be written as*

$$\int_0^{+\infty}\int_{\mathbb{R}} u(t,x)\,\partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty}\int_{\mathbb{R}} u(t,x)\,\partial^3_{xxx}u(t,x)\partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t = 0,$$
$$\text{for any } \varphi \in C_c^\infty((0,+\infty)\times\mathbb{R}). \tag{1.16}$$

Finally

$$\lim_{t\to 0^+} \|u(t) - u_0\|_{H^1(\mathbb{R})} = 0. \tag{1.17}$$

The proof of Theorem 1.2 is given in Sect. 5.

Comparison with other notion of solutions Although the problems studied in [6], [4], [7] in dimension $d = 1$ are set in a bounded interval and complemented with Neumann boundary conditions or with periodic conditions, it is interesting to compare the existence results of the quoted papers with the one obtained here by the gradient flow technique. In particular, the solutions obtained in [6] and in [4] as limit of classical solutions of non-degenerate regularized problems satisfy the analogous property of (1.12), (1.15), the weak formulation (1.16), and the initial datum is taken in the sense of (1.17). In [6] the non-negativity of the solutions and the regularity property (1.13) are stated [6, Remark 4.4], but proved in the case of the mobility u^n with $n > 1$ [6, Theorems 4.1 and 4.2]. The solutions in [6], being obtained by approximation by classical solutions of non degenerate problems, using parabolic Schauder estimates, satisfy the regularity $\partial_t u, \partial_x u, \partial^2_{xx}u, \partial^3_{xxx}u, \partial^4_{xxxx}u \in C(P)$ where $P := \{(t,x) \in (0,+\infty)\times\mathbb{R} : u(t,x) > 0\}$. This last property seems not simple to obtain with our completely different approximation.

With respect to the solutions in [6], the solutions found in [4] and [7] have an additional regularity, which is consequence of entropy estimates of power type. In particular in [4] it is proved that, for any $\beta \in (0,2)$, $u(t,\cdot)^{1/\beta}$ belongs to C^1 for a.e. $t \in (0,+\infty)$, which implies a zero contact angle condition of the following form:

$$\text{for a.e. } t \in (0,+\infty) \text{ exists } C(t): \quad u(t,x) \le C(t)|x - x_0|^\beta, \qquad \forall\, x_0 : u(t,x_0) = 0,$$

for any $\beta \in (0,2)$. A similar contact angle condition follows also by the regularity obtained in [7] by similar methods. In the present paper we used only the logarithmic entropy in order to obtain the entropy dissipation estimate (1.10) and the regularity in (3) of Theorem 1.1, and consequently (1.13). In our framework, we think that it could be possible, with some efforts, to obtain entropy inequalities similar to the ones in [4] and [7], both at the discrete level and at the continuous level. The proof and the extension of such inequalities to the fractional case seem not immediate. The regularity (1.13) implies only a zero contact angle condition of the following form:

$$\text{for a.e. } t \in (0,+\infty) \text{ exists } C(t): \quad u(t,x) \le C(t)|x - x_0|^{3/2}, \qquad \forall\, x_0 : u(t,x_0) = 0.$$

Of course, an improvement of the regularity of our solutions can be obtained using suitable integral estimates. Precisely, thanks to the properties (1.13) and (1.15), we can apply the Bernis type estimates of [2, Corollary 4.2] obtaining that

$$\partial_x(u^{1/2}) \in L^6((0,+\infty)\times\mathbb{R}). \tag{1.18}$$

The property (1.18) implies an improved zero contact angle condition of the following form:

$$\text{for a.e. } t \in (0,+\infty) \text{ exists } C(t): \quad u(t,x) \le C(t)|x - x_0|^{5/3}, \qquad \forall\, x_0 : u(t,x_0) = 0.$$

Moreover, since (1.15), (1.12) and (1.16) hold, we also obtain that

$$\partial_t u \in L^2(0, +\infty; H^{-1}(\mathbb{R})). \tag{1.19}$$

The properties (1.19), (1.18) and Theorem 1.2, show that by Theorem 1.2 we obtain the same notion of solution given in the paper [2] in the case $p = 2$ (the main results are written for $p > 2$ but the proofs could be carried out also in the simpler case $p = 2$).

Open problems

Qualitative properties The theory related to qualitative properties of solutions of thin film equations in the case $s = 1$ is very developed and sharp results are available.

An interesting property is the so-called "finite speed of propagation": roughly speaking the solutions preserve the compactness of the support when the initial datum is compactly supported, and precise rates of expansion of the support are established. In the classical case see [5] for dimension $d = 1$. In higher dimension see [8] for solutions in the sense of [12].

It seems natural that these qualitative properties should hold also for the fractional case (at least for $s \in (0, 1)$). This interesting problem is open. A possible proof could be carried out in dimension $d = 1$ using a similar result to Theorem 1.2 and proving additional suitable weighted entropy estimates. In [38] special self-similar solutions are exhibited (as the Barenblatt solutions for the case $s = 0$) that are compactly supported at any time t and with a precise speed of propagation of the support. This rate of expansion of the support should be expected for solutions starting from compact supported initial datum.

Another interesting qualitative property is the waiting-time phenomenon: it has been firstly proved in [13] and generalized to a larger class of equations in [18] (see also the results in [16]). Also this qualitative property should be very interesting to address for the fractional case.

Uniqueness The uniqueness of the solutions given by Theorem 1.1 is an open and challenging problem. Of course, the problem is still open in the classical case $s = 1$, since a comparison principle is not available. Only partial results are available: see [26] and [33]

Free boundary problem In the classical case $d = 1$ and $s = 1$ the equation (1.1) has been studied as a classical free boundary problem (see [20], [19] for the zero-contact angle case and [28], [29] for the non-zero contact angle case). For the fractional problem in $d = 1$, at least for $s \in (1/2, 1)$, it could be interesting to state the problem as a free boundary classical problem, for instance taking an initial datum close to a steady state of the form $u(x) = (C - C|x|^2)_+^{1+s}$. This problem is completely open.

Gradient flow solutions for non-linear mobilities A system of the type

$$\begin{cases} \partial_t u - \operatorname{div}(m(u)\nabla(\mathscr{L}_s u)) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^d, \end{cases}$$

where the mobility function $m : [0, +\infty) \to [0, +\infty)$ is concave and non-linear, is formally a gradient flow of the functional $\mathscr{F}_s$ in the space of Borel probability measures endowed with the mobility dependent distance introduced in [15] (see also [31]). This technique was used in [32] for the Cahn-Hilliard equation with $s = 1$ and it could be interesting to apply this technique also to this fractional case. The main difficulties arise from the new distance that is defined in a dynamical way and it is not linked to a relaxed transport problem as the classical Wasserstein distance.

2 Notation and preliminary results

2.1 Probability measures and Wasserstein distance

For a detailed treatment of this topic see [1] and [40]. We denote by $\mathscr{P}(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$. The narrow convergence in $\mathscr{P}(\mathbb{R}^d)$ is defined in duality with continuous and bounded functions on $\mathbb{R}^d$. Precisely, a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(\mathbb{R}^d)$ narrowly converges to $u \in \mathscr{P}(\mathbb{R}^d)$ if $\int_{\mathbb{R}^d} \phi \, \mathrm{d}u_n \to \int_{\mathbb{R}^d} \phi \, \mathrm{d}u$ as $n \to +\infty$ for every $\phi \in C_b(\mathbb{R}^d)$, where $C_b(\mathbb{R}^d)$ is the set of continuous and bounded real functions on $\mathbb{R}^d$.

Given $u \in \mathscr{P}(\mathbb{R}^k)$ and $G : \mathbb{R}^k \to \mathbb{R}^n$ a Borel measurable map, we define the push forward (or image measure) of u through G, denoted by $G_\# u \in \mathscr{P}(\mathbb{R}^n)$, by $G_\# u(B) := u(G^{-1}(B))$ for all Borel set $B \subset \mathbb{R}^n$, or equivalently,

$$\int_{\mathbb{R}^n} f(y) \, \mathrm{d}G_\# u(y) = \int_{\mathbb{R}^k} f(G(x)) \, \mathrm{d}u(x),$$

for every Borel positive function $f : \mathbb{R}^k \to \mathbb{R}$.

Since in this paper we use only measures $u \in \mathscr{P}(\mathbb{R}^d)$ absolutely continuous with respect to the Lebesgue measure, we identify the measure u with its density, and with abuse of notation we write $\mathrm{d}u(x) = u(x)\mathrm{d}x$.

We also recall that, when $u \in \mathscr{P}(\mathbb{R}^d)$ is absolutely continuous with respect to the Lebesgue measure and $G : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism, then $v := G_\# u$ is absolutely continuous with respect to the Lebesgue measure and

$$v(x) = u(G^{-1}(x)) \det(\nabla G^{-1}(x)). \tag{2.1}$$

We define $\mathscr{P}_2(\mathbb{R}^d) := \{u \in \mathscr{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \, \mathrm{d}u(x) < +\infty\}$ the set of Borel probability measures with finite second moment. The Wasserstein distance W in $\mathscr{P}_2(\mathbb{R}^d)$ is defined as

$$W(u, v) := \min_{\gamma \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d)} \left\{ \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, \mathrm{d}\gamma(x, y) \right)^{1/2} : (\pi_1)_\# \gamma = u, \ (\pi_2)_\# \gamma = v \right\} \tag{2.2}$$

where $\pi_i, i = 1, 2$, denote the canonical projections on the first and second factor respectively.

