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Calculus of Variations

Bubble decomposition for the harmonic map heat flow in the equivariant case

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Abstract

We consider the harmonic map heat flow for maps $\mathbb{R}^2 \to \mathbb{S}^2$, under equivariant symmetry. It is known that solutions to the initial value problem can exhibit bubbling along a sequence of times—the solution decouples into a superposition of harmonic maps concentrating at different scales and a body map that accounts for the rest of the energy. We prove that this bubble decomposition is unique and occurs continuously in time. The main new ingredient in the proof is the notion of a collision interval from Jendrej and Lawrie (J Amer Math Soc).

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Contents

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1 Introduction

1.1 Setting of the problem

Consider the harmonic map heat flow (HMHF) for maps $\Psi : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, that is, the heat flow associated to the Dirichlet energy

$$E(\Psi) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \Psi(x)|^2 \, \mathrm{d}x.$$

The initial value problem for the HMHF is given by

$$\begin{aligned} \partial_t \Psi - \Delta \Psi &= \Psi \, |\nabla \Psi|^2 \\ \Psi(0, x) &= \Psi_0(x). \end{aligned} \tag{1.1}$$

We say a solution to (1.1) is k-equivariant if it takes the form

$$\Psi(t, re^{i\theta}) = (\sin u(t,r) \cos k\theta, \sin u(t,r) \sin k\theta, \cos u(t,r)) \in \mathbb{S}^2 \subset \mathbb{R}^3,$$

where $k \in \mathbb{N}$ and (r, θ) are polar coordinates on $\mathbb{R}^2$. In this case the HMHF reduces to a scalar equation for the polar angle $u = u(t, r)$,

$$\begin{aligned} \partial_t u &= \partial_r^2 u + \frac{1}{r} \partial_r u - \frac{k^2}{r^2} \frac{\sin 2u}{2}, \\ u(0) &= u_0, \end{aligned} \tag{1.2}$$

and the energy $E = E(u)$ reduces to

$$E(u(t)) = 2\pi \int_0^\infty \tfrac{1}{2} \left((\partial_r u(t,r))^2 + k^2 \tfrac{\sin^2(u(t,r))}{r^2} \right) r \, \mathrm{d}r,$$

and formally satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t} E(u(t)) = -2\pi \int_0^\infty (\partial_t u(t,r))^2 \, r \, \mathrm{d}r = -2\pi \, \| \mathcal{T}(u(t)) \|_{L^2}^2,$$

where in the k-equivariant setting $\mathcal{T}(u) := \partial_r^2 u + \frac{1}{r} \partial_r u - \frac{k^2}{2r^2} \sin(2u)$ is called the tension of u. Integrating in time from t_0 to t gives,

$$E(u(t)) + 2\pi \int_{t_0}^t \| \mathcal{T}(u(s)) \|_{L^2}^2 \, \mathrm{d}s = E(u(t_0)). \tag{1.3}$$

The natural setting in which to consider the initial value problem for (1.2) is the space of initial data u_0 with finite energy, $E(u) < \infty$. This set is split into disjoint sectors, $\mathcal{E}_{\ell,m}$, which for $\ell, m \in \mathbb{Z}$, are defined by

$$\mathcal{E}_{\ell,m} := \big\{ u \mid E(u) < \infty, \quad \lim_{r \to 0} u(r) = \ell\pi, \quad \lim_{r \to \infty} u(r) = m\pi \big\}.$$

These sectors, which are preserved by the flow, are related to the topological degree of the full map $\Psi : \mathbb{R}^2 \to \mathbb{S}^2$: if $m - \ell$ is even and $u \in \mathcal{E}_{\ell,m}$, then the corresponding map Ψ with polar angle u is topologically trivial, whereas for odd $m - \ell$ the map has degree k.

The sets $\mathcal{E}_{\ell,m}$ are affine spaces, parallel to the linear space $\mathcal{E} := \mathcal{E}_{0,0}$, which we endow with the norm,

$$\|u_0\|_{\mathcal{E}}^2 := \int_0^\infty \Big((\partial_r u_0(r))^2 + k^2 \frac{(u_0(r))^2}{r^2} \Big) r\mathrm{d}r.$$

We make note of the embedding $\|u_0\|_{L^\infty} \le C\|u_0\|_{\mathcal{E}}$.

The unique k-equivariant harmonic map is given explicitly by

$$Q(r) := 2\arctan(r^k).$$

Here uniqueness means up to scaling, sign change, and adding a multiple of π, i.e., every finite energy stationary solution to (1.2) takes the form $Q_{\mu,\sigma,m}(r) = m\pi + \sigma Q(r/\mu)$ for some $\mu \in (0,\infty)$, $\sigma \in \{0,-1,1\}$ and $m \in \mathbb{Z}$. The map Q and its rescaled versions $Q_\lambda(r) := Q(\lambda^{-1}r)$ for $\lambda > 0$, are minimizers of the energy E within the class $\mathcal{E}_{0,1}$; in fact, $E(Q_\lambda) = 4\pi k$.

1.2 Statement of the results

We prove the following theorem.

Theorem 1 (Bubble decomposition) *Let $k \in \mathbb{N}$, let $\ell, m \in \mathbb{Z}$, and let $u(t)$ be the solution to* (1.2) *with initial data $u(0) = u_0 \in \mathcal{E}_{\ell,m}$, defined on its maximal interval of existence* $[0, T_+)$.

(Global solution) *If $T_+ = \infty$, there exist a time $T_0 > 0$, an integer $N \ge 0$, continuous functions $\lambda_1(t), \ldots, \lambda_N(t) \in C^0([T_0,\infty))$, signs $\iota_1, \ldots, \iota_N \in \{-1, 1\}$, and $g(t) \in \mathcal{E}$ defined by*

$$u(t) = m\pi + \sum_{j=1}^{N} \iota_j (Q_{\lambda_j(t)} - \pi) + g(t), \tag{1.4}$$

such that

$$\|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \ \text{ as } \ t \to \infty,$$

where above we use the convention that $\lambda_{N+1}(t) = \sqrt{t}$.

(Blow-up solution) *If $T_+ < \infty$, there exist a time $T_0 < T_+$, integers m_∞, m_Δ, a mapping $u^* \in \mathcal{E}_{0,m_\infty}$, an integer $N \ge 1$, continuous functions $\lambda_1(t), \ldots, \lambda_N(t) \in C^0([T_0, T_+))$, signs $\iota_1, \ldots, \iota_N \in \{-1, 1\}$, and $g(t) \in \mathcal{E}$ defined by*

$$u(t) = m_\Delta \pi + \sum_{j=1}^{N} \iota_j (Q_{\lambda_j(t)} - \pi) + u^* + g(t),$$

such that

$$\|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \ \text{ as } \ t \to T_+,$$

where above we use the convention that $\lambda_{N+1}(t) = \sqrt{T_+ - t}$.

Remark 1.1 Asymptotic decompositions of solutions to (1.2) (in fact for solutions to the Eq. (1.1) without symmetry assumptions) were proved along a *sequence of times* $t_n \to T_+$, in a series of works by Struwe [28], Qing [24], Ding-Tian [9], Wang [33], Qing-Tian [25], and Topping [31]. The main contribution of this paper is to show that the decomposition can be taken continuously in time for k-equivariant solutions.

Remark 1.2 In the non-equivariant setting, i.e., for (1.1), Topping [29, 30] made important progress on a related question in the global case, showing the uniqueness of the locations of the bubbling points under restrictions on the configurations of bubbles appearing in the sequential decomposition. His assumption, roughly, is that all of the bubbles concentrating at a certain point have to have the same orientation. We can contrast this assumption with the equivariant setting, where in the decomposition (1.4) subsequent bubbles have opposite orientations.

Remark 1.3 Given Theorem 1, it is natural to ask which configurations of bubbles are possible in the decomposition. Van der Hout [32] showed that there can only be one bubble in the decomposition in the case of equivariant finite time blow-up; see also [2]. In contrast, in the infinite time case, it is expected that there can be equivariant bubble trees of arbitrary size (see recent work of Del Pino, Musso, and Wei [8] for a construction in the case of the critical semi-linear heat equation).

Remark 1.4 There are solutions to the HMHF that develop a bubbling singularity in finite time, the first being the examples of Coron and Ghidaglia [5] (in dimension $d \geq 3$) and Chang, Ding, Ye [4] in the $2d$ case considered here. Guan, Gustafson, and Tsai [12] and Gustafson, Nakanishi, and Tsai [14] showed that the harmonic maps Q are asymptotically stable in equivariance classes $k \geq 3$, and thus there is no finite time blow up for energies close to Q in that setting. This asymptotic stability result was improved to energies up to $3E(Q)$ by Gustafson and Roxanas in [13] in equivariance classes $k \geq 4$. For $k = 2$, [14] gave examples of solutions exhibiting infnite time blow up and eternal oscillations. Raphaël and Schweyer constructed a stable blow-up regime for $k = 1$ in [26] and then blow up solutions with different rates in [27]. Recently, Davila, Del Pino, and Wei [7] constructed examples of solutions simultaneously concentrating a single copy of the ground state harmonic map at distinct points in space.

1.3 Summary of the proof

We give an informal description of the proof of Theorem 1 starting with a summary of the sequential bubbling results as in, e.g., [24, 31], adapted to our setting. A crucial ingredient is a sequential compactness lemma, which says that a sequence of maps with vanishing tension must converge (at least locally in space) to a multi-bubble, which we define as follows.

Definition 1.5 (Multi-bubble configuration) Given $M \in \{0, 1, \ldots\}$, $m \in \mathbb{Z}$, $\vec{\iota} = (\iota_1, \ldots, \iota_M) \in \{-1, 1\}^M$ and an increasing sequence $\vec{\lambda} = (\lambda_1, \ldots, \lambda_M) \in (0, \infty)^M$, a *multi-bubble configuration* is defined by the formula

$$\mathcal{Q}(m, \vec{\iota}, \vec{\lambda}; r) := m\pi + \sum_{j=1}^{M} \iota_j \big(Q_{\lambda_j}(r) - \pi \big).$$

Remark 1.6 If $M = 0$, it should be understood that $\mathcal{Q}(m, \vec{\iota}, \vec{\lambda}; r) = m\pi$ for all $r \in (0, \infty)$, where $\vec{\iota}$ and $\vec{\lambda}$ are 0-element sequences, that is the unique functions $\emptyset \to \{-1, 1\}$ and $\emptyset \to (0, \infty)$, respectively.

With this definition, we define a localized distance function to multi-bubble configurations by

$$\boldsymbol{\delta}_R(u) := \inf_{m, M, \vec{\iota}, \vec{\lambda}} \Big(\|u - \mathcal{Q}(m, \vec{\iota}, \vec{\lambda})\|^2_{\mathcal{E}(r \le R)} + \sum_{j=1}^{M} \Big(\frac{\lambda_j}{\lambda_{j+1}} \Big)^k \Big)^{\frac{1}{2}} \tag{1.5}$$

where the infimum is taken over all $m \in \mathbb{Z}$, $M \in \{0, 1, 2, \dots\}$, all vectors $\iota \in \{-1, 1\}^M$, $\vec{\lambda} \in (0, \infty)^M$, and we use the convention that the last scale $\lambda_{M+1} = R$.

The localized sequential compactness lemma (see Lemma 3.1) says the following: given a sequence of maps u_n with bounded energy, a sequence $\rho_n \in (0, \infty)$ of scales, and tension vanishing in L^2 relative to the scale ρ_n, i.e., $\lim_{n\to\infty} \rho_n \|\mathcal{T}(u_n)\|_{L^2} = 0$, there exists a subsequence of the u_n that converges to a multi-bubble configuration up to large scales relative to ρ_n, i.e., $\lim_{n\to\infty} \boldsymbol{\delta}_{R_n \rho_n}(u_n) = 0$ for some sequence $R_n \to \infty$. An analogous result with no symmetry assumptions was proved by Qing [24] using the local bubbling theory of Struwe [28] together with a delicate elliptic analysis showing that no energy can accumulate on the "neck" regions between the bubbles. Here we give a mostly self-contained proof of this compactness result in the simpler equivariant setting using the theory of profile decompositions of Gérard [11] and an approach in the spirit of Duyckaerts, Kenig, and Merle's work on nonlinear waves [10]. To control the energy on the neck regions we use a virial-type functional adapted from Jia and Kenig's proof of sequential soliton resolution for equivariant wave maps [19].

With the compactness lemma in place, we now consider the heat flow. To fix ideas, let $u(t)$ be a solution to (1.2) defined globally in time, i.e., $T_+ = \infty$. By the energy identity (1.3),

$$\int_0^\infty \|\mathcal{T}(u(t))\|^2_{L^2} \, \mathrm{d}t < \infty, \tag{1.6}$$

and thus we can find a sequence of times $t_n \to \infty$ so that $\lim_{n\to\infty} \sqrt{t_n} \|\mathcal{T}(u(t_n))\|_{L^2} = 0$. From the compactness lemma we deduce that after passing to a subsequence of the t_n, $u(t_n)$ converges to an N-bubble configuration up to the self-similar scale $r = \sqrt{t_n}$. In the exterior region $r \gtrsim \sqrt{t}$, we prove that $u(t)$ has vanishing energy (continuously in time) using a localized energy inequality due to Struwe [28]; see Proposition 4.3.

Let $\mathbf{d}(t)$ denote the distance to the particular N-bubble configuration obtained via the compactness lemma (which is defined analogously to (1.5), except without the spatial localization; see Definition 5.1). We have so far proved that

$$\lim_{n\to\infty} \mathbf{d}(t_n) = 0.$$

Theorem 1 follows from showing that in fact $\lim_{t\to\infty} \mathbf{d}(t) = 0$. We assume that continuous-in-time convergence of $\mathbf{d}(t)$ fails. To reach a contradiction we study time intervals on which bubbles come into collision (i.e., where $\mathbf{d}(t)$ grows), adapting the notion of a *collision interval* from our paper [17].

We say that an interval $[a, b]$ is a collision interval with parameters $0 < \epsilon < \eta$ and $N - K$ exterior bubbles for some $1 \le K \le N$, if $\mathbf{d}(a) \le \epsilon$, $\mathbf{d}(b) \ge \eta$, and there exists a curve $r = \rho_K(t)$ outside of which $u(t)$ is within ϵ of an $N - K$-bubble (in the sense of a localized version of $\mathbf{d}(t)$); see Defintion 5.4. We now define K to be the *smallest* non-negative integer for which there exists $\eta > 0$, a sequence $\epsilon_n \to 0$, and sequences $a_n, b_n \to \infty$, so that $[a_n, b_n]$ are collision intervals with parameters ϵ_n, η and $N - K$ exterior bubbles, and we write $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$; see Sect. 5.1 for the proof that K is well-defined and ≥ 1, under the contradiction hypothesis.

Consider a sequence of collision intervals $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$. Near the endpoint a_n, $u(t)$ is close to an N-bubble configuration and we denote the interior scales, which will come into collision, by $\vec{\lambda} = (\lambda_1, \ldots, \lambda_K)$ and the exterior scales, which stay coherent, by $\vec{\mu} = (\vec{\mu}_{K+1}, \ldots, \vec{\mu}_N)$. The crucial point is that the minimality of K allows us to relate the scale of the Kth bubble λ_K to the lengths of the collision intervals $b_n - a_n$. We prove, roughly, that for sufficiently large n the collision intervals $[a_n, b_n]$ contain subintervals $[c_n, d_n]$ on which (1) $\inf_{t\in[c_n,d_n]} \mathbf{d}(t) \geq \alpha$ for some $\alpha > 0$, (2) the scale $\lambda_K(t)$ stays roughly constant on $[c_n, d_n]$, and (3) the lower bound $d_n - c_n \gtrsim n^{-1}\lambda_K(c_n)^2$ holds. The compactness lemma and the lower bound $\mathbf{d}(t) \geq \alpha$ together yield a lower bound on the tension $\inf_{t\in[c_n,d_n]} \lambda_K(c_n)^2 \|\mathcal{T}(u(t))\|_{L^2}^2 \gtrsim 1$ where the scale λ_K appears again due to the definition of K. The last two sentences lead to an immediate contradiction from the boundedness of the integral (1.6), i.e.,

$$C \geq \int_0^\infty \|\mathcal{T}(u(t))\|_{L^2}^2 \,\mathrm{d}t \geq \sum_n \int_{c_n}^{d_n} \|\mathcal{T}(u(t))\|_{L^2}^2 \,\mathrm{d}t \gtrsim \sum_n n^{-1},$$

which proves that $\lim_{t\to\infty} \mathbf{d}(t) = 0$.

1.4 Notational conventions

The energy is denoted E, $\mathcal{E}$ is the energy space, $\mathcal{E}_{\ell,m}$ are the finite energy sectors. We use the notation $\mathcal{E}(r_1, r_2)$ to denote the local energy norm

$$\|g\|_{\mathcal{E}(r_1,r_2)}^2 := \int_{r_1}^{r_2} \Big((\partial_r g)^2 + \frac{k^2}{r^2} g^2\Big)\, r\mathrm{d}r,$$

By convention, $\mathcal{E}(r_0) := \mathcal{E}(r_0, \infty)$ for $r_0 > 0$. The local nonlinear energy is denoted $E(\boldsymbol{u}_0; r_1, r_2)$. We adopt similar conventions as for $\mathcal{E}$ regarding the omission of r_2, or both r_1 and r_2.

Given a function $\phi(r)$ and $\lambda > 0$, we denote by $\phi_\lambda(r) = \phi(r/\lambda)$, the $\mathcal{E}$-invariant re-scaling, and by $\phi_{\underline{\lambda}}(r) = \lambda^{-1}\phi(r/\lambda)$ the L^2-invariant re-scaling. We denote by $\Lambda := r\partial_r$ and $\underline{\Lambda} := r\partial r + 1$ the infinitesimal generators of these scalings. We denote $\langle\cdot \mid \cdot\rangle$ the radial $L^2(\mathbb{R}^2)$ inner product given by,

$$\big\langle \phi \mid \psi \big\rangle := \int_0^\infty \phi(r)\psi(r)\, r\,\mathrm{d}r.$$

We denote k the equivariance degree and $f(u) := \frac{1}{2}\sin 2u$ the nonlinearity in (1.2). We let χ be a smooth cut-off function, supported in $r \leq 2$ and equal 1 for $r \leq 1$.

We call a "constant" a number which depends only on the equivariance degree k and the number of bubbles N. Constants are denoted C, C_0, C_1, c, c_0, c_1. We write $A \lesssim B$ if $A \leq CB$ and $A \gtrsim B$ if $A \geq cB$. We write $A \ll B$ if $\lim_{n\to\infty} A/B = 0$.

For any sets X, Y, Z we identify $Z^{X\times Y}$ with $(Z^Y)^X$, which means that if $\phi : X \times Y \to Z$ is a function, then for any $x \in X$ we can view $\phi(x)$ as a function $Y \to Z$ given by $(\phi(x))(y) := \phi(x, y)$.

2 Preliminaries

2.1 Well-posedness

The starting point for our analysis is the following result of Struwe [28], which says that the initial value problem for the harmonic map flow is well-posed for data in the energy space.

Lemma 2.1 (Local well-posedness)*[28, Theorem 4.1] For each* $\ell, m \in \mathbb{Z}$ *and* $u_0 \in \mathcal{E}_{\ell,m}$ *there exists a maximal time of existence* $T_+ = T_+(u_0)$ *and a unique solution* $u(t) \in \mathcal{E}_{\ell,m}$ *to* (1.2) *on the time interval* $t \in [0, T_+)$ *with* $u(0) = u_0$*. The maximal time is characterized by the following condition: if* $T_+ < \infty$*, there exists* $\epsilon_0 > 0$ *such that*

$$\limsup_{t \to T_+} E(u(t); 0, r_0) \geq \epsilon_0, \tag{2.1}$$

for all $r_0 > 0$*. If there is no such* $T_+ < \infty$*, we say* $T_+ = \infty$ *and the flow is globally defined.*

The energy $E(u(t))$ *is absolutely continuous and non-increasing as a function of* $t \in [0, T]$ *for any* $T < T_+$*, and for any* $t_1 \leq t_2 \in [0, T_+)$*, there holds,*

$$E(u(t_2)) + 2\pi \int_{t_1}^{t_2} \int_0^\infty (\partial_t u(t, r))^2 \, r \, \mathrm{d}r \mathrm{d}t = E(u(t_1)).$$

In particular,

$$\int_0^{T_+} \int_0^\infty (\partial_t u(t, r))^2 \, r \, \mathrm{d}r \mathrm{d}t \leq E(u_0). \tag{2.2}$$

Remark 2.2 Local well-posedness is proved by Struwe for the HMHF without symmetry assumptions in the case of maps from a closed Riemann surface $\mathcal{M} \to \mathbb{S}^2$. For the case of maps from $\mathbb{R}^2$ we refer the reader to Lin and Wang [20, Theorem 5.2.1] for the short time existence of regular solutions. As equivariant symmetry is preserved by the flow, we obtain regular equivariant solutions to (1.2) by taking equivariant initial data. Solutions with finite energy initial data are then obtained as limits of smooth solutions, and in [28] Struwe proved these solutions are regular, e.g., C^2, on any compact time interval $[\tau, T] \subset (0, T_+)$. We note that in the equivariant case the energy can only concentrate at the origin $r = 0$, giving the form of the blow-up criterion in (2.1).

2.2 Basic estimates

Lemma 2.3 *Fix integers* ℓ, m*. For every* $\epsilon > 0$ *and* $R_0 > 1$*, there exists a* $\delta > 0$ *with the following property. Let* $0 \leq R_1 < R_2 \leq \infty$ *with* $R_2/R_1 \geq R_0$*, and* $u \in \mathcal{E}_{\ell,m}$ *be such that* $E(u; R_1, R_2) < \delta$*. Then, there exists* $\ell_0 \in \mathbb{Z}$ *such that* $|u(r) - \ell_0\pi| < \epsilon$ *for almost all* $r \in (R_1, R_2)$.

Moreover, there exist constants $C = C(R_0), \alpha = \alpha(R_0) > 0$ *such that if* $E(u; R_1, R_2) < \alpha$*, then*

$$\|u - \ell_0\pi\|_{\mathcal{E}(R_1, R_2)} \leq C E(u; R_1, R_2). \tag{2.3}$$

Proof By an approximation argument we can assume $u \in \mathcal{E}_{\ell,m}$ is smooth. First, we show that for any $\epsilon_0 > 0$, there exists $r_0 \in [R_1, R_2]$ such that $|u(r_0) - \ell_0\pi| < \epsilon_0$ for some $\ell_0 \in \mathbb{Z}$ as long as $E(u; R_1, R_2)$ is sufficiently small. If not, one could find $\epsilon_1 > 0$, $0 < R_1 < R_2$, and a sequence $u_n \in \mathcal{E}_{\ell,m}$ so that $E(u_n; R_1; R_2) \to 0$ as $n \to \infty$ but such that

$\inf_{r\in[R_1,R_2],\ell\in\mathbb{Z}} |u_n(r)-\ell\pi| \geq \epsilon_1$. The latter condition gives a constant $c(\epsilon_1)>0$ such that $\inf_{r\in[R_1,R_2]} |\sin(u_n(r))| \geq c(\epsilon_1)$. But then

$$E(u_n; R_1; R_2) \geq \frac{k^2}{2}\int_{R_1}^{R_2} \sin^2(u_n(r))\,\frac{\mathrm{d}r}{r} \geq \frac{k^2}{2}c(\epsilon_1)^2\log(R_2/R_1),$$

which is a contradiction. Next define the function, $G(u) = \int_0^u |\sin\rho|\,\mathrm{d}\rho$, and for $r_1 \in (R_1, R_2)$ note the inequality,

$$|G(u(r_0)) - G(u(r_1))| = \Big|\int_{u(r_1)}^{u(r_0)} |\sin\rho|\,\mathrm{d}\rho\Big| \leq \int_{r_1}^{r_0} |\sin u(r)|\,|\partial_r u(r)|\,\mathrm{d}r \lesssim E(u; R_1, R_2).$$

We conclude using that G is continuous and increasing that $|u(r)-\ell_0\pi| < \epsilon$ for all $r \in (R_1, R_2)$. As long as $\epsilon > 0$ is small enough we see that in fact, $\sin^2(u(r)) \geq \frac{1}{2}|u(r)-\ell_0\pi|^2$ for all $r \in (R_1, R_2)$ and (2.3) follows. □

Given a mapping $u : (0,\infty) \to \mathbb{R}$ we define its energy density,

$$\mathbf{e}(u(r), r) := \frac{1}{2}\left((\partial_r u(r))^2 + \frac{k^2}{r^2}\sin^2(u(r))\right).$$

Lemma 2.4 (Localized energy inequality) *Let $I \subset [0,\infty)$ be a time interval, and let $\phi : I\times(0,\infty) \to [0,\infty)$ be a smooth function. Let $u(t) \in \mathcal{E}_{\ell,m}$ be a solution to (1.2) on I. Then, for any $t_1 < t_2 \in I$,*

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^\infty (\partial_t u(t,r))^2\phi(t,r)^2\,r\,\mathrm{d}r\mathrm{d}t + \int_0^\infty \mathbf{e}(u(t_2,r),r)\phi(t_2,r)^2\,r\,\mathrm{d}r\\ &= \int_0^\infty \mathbf{e}(u(t_1,r),r)\phi(t_1,r)^2\,r\,\mathrm{d}r - 2\int_{t_1}^{t_2}\int_0^\infty \partial_t u(t,r)\partial_r u(t,r)\phi(t,r)\partial_r\phi(t,r)\,r\,\mathrm{d}r\mathrm{d}t\\ &\quad + 2\int_{t_1}^{t_2}\int_0^\infty \mathbf{e}(u(t,r),r)\phi(t,r)\partial_t\phi(t,r)\,r\,\mathrm{d}r\,\mathrm{d}t\end{aligned} \tag{2.4}$$

If $\phi(t,r)$ satisfies, $\partial_t\phi(t,r) \leq 0$ for all $t \in [t_1, t_2]$ then,

$$\begin{aligned}&\int_0^\infty \mathbf{e}(u(t_2),r)\phi(t_2,r)^2\,r\,\mathrm{d}r + \frac{1}{2}\int_{t_1}^{t_2}\int_0^\infty (\partial_t u(t,r))^2\phi(t,r)^2\,r\,\mathrm{d}r\mathrm{d}t\\ &\qquad \leq \int_0^\infty \mathbf{e}(u(t_1),r)\phi(t_1,r)^2\,r\,\mathrm{d}r + 2\int_{t_1}^{t_2}\int_0^\infty (\partial_r u(t,r))^2(\partial_r\phi(t,r))^2\,r\,\mathrm{d}r\mathrm{d}t,\end{aligned} \tag{2.5}$$

and,

$$\begin{aligned}&\int_0^\infty \mathbf{e}(u(t_2),r)\phi(t_2,r)^2\,r\,\mathrm{d}r + \int_{t_1}^{t_2}\int_0^\infty (\partial_t u(t,r))^2\phi(t,r)^2\,r\,\mathrm{d}r\mathrm{d}t\\ &\leq \int_0^\infty \mathbf{e}(u(t_1),r)\phi(t_1,r)^2\,r\,\mathrm{d}r + 2\sqrt{E(u(t_1))}(t_2-t_1)^{\frac{1}{2}}\\ &\quad \Big(\int_{t_1}^{t_2}\int_0^\infty (\partial_t u(t,r))^2(\partial_r\phi(t,r))^2(\phi(t,r))^2\,r\,\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{2}}.\end{aligned} \tag{2.6}$$

Proof By an approximation argument we may assume that u is smooth. Then (2.4) is obtained for smooth solutions to (1.2) by multiplying the equation by $\partial_t u\phi^2$ and integrating by parts. The subsequent inequalities follow from Cauchy-Schwarz. □

2.3 Profile decomposition

We state a profile decomposition in the sense of Gérard [11], adapted to sequences of functions in the affine spaces $\mathcal{E}_{\ell,m}$; see also [1, 3, 21–23]. We use the analysis of sequences in $\mathcal{E}_{\ell,m}$ by Jia and Kenig in [19], which synthesized Côte's analysis in [6].