If u is absolutely continuous with respect to the Lebesgue measure, then the minimum problem (2.2) has a unique solution γ induced by a transport map $T_u^v : \mathbb{R}^d \to \mathbb{R}^d$ by $\gamma = (I, T_u^v)_\# u$, where I denotes the identity map in $\mathbb{R}^d$. T_u^v is the unique solution (defined u-a.e.) of the Monge optimal transport problem

$$\min_{S:\mathbb{R}^d \to \mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} |S(x) - x|^2 \mathrm{d}u(x) : \ S_\# u = v \right\}.$$

of which (2.2) is the Kantorovich relaxed version. In particular

$$W^2(u, v) = \int_{\mathbb{R}^d} |T_u^v - I|^2 \, u \, \mathrm{d}x. \tag{2.3}$$

The function $W : \mathscr{P}_2(\mathbb{R}^d) \times \mathscr{P}_2(\mathbb{R}^d) \to [0, +\infty)$ is a distance and the metric space $(\mathscr{P}_2(\mathbb{R}^d), W)$ is complete and separable. The distance W is sequentially lower semi continuous with respect to the narrow convergence, i.e.,

$$u_n \to u, \quad v_n \to v, \ \text{narrowly} \implies \liminf_{n \to +\infty} W(u_n, v_n) \geq W(u, v), \tag{2.4}$$

and

$$\text{bounded sets in } (\mathscr{P}_2(\mathbb{R}^d), W) \text{ are narrowly sequentially relatively compact.} \tag{2.5}$$

2.2 Fourier transform and fractional Sobolev spaces

We denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz space of smooth functions with rapid decay at infinity and by $\mathscr{S}'(\mathbb{R}^d)$ the dual space of tempered distributions. The Fourier transform of $u \in \mathscr{S}(\mathbb{R}^d)$ is defined by $\hat{u}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x$. The Fourier transform is an automorphism of $\mathscr{S}(\mathbb{R}^d)$ and can be defined on $\mathscr{S}'(\mathbb{R}^d)$ by transposition. Moreover the Plancherel formula holds

$$\int_{\mathbb{R}^d} \hat{u}(\xi)\overline{\hat{w}(\xi)}\,\mathrm{d}\xi = (2\pi)^d \int_{\mathbb{R}^d} u(x)w(x)\,\mathrm{d}x, \qquad \forall u, w \in L^2(\mathbb{R}^d). \tag{2.6}$$

We observe that if u is real valued, then

$$\overline{\hat{u}(\xi)} = \hat{u}(-\xi) \qquad \forall \xi \in \mathbb{R}^d. \tag{2.7}$$

Moreover we recall the link between the Fourier transform and the differentiation, valid for tempered distributions,

$$\widehat{\partial_{x_k} u}(\xi) = i\xi_k \hat{u}(\xi), \qquad u \in \mathscr{S}'(\mathbb{R}^d). \tag{2.8}$$

Let $r \in \mathbb{R}$. For every tempered distribution $u \in \mathscr{S}'(\mathbb{R}^d)$ such that $\hat{u} \in L^1_{loc}(\mathbb{R}^d)$, we define

$$\|u\|^2_{H^r(\mathbb{R}^d)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\xi|^2)^r |\hat{u}(\xi)|^2\,\mathrm{d}\xi$$

and

$$\|u\|^2_{\dot{H}^r(\mathbb{R}^d)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2r} |\hat{u}(\xi)|^2\,\mathrm{d}\xi.$$

By (2.6) and (2.8) it holds

$$\|u\|^2_{H^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (|u(x)|^2 + |\nabla u(x)|^2)\,\mathrm{d}x, \qquad \|u\|^2_{\dot{H}^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\nabla u(x)|^2\,\mathrm{d}x.$$

The fractional Sobolev space $H^r(\mathbb{R}^d)$ is defined by

$$H^r(\mathbb{R}^d) := \{u \in \mathscr{S}'(\mathbb{R}^d) : \hat{u} \in L^1_{loc}(\mathbb{R}^d),\ \|u\|_{H^r(\mathbb{R}^d)} < +\infty\},$$

and the homogenous fractional Sobolev space $\dot{H}^r(\mathbb{R}^d)$ is defined by

$$\dot{H}^r(\mathbb{R}^d) := \{u \in \mathscr{S}'(\mathbb{R}^d) : \hat{u} \in L^1_{loc}(\mathbb{R}^d),\ \|u\|_{\dot{H}^r(\mathbb{R}^d)} < +\infty\}.$$

Using Plancherel's formula (2.6) and the definition of $\mathscr{L}_s$ in (1.2) it is immediate to show that

$$\|u\|^2_{\dot{H}^r(\mathbb{R}^d)} = \|\mathscr{L}_{r/2} u\|^2_{L^2(\mathbb{R}^d)}, \qquad \forall r > 0. \tag{2.9}$$

Moreover, it follows from the definition of $\mathscr{L}_s$ that

$$u \in \dot{H}^r(\mathbb{R}^d) \quad \Longrightarrow \quad \mathscr{L}_s u \in \dot{H}^{r-2s}(\mathbb{R}^d). \tag{2.10}$$

In this paper we use the following obvious relations:

$$\|u\|_{H^{r_1}(\mathbb{R}^d)} \le \|u\|_{H^{r_2}(\mathbb{R}^d)}, \qquad \text{if } r_1 < r_2,$$

$$\|u\|_{\dot{H}^r(\mathbb{R}^d)} \le \|u\|_{H^r(\mathbb{R}^d)}, \qquad \text{if } r > 0,$$
$$\|u\|_{\dot{H}^0(\mathbb{R}^d)} = \|u\|_{H^0(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)},$$

and the interpolation inequalities

$$\|u\|_{H^{r_1}(\mathbb{R}^d)} \le \|u\|^{1-\theta}_{H^{r_0}(\mathbb{R}^d)} \|u\|^{\theta}_{H^{r_2}(\mathbb{R}^d)} \quad \text{and} \quad \|u\|_{\dot{H}^{r_1}(\mathbb{R}^d)} \le \|u\|^{1-\theta}_{\dot{H}^{r_0}(\mathbb{R}^d)} \|u\|^{\theta}_{\dot{H}^{r_2}(\mathbb{R}^d)}, \quad \text{if } r_0 < r_1 < r_2 \text{ and } \theta \text{ satisfies } r_1 = (1-\theta) r_0 + \theta r_2, \tag{2.11}$$

(see for instance [3, Sections 1.3, 1.4]).

The following lemma and its Corollary will be useful in the sequel for proving convergence results.

Lemma 2.1 *Let $s > 0$ and $\{u_n\}_{n\in\mathbb{N}} \subset H^s(\mathbb{R}^d)$ a sequence such that $\sup_{n\in\mathbb{N}} \|u_n\|_{H^s(\mathbb{R}^d)} < +\infty$ and*

$$\sup_{n\in\mathbb{N}} \sup_{\xi\in B_R(0)} |\hat{u}_n(\xi)| < +\infty, \qquad \forall\, R > 0. \tag{2.12}$$

If u is a Borel signed measure such that $\hat{u}_n(\xi) \to \hat{u}(\xi)$ for every $\xi \in \mathbb{R}^d$, as $n \to +\infty$, then $u \in H^s(\mathbb{R}^d)$, $\|u_n - u\|_{H^r(\mathbb{R}^d)} \to 0$ as $n \to +\infty$, for any $r \in [0, s)$ and $u_n \to u$ weakly in $H^s(\mathbb{R}^d)$ as $n \to +\infty$.

Moreover, if $h \in [0, s/2)$, then $\|\mathscr{L}_h u_n - \mathscr{L}_h u\|_{H^r(\mathbb{R}^d)} \to 0$ for any $r \in [0, s-2h)$ and $\mathscr{L}_h u_n \to \mathscr{L}_h u$ weakly in $H^{s-2h}(\mathbb{R}^d)$. Finally, $\mathscr{L}_{s/2} u_n \to \mathscr{L}_{s/2} u$ weakly in $L^2(\mathbb{R}^d)$.

Proof We first prove that $u \in H^s(\mathbb{R}^d)$ and $u_n \to u$ weakly in $H^s(\mathbb{R}^d)$.
We define $U_n(\xi) := (1+|\xi|^2)^{s/2}\hat{u}_n(\xi)$. By assumption we have $\sup_n \|U_n\|_{L^2(\mathbb{R}^d)} < +\infty$ and $U_n(\xi) \to (1+|\xi|^2)^{s/2}\hat{u}(\xi)$ for every $\xi \in \mathbb{R}^d$ as $n \to +\infty$. By the L^2 weak compactness there exists a subsequence of $\{U_n\}$ weakly convergent in $L^2(\mathbb{R}^d)$ to some $U \in L^2(\mathbb{R}^d)$. By uniqueness of the weak limit we have that $U(\xi) = (1+|\xi|^2)^{s/2}\hat{u}(\xi)$. By the lower semicontinuity of the L^2 norm we obtain that $\|u\|_{H^s(\mathbb{R}^d)} \le \liminf_{n\to\infty} \|u_n\|_{H^s(\mathbb{R}^d)}$. Since the weak topology is metrizable in bounded sets and the limit is independent of the subsequence, all the sequence u_n converges weakly in $H^s(\mathbb{R}^d)$.

Let us fix $r \in [0, s)$ and we prove that u_n strongly converges to u in $H^r(\mathbb{R}^d)$.

For any $R > 0$ we write

$$\begin{aligned}
\|u_n - u\|^2_{H^r(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} (1+|\xi|^2)^r |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \,\mathrm{d}\xi \\
&= \int_{\{|\xi|\le R\}} (1+|\xi|^2)^r |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \,\mathrm{d}\xi \\
&\quad + \int_{\{|\xi|> R\}} (1+|\xi|^2)^r |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \,\mathrm{d}\xi.
\end{aligned} \tag{2.13}$$

Let C be a constant such that $\|u_n\|^2_{H^s(\mathbb{R}^d)} \le C$. Since, as observed in the first part of the proof, also $\|u\|^2_{H^s(\mathbb{R}^d)} \le C$, we have

$$\begin{aligned}
\int_{\{|\xi|>R\}} (1+|\xi|^2)^r |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \,\mathrm{d}\xi &= \int_{\{|\xi|>R\}} (1+|\xi|^2)^s (1+|\xi|^2)^{(r-s)} |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \,\mathrm{d}\xi \\
&\le (1+R^2)^{(r-s)} \int_{\{|\xi|>R\}} (1+|\xi|^2)^s |\hat{u}_n(\xi) - \hat{u}(\xi)|^2 \,\mathrm{d}\xi \\
&\le (1+R^2)^{(r-s)} \|u_n - u\|^2_{H^s(\mathbb{R}^d)} \le 4C(1+R^2)^{(r-s)}.
\end{aligned} \tag{2.14}$$

On the other hand, since $(1+|\xi|^2)^r|\hat{u}_n(\xi)-\hat{u}(\xi)|^2 \to 0$ for any $\xi \in \mathbb{R}^d$ and by (2.12) $(1+|\xi|^2)^r|\hat{u}_n(\xi)-\hat{u}(\xi)|^2 \leq (1+R^2)^r C_R^2$, where $C_R := \sup_{n\in\mathbb{N}} \sup_{\xi\in B_R(0)} |\hat{u}_n(\xi)|$, by dominated convergence Theorem we have

$$\lim_{n\to+\infty} \int_{\{|\xi|\leq R\}} (1+|\xi|^2)^r|\hat{u}_n(\xi)-\hat{u}(\xi)|^2 \, d\xi = 0. \tag{2.15}$$

Fixing $\varepsilon > 0$, by (2.14) there exists R such that $\int_{\{|\xi|>R\}}(1+|\xi|^2)^r|\hat{u}_n(\xi)-\hat{u}(\xi)|^2 \, d\xi < \varepsilon$ for any $n \in \mathbb{N}$. Then, by (2.13), (2.14) and (2.15) we obtain that $\limsup_{n\to+\infty} \|u_n - u\|^2_{H^r(\mathbb{R}^d)} \leq \varepsilon$. For the arbitrariness of ε we conclude.