Lemma 2.5 (Linear profile decomposition) *Let $\ell, m \in \mathbb{Z}$ and let u_n be a sequence in $\mathcal{E}_{\ell,m}$ with $\limsup_{n\to\infty} E(u_n) < \infty$. Then, there exists $K_0 \in \{0, 1, 2, \dots\}$, sequences $\lambda_{n,j} \in (0,\infty)$ for $j \in \{1, \dots, K_0\}$, $\sigma_{n,i} \in (0,\infty)$ for $i \in \mathbb{N}$, as well as mappings $\psi^j \in \mathcal{E}_{\ell_j,m_j}$ with $E(\psi^j) < \infty$, and mappings $v^i \in \mathcal{E}_{0,0}$ such that for each $J \geq 1$,*

$$u_n = m\pi + \sum_{j=1}^{K_0} (\psi^j\big(\tfrac{\cdot}{\lambda_{n,j}}\big) - m_j\pi) + \sum_{i=1}^{J} v^i\big(\tfrac{\cdot}{\sigma_{n,i}}\big) + w_n^J(\cdot)$$

so that,

- *the parameters $\lambda_{n,j}$ satisfy*

$$\lambda_{n,1} \ll \lambda_{n,2} \ll \cdots \ll \lambda_{n,K_0} \quad as \quad n \to \infty;$$

  and for each j one of $\lambda_{n,j} \to 0$, $\lambda_{n,j} = 1$ for all n, or $\lambda_{n,j} \to \infty$ as $n \to \infty$, holds;
- *for each i either $\sigma_{n,i} \to 0$, $\sigma_{n,i} = 1$ for all n, or $\sigma_{n,i} \to \infty$ as $n \to \infty$;*
- *for each $i \in \mathbb{N}$,*

$$\frac{\lambda_{n,j}}{\sigma_{n,i}} + \frac{\sigma_{n,i}}{\lambda_{n,j}} \to \infty \quad as \quad n \to \infty \quad \forall j = 1, \dots, K_0;$$

- *the scales $\sigma_{n,i}$ satisfy,*

$$\frac{\sigma_{n,i}}{\sigma_{n,i'}} + \frac{\sigma_{n,i'}}{\sigma_{n,i}} \to \infty \quad as \quad n \to \infty;$$

- *the integers ℓ_j and m_j satisfy, $\left|\ell_j - m_j\right| \geq 1$, and,*

$$\ell = m + \sum_{j=1}^{K_0} (\ell_j - m_j);$$

- *the error term w_n^J satisfies,*

$$w_n^J(\lambda_{n,j}\cdot) \rightharpoonup 0 \in \mathcal{E} \quad as \quad n \to \infty$$
$$w_n^J(\sigma_{n,i}\cdot) \rightharpoonup 0 \in \mathcal{E} \quad as \quad n \to \infty$$

  for each $J \geq 1$, each $j = 1, \dots, K_0$, and $i \in \mathbb{N}$, and vanishes strongly in the sense that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|w_n^J\|_{L^\infty} = 0; \tag{2.7}$$

- *the following pythagorean decomposition of the nonlinear energy holds: for each $J \geq 1$,*

$$E(u_n) = \textstyle\sum_{j=1}^{K_0} E(\psi^j) + \sum_{i=1}^{J} E(v^j) + E(w_n^J) + o_n(1)$$

  as $n \to \infty$.

Proof *(Sketch of Proof)* We follow Jia and Kenig's argument [19, Proof of Lemma 5.5] to first extract the profiles $\psi^j \in \mathcal{E}_{\ell_j,m_j}$ at the scales $\lambda_{n,j}$, see [19, Pages 1594–1600]. Since these all have energy $\geq E(Q)$, there can only be finitely many of them, which defines the non-negative integer K_0. The conclusion of their argument yields a sequence,

$$h_n := u_n - m\pi - \sum_{j=1}^{K_0} (\psi^j_{\lambda_{n,j}} - m_j\pi) \in \mathcal{E}_{0,0}$$

with $\limsup_{n\to\infty} \|h_n\|_{\mathcal{E}} < \infty$. Setting $H_n := r^{-k} h_n$ we see that $\limsup_{n\to\infty} \|H_n\|_{\dot{H}^1(\mathbb{R}^d)} < \infty$ for $d = 2k+2$ (here we view H_n as a sequence of radially symmetric functions on $\mathbb{R}^d$). Thus we may apply Gérard's profile decomposition [11, Theorem 1.1] for sequences in $\dot{H}^1(\mathbb{R}^d)$ to the sequence H_n obtaining sequences of scales $\sigma_{n,i}$ and profiles V^i so that for W_n^J defined by

$$H_n = \sum_{i=1}^{J} \sigma_{n,i}^{-\frac{d}{p^*}} V\Big(\frac{\cdot}{\sigma_{n,i}}\Big) + W_n^J$$

we have

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|W_n^J\|_{L^{p^*}} = 0,$$

along with the usual orthogonality of the scales and the pythagorean expansion of the $\dot{H}^1$ norm. Note that here $p^* := \frac{2d}{d-2}$ is the critical Sobolev exponent. We set $v^i(r) := r^k V^i(r)$ and $w_n^J(r) := r^k W_n^J(r)$ for each i, n, J. Note that $w_n^J \in \mathcal{E}$ and

$$\lim_{J\to\infty} \limsup_{n\to\infty} \int_0^\infty (w_n^J(r))^{p^*} \frac{\mathrm{d}r}{r} = 0. \tag{2.8}$$

We conclude by observing the inequality

$$\sup_{r>0} |w(r)|^{\frac{p^*}{2}+1} \leq C(p^*) \left(\int_0^\infty (w(r))^{p^*} \frac{\mathrm{d}r}{r}\right)^{\frac{1}{2}} \left(\int_0^\infty (\partial_r w(r))^2\, r\, \mathrm{d}r\right)^{\frac{1}{2}},$$

which holds for all $w \in \mathcal{E}$. Thus (2.8) combined with the above gives the vanishing of the error as in (2.7). □

2.4 Multi-bubble configurations

We study properties of finite energy maps near a multi-bubble configuration as in Definition 1.5. We record here several lemmas proved in [17].

The operator $\mathcal{L}_{\mathcal{Q}}$ obtained by linearization of (1.2) about an M-bubble configuration $\mathcal{Q}(m, \vec{\iota}, \vec{\lambda})$ is given by,

$$\mathcal{L}_{\mathcal{Q}}\, g := \mathrm{D}^2 E(\mathcal{Q}(m, \vec{\iota}, \vec{\lambda}))g = -\partial_r^2 g - \frac{1}{r}\partial_r g + \frac{k^2}{r^2} f'(\mathcal{Q}(m, \vec{\iota}, \vec{\lambda}))g,$$

where $f'(z) = \cos 2z$. Given $g \in \mathcal{E}$,

$$\left\langle \mathrm{D}^2 E(\mathcal{Q}(m, \vec{\iota}, \vec{\lambda}))g \mid g\right\rangle = \int_0^\infty \left((\partial_r g(r))^2 + \frac{k^2}{r^2} f'(\mathcal{Q}(m, \vec{\iota}, \vec{\lambda}))g(r)^2\right) r\mathrm{d}r.$$

An important instance of the operator $\mathcal{L}_{\mathcal{Q}}$ is given by linearizing (1.2) about a single harmonic map $\mathcal{Q}(m, M, \vec{\iota}, \vec{\lambda}) = Q_\lambda$. In this case we use the short-hand notation,

$$\mathcal{L}_\lambda := \left(-\Delta + \frac{k^2}{r^2}\right) + \frac{k^2}{r^2}\left(f'(Q_\lambda) - 1\right)$$

We write $\mathcal{L} := \mathcal{L}_1$. For each $k \geq 1$,

$$\Lambda Q(r) := r\partial_r Q(r) = k \sin Q = 2k\frac{r^k}{1+r^{2k}}$$

When $k \geq 2$, ΛQ is a zero energy eigenfunction for $\mathcal{L}$, i.e.,

$$\mathcal{L}\Lambda Q = 0, \quad \text{and} \quad \Lambda Q \in L^2_{\mathrm{rad}}(\mathbb{R}^2).$$

When $k = 1$, $\mathcal{L}\Lambda Q = 0$ holds but $\Lambda Q \notin L^2$ due to slow decay as $r \to \infty$ and 0 is called a threshold resonance.

We define a smooth non-negative function $\mathcal{Z} \in C^\infty(0,\infty) \cap L^1((0,\infty), r\,\mathrm{d}r)$ by

$$\mathcal{Z}(r) := \begin{cases} \chi(r)\Lambda Q(r) & \text{if } k = 1, 2 \\ \Lambda Q(r) & \text{if } k \geq 3 \end{cases} \tag{2.9}$$

and note that

$$\langle \mathcal{Z} \mid \Lambda Q \rangle > 0.$$

The precise form of $\mathcal{Z}$ is not so important, rather only that it is not perpendicular to ΛQ and has sufficient decay and regularity. We fix it as above because of the convenience of setting $\mathcal{Z} = \Lambda Q$ if $k \geq 3$. We record the following localized coercivity lemma proved in [18].

Lemma 2.6 (Localized coercivity for $\mathcal{L}$)*[18, Lemma 5.4] Fix $k \geq 1$. There exist uniform constants $c < 1/2$, $C > 0$ with the following properties. Let $g \in \mathcal{E}$. Then,*

$$\langle \mathcal{L}g \mid g \rangle \geq c\|g\|_H^2 - C\langle \mathcal{Z} \mid g \rangle^2$$

If $R > 0$ is large enough then,

$$\begin{aligned}(1-2c)\int_0^R \Big((\partial_r g)^2 + k^2\frac{g^2}{r^2}\Big) r\mathrm{d}r + c\int_R^\infty \Big((\partial_r g)^2 + k^2\frac{g^2}{r^2}\Big) r\mathrm{d}r + \Big\langle \frac{k^2}{r^2}(f'(Q)-1)g \mid g\Big\rangle \\ \geq -C\langle \mathcal{Z} \mid g\rangle^2.\end{aligned}$$

If $r > 0$ is small enough, then

$$\begin{aligned}(1-2c)\int_r^\infty \Big((\partial_r g)^2 + k^2\frac{g^2}{r^2}\Big) r\mathrm{d}r + c\int_0^r \Big((\partial_r g)^2 + k^2\frac{g^2}{r^2}\Big) r\mathrm{d}r + \Big\langle \frac{k^2}{r^2}(f'(Q)-1)g \mid g\Big\rangle \\ \geq -C\langle \mathcal{Z} \mid g\rangle^2.\end{aligned}$$

As a consequence, (see for example [16, Proof of Lemma 2.4] for an analogous argument) one obtains the following coercivity property of the operator $\mathcal{L}_{\mathcal{Q}}$.

Lemma 2.7 *[17, Lemma 2.19] Fix $k \geq 1$, $M \in \mathbb{N}$. There exist $\eta, c_0 > 0$ with the following properties. Consider the subset of M-bubble configurations $\mathcal{Q}(m, \vec{\iota}, \vec{\lambda})$ for $\vec{\iota} \in \{-1,1\}^M$, $\vec{\lambda} \in (0,\infty)^M$ such that,*

$$\sum_{j=1}^{M-1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^k \leq \eta^2. \tag{2.10}$$

Let $g \in H$ be such that

$$0 = \Big\langle \mathcal{Z}_{\underline{\lambda_j}} \mid g \Big\rangle \;\; \textit{for} \;\; j = 1, \dots M.$$

for some $\vec{\lambda}$ *as in* (2.10). *Then,*

$$\left\langle \mathrm{D}^2 E(\mathcal{Q}(m,\vec{\iota},\vec{\lambda}))g \mid g\right\rangle \ge c_0\|g\|_{\mathcal{E}}^2.$$

The following technical lemma is useful when computing interactions between bubbles at different scales.

Lemma 2.8 *For any* $\lambda \le \mu$ *and* $\alpha, \beta > 0$ *with* $\alpha \neq \beta$ *the following bound holds:*

$$\int_0^\infty \max\Big(1, \frac{r}{\lambda}\Big)^{-\alpha} \max\Big(1, \frac{\mu}{r}\Big)^{-\beta} \frac{\mathrm{d}r}{r} \lesssim_{\alpha,\beta} \Big(\frac{\lambda}{\mu}\Big)^{\min(\alpha,\beta)}.$$

For any $\alpha > 0$ *the following bound holds:*

$$\int_0^\infty \max\Big(1, \frac{r}{\lambda}\Big)^{-\alpha} \max\Big(1, \frac{\mu}{r}\Big)^{-\alpha} \frac{\mathrm{d}r}{r} \lesssim_{\alpha} \Big(\frac{\lambda}{\mu}\Big)^{\alpha}\Big(1+\log\Big(\frac{\mu}{\lambda}\Big)\Big).$$

Proof This is a straightforward computation, considering separately the regions $0 < r \le \lambda$, $\lambda \le r \le \mu$, and $r \ge \mu$. □

Using the above, along with the formula for $\mathcal{Z}$ in (2.9) we obtain the following.

Corollary 2.9 *Let* $\mathcal{Z}$ *be as in* (2.9) *and suppose that* $\lambda, \mu > 0$ *satisfy* $\lambda/\mu \le 1$. *Then,*

$$\left\langle \mathcal{Z}_{\underline{\lambda}} \mid \Lambda Q_{\underline{\mu}}\right\rangle \lesssim \begin{cases} (\lambda/\mu)^{k+1} & \text{if } k = 1,2 \\ (\lambda/\mu)^{k-1} & \text{if } k \ge 3 \end{cases}, \quad \left\langle \mathcal{Z}_{\underline{\mu}} \mid \Lambda Q_{\underline{\lambda}}\right\rangle \lesssim \begin{cases} 1 & \text{if } k = 1 \\ (\lambda/\mu)^{k-1} & \text{if } k \ge 2 \end{cases}$$

Lemma 2.8 is also used to prove the following lemma from [17] giving leading order terms in an expansion of the nonlinear energy functional about an M-bubble configuration. We refer the reader to [17] for the proof.