The last assertions are consequence of the proved convergences and the following relation

$$\|\mathscr{L}_h u_n\|_{\dot{H}^r(\mathbb{R}^d)} = \|u_n\|_{\dot{H}^{r+2h}(\mathbb{R}^d)} \leq \|u_n\|_{H^{r+2h}(\mathbb{R}^d)}.$$

□

Corollary 2.2 *Let $s > 0$ and $\{u_n\}_{n\in\mathbb{N}} \subset \dot{H}^s(\mathbb{R}^d) \cap \mathscr{P}(\mathbb{R}^d)$ a sequence such that $\sup_{n\in\mathbb{N}} \|u_n\|_{\dot{H}^s(\mathbb{R}^d)} < +\infty$. If $u \in \mathscr{P}(\mathbb{R}^d)$ and u_n narrowly converges to u, then $u_n, u \in H^s(\mathbb{R}^d)$, and all the conclusions of Lemma 2.1 hold.*

Proof We prove that the assumptions of Lemma 2.1 hold. Since u_n is a density of a probability measure, then $|\hat{u}_n(\xi)| \leq 1$ for any $\xi \in \mathbb{R}^d$. Then (2.12) holds. In order to prove that $\sup_{n\in\mathbb{N}} \|u_n\|_{H^s(\mathbb{R}^d)} < +\infty$, by definition of the $H^s(\mathbb{R}^d)$ norm, it is sufficient to prove that $\sup_{n\in\mathbb{N}} \|u_n\|_{L^2(\mathbb{R}^d)} < +\infty$. Denoting by B_1 the unitary ball in $\mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} |\hat{u}_n(\xi)|^2 \, d\xi = \int_{B_1} |\hat{u}_n(\xi)|^2 \, d\xi + \int_{\mathbb{R}^d\setminus B_1} |\hat{u}_n(\xi)|^2 \, d\xi \leq |B_1| + \int_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}_n(\xi)|^2 \, d\xi,$$

and by Plancherel formula we obtain

$$(2\pi)^d \|u_n\|^2_{L^2(\mathbb{R}^d)} \leq |B_1| + \|u_n\|^2_{\dot{H}^s(\mathbb{R}^d)}. \tag{2.16}$$

Moreover the narrow convergence of u_n to u implies the pointwise convergence $\hat{u}_n(\xi) \to \hat{u}(\xi)$ for any $\xi \in \mathbb{R}^d$. □

3 Energy functional and first convergence result

3.1 Energy functional

After noticing that a Borel probability measure u is a tempered distribution with $\hat{u}$ in $L^1_{loc}(\mathbb{R}^d)$, we define the energy functional $\mathscr{F}_s : \mathscr{P}_2(\mathbb{R}^d) \to [0, +\infty]$ by

$$\mathscr{F}_s(u) := \frac{1}{2}\|u\|^2_{\dot{H}^s(\mathbb{R}^d)}.$$

We denote by $D(\mathscr{F}_s) = \{u \in \mathscr{P}_2(\mathbb{R}^d) : \mathscr{F}_s(u) < +\infty\}$. Using Corollary 2.2, it is immediate to prove the following Proposition.

Proposition 3.1 *The following assertions hold:*

- $D(\mathscr{F}_s) = H^s(\mathbb{R}^d) \cap \mathscr{P}_2(\mathbb{R}^d)$.
- *$\mathscr{F}_s$ is sequentially lower semicontinuous w.r.t. the narrow convergence.*

3.2 Wasserstein gradient flow, minimizing movements

We consider, for $k = 1, 2, \ldots$, the problem

$$\min_{u \in \mathscr{P}_2(\mathbb{R}^d)} \mathscr{F}_s(u) + \frac{1}{2\tau} W^2(u, u_\tau^{k-1}). \tag{3.1}$$

Proposition 3.2 *For every $\tau > 0$ and every $u_0 \in D(\mathscr{F}_s)$ there exists a unique sequence $\{u_\tau^k : k = 0, 1, 2, \ldots\} \subset D(\mathscr{F}_s)$ satisfying $u_\tau^0 = u_0$ and such that u_τ^k is a solution to problem (3.1) for $k = 1, 2, \ldots$.*

Proof Let $\tau > 0$ and $k \in \mathbb{N}$. By Proposition 3.1 and the properties of the Wasserstein distance (2.4) (2.5), the functional $u \mapsto \mathscr{F}_s(u) + \frac{1}{2\tau} W^2(u, u_\tau^{k-1})$ is nonnegative, lower semicontinuous with respect to the narrow convergence and with narrowly compact sublevels. The existence of minimizers follows by standard direct methods in calculus of variations. The uniqueness of minimizers follows from the strict convexity of the functional $u \mapsto \mathscr{F}_s(u) + \frac{1}{2\tau} W^2(u, u_\tau^{k-1})$ with respect to linear convex combinations in $\mathscr{P}_2(\mathbb{R}^d)$. □

By Proposition 3.2, the piecewise constant curve

$$u_\tau(t) := u_\tau^k, \qquad \text{if } t \in ((k-1)\tau, k\tau], \quad k = 1, 2, \ldots, \qquad u_\tau(0) := u_\tau^0 = u_0, \tag{3.2}$$

is uniquely defined.

We say that a curve $u : [0, +\infty) \to \mathscr{P}_2(\mathbb{R}^d)$ is absolutely continuous with finite energy, and we use the notation $u \in AC^2([0, +\infty); (\mathscr{P}_2(\mathbb{R}^d), W))$, if there exists $m \in L^2([0, +\infty))$ such that $W(u(t_1), u(t_2)) \leq \int_{t_1}^{t_2} m(r)\, dr$ for any $t_1, t_2 \in [0, +\infty)$, $t_1 < t_2$.

Theorem 3.3 *[First convergence result] Let $u_0 \in D(\mathscr{F}_s)$ and u_τ the piecewise constant curve defined in (3.2). For every vanishing sequence τ_n there exists a subsequence (not relabeled) τ_n and a curve $u \in AC^2([0, +\infty); (\mathscr{P}_2(\mathbb{R}^d), W))$ such that*

$$u_{\tau_n}(t) \to u(t) \quad \textit{narrowly as } n \to \infty, \textit{ for any } t \in [0, +\infty). \tag{3.3}$$

Proof The first estimate given by the scheme (3.1), is the following

$$\mathscr{F}_s(u_\tau^N) + \frac{1}{2} \sum_{k=1}^{N} \tau \frac{W^2(u_\tau^k, u_\tau^{k-1})}{\tau^2} \leq \mathscr{F}_s(u_\tau^0) = \mathscr{F}_s(u_0), \qquad \forall N \in \mathbb{N}. \tag{3.4}$$

We show that for any $T > 0$ the set $\text{Å}_T := \{u_\tau^N : \tau > 0, N \in \mathbb{N}, N\tau \leq T\}$ is bounded in $(\mathscr{P}_2(\mathbb{R}^d), W)$ and by (2.5) sequentially narrowly compact.
Indeed, recalling that $\int_{\mathbb{R}^d} |x|^2 \,\mathrm{d}u(x) = W^2(u, \delta_0)$ for any $u \in \mathscr{P}_2(\mathbb{R}^d)$, using the triangle inequality for W and Jensen's discrete inequality we have

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^2 u_\tau^N(x)\,\mathrm{d}x = W^2(u_\tau^N, \delta_0) &\leq \Big(\sum_{k=1}^{N} W(u_\tau^k, u_\tau^{k-1}) + W(u_\tau^0, \delta_0) \Big)^2 \\ &\leq 2 \Big(\sum_{k=1}^{N} \tau \frac{W(u_\tau^k, u_\tau^{k-1})}{\tau} \Big)^2 + 2W^2(u_\tau^0, \delta_0) \qquad (3.5) \\ &\leq 2N\tau \sum_{k=1}^{N} \tau \frac{W^2(u_\tau^k, u_\tau^{k-1})}{\tau^2} + 2W^2(u_\tau^0, \delta_0). \end{aligned}$$

Since $\mathscr{F}_s \geq 0$, from (3.4) and (3.5) it follows that

$$\int_{\mathbb{R}^d} |x|^2 u_\tau^N(x)\,\mathrm{d}x \leq 2T\mathscr{F}_s(u_0) + 2\int_{\mathbb{R}^d} |x|^2 u_0(x)\,\mathrm{d}x, \qquad \forall N \in \mathbb{N}: \ N\tau \leq T \quad (3.6)$$

and the boundedness of Å_T follows.

We define the piecewise constant function $m_\tau : [0, +\infty) \to [0, +\infty)$ as

$$m_\tau(t) := \frac{W(u_\tau(t), u_\tau(t-\tau))}{\tau}$$

with the convention that $u_\tau(t-\tau) = u_\tau(0)$ if $t - \tau < 0$. Since $\mathscr{F}_s \geq 0$, from (3.4) it follows that

$$\frac{1}{2}\int_0^{+\infty} m_\tau^2(t)\,\mathrm{d}t \leq \mathscr{F}_s(u_0).$$

It follows that there exists $m \in L^2(0, +\infty)$ such that m_τ weakly converges to m in $L^2(0, +\infty)$. Moreover for any $t_1, t_2 \in [0, +\infty)$, $t_1 < t_2$, setting $k_1(\tau) = [t_1/\tau]$ and $k_2(\tau) = [t_2/\tau]$, by triangle inequality it holds

$$W(u_\tau(t_1), u_\tau(t_2)) \leq \sum_{k=k_1(\tau)}^{k_2(\tau)-1} W(u_\tau^k, u_\tau^{k-1}) \leq \int_{k_1(\tau)\tau}^{k_2(\tau)\tau} m_\tau(t)\,\mathrm{d}t.$$

By the L^2 weak convergence of m_τ the following equicontinuity estimate holds

$$\limsup_{\tau\to 0} W(u_\tau(t_1), u_\tau(t_2)) \leq \lim_{\tau\to 0}\int_{k_1(\tau)\tau}^{k_2(\tau)\tau} m_\tau(t)\,\mathrm{d}t = \int_{t_1}^{t_2} m(t)\,\mathrm{d}t. \quad (3.7)$$

Applying Proposition 3.3.1 of [1] we obtain the convergence (3.3). Passing to the limit in (3.7) we obtain

$$W(u(t_1), u(t_2)) \leq \int_{t_1}^{t_2} m(t)\,\mathrm{d}t, \qquad \forall t_1, t_2 \in [0, +\infty), \quad t_1 < t_2,$$

and then $u \in AC^2([0, +\infty); (\mathscr{P}_2(\mathbb{R}^d), W))$. □

4 Estimates on discrete solutions, convergence and weak solution

In this Section we briefly review the "flow interchange estimate" introduced by Matthes-McCann-Savaré in [34]. Using this estimate with the entropy functional, we obtain a suitable regularity estimate for the family of discrete solutions u_τ. Moreover, using this estimate with a family of suitable potential energy functionals, we obtain that u_τ satisfies an approximate weak formulation of the equation in (1.1).

4.1 Flow interchange technique

We say that a lower semi continuous functional $\mathscr{V} : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$, with proper domain $D(\mathscr{V}) = \{u \in \mathscr{P}_2(\mathbb{R}^d) : \mathscr{V}(u) < +\infty\} \neq \emptyset$, generates a λ-flow, for $\lambda \in \mathbb{R}$, if there exists a continuous semigroup $S_t : D(\mathscr{V}) \to D(\mathscr{V})$ such that the following family of

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$$\limsup_{t\to 0^+} \frac{W^2(S_t(u), v) - W^2(u, v)}{2t} + \frac{\lambda}{2} W^2(u, v) \le \mathscr{V}(v) - \mathscr{V}(u), \quad \forall u \in D(\mathscr{V}), \tag{4.1}$$

hold. We recall that a continuous semigroup is a family of maps $S_t : D(\mathscr{V}) \to D(\mathscr{V}), t \ge 0$, such that

$$S_t(S_r(u)) = S_{t+r}(u), \quad \forall\, t, r \ge 0, \qquad \lim_{t\to 0^+} W(S_t(u), u) = 0, \quad \forall u \in D(\mathscr{V}).$$

If $u \in D(\mathscr{F}_s)$ we define the dissipation of $\mathscr{F}_s$ along the flow S_t of $\mathscr{V}$ by

$$\mathfrak{D}_{\mathscr{V}} \mathscr{F}_s(u) := \limsup_{t\to 0^+} \frac{\mathscr{F}_s(u) - \mathscr{F}_s(S_t(u))}{t}. \tag{4.2}$$

Proposition 4.1 [Flow interchange] *Let $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ be the sequence given by* Proposition 3.2, *$\lambda \in \mathbb{R}$ and $\mathscr{V}$ a functional generating a λ-flow. If $u_\tau^k \in D(\mathscr{V})$ then*

$$\mathfrak{D}_{\mathscr{V}} \mathscr{F}_s(u_\tau^k) + \frac{\lambda}{2\tau} W^2(u_\tau^k, u_\tau^{k-1}) \le \frac{\mathscr{V}(u_\tau^{k-1}) - \mathscr{V}(u_\tau^k)}{\tau}, \qquad k = 1, 2, \ldots. \tag{4.3}$$

Proof For $t > 0$ and $k \ge 1$, by definition of minimizer there holds

$$\mathscr{F}_s(u_\tau^k) + \frac{1}{2\tau} W^2(u_\tau^k, u_\tau^{k-1}) \le \mathscr{F}_s(S_t(u_\tau^k)) + \frac{1}{2\tau} W^2(S_t(u_\tau^k), u_\tau^{k-1}),$$

that is,

$$\tau \frac{\mathscr{F}_s(u_\tau^k) - \mathscr{F}_s(S_t(u_\tau^k))}{t} \le \frac{W^2(S_t(u_\tau^k), u_\tau^{k-1}) - W^2(u_\tau^k, u_\tau^{k-1})}{2t}.$$

By using (4.1) and the definition (4.2) we obtain (4.3). □

The next two propositions summarize well known results (see [1] Theorems 11.2.5 and 11.2.3).

Proposition 4.2 *The entropy functional $\mathscr{H} : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ defined by*

$$\mathscr{H}(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} u \log u \, \mathrm{d}x & \textit{if } u \textit{ is absolutely continuous w.r.t. Lebesgue measure,} \\ +\infty & \textit{otherwise,} \end{cases}$$

generates a 0-flow. The semigroup associated $u_t := S_t(\bar{u})$ is the unique solution of the Cauchy problem for the heat equation

$$\begin{cases} \partial_t u_t = \Delta u_t, & \textit{in } (0, +\infty) \times \mathbb{R}^d \\ u_0 = \bar{u} & \textit{in } \mathbb{R}^d. \end{cases}$$

Proposition 4.3 *Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\lambda \ge \|D^2\varphi\|_\infty$. The functional $\mathscr{V} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ defined by*

$$\mathscr{V}(u) := \int_{\mathbb{R}^d} \varphi(x) \, \mathrm{d}u(x)$$

generates a $(-\lambda)$*-flow* S_t *and* $S_t(u) = (X_t)_\# u$*, where* $x \mapsto X_t(x)$*,* $x \in \mathbb{R}^d$*, is the map defined by the system*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X_t(x) = -\nabla\varphi(X_t(x)), & t \in \mathbb{R} \\ X_0(x) = x. \end{cases} \tag{4.4}$$

We observe that under the assumptions of Proposition 4.3, X_t is defined also for $t < 0$.

4.2 Improved regularity and dissipation along the Heat flow

The following result makes use of flow interchange with the choice $\mathscr{V} = \mathscr{H}$, the entropy functional.

Lemma 4.4 *Let* $u_0 \in D(\mathscr{F}_s)$ *and* $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ *the sequence given by* Proposition 3.2*. Then* $u_\tau^k \in H^{1+s}(\mathbb{R}^d)$ *for any* $k \geq 1$ *and*

$$\|u_\tau^k\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)} \leq \frac{\mathscr{H}(u_\tau^{k-1}) - \mathscr{H}(u_\tau^k)}{\tau}, \qquad k = 1, 2, \ldots. \tag{4.5}$$

Proof Since $u_\tau^k \in D(\mathscr{F}_s) \subset L^2(\mathbb{R}^d)$ and $(u \log u)_+ \leq u^2$, then $u_\tau^k \in D(\mathscr{H})$ for any $k \geq 0$.

Let us fix $k \geq 1$. For $t \geq 0$, we denote by S_t the 0-flow generated by the entropy $\mathscr{H}$, and we define $w_t := S_t(u_\tau^k)$. By Proposition 4.2, S_t coincides with the heat semigroup on $\mathbb{R}^d$. By uniqueness of the solution of the Cauchy problem for the heat equation, we have the representation

$$w_t = \Gamma_t * u_\tau^k, \qquad \Gamma_t(x) := \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/(4t)}, \tag{4.6}$$

where $*$ denotes the convolution with respect to the space variable x. For the relation with convolution and Fourier transform, by (4.6) we have

$$\hat{w}_t(\xi) = \hat{\Gamma}_t(\xi)\hat{u}_\tau^k(\xi). \tag{4.7}$$

We also recall that the Fourier transform of Γ_t has the expression

$$\hat{\Gamma}_t(\xi) = e^{-t|\xi|^2}. \tag{4.8}$$

The Cauchy problem for the heat equation in the Fourier setting can be written as a family depending on $\xi \in \mathbb{R}^d$ of Cauchy problems

$$\begin{cases} \partial_t \hat{w}_t(\xi) = -|\xi|^2 \hat{w}_t(\xi) & t \in (0, +\infty), \\ \lim_{t\to 0} \hat{w}_t(\xi) = \hat{u}_\tau^k(\xi). \end{cases} \tag{4.9}$$

It is easy to prove that $w_t \in \dot{H}^{1+s}(\mathbb{R}^d)$ for any $t > 0$. Indeed, by (4.7) we have

$$\begin{aligned} \|w_t\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)} &= (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^{2(1+s)} |\hat{w}_t(\xi)|^2 \,\mathrm{d}\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} |\xi|^{2(1+s)} |\hat{\Gamma}_t(\xi)|^2 |\hat{u}_\tau^k(\xi)|^2 \,\mathrm{d}\xi \\ &\leq C_t \|u_\tau^k\|^2_{\dot{H}^s(\mathbb{R}^d)} = 2C_t \mathscr{F}_s(u_\tau^k) < +\infty, \end{aligned}$$

where, using (4.8),

$$C_t := \max_{\xi \in \mathbb{R}^d} |\xi|^2 |\hat{\Gamma}_t(\xi)|^2 = e^{-2} t^{-2}. \tag{4.10}$$

We define the function $g : [0,+\infty) \to \mathbb{R}$ by $g(t) := \mathscr{F}_s(w_t)$. We prove that g is differentiable in $(0,+\infty)$, continuous at $t = 0$, and

$$g'(t) = -\|w_t\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)} \qquad \forall\, t \in (0,+\infty). \tag{4.11}$$

Indeed, taking into account that $|\hat{w}_t(\xi)|^2 = \hat{w}_t(\xi)\overline{\hat{w}_t(\xi)} = \hat{w}_t(\xi)\hat{w}_t(-\xi)$, by (4.9) we have that

$$\partial_t|\hat{w}_t(\xi)|^2 = -2|\xi|^2|\hat{w}_t(\xi)|^2 \qquad \forall\,(t,\xi) \in (0,+\infty)\times\mathbb{R}^d.$$

Since for any $\xi \in \mathbb{R}^d$ the function $t \mapsto |\hat{w}_t(\xi)|^2$ belongs to $C^1(0,+\infty)$ and

$$\left|\partial_t|\xi|^{2s}|\hat{w}_t(\xi)|^2\right| = 2|\xi|^{2s+2}|\hat{w}_t(\xi)|^2 \le 2C_t|\xi|^{2s}|\hat{u}^k_\tau(\xi)|^2,$$

we can differentiate under the integral sign obtaining that

$$\begin{aligned} g'(t) &= \frac{1}{2(2\pi)^d}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|\xi|^{2s}|\hat{w}_t(\xi)|^2\,\mathrm{d}\xi \\ &= -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2s}|\xi|^2|\hat{w}_t(\xi)|^2\,\mathrm{d}\xi = -\|w_t\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)} \end{aligned}$$

and (4.11) is proved. Since $0 < \hat{\Gamma}_t(\xi) \le 1$ we have $|\hat{w}_t(\xi)|^2 = |\hat{\Gamma}_t(\xi)\hat{u}^k_\tau(\xi)|^2 \le |\hat{u}^k_\tau(\xi)|^2$ and then $\mathscr{F}_s(w_t) \le \mathscr{F}_s(u^k_\tau)$, i.e., $g(t) \le g(0)$ for any $t \in (0,+\infty)$. Since $\mathscr{F}_s$ is lower semi continuous with respect to the narrow convergence (Proposition 3.1), we have that $\liminf_{t\to0^+} g(t) \ge g(0)$ and the continuity of g at $t = 0$ is proved.