Lemma 2.10 *[17, Lemma 2.22] Fix* $k \ge 1$, $M \in \mathbb{N}$. *For any* $\theta > 0$, *there exists* $\eta > 0$ *with the following property. Consider the subset of* M*-bubble* $\mathcal{Q}(m,\iota,\vec{\lambda})$ *configurations such that*

$$\sum_{j=1}^{M-1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^k \le \eta.$$

Then,

$$\Big|E(\mathcal{Q}(m,\vec{\iota},\vec{\lambda})) - ME(Q) - 16k\pi\sum_{j=1}^{M-1}\iota_j\iota_{j+1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^k\Big| \le \theta\sum_{j=1}^{M-1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^k.$$

Moreover, there exists a uniform constant $C > 0$ *such that for any* $g \in H$,

$$\Big|\left\langle \mathrm{D} E(\mathcal{Q}(m,\vec{\iota},\vec{\lambda})) \mid g\right\rangle\Big| \le C\|g\|_{\mathcal{E}}\sum_{j=1}^{M}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^k.$$

The following (standard) modulation lemma plays an important role and we refer the reader to [17, Lemma 2.25] for its proof. Before stating it, we define a proximity function to M-bubble configurations. Fixing m, M we observe that $\mathcal{Q}(m,\vec{\iota},\vec{\lambda}; r)$ is an element of $\mathcal{E}_{\ell,m}$, where

$$\ell = \ell(m, M, \vec{\iota}) := m - \sum_{j=1}^{M}\iota_j \tag{2.11}$$

Definition 2.11 Fix m, M as in Definition 1.5 and let $v \in \mathcal{E}_{\ell,m}$ for some $\ell \in \mathbb{Z}$. Define,

$$\mathbf{d}(v) = \mathbf{d}_{m,M}(v) := \inf_{\vec{\iota},\vec{\lambda}} \left(\|v - \mathcal{Q}(m, \vec{\iota}, \vec{\lambda})\|_{\mathcal{E}}^2 + \sum_{j=1}^{M-1} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k \right)^{\frac{1}{2}}. \tag{2.12}$$

where the infimum is taken over all vectors $\vec{\lambda} = (\lambda_1, \dots, \lambda_M) \in (0, \infty)^M$ and all $\vec{\iota} = \{\iota_1, \dots, \iota_M\} \in \{-1, 1\}^M$ satisfying (2.11).

Lemma 2.12 (Static modulation lemma) *[17, Lemma 2.25] Fix $k \ge 1$ and $M \in \mathbb{N}$. There exists $\eta \in (0, 1)$, $C > 0$ with the following properties. Let m be as in Definition 1.5 and $\mathbf{d}_{m,M}$ as in Definition 2.11. Let $\theta > 0$, $\ell \in \mathbb{Z}$, and let $v \in \mathcal{E}_{\ell,m}$ be such that*

$$\mathbf{d}_{m,M}(v) \le \eta, \quad \text{and} \quad E(v) \le ME(Q) + \theta^2,$$

Then, there exists a unique choice of $\vec{\lambda} = (\lambda_1, \dots, \lambda_M) \in (0, \infty)^M$, $\vec{\iota} \in \{-1, 1\}^M$, and $g \in H$, such that

$$\begin{aligned} v &= \mathcal{Q}(m, \vec{\iota}, \vec{\lambda}) + g, \\ 0 &= \left\langle \mathcal{Z}_{\underline{\lambda_j}} \mid g \right\rangle, \quad \forall j = 1, \dots, M, \end{aligned}$$

along with the estimates,

$$\mathbf{d}_{m,M}(v)^2 \le \|g\|_{\mathcal{E}}^2 + \sum_{j=1}^{M-1} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k \le C\mathbf{d}_{m,M}(v)^2,$$

and,

$$\|g\|_{\mathcal{E}}^2 + \sum_{j \notin \mathcal{A}} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k \le C \max_{j \in \mathcal{A}} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k + \theta^2, \tag{2.13}$$

where $\mathcal{A} := \{j \in \{1, \dots, M-1\} : \iota_j \ne \iota_{j+1}\}$.

We also make use of the following lemma proved from [17] which says that a finite energy map cannot be close to two distinct multi-bubble configurations.

Lemma 2.13 *[17, Lemma 2.27] Let $k \ge 1$. There exists $\eta > 0$ sufficiently small with the following property. Let $m, \ell \in \mathbb{Z}$, $M, L \in \mathbb{N}$, $\vec{\iota} \in \{-1, 1\}^M$, $\vec{\sigma} \in \{-1, 1\}^L$, $\vec{\lambda} \in (0, \infty)^M$, $\vec{\mu} \in (0, \infty)^L$, and w be such that $E_{\mathbf{p}}(w) < \infty$ and,*

$$\begin{aligned} \|w - \mathcal{Q}(m, \vec{\iota}, \vec{\lambda})\|_{\mathcal{E}}^2 + \sum_{j=1}^{M-1} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k &\le \eta, \\ \|w - \mathcal{Q}(\ell, \vec{\sigma}, \vec{\mu})\|_{\mathcal{E}}^2 + \sum_{j=1}^{L-1} \left(\frac{\mu_j}{\mu_{j+1}} \right)^k &\le \eta. \end{aligned}$$

Then, $m = \ell$, $M = L$, $\vec{\iota} = \vec{\sigma}$. Moreover, for every $\theta > 0$ the number $\eta > 0$ above can be chosen small enough so that

$$\max_{j=1,\dots M} \left| \frac{\lambda_j}{\mu_j} - 1 \right| \le \theta.$$

3 Localized sequential bubbling

We define a localized distance function

$$\boldsymbol{\delta}_R(u) := \inf_{m,M,\vec{\iota},\vec{\lambda}} \left(\|u - \mathcal{Q}(m, \vec{\iota}, \vec{\lambda})\|^2_{\mathcal{E}(r \le R)} + \sum_{j=1}^{M} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k \right)^{\frac{1}{2}} \tag{3.1}$$

where the infimum is taken over all $m \in \mathbb{Z}$, $M \in \{0, 1, 2, \dots\}$, all vectors $\iota \in \{-1, 1\}^M$, $\vec{\lambda} \in (0, \infty)^M$, and we use the convention that the last scale $\lambda_{M+1} = R$.

Lemma 3.1 *Let $\ell, m \in \mathbb{Z}$ and let $u_n \in \mathcal{E}_{\ell,m}$ be a sequence of maps with* $\limsup_{n\to\infty} E(u_n) < \infty$. *Let $\rho_n \in (0, \infty)$ be a sequence and suppose that*

$$\lim_{n\to\infty} (\rho_n \|\mathcal{T}(u_n)\|_{L^2}) = 0. \tag{3.2}$$

Then, there exists a sequence $R_n \to \infty$ so that, up to passing to a subsequence of the u_n, we have,

$$\lim_{n\to\infty} \boldsymbol{\delta}_{R_n \rho_n}(u_n) = 0.$$

The subsequence of the u_n can be chosen so that there are fixed $(M, m, \vec{\iota}) \in \mathbb{N} \cup \{0\} \times \mathbb{Z} \times \{-1, 1\}^M$, a sequence $\vec{\lambda}_n \in (0, \infty)^M$, and $C_0 > 0$ with

$$\lim_{n\to\infty} \left(\|u_n - \mathcal{Q}(m, \vec{\iota}, \vec{\lambda}_n)\|^2_{\mathcal{E}(r \le R_n \rho_n)} + \sum_{j=1}^{M-1} \left(\frac{\lambda_{n,j}}{\lambda_{n,j+1}} \right)^k \right) = 0,$$

and,

$$\lambda_{n,M} \le C_0 \rho_n, \quad \forall n.$$

Remark 3.2 Lemma 3.1 is proved in the general (non-equivariant) setting by Qing [24]. Here we give a different (but related) treatment adapted to the equivariant setting using explicitly the notion of a profile decomposition of Gérard [11]. The proof that no energy can accumulate in the "neck" regions between the bubbles can be simplified in the equivariant setting and here we use an argument due to Jia and Kenig [19] from their proof of an analogous result for equivariant wave maps; see Lemma 3.4 below.

Lemma 3.3 *If $a_{k,n}$ are positive numbers such that* $\lim_{n\to\infty} a_{k,n} = \infty$ *for all $k \in \mathbb{N}$, then there exists a sequence of positive numbers b_n such that* $\lim_{n\to\infty} b_n = \infty$ *and* $\lim_{n\to\infty} a_{k,n}/b_n = \infty$ *for all $k \in \mathbb{N}$.*

Proof For each k and each n define $\widetilde{a}_{k,n} = \min\{a_{1,n}, \dots, a_{k,n}\}$. Then the sequences $\widetilde{a}_{k,n} \to \infty$ as $n \to \infty$ for each k, but also satisfy $\widetilde{a}_{k,n} \le a_{k,n}$ for each k, n, as well as $\widetilde{a}_{j,n} \le \widetilde{a}_{k,n}$ if $j > k$. Next, choose a strictly increasing sequence $\{n_k\}_k \subset \mathbb{N}$ such that $\widetilde{a}_{k,n} \ge k^2$ as long as $n \ge n_k$. For n large enough, let $b_n \in \mathbb{N}$ be determined by the condition $n_{b_n} \le n < n_{b_n+1}$. Observe that $b_n \to \infty$ as $n \to \infty$. Now fix any $\ell \in \mathbb{N}$ and let n be such that $b_n > \ell$. We then have

$$a_{\ell,n} \ge \widetilde{a}_{\ell,n} \ge \widetilde{a}_{b_n,n} \ge b_n^2 \gg b_n.$$

Thus the sequence b_n has the desired properties. □

The proof of the Lemma 3.1 consists of several steps, which are designed to reduce the proof to a scenario already considered by Côte in [6, Proof of Lemma 3.5] and then by Jia-Kenig in [19, Proof of Theorem 3.2], albeit in a different context. In particular, we will seek to apply the following result from [19].

Lemma 3.4 *[19, Theorem 3.2] Let v_n be a sequence of maps such that* $\limsup_{n\to\infty} E(v_n) < \infty$*. Suppose that there exists a sequence an integer $M \geq 0$ and scales $\lambda_{n,1} \ll \cdots \ll \lambda_{n,M} \lesssim 1$ such that*

$$v_n = m_1\pi + \sum_{j=1}^{M} \iota_j (Q(\tfrac{\cdot}{\lambda_{n,j}}) - \pi) + w_{n,0},$$

where $\|w_n\|_{L^\infty} \to 0$ and $\|w_n\|_{\mathcal{E}(r \geq r_n^{-1})} \to 0$ as $n \to \infty$ for some sequence $r_n \to \infty$. Suppose in addition that, $\|w_n\|_{\mathcal{E}(A^{-1}\lambda_n \leq r \leq A\lambda_n)} \to 0$ as $n \to \infty$ for any sequence $\lambda_n \lesssim 1$ and any $A > 1$, and finally, that

$$\limsup_{n\to\infty} \int_0^\infty \left(k^2 \frac{\sin^2(2v_n)}{2r^2} + (\partial_r v_n)^2 2\cos(2v_n) \right) r\, \mathrm{d}r \leq 0. \tag{3.3}$$

Then,

$$\|w_n\|_{\mathcal{E}} \to 0 \ \text{ as } \ n \to \infty.$$

Remark 3.5 Lemma 3.4 is not stated in [19] exactly as given above. However, an examination of [19, Proof of Theorem 3.2] shows that this is precisely what is established. The heart of the matter lies in the fact that the Jia-Kenig virial functional (3.3) vanishes at Q, i.e.,

$$\int_0^\infty \left(k^2 \frac{\sin^2(2Q)}{2r^2} + (\partial_r Q)^2 2\cos(2Q) \right) r\, \mathrm{d}r = 0,$$

but gives coercive control of the energy in regions where $v_n(r)$ is near integer multiples of π.

Proof of Lemma 3.1 By rescaling we may assume that $\rho_n = 1$ for each n.

First, we observe that after passing to a subsequence, u_n admits a profile decomposition,

$$u_n = m\pi + \sum_{j=1}^{K_0} \left(\psi^j(\tfrac{\cdot}{\lambda_{n,j}}) - m_j\pi \right) + \sum_{i=1}^{J} v^i(\tfrac{\cdot}{\sigma_{n,i}}) + w_n^J(\cdot).$$

where the profiles $(\psi^j, \lambda_{n,j})$, $(v^j, \sigma_{n,j})$ and the error satisfy the conclusions of Lemma 2.5.

Step 1 We make an initial restriction on the sequence $R_n \to \infty$, refining our choice of this sequence later in the proof. Consider the sets of indices

$$\mathcal{J}_\infty := \left\{ j \in \{1, \ldots, K_0\} \mid \lim_{n\to\infty} \lambda_{n,j} = \infty \right\}, \quad \mathcal{I}_\infty := \left\{ i \in \mathbb{N} \mid \lim_{n\to\infty} \sigma_{n,i} = \infty \right\}$$

By Lemma 3.3 we choose a sequence $R_{n,1} \to \infty$ so that $R_{n,1} \ll \lambda_{n,j}, \sigma_{n,i}$ for each $\lambda_{n,j}$ with $j \in \mathcal{J}_\infty$ and each $\sigma_{n,i}$ with $i \in \mathcal{I}_\infty$. If follows that

$$\lim_{n\to\infty} E(\psi^j(\cdot/\lambda_{n,j}); 0, R_{n,1}) = 0, \quad \lim_{n\to\infty} E(v^j(\cdot/\sigma_{n,i}); 0, R_{n,1}) = 0$$

for any of the indices $j \in \mathcal{J}_\infty$ or $i \in \mathcal{I}_\infty$, and thus these profiles do not factor into the distance $\boldsymbol{\delta}_{R_n}(u_n)$ for any sequence $R_n \leq R_{n,1}$.

Step 2 Next we perform a bubbling analysis on the profiles with bounded scale. Define

$$\mathcal{J}_0 := \left\{ j \in \{1, \ldots, K_0\} \mid \lim_{n\to\infty} \lambda_{n,j} < \infty \right\}, \quad \mathcal{I}_0 := \left\{ i \in \mathbb{N} \mid \lim_{n\to\infty} \sigma_{n,i} < \infty \right\}$$

First, for $j \in \mathcal{J}_0$ and $i \in \mathcal{I}_0$, denote

$$u_n^j(r) := u_n(\lambda_{n,j} r), \quad u_n^i(r) := u_n(\sigma_{n,i} r)$$

Then we have $u_n^j \to \psi^j$ as $n \to \infty$ locally uniformly in $(0, \infty)$ and weakly in $\dot{H}^1(\mathbb{R}^2)$ (that is, if we view each u_n^j as a radially symmetric function on $\mathbb{R}^2$). These convergence properties are by construction, see [19, pg. 1594]). Moreover, since $\lim_{n\to\infty} \lambda_{n,j} < \infty$ we have,

$$\|\mathcal{T}(u_n^j)\|_{L^2} = \lambda_{n,j} \|\mathcal{T}(u_n)\|_{L^2} \to 0 \text{ as } n \to \infty$$

It follows that,

$$\langle \mathcal{T}(\psi^j) \mid \phi \rangle_{L^2} = 0$$

for all $\phi \in C_0^\infty(0, \infty)$, i.e., ψ^j is a weak harmonic map, and hence a smooth harmonic map by Hélein [15]. Since $|m_j - \ell_j| \geq 1$ we see that $E(\psi^j) \geq E(Q)$, and thus $\psi^j = \ell_j \pi + \iota_j Q_{\lambda_{j,0}}$ for some $\iota_j \in \{-1, 1\}$ and some fixed scale $\lambda_{j,0}$ and $m_j = \ell_j + \iota\pi$. We will abuse notation and replace $\lambda_{n,j}$ with $\lambda_{n,j}\lambda_{j,0}$ while still calling this sequence $\lambda_{n,j}$.

We perform the same analysis with the u_n^i and v^i, concluding that each v^i is a smooth harmonic map. But since $v^i \in \mathcal{E}_{0,0}$ we find that $v^i \equiv 0$ for every $i \in \mathcal{I}_0$.

Step 3: Next, by (3.2) and recalling that we have rescaled so that $\rho_n = 1$, we let $R_{2,n} \to \infty$ be a sequence such that

$$1 \ll R_{2,n} \ll \|\mathcal{T}(u_n)\|_{L^2}^{-1}.$$

Then, by Cauchy-Schwarz

$$\left| \langle \mathcal{T}(u_n) \mid \sin(2u_n)\chi_{\widetilde{R}_n} \rangle \right| \leq \|\mathcal{T}(u_n)\|_{L^2} \widetilde{R}_n \to 0 \text{ as } n \to \infty \tag{3.4}$$

for any sequence $\widetilde{R}_n \leq R_{2,n}$. We define $R_{3,n} := \min(R_{1,n}, R_{2,n})$.

Step 4: We close in on the final selection of the sequence R_n, choosing first $\sqrt{R_{3,n}} \leq R_{4,n} \leq (1/2)R_{3,n}$ so that

$$E\left(u_n; \frac{1}{4}R_n, 4R_n\right) \to 0 \text{ as } n \to \infty$$

The existence of such a sequence is proved by pigeonholing; see for example [17, Eq. (3.12)]. Using Lemma 2.3 we can, after passing to a subsequence, find an integer $m_1 \in \mathbb{Z}$ so that $|u_n(r) - m_1\pi| \to 0$ for a.e., $r \in [\frac{1}{4}R_n, 4R_n]$, and we define a truncated sequence

$$\widetilde{u}_n := \chi_{R_{4,n}} u_n + (1 - \chi_{R_{4,n}})m_1\pi$$

By construction we have the following decomposition for $\widetilde{u}_n$,

$$\widetilde{u}_n = m_1\pi + \sum_{j\in\mathcal{J}_0} (\iota_j Q_{\lambda_j} - \pi) + \widetilde{w}_n$$

where the error $\widetilde{w}_n := \chi_{R_{4,n}} w_n^J + o_n(1)$ (note we can drop the index J since any nontrivial profiles from the index sets $\mathcal{J}_\infty$ or $\mathcal{I}_\infty$ contribute a vanishing error in the region $r \leq R_{4,n}$ by Step 1 and there are no nontrivial profiles from the index set $\mathcal{I}_0$ from Step 2). We define $M := \#\mathcal{J}_0$ and we reorder/relabel the profiles so that $\lambda_{n,1} \ll \lambda_{n,2} \ll \dots \lambda_{n,M}$ for the indices $j \in \mathcal{J}_0$. Note that we have proved that

$$\lim_{n\to\infty} \|\widetilde{w}_n\|_{L^\infty} = 0 \tag{3.5}$$

After passing to a subsequence of the u_n, we claim there is a sequence $R_n \to \infty$ with the properties,

$$1 \ll R_n \le R_{4,n}, \quad \|\widetilde{w}_n\|_{\mathcal{E}(\frac{1}{4}R_n^{-1} \le r \le 4R_n)} \to 0 \ \text{ as } \ n \to \infty. \tag{3.6}$$

The existence of such a sequence is a consequence of the following property about $\widetilde{w}_n$: for any sequence $\lambda_n \lesssim 1$ and any $A > 1$ we have,

$$\|w_n\|_{\mathcal{E}(\lambda_n A^{-1} \le r \le \lambda_n A)} \to 0 \ \text{ as } \ n \to \infty. \tag{3.7}$$

The property (3.7) was proved in [6, Step 2., p.1973–1975, Proof of Theorem 3.5] and [19, Proof of (5.29) in Theorem 5.1] and we refer the reader to those works for details of the argument, which also applies in the current setting. The intuition is that at any scale $\lambda_n \lesssim 1$ at which $\widetilde{u}_n$ carries energy we have already extracted a profile $Q_{\lambda_{n,j}}$ with $\lambda_{n,j} \simeq \lambda_n$. To prove (3.6) we consider the case $\lambda_n = 1$ in (3.7), and passing to a subsequence of the $\widetilde{u}_n$, we obtain a sequence as in (3.6).

We truncate to the region $r \le R_n$, following the same procedure used to define $\widetilde{u}_n$, using now R_n in place of $R_{4,n}$. Indeed, set

$$\breve{u}_n(t_n, r) := \chi_{R_n}(r)\widetilde{u}_n(t, r) + (1 - \chi_{R_n}(r))m_1\pi.$$

Defining $\breve{w}_{n,0} := \chi_{R_n}(r)\widetilde{w}_n + (\chi_{R_n}(r) - 1)\sum_{j=1}^M \iota_j(Q(\frac{\cdot}{\lambda_{n,j}}) - \pi)$ and using that $\lambda_{n,1} \ll \cdots \ll \lambda_{n,M} \lesssim 1$ along with (3.5) and (3.6) we see that,

$$\begin{aligned} &\breve{u}_n(t_n) = m_1\pi + \sum_{j=1}^M \iota_j\left(Q\left(\frac{\cdot}{\lambda_{n,j}}\right) - \pi\right) + \breve{w}_{n,0}, \quad \text{and} \\ &\lim_{n\to\infty}\left(\|\breve{w}_n\|_{\mathcal{E}(R_n^{-1} \le r < \infty)} + \|\breve{w}_n\|_{L^\infty}\right) = 0. \end{aligned} \tag{3.8}$$

Moreover, by (3.7) we see that for any sequence $\lambda_n \lesssim 1$ and any $A > 1$ that,

$$\lim_{n\to\infty}\|\breve{w}_n\|_{\mathcal{E}(\lambda_n A^{-1} \le r \le \lambda_n A)} = 0.$$

Note that since $\breve{u}_n(r) = u_n(r)$ for $r \le R_n$, we deduce from (3.4) that,

$$\left|\left\langle \mathcal{T}(\breve{u}_n) \mid \sin(2\breve{u}_n)\chi_{\frac{1}{4}R_n}\right\rangle\right| \to 0 \ \text{ as } \ n \to \infty$$

We claim that

$$\left|\left\langle \mathcal{T}(\breve{u}_n) \mid \sin(2\breve{u}_n)(1 - \chi_{\frac{1}{4}R_n})\right\rangle\right| \to 0 \ \text{ as } \ n \to \infty$$

as well. To see this, note that by (3.8)

$$\lim_{n\to\infty} E(\breve{u}_n; r_n, \infty) = 0$$

for any sequence $r_n \to \infty$. And after integration by parts we deduce the bound,

$$\left|\left\langle \mathcal{T}(\breve{u}_n) \mid \sin(2\breve{u}_n)(1 - \chi_{\frac{1}{4}R_n})\right\rangle\right| \lesssim E(\breve{u}_n; 1/8R_n, \infty) \to 0 \ \text{ as } \ n \to \infty$$

Hence,

$$\left|\left\langle \mathcal{T}(\breve{u}_n) \mid \sin(2\breve{u}_n)\right\rangle\right| \to 0 \ \text{ as } \ n \to \infty$$

Integrating by parts on the left hand side, we see that

$$\lim_{n\to\infty}\int_0^\infty\left(k^2\frac{\sin^2(2\breve{u}_n)}{2r^2}+(\partial_r\breve{u}_n)^2 2\cos(2\breve{u}_n)\right)r\,\mathrm{d}r=0.$$

The sequence $\breve{u}_n$ then satisfies all the conditions of Lemma 3.4 and we conclude that $\lim_{n\to\infty}\|\breve{w}_n\|_{\mathcal{E}}=0$. Since $\breve{u}_n(r)=u_n(r)$ for $r\leq R_n$ we conclude that $\lim_{n\to\infty}\boldsymbol{\delta}_{R_n}(u_n)=0$. An examination of the decomposition (3.8) yields the remaining claims in from Lemma 3.1. □

4 Sequential bubbling

4.1 Sequential bubbling for finite time blow-up solutions

Proposition 4.1 (Sequential bubbling for solutions that blow up in finite time) *Let $\ell, m\in\mathbb{Z}$, $u_0\in\mathcal{E}_{\ell,m}$, and let $u(t)$ denote the solution to (1.2) with initial data u_0. Suppose that $T_+(u_0)<\infty$. There exist integers m_∞, m_Δ, a mapping $u^*\in\mathcal{E}_{0,m_\infty}$, an integer $N\geq 1$, a sequence of times $t_n\to T_+$, signs $\vec{\iota}\in\{-1,1\}^N$, a sequence of scales $\vec{\lambda}_n\in(0,\infty)^N$, and an error g_n defined by*

$$u(t_n)=m_\Delta\pi+\sum_{j=1}^N\iota_j(Q_{\lambda_n}-\pi)+u^*+g_n,$$

with the following properties:

(i) The integer $N\geq 1$ and the body map u^* satisfy,

$$\lim_{t\to T_+}E(u(t))=NE(Q)+E(u^*);\tag{4.1}$$

(i) for any $\alpha>0$,

$$\lim_{t\to T_+}E\left(u(t);0,\alpha(T_+-t)^{\frac{1}{2}}\right)=NE(Q),\tag{4.2}$$

$$\lim_{t\to T_+}E\left(u(t)-u^*;\alpha(T_+-t)^{\frac{1}{2}},\infty\right)=0,\tag{4.3}$$

and there exists $0<T_0<T_+$ and function $\rho:[T_0,T_+)\to(0,\infty)$ satisfying,

$$\lim_{t\to T_*}\left((\rho(t)/\sqrt{T_+-t})+\|u(t)-u^*-m_\Delta\pi\|_{\mathcal{E}(\rho(t))}\right)=0;\tag{4.4}$$

(ii) the error g_n and the scales $\vec{\lambda}_n$ satisfy,

$$\lim_{n\to\infty}\left(\|g_n\|_{\mathcal{E}}^2+\sum_{j=1}^N\left(\frac{\lambda_{n,j}}{\lambda_{n,j+1}}\right)^k\right)^{\frac{1}{2}}=0,\tag{4.5}$$

where here we adopt the convention that $\lambda_{n,N+1}:=(T_+-t_n)^{\frac{1}{2}}$.

Lemma 4.2 (Identification of the body map) *Let $u_0\in\mathcal{E}_{\ell,m}$ and let $u(t)$ be the solution to (1.2). Suppose that $T_+(u_0)<\infty$ and let $I_*=[0,T_+)$. There exist $m_\infty, m_\Delta\in\mathbb{Z}$ and a mapping $u^*\in\mathcal{E}_{0,m_\infty}$ such that for any $r_0>0$,*

$$\lim_{t\to T_*}\|u(t)-u^*-m_\Delta\pi\|_{\mathcal{E}(r\geq r_0)}=0.\tag{4.6}$$

Moreover, there exists $L > 0$ such that for each $r_0 \in (0, \infty]$,

$$\lim_{t \to T_+} E(u(t); 0, r_0) = L + E(u^*; 0, r_0), \tag{4.7}$$

and in particular, $\lim_{r_0 \to 0} \lim_{t \to T_+} E(u(t); 0, r_0) = L$.