Applying Lagrange's mean value Theorem to g in the interval $[0,t]$, for any $t > 0$ there exists $\theta(t) \in (0,t)$ such that, recalling the definition of g and (4.11),

$$\frac{\mathscr{F}_s(u^k_\tau) - \mathscr{F}_s(S_t(u^k_\tau))}{t} = \|S_{\theta(t)}(u^k_\tau)\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)}.$$

From this equality and the definition (4.2), by the lower semicontinuity of the $\dot{H}^{1+s}(\mathbb{R}^d)$ semi-norm with respect to the narrow convergence it follows that

$$\|u^k_\tau\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)} \le \mathfrak{D}_{\mathscr{H}}\mathscr{F}_s(u^k_\tau).$$

Finally, by Propositions 4.1 and 4.2, we obtain the estimate (4.5) and $u^k_\tau \in H^{1+s}(\mathbb{R}^d)$. □

Integrating the estimate (4.5) with respect to time, we obtain the following space-time bound on the discrete solution u_τ.

Corollary 4.5 *Let $u_0 \in D(\mathscr{F}_s)$, $\tau > 0$, $\{u^k_\tau : k = 0,1,2,\ldots\}$ the sequence given by* Proposition 3.2 *and u_τ the corresponding discrete piecewise constant approximate solution defined in* (3.2). *Then $u_\tau(t) \in H^{1+s}(\mathbb{R}^d)$ for every $t > 0$ and there exists $C > 0$ depending only on the dimension d such that*

$$\int_0^T \|u_\tau(t)\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)}\,\mathrm{d}t \le \mathscr{H}(u_0) + C\left(1 + T\mathscr{F}_s(u_0) + \int_{\mathbb{R}^d}|x|^2u_0(x)\,\mathrm{d}x\right) \tag{4.12}$$

for any $T > 0$.

Proof Let $T > 0$ and $N := [T/\tau] + 1$. Using (4.5) and the definition of u_τ we obtain

$$\int_0^T \|u_\tau(t)\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)}\,\mathrm{d}t \le \sum_{k=1}^N \tau\|u^k_\tau\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)} \le \mathscr{H}(u_0) - \mathscr{H}(u^N_\tau). \tag{4.13}$$

Using Jensen's inequality, it is not difficult to prove that (see for instance [10]),

$$\mathscr{H}(u) \geq -\frac{1}{e} - \frac{d}{2}\log(4\pi) - \frac{1}{4}\int_{\mathbb{R}^d} |x|^2 u(x)\,\mathrm{d}x, \qquad \forall\, u \in D(\mathscr{H}). \tag{4.14}$$

By (4.14) and (3.6) we obtain

$$-\mathscr{H}(u_\tau^N) \leq C\Big(1 + T\mathscr{F}_s(u_0) + \int_{\mathbb{R}^d} |x|^2 u_0(x)\,\mathrm{d}x\Big)$$

for C depending only on the dimension d. By the last inequality and (4.13) we have (4.12). □

4.3 Improved convergence

Thanks to the estimate of Corollary 4.5 we obtain the following result of convergence. This convergence will be fundamental in order to obtain the weak formulation of the equation in (1.1).

Lemma 4.6 *Let $u_0 \in D(\mathscr{F}_s)$, u_τ the piecewise constant curve defined in (3.2) for any $\tau > 0$. Given a vanishing sequence τ_n, let u_{τ_n} be a convergent subsequence (not relabeled) given by* Theorem 3.3 *and u its limit curve. Then, for any $T > 0$ we have $u \in L^2((0,T); H^{1+s}(\mathbb{R}^d))$ and*

$$u_{\tau_n} \to u \quad \text{strongly in } L^2((0,T); H^{1+r}(\mathbb{R}^d)) \quad \text{as } n \to \infty,\ \forall\, r < s. \tag{4.15}$$

Proof Let $r < s$. By (3.4) and lower semicontinuity we have

$$\|u_{\tau_n}(t)\|^2_{\dot{H}^s(\mathbb{R}^d)} \leq 2\mathscr{F}_s(u_0), \quad \|u(t)\|^2_{\dot{H}^s(\mathbb{R}^d)} \leq 2\mathscr{F}_s(u_0) \quad \forall\, t \in [0,+\infty). \tag{4.16}$$

By (3.3) and Corollary 2.2 we obtain

$$\lim_{n\to+\infty} \|u_{\tau_n}(t) - u(t)\|^2_{H^r(\mathbb{R}^d)} = 0, \qquad \forall t \in [0,+\infty). \tag{4.17}$$

By Corollary 4.5 and lower semicontinuity we have

$$\int_0^T \|u(t)\|^2_{\dot{H}^{1+s}(\mathbb{R}^d)}\,\mathrm{d}t \leq \mathscr{H}(u_0) + C\Big(1 + T\mathscr{F}_s(u_0) + \int_{\mathbb{R}^d} |x|^2 u_0(x)\,\mathrm{d}x\Big) \tag{4.18}$$

for any $T > 0$.

Using the interpolation (2.11), we can write

$$\|u_\tau(t) - u(t)\|_{H^{1+r}(\mathbb{R}^d)} \leq \|u_\tau(t) - u(t)\|^{1-\theta}_{H^r(\mathbb{R}^d)} \|u_\tau(t) - u(t)\|^{\theta}_{H^{1+s}(\mathbb{R}^d)},$$

for $\theta = 1/(1+s-r) \in (0,1)$ and for a.e. $t \in (0,+\infty)$. Fixing $T > 0$, by Hölder's inequality we obtain

$$\begin{aligned}
&\int_0^T \|u_{\tau_n}(t) - u(t)\|^2_{H^{1+r}(\mathbb{R}^d)}\,\mathrm{d}t \\
&\leq \int_0^T \|u_{\tau_n}(t) - u(t)\|^{2(1-\theta)}_{H^r(\mathbb{R}^d)} \|u_{\tau_n}(t) - u(t)\|^{2\theta}_{H^{1+s}(\mathbb{R}^d)}\,\mathrm{d}t \\
&\leq \Big(\int_0^T \|u_{\tau_n}(t) - u(t)\|^2_{H^r(\mathbb{R}^d)}\,\mathrm{d}t\Big)^{1-\theta} \Big(\int_0^T \|u_{\tau_n}(t) - u(t)\|^2_{H^{1+s}(\mathbb{R}^d)}\,\mathrm{d}t\Big)^{\theta}.
\end{aligned}$$

By estimate (4.12) and (4.18) the factor $\int_0^T \|u_{\tau_n}(t)-u(t)\|^2_{H^{1+s}(\mathbb{R}^d)}\,\mathrm{d}t$ is bounded. Finally, by the previous inequality, (4.17) and (4.16) we obtain (4.15) using dominated convergence. □

4.4 Weak formulation of the equation for the discrete solution

In order to obtain a sort of weak formulation of the equation for the discrete solution, we use the flow interchange estimate with the $(-\lambda)$-flow generated by a potential energy as in Proposition 4.3. Preliminarily we compute the derivative of the energy functional $\mathscr{F}_s$ along the flow of a smooth vector field.

Lemma 4.7 *Let $\eta \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ and $X_t:\mathbb{R}^d\to\mathbb{R}^d$, $t\in\mathbb{R}$, be the flow associated to η defined, for any $x\in\mathbb{R}^d$ as the unique global solution of the problem*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X_t(x)=\eta(X_t(x)), & t\in\mathbb{R}\\ X_0(x)=x. \end{cases} \tag{4.19}$$

Let $u\in H^{1+s}(\mathbb{R}^d)\cap\mathscr{P}_2(\mathbb{R}^d)$ and $u_t:=(X_t)_\# u$. Then the map $t\mapsto\mathscr{F}_s(u_t)$ is differentiable at $t=0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{F}_s(u_t)_{|t=0}=-N(u,\eta), \tag{4.20}$$

where $N: H^{1+s}(\mathbb{R}^d)\times C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)\to\mathbb{R}$ is defined in (1.9).

Proof Since $\eta\in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, then for any $t\in\mathbb{R}$, the map X_t is a C^∞ diffeomorphism of $\mathbb{R}^d$ and $X_t^{-1}=X_{-t}$. Moreover if $x\notin\operatorname{supp}\eta$, then $X_t(x)=x$. Since

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\nabla X_t=\nabla\eta(X_t)\nabla X_t, & t\in\mathbb{R}\\ \nabla X_0=I, \end{cases}$$

where we used the notation $\nabla\eta$ and ∇X_t for the Jacobian matrices of η and X_t, there exists a constant $L>0$ such that

$$|X_t(x)-X_t(y)|\le L|x-y|, \qquad \forall x,y\in\mathbb{R}^d, \quad \forall t\in[-1,1].$$

Recalling the formula (2.1), $u_t(x)=u(X_{-t}(x))\det(\nabla X_{-t}(x))$. Observing that the map $x\mapsto\det(\nabla X_{-t}(x))$ belongs to $C^\infty(\mathbb{R}^d;\mathbb{R})$ and $\det(\nabla X_{-t}(x))=1$ for any $x\notin\operatorname{supp}\eta$, there exists a constant $C>0$ such that

$$\|u_t\|_{H^{1+s}(\mathbb{R}^d)}\le C\|u\|_{H^{1+s}(\mathbb{R}^d)}, \qquad \forall t\in[-1,1]. \tag{4.21}$$

See, for instance, [3, Corollary 1.60 and Theorem 1.62].

Using the formula $|a|^2-|b|^2=(\bar a+\bar b)(a-b)+\bar a b-\bar b a$ valid for $a,b\in\mathbb{C}$, by (2.7) we have

$$\mathscr{F}_s(u_t)-\mathscr{F}_s(u)=\frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\xi|^{2s}\big(\hat u_t(-\xi)+\hat u(-\xi)\big)\big(\hat u_t(\xi)-\hat u(\xi)\big)\,\mathrm{d}\xi, \tag{4.22}$$

because

$$\int_{\mathbb{R}^d}|\xi|^{2s}\hat u_t(-\xi)\hat u(\xi)\,\mathrm{d}\xi=\int_{\mathbb{R}^d}|\xi|^{2s}\hat u_t(\xi)\hat u(-\xi)\,\mathrm{d}\xi.$$

Let $m = [s/2]$. If $s \in [2m, 2m+1]$, by (4.22), using the Plancherel identity (2.6) and the definition (1.2), we obtain

$$
\begin{aligned}
\frac{\mathscr{F}_s(u_t) - \mathscr{F}_s(u)}{t} &= \frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{2(s-m)}\big(\hat{u}_t(-\xi) + \hat{u}(-\xi)\big)\frac{|\xi|^{2m}(\hat{u}_t(\xi) - \hat{u}(\xi))}{t}\,\mathrm{d}\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \mathscr{L}_{s-m}(u_t + u)\mathscr{L}_m\left(\frac{u_t - u}{t}\right)\mathrm{d}x.
\end{aligned}
\tag{4.23}
$$

Analogously, if $s \in (2m+1, 2m+2)$, we write

$$
\begin{aligned}
\frac{\mathscr{F}_s(u_t) - \mathscr{F}_s(u)}{t} &= \frac{1}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \xi|\xi|^{2(s-m-1)}\big(\hat{u}_t(-\xi) + \hat{u}(-\xi)\big)\cdot\xi\,\frac{|\xi|^{2m}(\hat{u}_t(\xi) - \hat{u}(\xi))}{t}\,\mathrm{d}\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \nabla\mathscr{L}_{s-m-1}(u_t + u)\cdot\nabla\mathscr{L}_m\left(\frac{u_t - u}{t}\right)\mathrm{d}x.
\end{aligned}
\tag{4.24}
$$

Moreover $u_t \to u$ narrowly as $t \to 0$. Indeed, for $\varphi : \mathbb{R}^d \to \mathbb{R}$ continuous and bounded, by the definition of $(X_t)_{\#}u$ and dominated convergence theorem we have that $\int_{\mathbb{R}^d} \varphi(x)u_t(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(X_t(x))u(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \varphi(x)u(x)\,\mathrm{d}x$ as $t \to 0$.