Proof of Lemma 4.2 In the general (non-equivariant) setting Struwe [28] proves the existence of the body map as the weak limit of the flow in H^1 as $t \to T_+$ and moreover that one has strong C^2 convergence on compact sets not containing the bubbling points (the origin in our case); see for example [20, Step 3, Proof of Theorem 6.16]. The existence of the limit L is proved by Qing in [24, Proposition 2.1], and an identical argument can be used in the equivariant setting. □

Proof of Proposition 4.1 We follow, roughly, the arguments by Qing in [24, Proof of Theorem 1.1] and Topping in [31, Proof of Theorem 1.4]. The main ingredient is the compactness result, Lemma 3.1. Let $u(t) \in \mathcal{E}_{\ell,m}$ be a heat flow blowing up at time $T_+ > 0$. By (2.2) we can find a sequence $t_n \to T_+$ so that,

$$(T_+ - t_n)^{\frac{1}{2}} \|\mathcal{T}(u(t_n))\|_{L^2} \to 0 \text{ as } n \to \infty.$$

We can now apply Lemma 3.1 with $\rho_n := (T_+ - t_n)^{\frac{1}{2}}$, which yields $N \geq 0$, $m_0 \in \mathbb{Z}$, $\vec{\iota} \in \{-1, 1\}^N$, $\vec{\lambda}_n \in (0, \infty)^N$ such that after passing to a subsequence, we have

$$\lim_{n \to \infty} \left(\|u(t_n) - \mathcal{Q}(m_0, \vec{\iota}, \vec{\lambda}_n)\|^2_{\mathcal{E}(r \leq A(T_+ - t_n)^{\frac{1}{2}})} + \sum_{j=1}^{N-1} \left(\frac{\lambda_{n,j}}{\lambda_{n,j+1}} \right)^k \right) = 0 \tag{4.8}$$

for each $A > 0$, and moreover that $\lambda_{n,N} \lesssim (T_+ - t_n)^{\frac{1}{2}}$. Next, for each $R > 0$ define the localized energy,

$$\Theta_R(t) := \int_0^\infty \chi_R(r)^2 \mathbf{e}(u(t, r))\, r \,\mathrm{d}r.$$

along with the localized energy of the body map,

$$\Theta_R^* := \int_0^\infty \chi_R(r)^2 \mathbf{e}(u^*(r))\, r \,\mathrm{d}r.$$

From (2.4) we see that for each $0 < s < \tau < T_+$ we have,

$$\begin{aligned} \left|\Theta_R(\tau) - \Theta_R(s)\right| &\lesssim \int_s^\tau \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t + \frac{(\tau - s)^{\frac{1}{2}}}{R} \left(\int_s^\tau \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \\ &\lesssim \int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t + \frac{(T_+ - s)^{\frac{1}{2}}}{R} \left(\int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \end{aligned} \tag{4.9}$$

Since the right-hand side tends to zero as $s \to T_+$, it follows that $\lim_{t \to T_+} \Theta_R(t) := \ell_R$ exists. Define,

$$\frac{1}{2\pi} L_R := \ell_R - \Theta_R^*$$

and we claim that in fact, $L_R = L := \lim_{r_0 \to 0} \lim_{t \to T_+} E(u(t); 0, r_0)$, which is independent of $R > 0$. To see this we write, for any $0 < r_0 < R$,

$$\Theta_R(t) - \Theta_R^* = \int_{r_0}^{4R} \chi_R(r)^2 (\mathbf{e}(u(t, r)) - \mathbf{e}(u^*(r)))\, r \,\mathrm{d}r + \tfrac{1}{2\pi} E(u(t); 0, r_0) - \tfrac{1}{2\pi} E(u^*; 0, r_0)$$

Letting $t \to T_+$, the right hand side tends to $\frac{1}{2\pi} L_R$. By (4.6) the first term on the left vanishes as $t \to T_+$. Sending $r_0 \to 0$ after letting $t \to T_+$ on the right, we see from (4.7) that $L_R = L = \lim_{r_0 \to 0} \lim_{t \to T_+} E(u(t); 0, r_0)$.

Next, let $\gamma > 0$ and set $R = \gamma(T_+ - s)^{\frac{1}{2}}$ in (4.9) we obtain, after letting $\tau \to T_+$,

$$\left| \frac{1}{2\pi} L + \Theta^*_{\gamma(T_+-s)^{\frac{1}{2}}} - \Theta_{\gamma(T_+-s)^{\frac{1}{2}}}(s) \right| \lesssim \int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t + \frac{1}{\gamma} \Big(\int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t \Big)^{\frac{1}{2}}$$

Letting $s \to T_+$ above we see that $\lim_{s \to T_+} \Theta_{\gamma(T_+-s)^{\frac{1}{2}}}(s) = \frac{1}{2\pi} L$ for all $\gamma > 0$.

Let $\alpha > 0$ and note the inequality,

$$2\pi \Theta_{\frac{\alpha}{2}(T_+-s)^{\frac{1}{2}}}(s) \le E(u(s); 0, \alpha(T_+ - s)^{\frac{1}{2}}) \le 2\pi \Theta_{\alpha(T_+-s)^{\frac{1}{2}}}(s)$$

which implies that $\lim_{s \to T_+} E(u(s); 0, \alpha(T_+ - s)^{\frac{1}{2}}) = L$ for any $\alpha > 0$. Hence, for any $0 < \alpha < A < \infty$, $\lim_{s \to T_+} E(u(s); \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) = 0$. Returning to the decomposition (4.8) we find that

$$\frac{\lambda_{n,N}}{(T_+ - t_n)^{\frac{1}{2}}} \to 0 \ \text{ as } \ n \to \infty, \tag{4.10}$$

and as a consequence, $L = NE(Q)$ and (4.2) is proved. Further, we see from (4.7) that for every $r_0 > 0$,

$$\lim_{t \to T_+} E(u(t); 0, r_0) = NE(Q) + E(u^*; 0, r_0).$$

and we see from (2.1) that $N \ge 1$. Combining the above with (4.2) we see that for every $\alpha > 0$, $r_0 \in (0, \infty]$,

$$\lim_{t \to T_+} E(u(t); \alpha(T_+ - t)^{\frac{1}{2}}, r_0) = E(u^*; 0, r_0) \tag{4.11}$$

and (4.1) now follows. Next, if (4.3) were to fail, we could find $\alpha_1, \epsilon_1 > 0$ and a sequence $s_n \to T_+$ such that

$$E(u(s_n) - u^*; \alpha_1(T_+ - s_n)^{\frac{1}{2}}, \infty) \ge \epsilon_1, \quad \forall n.$$

To reach a contradiction, we choose $r_0 > 0$ sufficiently small so that $E(u^*; 0, r_0) \le \epsilon_1/8$, and then, using (4.6) and (4.11), n sufficiently large so that $E(u(s_n) - u^*; r_0, \infty) \le \epsilon_1/8$ and $E(u(s_n); \alpha_1(T_+ - s_n)^{\frac{1}{2}}, r_0) \le \epsilon_1/4$. We then estimate,

$$\begin{aligned} E(u(s_n) - u^*;\ \alpha_1(T_+ - s_n)^{\frac{1}{2}}, \infty) &\le E(u(s_n) - u^*; \alpha_1(T_+ - s_n)^{\frac{1}{2}}, r_0) + E(u(s_n) - u^*; r_0, \infty) \\ &\le 2E(u(s_n); \alpha_1(T_+ - s_n)^{\frac{1}{2}}, r_0) + 2E(u^*; \alpha_1(T_+ - s_n)^{\frac{1}{2}}, r_0) + \epsilon_1/8 \le 7\epsilon_1/8, \end{aligned}$$

a contradiction, proving (4.3). We see from (4.6) and (4.8) that $m_0 = m_\Delta$ and from Lemma 2.3 we have,

$$\lim_{t \to T_+} \|u(t) - u^* - m_\Delta \pi\|_{\mathcal{E}(r \ge \alpha(T_+ - t))} = 0,$$

which implies (4.4). Finally, the above together with (4.8) and (4.10) yield (4.5). □

4.2 Sequential bubbling for global solutions

Proposition 4.3 (Sequential bubbling for global-in-time solutions) *Let $\ell, m \in \mathbb{Z}$. Let $u_0 \in \mathcal{E}_{\ell,m}$ and let $u(t)$ denote the solution to (1.2) with initial data u_0. Suppose that $T_+(u_0) = \infty$. Then there exist $T_0 > 0$, an integer $N \geq 0$, a sequence of times $t_n \to \infty$, signs $\vec{\iota} \in \{-1, 1\}^N$, a sequence of scales $\vec{\lambda}_n \in (0, \infty)^N$, and an error g_n defined by*

$$u(t_n) = m\pi + \sum_{j=1}^{N} \iota_j (Q_{\lambda_n} - \pi) + g_n$$

with the following properties:

(i) the integer $N \geq 0$ satisfies,

$$\lim_{t\to\infty} E(u(t)) = NE(Q); \tag{4.12}$$

(ii) for every $\alpha > 0$,

$$\lim_{t\to\infty} E(u(t); \alpha\sqrt{t}, \infty) = 0, \tag{4.13}$$

and there exists $T_0 > 0$ and a function $\rho : [T_0, \infty) \to (0, \infty)$ such that

$$\lim_{t\to\infty} \Big(\frac{\rho(t)}{\sqrt{t}} + \|u(t) - m\pi\|_{\mathcal{E}(r\geq\rho(t))} \Big) = 0; \tag{4.14}$$

(iii) the scales $\vec{\lambda}_n$ and the sequence g_n satisfy,

$$\lim_{n\to\infty} \left(\|g_n\|_{\mathcal{E}}^2 + \sum_{j=1}^{N} \Big(\frac{\lambda_{n,j}}{\lambda_{n,j+1}} \Big)^k \right)^{\frac{1}{2}} = 0 \tag{4.15}$$

where here we adopt the convention that $\lambda_{n,j+1} := t_n^{\frac{1}{2}}$.

Proof Let $u(t) \in \mathcal{E}_{\ell,m}$ be a heat flow defined globally in time. By (2.2) we can find a sequence $t_n \to \infty$ so that,

$$t_n^{\frac{1}{2}} \|\mathcal{T}(u(t_n))\|_{L^2} \to 0 \ \text{ as } \ n \to \infty.$$

We can now apply Lemma 3.1 with $\rho_n := t_n^{\frac{1}{2}}$, which yields $N \geq 0$, $m_0 \in \mathbb{Z}$, $\vec{\iota} \in \{-1, 1\}^N$, $\vec{\lambda}_n \in (0, \infty)^N$ such that after passing to a subsequence, we have

$$\lim_{n\to\infty} \left(\|u(t_n) - \mathcal{Q}(m_0, \vec{\iota}, \vec{\lambda}_n)\|^2_{\mathcal{E}(r\leq At_n^{\frac{1}{2}})} + \sum_{j=1}^{N-1} \Big(\frac{\lambda_{n,j}}{\lambda_{n,j+1}} \Big)^k \right) = 0 \tag{4.16}$$

for each $A > 0$, and moreover that $\lambda_{n,N} \lesssim t_n^{\frac{1}{2}}$.

Fix $\alpha > 0$ and let $\epsilon > 0$. By (2.2) and the fact that $E(u(0)) < \infty$ we can find $T_0 = T_0(\epsilon) > 0$ such that,

$$\frac{4\sqrt{E(u(0))}}{\alpha} \Big(\int_{T_0}^{\infty} \int_0^{\infty} (\partial_t u(t, r))^2 \, r \, \mathrm{d}r \, \mathrm{d}t \Big)^{\frac{1}{2}} \leq \epsilon \tag{4.17}$$

Next, choose $T_1 \geq T_0$ so that

$$E(u(T_0); \alpha\sqrt{T}/4, \infty) \leq \epsilon \tag{4.18}$$

for all $T \geq T_1$. Fixing any such T, we set

$$\phi(t,r) = \phi_T(r) = 1 - \chi(4r/\alpha\sqrt{T}) \text{ for } t \in [T_0, T]$$

where $\chi(r)$ is a smooth function on $(0,\infty)$ such that $\chi(r) = 1$ for $r \leq 1$, $\chi(r) = 0$ if $r \geq 4$, and $\left|\chi'(r)\right| \leq 1$ for all $r \in (0,\infty)$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,r) = 0$ for $t \in [T_0, T]$ it follows from (2.6) that,

$$\int_0^\infty \mathbf{e}(u(T,r))\,\phi_T(r)^2\, r\,\mathrm{d}r \leq \int_0^\infty \mathbf{e}(u(T_0,r))\,\phi_T(r)^2\, r\,\mathrm{d}r + \frac{4\sqrt{E(u(0))}}{\alpha}\left(\int_{T_0}^T \int_0^\infty (\partial_t u(t,r))^2\, r\,\mathrm{d}r\,\mathrm{d}t\right)^{\frac{1}{2}}$$

Using the above together with (4.17) and (4.18) we find that

$$E(u(T); \alpha\sqrt{T}, \infty) \leq \epsilon.$$

for all $T \geq T_1$, completing the proof of (4.13). It follows from (4.13) that there exists $T_0 > 0$ and a function $\rho : [T_0,\infty) \to (0,\infty)$ with $\rho(t) \ll \sqrt{t}$ and $\lim_{t\to\infty} E(u(t); \rho(t), \infty) = 0$. Thus, (4.14) is a consequence of Lemma 2.3.

Returning to the sequential decomposition we see from (4.16), the fact that $\lambda_{n,N} \lesssim t_n^{\frac{1}{2}}$, and from (4.13) that we must have

$$\lim_{n\to\infty} \frac{\lambda_{n,N}}{t_n^{\frac{1}{2}}} = 0.$$

Then, (4.15) follows from the above, (4.14) and (4.16). Moreover we see that $\lim_{n\to\infty} E(u(t_n)) = NE(Q)$ and the continuous limit (4.12) then follows from the fact that $E(u(t))$ is non-increasing. □

5 Decomposition of the solution and collision intervals

For the remainder of the paper we fix a solution $u(t) \in \mathcal{E}_{\ell,m}$ of (1.2), defined on the time interval $I_* = [0, T_*)$ where $T_* := T_+ < \infty$ in the finite time blow-up case and $T_* = \infty$ in the global case. Let $u^* \in \mathcal{E}_{0,m_\infty}$ be the body map as defined in Proposition 4.1 and in the case of a global solution we adopt the convention that $u^* = 0$. Note that $m_\infty = 0$ if $T_* = \infty$. We let m_Δ be as in Proposition 4.1 so that $u(t) \sim m_\Delta\pi + u^*$ in the region $r \gtrsim (T_+ - t)^{\frac{1}{2}}$. To unify notation, we adopt the convention that $m_\Delta = m$ in the case of a global solution, so that we may again view $u(t) \sim m_\Delta\pi + u^*$ in the region $r \gtrsim \sqrt{t}$. By Propositions 4.1 and 4.3 there exists an integer $N \geq 0$ and a sequence of times $t_n \to T_*$ so that $u(t_n) - u^*$ approaches an N-bubble as $n \to \infty$.

We define a localized distance to an N-bubble.

Definition 5.1 (*Proximity to a multi-bubble*) For all $t \in I$, $\rho \in (0,\infty)$, and $K \in \{0, 1, \ldots, N\}$, we define the *localized multi-bubble proximity function* as

$$\mathbf{d}_K(t;\rho) := \inf_{\vec{\iota},\vec{\lambda}} \left(\|u(t) - u^* - \mathcal{Q}(m_\Delta, \vec{\iota}, \vec{\lambda})\|^2_{\mathcal{E}(\rho,\infty)} + \sum_{j=K}^{N} \left(\frac{\lambda_j}{\lambda_{j+1}}\right)^k \right)^{\frac{1}{2}},$$

where $\vec{\iota} := (\iota_{K+1}, \ldots, \iota_N) \in \{-1, 1\}^{N-K}$, $\vec{\lambda} := (\lambda_{K+1}, \ldots, \lambda_N) \in (0, \infty)^{N-K}$, $\lambda_K := \rho$ and $\lambda_{N+1} := \sqrt{T_+ - t}$ in the finite time blow-up case and $\lambda_{N+1} := \sqrt{t}$ in the case of a global solution.