Thanks to (4.21) and the narrow convergence of u_t to u we can apply Lemma 2.1 obtaining that

$$
\begin{aligned}
&\|\mathscr{L}_{s-m}u_t - \mathscr{L}_{s-m}u\|_{L^2(\mathbb{R}^d)} \to 0 \qquad \text{if } s \in [2m, 2m+1), \\
&\mathscr{L}_{s-m}u_t \to \mathscr{L}_{s-m}u \qquad \text{weakly in } L^2(\mathbb{R}^d) \quad \text{if } s = 2m+1, \\
&\|\mathscr{L}_{s-m-1}u_t - \mathscr{L}_{s-m-1}u\|_{H^1(\mathbb{R}^d)} \to 0 \qquad \text{if } s \in (2m+1, 2m+2),
\end{aligned}
\tag{4.25}
$$

as $t \to 0$.

For every $\xi \in \mathbb{R}^d$ we define $g_\xi : \mathbb{R} \to \mathbb{R}$ by $g_\xi(t) := \hat{u}_t(\xi)$. We prove that $g_\xi \in C^1(\mathbb{R})$ and

$$
g'_\xi(t) = -\widehat{\mathrm{div}(\eta u_t)}(\xi). \tag{4.26}
$$

Indeed, by definition of image measure, we have

$$
g_\xi(t) = \hat{u}_t(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot X_t(x)}u(x)\,\mathrm{d}x.
$$

Using this expression, by dominated convergence Theorem, we have that

$$
\begin{aligned}
\frac{g_\xi(t+h) - g_\xi(t)}{h} &= \int_{\mathbb{R}^d} \frac{1}{h}(e^{-i\xi\cdot X_{t+h}(x)} - e^{-i\xi\cdot X_t(x)})u(x)\,\mathrm{d}x \\
&\to \int_{\mathbb{R}^d} e^{-i\xi\cdot X_t(x)}(-i\xi\cdot\eta(X_t(x)))u(x)\,\mathrm{d}x
\end{aligned}
$$

as $h \to 0$. Moreover, taking into account the definition of image measure and (2.8),

$$
\int_{\mathbb{R}^d} e^{-i\xi\cdot X_t(x)}(-i\xi\cdot\eta(X_t(x)))u(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}(-i\xi\cdot\eta(x))u_t(x)\,\mathrm{d}x = -\widehat{\mathrm{div}(\eta u_t)}(\xi).
\tag{4.27}
$$

The continuity of g'_ξ follows from the expression above and the regularity of the maps $t \mapsto X_t(x)$, using dominated convergence Theorem.

Using the fundamental theorem of calculus and Jensen's inequality we have

$$\begin{aligned}\left\|\frac{u_t-u}{t}\right\|^2_{H^s(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}(1+|\xi|^2)^s\left|\frac{\hat{u}_t(\xi)-\hat{u}(\xi)}{t}\right|^2 \mathrm{d}\xi \\ &= \int_{\mathbb{R}^d}(1+|\xi|^2)^s\left|\frac{g_\xi(t)-g_\xi(0)}{t}\right|^2 \mathrm{d}\xi \\ &= \int_{\mathbb{R}^d}(1+|\xi|^2)^s\left|\frac{1}{t}\int_0^t g'_\xi(r)\,\mathrm{d}r\right|^2 \mathrm{d}\xi \\ &\le \int_{\mathbb{R}^d}(1+|\xi|^2)^s\frac{1}{t}\int_0^t \left|g'_\xi(r)\right|^2 \mathrm{d}r\,\mathrm{d}\xi \\ &= \frac{1}{t}\int_0^t\int_{\mathbb{R}^d}(1+|\xi|^2)^s\left|g'_\xi(r)\right|^2 \mathrm{d}\xi\,\mathrm{d}r \\ &= \frac{1}{t}\int_0^t\int_{\mathbb{R}^d}(1+|\xi|^2)^s\left|\widehat{\mathrm{div}(\eta u_r)}(\xi)\right|^2 \mathrm{d}\xi\,\mathrm{d}r \\ &= \frac{1}{t}\int_0^t \|\mathrm{div}(\eta u_r)\|^2_{H^s(\mathbb{R}^d)}\mathrm{d}r.\end{aligned} \tag{4.28}$$

From the estimate (4.21) it follows that there exists $C>0$, depending on η, such that

$$\|\mathrm{div}(\eta u_r)\|_{H^s(\mathbb{R}^d)} \le \tilde{C}\|u_r\|_{H^{1+s}(\mathbb{R}^d)} \le C\|u\|_{H^{1+s}(\mathbb{R}^d)}, \qquad \forall r\in[-1,1]. \tag{4.29}$$

By (4.28) and (4.29) it follows that

$$\left\|\frac{u_t-u}{t}\right\|_{H^s(\mathbb{R}^d)} \le C\|u\|_{H^{1+s}(\mathbb{R}^d)}, \qquad \forall t\in[-1,1],\ t\neq 0. \tag{4.30}$$

Moreover, by Lagrange mean value, (4.26) and (4.27) we obtain

$$\left|\frac{\hat{u}_t(\xi)-\hat{u}(\xi)}{t}\right| \le |\xi|\|\eta\|_\infty \qquad \forall\xi\in\mathbb{R}^d,\ \forall t\in[-1,1],\ t\neq 0. \tag{4.31}$$

Since, by (4.26), $\lim_{t\to 0}\frac{\hat{u}_t(\xi)-\hat{u}(\xi)}{t} = -\widehat{\mathrm{div}(\eta u)}(\xi)$ for any $\xi\in\mathbb{R}^d$, and (4.30) (4.31) hold, we can apply Lemma 2.1 and we obtain

$$\begin{aligned}&\mathscr{L}_m\left(\frac{u_t-u}{t}\right) \to \mathscr{L}_m(-\mathrm{div}(\eta u)) \qquad \text{weakly in } L^2(\mathbb{R}^d) \quad \text{if } s=2m, \\ &\left\|\mathscr{L}_m\left(\frac{u_t-u}{t}\right) - \mathscr{L}_m(-\mathrm{div}(\eta u))\right\|_{L^2(\mathbb{R}^d)} \to 0 \qquad \text{if } s\in(2m,2m+1], \\ &\left\|\nabla\mathscr{L}_m\left(\frac{u_t-u}{t}\right) - \nabla\mathscr{L}_m(-\mathrm{div}(\eta u))\right\|_{L^2(\mathbb{R}^d)} \to 0 \qquad \text{if } s\in(2m+1,2m+2),\end{aligned} \tag{4.32}$$

as $t\to 0$.

Finally, using (4.25) and (4.32) we pass to the limit in (4.23) and (4.24) and we obtain

$$\lim_{t\to 0}\frac{1}{t}\left(\mathscr{F}_s(u_t)-\mathscr{F}_s(u)\right) = -\int_{\mathbb{R}^d}(\mathscr{L}_{s-m}u)\mathscr{L}_m(\mathrm{div}(\eta\, u))\,\mathrm{d}x \tag{4.33}$$

if $s\in[2m,2m+1]$ and

$$\lim_{t\to 0}\frac{1}{t}\left(\mathscr{F}_s(u_t)-\mathscr{F}_s(u)\right) = -\int_{\mathbb{R}^d}\nabla((\mathscr{L}_{s-m-1}u)\cdot\nabla(\mathscr{L}_m(\mathrm{div}(\eta\, u))\,\mathrm{d}x \tag{4.34}$$

if $s \in (2m+1, 2m+2)$.

By (4.33) and (4.34) we obtain (4.20). □

The application of the Flow interchange estimate with the flow generated by a potential energy yields the following Proposition. We observe that the inequality (4.35) is a sort of discrete weak formulation of the equation in (1.1) (see (4.40) and (4.41)).

Proposition 4.8 *Let* $u_0 \in D(\mathscr{F}_s)$, $\tau > 0$, $\{u_\tau^k : k = 0, 1, 2, \dots\}$ *the sequence given by* Proposition 3.2. *Let* $\varphi \in C_c^\infty(\mathbb{R}^d)$ *and* $\lambda \geq \|D^2\varphi\|_\infty$. *Then*

$$\begin{aligned} -\frac{\lambda}{2} W^2(u_\tau^n, u_\tau^{n-1}) &\leq \int_{\mathbb{R}^d} \varphi(x) u_\tau^n(x)\, \mathrm{d}x - \int_{\mathbb{R}^d} \varphi(x) u_\tau^{n-1}(x)\, \mathrm{d}x - \tau N(u_\tau^n, \nabla\varphi) \\ &\leq \frac{\lambda}{2} W^2(u_\tau^n, u_\tau^{n-1}), \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{4.35}$$

where N is defined in (1.9).

Proof Let us define the functional $\mathscr{V} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ by

$$\mathscr{V}(u) := \int_{\mathbb{R}^d} \varphi(x)\, \mathrm{d}u(x).$$

Let $n \in \mathbb{N}$. Since by Lemma 4.4 $u_\tau^n \in H^{1+s}(\mathbb{R}^d)$, by Proposition 4.3 and Lemma 4.7 for $\eta = -\nabla\varphi$, we have

$$\mathfrak{D}_{\mathscr{V}} \mathscr{F}_s(u_\tau^n) := \limsup_{t \downarrow 0} \frac{\mathscr{F}_s(u_\tau^n) - \mathscr{F}_s(S_t(u_\tau^n))}{t} = -\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{F}_s(S_t(u_\tau^n))_{|t=0} = N(u_\tau^n, -\nabla\varphi),$$

Applying Proposition 4.1 to $\mathscr{V}$ and observing that $N(u_\tau^n, -\nabla\varphi) = -N(u_\tau^n, \nabla\varphi)$, we obtain

$$-\frac{\lambda}{2} W^2(u_\tau^n, u_\tau^{n-1}) - \tau N(u_\tau^n, \nabla\varphi) \leq \mathscr{V}(u_\tau^{n-1}) - \mathscr{V}(u_\tau^n). \tag{4.36}$$

Analogously, applying Proposition 4.1 to $-\mathscr{V}$ instead of $\mathscr{V}$ and observing that $-\mathscr{V}$ still generates a $-\lambda$-flow we obtain

$$-\frac{\lambda}{2} W^2(u_\tau^n, u_\tau^{n-1}) + \tau N(u_\tau^n, \nabla\varphi) \leq -\mathscr{V}(u_\tau^{n-1}) + \mathscr{V}(u_\tau^n). \tag{4.37}$$

Finally, the inequality (4.35) follows by (4.36) and (4.37). □

4.5 Solution of the problem

In this Section we prove that the limit curve given by Theorem 3.3 is a weak solution of problem (1.1) and we conclude the proof of Theorem 1.1.

Theorem 4.9 *If* $u \in AC^2([0, +\infty); (\mathscr{P}_2(\mathbb{R}^d), W))$ *is a limit curve given by* Theorem 3.3, *then u is a solution of the equation in* (1.1) *in the following weak form: for any* $\varphi \in C_c^\infty((0, +\infty) \times \mathbb{R}^d)$

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \partial_t \varphi u \, \mathrm{d}x\, \mathrm{d}t + \int_0^{+\infty} N(u(t), \nabla\varphi(t, \cdot))\, \mathrm{d}t = 0, \tag{4.38}$$

where N is defined in (1.9).

Proof Let $\varphi \in C_c^\infty((0,+\infty)\times\mathbb{R}^d)$, $T>0$ such that $\varphi(t,\cdot)=0$ for any $t>T$. Let $\lambda \geq \max_{t\in[0,T]}\|D^2\varphi(t,\cdot)\|_\infty$.

Using the notation $u_\tau(t,x) := u_\tau(t)(x)$ and the convention $u_\tau(t) := u_0$ if $t<0$, the inequality (4.35) can be rewritten as

$$\begin{aligned} &-\frac{\lambda}{2}W^2(u_\tau(t),u_\tau(t-\tau)) \\ &\leq \int_{\mathbb{R}^d}\varphi(t,x)(u_\tau(t,x)-u_\tau(t-\tau,x))\,\mathrm{d}x - \tau N(u_\tau(t),\nabla\varphi(t,\cdot)) \\ &\leq \frac{\lambda}{2}W^2(u_\tau(t),u_\tau(t-\tau)), \qquad \forall t\in[0,+\infty), \quad \forall \tau>0. \end{aligned} \tag{4.39}$$

Dividing the inequality in (4.39) by $\tau>0$ and integrating in time, we obtain

$$\begin{aligned} &\left|\int_0^T\int_{\mathbb{R}^d}\frac{\varphi(t,x)-\varphi(t+\tau,x)}{\tau}u_\tau(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T N(u_\tau(t),\nabla\varphi(t,\cdot))\,\mathrm{d}t\right| \\ &\leq \frac{\lambda}{2\tau}\int_0^T W^2(u_\tau(t),u_\tau(t-\tau))\,\mathrm{d}t. \end{aligned} \tag{4.40}$$

We observe that the inequality (4.40) is a discrete weak formulation of the equation (1.1).

Let τ_n be a vanishing sequence given by Theorem 3.3.

First of all we show that

$$\lim_{n\to+\infty}\frac{\lambda}{2\tau_n}\int_0^T W^2(u_{\tau_n}(t),u_{\tau_n}(t-\tau_n))\,\mathrm{d}t = 0. \tag{4.41}$$

Indeed, by (3.4)

$$\frac{1}{2\tau_n}\int_0^T W^2(u_{\tau_n}(t),u_{\tau_n}(t-\tau_n))\,\mathrm{d}t \leq \frac{1}{2}\sum_{k=1}^{[T/\tau_n]+1}W^2(u_{\tau_n}^k,u_{\tau_n}^{k-1}) \leq \tau_n\mathscr{F}_s(u_0)$$

and (4.41) follows.

We pass to the limit in the other two terms in (4.40). By the convergence (4.15) and the regualrity of φ it follows that

$$\lim_{n\to+\infty}\int_0^T\int_{\mathbb{R}^d}\frac{\varphi(t,x)-\varphi(t+\tau_n)}{\tau_n}u_{\tau_n}(t,x)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_{\mathbb{R}^d}\partial_t\varphi(t,x)u(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.42}$$

Let $m=[s/2]$. For $s\in[2m,2m+1]$, by definition of N,

$$\int_0^T N(u_{\tau_n}(t),\nabla\varphi(t,\cdot))\,\mathrm{d}t = \int_0^T\int_{\mathbb{R}^d}\mathscr{L}_{s-m}(u_{\tau_n})\mathscr{L}_m(\mathrm{div}(\nabla\varphi\, u_{\tau_n}))\,\mathrm{d}x\,\mathrm{d}t. \tag{4.43}$$

We observe that $\|\mathscr{L}_{s-m}(u_{\tau_n}-u)\|_{L^2(\mathbb{R}^d)} = \|u_{\tau_n}-u\|_{\dot H^{2s-2m}(\mathbb{R}^d)}$. Let $s\in[2m,2m+1)$, defining r such that $1+r=2s-2m$, it holds $r<s$. By Lemma 4.6, we have $\mathscr{L}_{s-m}u_{\tau_n}\to\mathscr{L}_{s-m}u$ strongly in $L^2((0,T);L^2(\mathbb{R}^d))$. If $s=2m+1$ we have $\mathscr{L}_{m+1}u_{\tau_n}\to\mathscr{L}_{m+1}u$ weakly in $L^2((0,T);L^2(\mathbb{R}^d))$. By Lemma 4.6, we have also that $\mathrm{div}(\nabla\varphi\, u_{\tau_n})\to\mathrm{div}(\nabla\varphi\, u)$ strongly in $L^2((0,T);H^r(\mathbb{R}^d))$ for any $r<s$. $\mathscr{L}_m\mathrm{div}(\nabla\varphi\, u_{\tau_n})\to\mathscr{L}_m\mathrm{div}(\nabla\varphi\, u)$ strongly in $L^2((0,T);H^{r-2m}(\mathbb{R}^d))$. In particular, for $r=2m$ we obtain $\mathscr{L}_m\mathrm{div}(\nabla\varphi\, u_{\tau_n})\to\mathscr{L}_m\mathrm{div}(\nabla\varphi\, u)$ strongly in $L^2((0,T);L^2(\mathbb{R}^d))$. The convergences above and (4.43) show that

$$\lim_{n\to+\infty}\int_0^T N(u_{\tau_n}(t),\nabla\varphi(t,\cdot))\,\mathrm{d}t = \int_0^T N(u(t),\nabla\varphi(t,\cdot))\,\mathrm{d}t \tag{4.44}$$

when $s \in [2m, 2m+1)$. Analogously we can prove (4.44) also in the case $s \in (2m+1, 2m+2)$.

The proof of (4.38) follows by (4.40), (4.42), (4.44) and (4.41). □

We conclude this Section with the

Proof of Theorem 1.1. The part i) is exactly Proposition 3.2. The part ii) follows by Theorem 3.3, the inequality (4.16), Corollary 2.2 and Lemma 4.6. The limit $\lim_{t\to 0^+} \|u(t) - u_0\|_{H^s(\mathbb{R})} = 0$ follows by the second inequality in (4.16) which implies that $\lim_{t\to 0^+} \|u(t,\cdot)\|^2_{\dot{H}^s(\mathbb{R})} = \|u_0\|^2_{\dot{H}^1(\mathbb{R})}$. The part iii) is exactly Theorem 4.9. The inequality (1.10) in part iv) follows by (4.13) passing to limit by a lower semicontinuity argument. □

5 The case $d = 1$ and $s = 1$

In this Section we use the notation u' to denote the derivative of u with respect to the one-dimensional space variable.

Lemma 5.1 *Let $u \in H^2(\mathbb{R})$ such that $u \geq 0$ and $u\,u'' \in H^1(\mathbb{R})$. Then u'' is differentiable a.e. in $P := \{x \in \mathbb{R} : u(x) > 0\}$, $u\,u''' \in L^2(\mathbb{R})$, where we use the convention $u\,u''' = 0$ on $\mathbb{R} \setminus P$, and*

$$\int_{\mathbb{R}} u''u\,\eta'\,\mathrm{d}x + \int_{\mathbb{R}} u''u'\,\eta\,\mathrm{d}x = -\int_{\mathbb{R}} u'''u\,\eta\,\mathrm{d}x \qquad \forall\,\eta \in C_c^\infty(\mathbb{R}). \tag{5.1}$$

Proof Let us fix $u \in C^1(\mathbb{R})$ a representative of $u \in H^2(\mathbb{R})$. We denote by $P = \{x \in \mathbb{R} : u(x) > 0\}$ the open set of positivity of u. We observe that $u(x) = 0$ and $u'(x) = 0$ for each $x \in \mathbb{R} \setminus P$ (since are minimum points of u).

Since $u(x)u''(x) = 0$ for each $x \in \mathbb{R} \setminus P$ and $u\,u'' \in H^1(\mathbb{R})$ we have that

$$(uu'')'(x) = 0 \qquad \text{for a.e. } x \in \mathbb{R} \setminus P. \tag{5.2}$$

Denoting by $f := uu'' \in H^1(\mathbb{R})$, then $u'' = \frac{f}{u}$ is differentiable in every point of differentiability of f in P. In particular it holds that

$$(uu'')'(x) = (u'''u + u''u')(x) \qquad \text{for a.e. } x \in P. \tag{5.3}$$

Fixing $\eta \in C_c^\infty(\mathbb{R})$, taking into account (5.2) and (5.3) and using the convention $u\,u''' = 0$ on $\mathbb{R} \setminus P$, it holds

$$\begin{aligned} \int_{\mathbb{R}} u''u\,\eta'\,\mathrm{d}x + \int_{\mathbb{R}} u''u'\,\eta\,\mathrm{d}x &= -\int_{\mathbb{R}} (u''u)'\,\eta\,\mathrm{d}x + \int_{\mathbb{R}} u''u'\,\eta\,\mathrm{d}x \\ &= -\int_{\mathbb{R}} u'''u\,\eta\,\mathrm{d}x - \int_{\mathbb{R}} u''u'\,\eta\,\mathrm{d}x + \int_{\mathbb{R}} u''u'\,\eta\,\mathrm{d}x = -\int_{\mathbb{R}} u'''u\,\eta\,\mathrm{d}x. \end{aligned}$$

Finally, by the last equality and the assumptions $u \in H^2(\mathbb{R})$ and $u\,u'' \in H^1(\mathbb{R})$, it follows that $u\,u''' \in L^2(\mathbb{R})$. □

Proposition 5.2 *Let $d = 1$, $s = 1$, $u_0 \in \dot{H}^1(\mathbb{R}) \cap \mathscr{P}_2(\mathbb{R})$, $\tau > 0$ and $\{u_\tau^k : k = 0, 1, 2, \ldots\}$ the sequence given by* Proposition 3.2.