The *multi-bubble proximity function* is defined by $\mathbf{d}(t) := \mathbf{d}_0(t; 0)$.

Remark 5.2 We emphasize that if $\mathbf{d}_K(t; \rho)$ is small, this means that $u(t) - u^*$ is close to $N - K$ bubbles in the exterior region $r \in (\rho, \infty)$.

We can now rephrase a consequence of Propositions 4.1 and 4.3 in this notation: there exists a monotone sequence $t_n \to T_*$ such that

$$\lim_{n \to \infty} \mathbf{d}(t_n) = 0. \tag{5.1}$$

We state and prove some simple consequences of the set-up above. We always assume $N \geq 1$, since Theorem 1 in the case $N = 0$ is immediate from (4.12).

A direct consequence of (4.14) is that $u(t)$ always approaches a 0-bubble in some exterior region. With $\rho_N(t) = \rho(t)$ given by the function in Proposition 4.1 or 4.3 the following lemma is immediate from the conventions of Definition 5.1.

Lemma 5.3 *There exists* $T_0 > 0$ *and function* $\rho_N : [T_0, T_*) \to (0, \infty)$ *such that*

$$\lim_{t \to T_*} \mathbf{d}_N(t; \rho_N(t)) = 0. \tag{5.2}$$

5.1 Collision intervals

Theorem 1 will follow from showing that,

$$\lim_{t \to T_*} \mathbf{d}(t) = 0. \tag{5.3}$$

The approach which we adopt in order to prove (5.3) is to study colliding bubbles. A collision is defined as follows.

Definition 5.4 (*Collision interval*) Let $K \in \{0, 1, \ldots, N\}$. A compact time interval $[a, b] \subset I_*$ is a *collision interval* with parameters $0 < \epsilon < \eta$ and $N - K$ exterior bubbles if

- $\mathbf{d}(a) \leq \epsilon$ and $\mathbf{d}(b) \geq \eta$,
- there exists a function $\rho_K : [a, b] \to (0, \infty)$ such that $\mathbf{d}_K(t; \rho_K(t)) \leq \epsilon$ for all $t \in [a, b]$.

In this case, we write $[a, b] \in \mathcal{C}_K(\epsilon, \eta)$.

Definition 5.5 (*Choice of* K) We define K as the *smallest* nonnegative integer having the following property. There exist $\eta > 0$, a decreasing sequence $\epsilon_n \to 0$, and sequences $(a_n), (b_n)$ such that $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$ for all $n \in \{1, 2, \ldots\}$.

Lemma 5.6 (Existence of $K \geq 1$) *If* (5.3) *is false, then* K *is well defined and* $K \in \{1, \ldots, N\}$.

Remark 5.7 The fact that $K \geq 1$ means that at least one bubble must lose its shape if (5.3) is false.

Proof of Lemma 5.6 Assume (5.3) does not hold, so that there exist $\eta > 0$ and a monotone sequence $b_n \to T_*$ such that

$$\mathbf{d}(b_n) \geq \eta, \qquad \text{for all } n.$$

We claim that there exist sequences (ϵ_n), (a_n) such that $[a_n, b_n] \in \mathcal{C}_N(\epsilon_n, \eta)$. Indeed, (5.1) implies that there exist $\epsilon_n \to 0$ and $a_n \le b_n$ such that $\mathbf{d}(a_n) \le \epsilon_n$. Note that $a_n \to T_*$ and $b_n \to T_*$. Let $\rho_N : [a_n, b_n] \to (0, \infty)$ be the function given by Lemma 5.3, restricted to the time interval $[a_n, b_n]$. Then (5.2) yields

$$\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} \mathbf{d}_N(t; \rho_N(t)) = 0.$$

Upon adjusting the sequence ϵ_n, we obtain that all the requirements of Definition 5.4 are satisfied for $K = N$.

We now prove that $K \ge 1$. Suppose $K = 0$. By Definition 5.4 of a collision interval, there exist $\eta > 0$, and sequences $a_n, b_n \to T_*$ and $\rho_0(b_n) \ge 0$ such that $\mathbf{d}_0(b_n; \rho_0(b_n)) \le \epsilon_n$ and at the same time $\mathbf{d}(b_n) \ge \eta$. We show that this is impossible.

Define $v_n := u(b_n) - u^*$. Since $\mathbf{d}_0(b_n; \rho_0(b_n)) \le \epsilon_n$ we can find parameters, $\rho_0(b_n) \ll \lambda_{n,1} \ll \cdots \ll \lambda_{n,N}$ and signs $\vec{\iota}_n$ such that defining $g_n = v_n - \mathcal{Q}(m_\Delta, \vec{\iota}_n, \vec{\lambda}_n)$ we have

$$\mathbf{d}_0(c_n; \rho_0(b_n)) \simeq \|g_n\|^2_{\mathcal{E}(\rho_0(b_n),\infty)} + \sum_{j=0}^{N} \Big(\frac{\lambda_{n,j}}{\lambda_{n,j+1}}\Big)^k \lesssim \epsilon_n^2. \tag{5.4}$$

If $T_* < \infty$, with $\rho(t)$ as in (4.4) we see that we must have $\lambda_{n,N} \ll \rho(b_n) \ll (T_* - b_n)^{\frac{1}{2}}$, and thus using (4.4) along with (5.4) and Lemma 2.10 we have

$$\begin{aligned} E(u(b_n); \rho_0(b_n), \infty) &= E(g_n + u^* + \mathcal{Q}(m_\Delta, \vec{\iota}_n, \vec{\lambda}_n); \rho_0(b_n), \rho(b_n)) \\ &\quad + E(g_n + u^* + \mathcal{Q}(m_\Delta, \vec{\iota}_n, \vec{\lambda}_n); \rho(b_n), \infty) \\ &= NE(Q) + E(u^*) + o_n(1). \end{aligned}$$

A similar argument in the case $T_* = \infty$ shows that

$$E(u(b_n); \rho_0(b_n), \infty) = NE(Q) + o_n(1).$$

Since by (4.1) and (4.12) we know that $\lim_{n\to\infty} E(u(b_n)) = NE(Q) + E(u^*)$, we conclude from the previous line that,

$$E(u(b_n); 0, \rho_0(b_n)) = o_n(1) \ \text{ as } \ n \to \infty.$$

Using the fact that $\rho_0(b_n) \ll \rho(b_n)$ it follows that $E(v_n; 0, \rho_0(b_n)) = o_n(1)$, and hence by (2.3) we conclude that

$$\|v_n - \ell\pi\|_{\mathcal{E}(0,\rho_0(b_n))} \lesssim E(v_n; 0, \rho_0(b_n)) = o_n(1) \ \text{ as } \ n \to \infty$$

Thus, combining the above with (5.4) we have $\mathbf{d}(b_n) = o_n(1)$ as $n \to \infty$, a contradiction. □

Remark 5.8 For each collision interval we may assume without loss of generality that $\mathbf{d}(a_n) = \epsilon_n$, $\mathbf{d}(b_n) = \eta$, and $\mathbf{d}(t) \in [\epsilon_n, \eta]$ for each $t \in [a_n, b_n]$. Indeed, given some initial choice of $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$, just set $a_n \le \widetilde{a}_n := \sup\{t \in [a_n, b_n] \mid \mathbf{d}(t) \le \epsilon_n\}$ and $\widetilde{b}_n := \inf\{t \in [\widetilde{a}_n, b_n] \mid \mathbf{d}(t) \ge \eta\}$.

Similarly, given some initial choice $\epsilon_n \to 0$, $\eta > 0$ and intervals $[a_n, b_n] \in \mathcal{C}_K(\eta, \epsilon_n)$ we are free to "enlarge" ϵ_n or "shrink" $\eta > 0$, by choosing some other sequence $\epsilon_n \le \widetilde{\epsilon}_n \to 0$, and $0 < \widetilde{\eta} \le \eta$, and new collision subintervals $[\widetilde{a}_n, \widetilde{b}_n] \subset [a_n, b_n] \cap \mathcal{C}_K(\widetilde{\eta}, \widetilde{\epsilon}_n)$ as in the previous paragraph. We will enlarge our initial choice of ϵ_n and shrink η in this fashion over the course of the proof.

5.2 Decomposition of the solution

Lemma 5.9 (Basic modulation) *Let $K \geq 1$ be the number given by Lemma 5.6. There exist $\eta > 0$, a sequence $\epsilon_n \to 0$, and sequences $a_n, b_n \to \infty$ satisfying the requirements of Definition 5.5, and such that $\mathbf{d}(a_n) = \epsilon_n$, $\mathbf{d}(b_n) = \eta$ and $\mathbf{d}(t) \in [\epsilon_n, \eta]$ for all $t \in [a_n, b_n]$ and so that the following properties hold. There exist signs $\vec{\iota} \in \{-1, 1\}^N$, a function $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \in C^1(\cup_{n\in\mathbb{N}}[a_n, b_n]; (0, \infty)^N)$, sequences $\alpha_n \to 0$ and $\nu_n \to 0$, such that defining the functions,*

$$\nu : \cup_{n\in\mathbb{N}}[a_n, b_n] \to (0, \infty), \quad \nu(t) := \nu_n \lambda_{K+1}(t), \ \textit{for} \quad t \in [a_n, b_n],$$

$$\alpha \ \cup_{n\in\mathbb{N}} [a_n, b_n] \to (0, \infty), \quad \alpha(t) := \begin{cases} \alpha_n \sqrt{T_+ - t_n} \ \textit{if} \ T_+ < \infty \\ \alpha_n \sqrt{t} \ \textit{if} \ T_+ = \infty \end{cases}, \ \textit{for} \quad t \in [a_n, b_n],$$

$$u^*(t) := \begin{cases} (1 - \chi_{\alpha(t)})\big(u(t) - m_\Delta \pi\big) \ \textit{if} \ T_+ < \infty \\ 0 \ \textit{if} \ T_+ = \infty \end{cases}$$

and

$$g : \cup_{n\in\mathbb{N}}[a_n, b_n] \to \mathcal{E}; \quad g(t) := u(t) - u^*(t) - \mathcal{Q}(m_\Delta, \vec{\iota}, \vec{\lambda}(t)),$$

there hold,

(i) the orthogonality conditions,

$$0 = \big\langle \mathcal{Z}_{\underline{\lambda(t)}} \mid g(t) \big\rangle, \quad \forall\, t \in [a_n, b_n], \quad \forall n; \tag{5.5}$$

(ii) and the estimates,

$$\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} \left(\frac{\nu(t)}{\lambda_{K+1}(t)} + \sum_{j=K+1}^{N-1} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} + \frac{\lambda_N(t)}{\alpha(t)} + E(u(t); \frac{1}{4}\nu(t), 4\nu(t)) \right) = 0, \tag{5.6}$$

$$C_0^{-1}\mathbf{d}(t) \leq \|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N-1} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\right)^{\frac{k}{2}} \leq C_0 \mathbf{d}(t), \tag{5.7}$$

$$\|g(t)\|_{\mathcal{E}} + \sum_{j\notin\mathcal{A}} \left(\frac{\lambda_j}{\lambda_{j+1}}\right)^{\frac{k}{2}} \leq C_0 \sum_{j\in\mathcal{A}} \left(\frac{\lambda_j}{\lambda_{j+1}}\right)^{\frac{k}{2}} \tag{5.8}$$

$$\left|\lambda_j'(t)\right| \leq C_0 \frac{1}{\lambda_j(t)} \mathbf{d}(t), \tag{5.9}$$

for all $t \in [a_n, b_n]$ and all $n \in \mathbb{N}$;

(iii) for any sequence $s_n \in [a_n, b_n]$ and any sequence R_n such that $\nu(s_n) \leq R_n \ll \lambda_{K+1}(s_n)$ if $K < N$ and $\nu(s_n) \leq R_n \leq \alpha(s_n)$ if $K = N$, then,

$$\lim_{n\to\infty} E(u(s_n); R_n, \infty) = (N - K)E(Q) + E(u^*). \tag{5.10}$$

and,

$$\lim_{n\to\infty} \Big(\|u(s_n) - u^*(s_n) - \mathcal{Q}(m_\Delta, \iota_{K+1}, \ldots, \iota_N, \lambda_{K+1}(s_n), \ldots, \lambda_N(s_n))\|_{\mathcal{E}(r\geq R_n)} + \sum_{j=K+1}^{N} \left(\frac{\lambda_j(s_n)}{\lambda_{j+1}(s_n)}\right)^{\frac{k}{2}} \Big) = 0. \tag{5.11}$$

Remark 5.10 One should think of $\nu(t)$ as the scale that separates the $N-K$ "exterior" bubbles, which stay coherent on the union of the collision intervals $[a_n, b_n]$ from the K "interior" bubbles that are coherent at the left endpoint $[a_n, b_n]$, but come into collision inside the interval and lose their shape. In the case $K = N$, there are no exterior bubbles, we set $\lambda_{K+1}(t) := \sqrt{T_+ - t}$ and $\nu_n \to 0$ is chosen using (4.4) in the blow up case, and $\lambda_{K+1}(t) := \sqrt{t}$ and $\nu_n \to 0$ is chosen using (4.14) in the global case.

Proof of Lemma 5.9 We carry out the argument in the case $T_+ < \infty$, and note that the global case is similar, and in fact, slightly less involved since $u^* = 0$ in that case. Let $a_n, b_n, \epsilon_n, \eta$, and $K \in \{1, \dots, N\}$ be some initial choice of parameters given by Definition 5.5 and Lemma 5.6. Over the course of the proof we will shrink η and enlarge ϵ_n as in Remark 5.8, but abuse notation by still denoting the resulting subintervals by $[a_n, b_n]$ after these modifications.