Then, for any $k \geq 1$, $(u_\tau^k)''$ is differentiable a.e. in $\{x \in \mathbb{R} : u_\tau^k(x) > 0\}$, $u_\tau^k\,(u_\tau^k)''' \in L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}} |(u_\tau^k)'''|^2 u_\tau^k\,\mathrm{d}x = \frac{1}{\tau^2} W^2(u_\tau^k, u_\tau^{k-1}). \tag{5.4}$$

Proof Let us fix $k \geq 1$ and we denote by T the optimal transport map from u_τ^k and u_τ^{k-1}. Fixing $\eta \in C_c^\infty(\mathbb{R})$, using the flow in (4.19) and defining $u_t := (X_t)_\# u_\tau^k$, by a standard computation, see for instance [40] or [1], it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\, W^2(u_t, u_\tau^{k-1})_{|t=0} = -\int_{\mathbb{R}} (T - I) u_\tau^k \eta \,\mathrm{d}x. \tag{5.5}$$

Taking into account that $u_\tau^k \in H^2(\mathbb{R})$ and it is a minimizer of the functional $u \mapsto \mathscr{F}_s(u) + \frac{1}{2\tau}\, W^2(u, u_\tau^{k-1})$, using (5.5) and (4.20), it follows that

$$\frac{1}{\tau}\int_{\mathbb{R}} (T - I) u_\tau^k \,\eta \,\mathrm{d}x + N(u_\tau^k, \eta) = 0,$$

which can be written as

$$-\frac{1}{\tau}\int_{\mathbb{R}} (T - I) u_\tau^k \,\eta \,\mathrm{d}x = \int_{\mathbb{R}} (u_\tau^k)'' u_\tau^k \,\eta' \,\mathrm{d}x + \int_{\mathbb{R}} (u_\tau^k)''(u_\tau^k)' \,\eta \,\mathrm{d}x. \tag{5.6}$$

Since

$$\int_{\mathbb{R}} |T - I|^2 \, u_\tau^k \,\mathrm{d}x = W^2(u_\tau^k, u_\tau^{k-1}) < +\infty \tag{5.7}$$

and u_τ^k is bounded we have that $(T - I)u_\tau^k \in L^2(\mathbb{R})$. Moreover $(u_\tau^k)''(u_\tau^k)' \in L^2(\mathbb{R})$ since also $(u_\tau^k)'$ is bounded. Then by (5.6) it follows that $(u_\tau^k)'' u_\tau^k \in H^1(\mathbb{R})$. We can apply Lemma 5.1 to u_τ^k obtaining

$$\frac{1}{\tau}\int_{\mathbb{R}} (T - I) u_\tau^k \eta \,\mathrm{d}x = \int_{\mathbb{R}} (u_\tau^k)''' u_\tau^k \,\eta \,\mathrm{d}x. \tag{5.8}$$

Since (5.8) holds for any $\eta \in C_c^\infty(\mathbb{R})$, it follows that $\frac{1}{\tau}(T - I)u_\tau^k = (u_\tau^k)''' u_\tau^k$ in $L^2(\mathbb{R})$ and $\frac{1}{\tau}(T - I)(x) = (u_\tau^k)'''(x)$ for u_τ^k-a.e. $x \in \mathbb{R}$. Then

$$\frac{1}{\tau^2}\int_{\mathbb{R}} |T - I|^2 u_\tau^k \,\mathrm{d}x = \int_{\mathbb{R}} |(u_\tau^k)'''|^2 u_\tau^k \,\mathrm{d}x$$

and by (5.7) we have (5.4).

□

Proof of Theorem 1.2. In this proof we use the notation u' to denote the partial derivative of u with respect to the one-dimensional space variable and $\partial_t u$ to denote the partial derivative of u with respect to the time variable.

The property (1.13) follows from property (3) of Theorem 1.1 and the embedding of $H^2(\mathbb{R})$ in $C^{1,1/2}(\mathbb{R})$.

We observe that, from the second inequality of (4.16), it follows that there exists a constant C depending only on u_0 such that

$$0 \leq u(t, x) \leq C \qquad \forall\, (t, x) \in (0, +\infty) \times \mathbb{R}. \tag{5.9}$$

Denoting by u_τ the discrete solution of step $\tau > 0$, from the basic estimate (3.4) and (5.4) it follows that

$$\int_0^{+\infty} \int_{\mathbb{R}} |(u_\tau)'''|^2 \, u_\tau \,\mathrm{d}x \,\mathrm{d}t \leq \int_{\mathbb{R}} |(u_0)'|^2 \,\mathrm{d}x. \tag{5.10}$$

Let τ_n be a vanishing sequence given by Theorem 3.3. By Theorem 5.4.4 of [1], thanks to the bound (5.10), there exists $v : (0,+\infty)\times\mathbb{R}\to\mathbb{R}$ such that

$$\int_0^{+\infty}\int_{\mathbb{R}} |v|^2\, u\,\mathrm{d}x\,\mathrm{d}t \le \int_{\mathbb{R}} |(u_0)'|^2\,\mathrm{d}x \tag{5.11}$$

and

$$\lim_{n\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}} (u_{\tau_n})'''\eta\, u_{\tau_n}\,\mathrm{d}x\,\mathrm{d}t = \int_0^{+\infty}\int_{\mathbb{R}} v\eta\, u\,\mathrm{d}x\,\mathrm{d}t, \quad \forall\,\eta\in C_c^\infty((0,+\infty)\times\mathbb{R};\mathbb{R}). \tag{5.12}$$

Applying Lemma 5.1 for any $t\in(0,+\infty)$ to $u_\tau(t)$ it holds that

$$\begin{aligned} &-\int_0^{+\infty}\int_{\mathbb{R}} (u_{\tau_n})'''\eta\, u_{\tau_n}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^{+\infty}\int_{\mathbb{R}} (u_{\tau_n})'' u_{\tau_n}\,\eta'\,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty}\int_{\mathbb{R}} (u_{\tau_n})''(u_{\tau_n})'\,\eta\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.13}$$

Since $(u_{\tau_n})''$ weakly converges in $L^2(0,T;L^2(\mathbb{R}))$ to u'', using Lemma 4.6, it follows that

$$\begin{aligned} &\lim_{n\to+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}} (u_{\tau_n})'' u_{\tau_n}\,\eta'\,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty}\int_{\mathbb{R}} (u_{\tau_n})''(u_{\tau_n})'\,\eta\,\mathrm{d}x\,\mathrm{d}t\right) \\ &= \int_0^{+\infty}\int_{\mathbb{R}} u'' u\,\eta'\,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty}\int_{\mathbb{R}} u''u'\,\eta\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.14}$$

By (5.12), (5.13) and (5.14) it follows that

$$-\int_0^{+\infty}\int_{\mathbb{R}} v\,u\,\eta\,\mathrm{d}x\,\mathrm{d}t = \int_0^{+\infty}\int_{\mathbb{R}} u''u\,\eta'\,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty}\int_{\mathbb{R}} u''u'\,\eta\,\mathrm{d}x\,\mathrm{d}t \tag{5.15}$$

Using test functions η in (5.15) of the form $\eta(t,x)=\tilde\eta(x)\phi(t)$ for $\tilde\eta\in C_c^\infty(\mathbb{R})$ and $\phi\in C_c^\infty((0,+\infty))$, it follows that,

$$-\int_{\mathbb{R}} v(t,x)\,u(t,x)\,\tilde\eta(x)\,\mathrm{d}x = \int_{\mathbb{R}} u''(t,x)u(t,x)\,\tilde\eta'(x)\,\mathrm{d}x + \int_{\mathbb{R}} u''(t,x)u'(t,x)\,\tilde\eta(x)\,\mathrm{d}x \tag{5.16}$$

for a.e. $t\in(0,+\infty)$ and for any $\tilde\eta\in C_c^\infty(\mathbb{R})$. From (5.11) and the boundedness (5.9) of u, it follows that $v(t,\cdot)u(t,\cdot)\in L^2(\mathbb{R})$ for a.e. $t\in(0,+\infty)$. By (5.16) and the boundedness of $u'(t,\cdot)$ we obtain that $u''(t,\cdot)u(t,\cdot)\in H^1(\mathbb{R})$ for a.e. $t\in(0,+\infty)$. Then, taking into account that $u(t,\cdot)\in H^2(\mathbb{R})$ for a.e. $t\in(0,+\infty)$, we can apply Lemma 5.1 obtaining (1.14) and

$$\int_{\mathbb{R}} v(t,x)\,u(t,x)\,\tilde\eta(x)\,\mathrm{d}x = \int_{\mathbb{R}} u'''(t,x)u(t,x)\,\tilde\eta(x)\,\mathrm{d}x \tag{5.17}$$

for a.e. $t\in(0,+\infty)$ and for any $\tilde\eta\in C_c^\infty(\mathbb{R})$. By this last relation and (5.11) we obtain (1.15).

The weak formulation (1.8) in dimension 1 obtained in Theorem 1.1 can be written as (see (1.11))

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}} u\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_0^{+\infty}\int_{\mathbb{R}} (u''\,\varphi''\,u + u''\,\varphi'u')\,\mathrm{d}x\,\mathrm{d}t = 0, \\ &\forall\,\varphi\in C_c^\infty((0,+\infty)\times\mathbb{R}). \end{aligned} \tag{5.18}$$

Using the relations in (5.16) and (5.17) in (5.18) we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}} u\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^{+\infty}\int_{\mathbb{R}} u'''\,\varphi' u\,\mathrm{d}x\,\mathrm{d}t = 0, \qquad \forall\,\varphi\in C_c^\infty((0,+\infty)\times\mathbb{R}). \tag{5.19}$$

which is (1.16).

In order to prove the regularity (1.12) we observe that a uniform, with respect to t, $C^{1/2}$ Hölder estimate for the space variable follows from the estimate $\|u(t,\cdot)\|^2_{\dot H^1(\mathbb{R})} \le \|u_0\|^2_{\dot H^1(\mathbb{R})}$ for any $t\in[0,+\infty)$, see (4.16), and the classical $C^{1/2}$ embedding; the proof of the uniform, with respect to x, $C^{1/8}$ Hölder estimate for the time variable can be carried out as in the proof of Lemma 2.1 of [6] using the weak formulation (1.16). □

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