We first define the function $\alpha(t)$ and choose the sequence $\nu_n \to 0$. By Defintion 5.1, for each n we can find scales $\rho_K(t) \ll \mu_{K+1}(t) \ll \cdots \ll \mu_N(t) \ll (T_+ - t)^{\frac{1}{2}}$ and signs $\vec{\sigma}(t) \in \{-1, 1\}^{N-K}$ for $t \in [a_n, b_n]$, such that defining $h_{\rho_K}(t)$ for $r \in (\rho_K(t), \infty)$ by

$$u(t) - u^* = \mathcal{Q}(m_\Delta, \vec{\sigma}(t), \vec{\mu}(t)) + h_{\rho_K}(t)$$

we have,

$$\mathbf{d}(t; \rho_K(t)) \simeq \|h_{\rho_K}(t)\|^2_{\mathcal{E}(\rho_K(t),\infty)} + \sum_{j=K}^{N} \left(\frac{\mu_j(t)}{\mu_{j+1}(t)}\right)^k \lesssim \epsilon_n^2, \tag{5.12}$$

keeping the convention $\mu_K(t) := \rho_K(t), \mu_{N+1}(t) := (T_+ - t)^{\frac{1}{2}}$. Using $\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} \mathbf{d}_K(t; \rho_K(t)) = 0$ and the fact that

$$\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} E(\mathcal{Q}(m_\Delta, \vec{\sigma}(t), \vec{\mu}(t)); \nu_{n,1}\widetilde{\mu}_{K+1}(t), \nu_{n,2}\widetilde{\mu}_{K+1}(t)) = 0, \tag{5.13}$$

for any two sequence $\nu_{n,1} \ll \nu_{n,2} \ll 1$, we can choose a sequence $\nu_n \to 0$ such that for any $A > 1$,

$$\rho_K(t) \le \nu_n \mu_{K+1}(t), \quad \text{and} \quad \lim_{n\to\infty} \sup_{t\in[a_n,b_n]} E(u(t) - u^*; \frac{1}{A}\nu_n\mu_{K+1}(t), A\nu_n\mu_{K+1}(t)) = 0. \tag{5.14}$$

Next, letting $\rho(t)$ be as in (4.4), we can use (5.12) to choose $\alpha_n \to 0$ to be a sequence such that,

$$\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} \left(\frac{\mu_N(t)}{\alpha_n(T_+ - t)^{\frac{1}{2}}} + \frac{\rho(t)}{\alpha_n(T_+ - t)^{\frac{1}{2}}}\right) = 0, \tag{5.15}$$

and we define $\alpha(t) := \alpha_n(T_+ - t)^{\frac{1}{2}}$ for $t \in [a_n, b_n]$. If $K = N$ we may assume that $\alpha_n \ge \nu_n$. Setting,

$$u^*(t) := (1 - \chi_{\alpha(t)})\big(u(t) - m_\Delta\pi\big) \tag{5.16}$$

we see from (4.4) and the fact that $\lim_{t\to T_+} E(u^*; \gamma(t)) = 0$ for any $\gamma(t) \to 0$ as $t \to T_+$, that

$$\lim_{t\to T_+} \|u^*(t) - u^*\|_{\mathcal{E}} = 0, \tag{5.17}$$

and by definition,

$$u(t) - u^*(t) = \chi_{\alpha(t)} u(t) + (1 - \chi_{\alpha(t)}) m_\Delta \pi$$

and by (4.3) and (4.2) we have,

$$\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} \left| E(u(t) - u^*(t)) - NE(Q) \right| = 0 \tag{5.18}$$

Now that $u^*(t)$ is defined, we find the parameters $\vec{\iota} \in \{-1, 1\}^N$ and $\vec{\lambda}(t) \in (0, \infty)^N$. By the definition of $\mathbf{d}(t)$ we make an initial choice of signs $\vec{\widetilde{\iota}}(t) \in \{-1, 1\}$ and scales $\vec{\widetilde{\lambda}}(t) \in (0, \infty)^N$ such that defining

$$\widetilde{g}(t) := u(t) - u^* - \mathcal{Q}(m_\Delta, \vec{\widetilde{\iota}}(t), \vec{\widetilde{\lambda}}(t)) \tag{5.19}$$

we have,

$$\mathbf{d}(t) \le \|\widetilde{g}(t)\|_{\mathcal{E}} + \sum_{j=1}^{N} \left(\frac{\widetilde{\lambda}_j(t)}{\widetilde{\lambda}_{j+1}(t)} \right)^{\frac{k}{2}} \le 2\mathbf{d}(t) \le 2\eta \tag{5.20}$$

keeping the convention that $\lambda_{N+1}(t) = (T_+ - t)^{\frac{1}{2}}$.

By (5.17) (5.19), and (5.20) we see that $\mathbf{d}(t) \le \eta$ implies that

$$\mathbf{d}_{m_\Delta,N}(u(t) - u^*(t)) \le C_0 \mathbf{d}(t) + o_n(1) \le 2C_0\eta, \tag{5.21}$$

where $\mathbf{d}_{m_\Delta,N}$ is as in (2.12) and $o_n(1)$ denotes a term that tends to zero as $n \to \infty$. We may then shrink $\eta > 0$ as in Remark 5.8 small enough so that we can apply Lemma 2.12 to $u(t) - u^*(t)$, obtaining $\vec{\lambda}(t) \in (0, \infty)^N$ defined on $\cup_n [a_n, b_n]$, and signs $\vec{\iota} \in \{-1, 1\}^N$ (which can be taken independent of $t \in [a_n, b_n]$ using continuity of the flow and independently of n after passing to a subsequence of the $[a_n, b_n]$), and $g(t)$ so that

$$u(t) - u^*(t) = m_\Delta \pi + \sum_{j=1}^{N} \iota_j (Q_{\lambda_j(t)} - \pi) + g(t), \quad \left\langle \mathcal{Z}_{\underline{\lambda(t)}} \mid g(t) \right\rangle = 0, \quad \forall t \in [a_n, b_n],$$

and,

$$\mathbf{d}_{m_\Delta,N}(u(t) - u^*(t))) \le \|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N-1} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \right)^{\frac{k}{2}} \le C_0 \mathbf{d}_{m_\Delta,N}(u(t) - u^*(t))$$

Using again (5.17) along with (5.21) we see that in fact,

$$\mathbf{d}(t) - \zeta_{1,n} \le \|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \right)^{\frac{k}{2}} \le C_0 \mathbf{d}(t) + \zeta_{1,n}$$

where $\zeta_{1,n}$ is a sequence tending to zero as $n \to \infty$. By enlarging ϵ_n so that $\epsilon_n \ge 2\zeta_{1,n}$ for all n as in Remark 5.8 we prove (5.7) (note here that because of Remark 5.8, the act of "enlarging" ϵ_n does not affect the sequence $\zeta_{1,n}$).

Next, we compare the scales $\lambda_{K+1}, \dots, \lambda_N$ to $\mu_{K+1}, \dots, \mu_N$. Denoting by $\widetilde{\nu}(t) := \nu_n \mu_{K+1}(t)$ we claim that for each $j = 1, \dots, N$,

$$\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} \left(\frac{\widetilde{\nu}(t)}{\lambda_j(t)} + \frac{\lambda_j(t)}{\widetilde{\nu}(t)} \right) = 0. \tag{5.22}$$

If not, we could find $C > 0$, $j \in \{1, \dots, N\}$, a subsequence of the $[a_n, b_n]$ and a sequence $s_n \in [a_n, b_n]$ such that

$$C^{-1}\widetilde{\nu}(s_n) \le \lambda_j(s_n) \le C\widetilde{\nu}(s_n)$$

By (5.7) for all $\eta > 0$ sufficiently small we can find $\delta = \delta(\eta)$, $R = R(\eta) > 0$ so that for all n,

$$\delta \le E(u(s_n) - u^*(s_n); R^{-1}\lambda_j(s_n), R\lambda_j(s_n)) \le E(u(s_n) - u^*(s_n); C^{-1}R^{-1}\widetilde{\nu}(s_n), RC\widetilde{\nu}(s_n))$$

which contradicts (5.14).

By (5.14) and Lemma 2.3 we can find integers m_n so that denoting

$$w(t) = m_n\pi\chi_{\widetilde{\nu}(t)} + (1 - \chi_{\widetilde{\nu}(t)})(u(t) - u^*(t))$$

we have,

$$\|w(t) - \mathcal{Q}(m_\Delta, \vec{\sigma}(t), \vec{\mu}(t))\|_{\mathcal{E}}^2 + \sum_{j=K+1}^{N-1}\left(\frac{\mu_j(t)}{\mu_{j+1}(t)}\right)^k = o_n(1) \tag{5.23}$$

On the other hand, by (5.22) we can find $j_0 \in \{1, \dots, N-1\}$ so that

$$\|w(t) - \mathcal{Q}(m_\Delta, \iota_{j_0}, \dots, \iota_N, \lambda_{j_0}(t), \dots, \lambda_N(t))\|_{\mathcal{E}}^2 + \sum_{j=j_0}^{N-1}\left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\right)^k \le C_0\eta$$

An application of Lemma 2.13 yields $j_0 = K + 1$, $\vec{\sigma}(t) = \{\iota_{K+1}, \dots, \iota_K\}$ and moreover, by shrinking $\eta > 0$, we can ensure that

$$\sup_{t\in[a_n,b_n]}\left|\frac{\lambda_j(t)}{\mu_j(t)} - 1\right| \le \frac{1}{4}$$

and thus, defining $\nu(t) := \nu_n\lambda_{K+1}(t)$ we see that (5.6) follows from (5.12) (5.14), and (5.15). Let $s_n \in [a_n, b_n]$ and R_n so that $\nu(s_n) \le R_n \ll \lambda_{K+1}(s_n)$. If $K < N$ then $R_n \ll \alpha(s_n)$, thus, using (5.23) and (5.15), we see that

$$E(u(s_n); R_n, \alpha(s_n)) \to (N - K)E(Q) \;\text{ as }\; n \to \infty$$

Since by (5.15), (5.16) and (5.17),

$$E(u(s_n); \alpha(s_n), \infty) \to E(u^*) \;\text{ as }\; n \to \infty$$

we see that (5.10) follows. If $K = N$ then $E(u(s_n); R_n, \infty) \to E(u^*)$. Similarly $N - K$ converge now follows from (5.23).

Next we prove (5.8). An application of (2.13) together with (5.18) gives,

$$\|g(t)\|_{\mathcal{E}} + \sum_{j\notin\mathcal{A}}\left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\right)^{\frac{k}{2}} \le C_0\sum_{j\in\mathcal{A}}\left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\right)^{\frac{k}{2}} + \zeta_{2,n}$$

for some sequence $\zeta_{2,n} \to 0$, which is independent of $t \in [a_n, b_n]$. But then by enlarging $\epsilon_n \to 0$ as in Remark 5.8 so that $\epsilon_n \gg \zeta_{2,n}$ we obtain (5.8) via the above and (5.7) (note again here that because of Remark 5.8, the act of "enlarging" ϵ_n does not affect the sequence $\zeta_{2,n}$).

Lastly, we prove the modulation estimate (5.9). Differentiating in time the orthogonality conditions (5.5) yields, for each $j = 1, \dots, N$, the identity,

$$\left\langle \partial_t g \mid \mathcal{Z}_{\underline{\lambda_j}} \right\rangle = \frac{\lambda_j'}{\lambda_j} \left\langle \underline{\mathcal{Z}_{\lambda}} \mid g \right\rangle \tag{5.24}$$

Next, differentiating in time the expression for $g(t)$ in (5.2) and recalling the definition of $u^*(t)$ gives,

$$\begin{aligned}
\partial_t g &= \partial_t \chi_\alpha - \frac{\alpha'}{\alpha} \Lambda \chi_\alpha (u(t) - m_\Delta \pi) + \sum_{j=1}^N \iota_j \lambda_j' \Lambda Q_{\underline{\lambda_j}} \\
&= (\Delta u)\chi_\alpha - \frac{k^2}{r^2} f(u)\chi_\alpha - \frac{\alpha'}{\alpha} \Lambda \chi_\alpha (u(t) - m_\Delta \pi) + \sum_{j=1}^N \iota_j \lambda_j' \Lambda Q_{\underline{\lambda_j}} \\
&= \Delta(\chi_\alpha u + (1-\chi_\alpha) m_\Delta \pi) - \frac{k^2}{r^2} f\big(\chi_\alpha u + (1-\chi_\alpha) m_\Delta \pi\big) + \sum_{j=1}^N \iota_j \lambda_j' \Lambda Q_{\underline{\lambda_j}} \\
&\quad - (u - m_\Delta \pi)\Delta \chi_\alpha - 2\partial_r u \partial_r \chi_\alpha - \frac{\alpha'}{\alpha} \Lambda \chi_\alpha (u(t) - m_\Delta \pi) \\
&\quad - \frac{k^2}{r^2}\Big(f(u)\chi_\alpha - f(\chi_\alpha u + (1-\chi_\alpha) m_\Delta \pi)\Big),
\end{aligned}$$

and we see that

$$\partial_t g = -\mathcal{L}_{\mathcal{Q}} g + \sum_{j=1}^N \iota_j \lambda_j' \Lambda Q_{\underline{\lambda_j}} + f_{\mathbf{i}}(m_\Delta, \vec{\iota}, \vec{\lambda}) + f_{\mathbf{q}}(m_\Delta, \vec{\iota}, \vec{\lambda}, g) + \phi(u, \alpha) \tag{5.25}$$

where

$$\begin{aligned}
\phi(u, \alpha) := &-(u - m_\Delta \pi)\Delta \chi_\alpha - 2\partial_r u \partial_r \chi_\alpha - \frac{\alpha'}{\alpha} \Lambda \chi_\alpha (u(t) - m_\Delta \pi) \\
&- \frac{k^2}{r^2}\Big(f(u)\chi_\alpha - f(\chi_\alpha u + (1-\chi_\alpha) m_\Delta \pi)\Big)
\end{aligned}$$

and

$$\begin{aligned}
f_{\mathbf{i}}(m_\Delta, \vec{\iota}, \vec{\lambda}) &:= -\,\mathrm{D}\, E(\mathcal{Q}(m, \vec{\iota}, \vec{\lambda})) = -\frac{k^2}{r^2}\left(f\big(\mathcal{Q}(m_\Delta, \vec{\iota}, \vec{\lambda})\big) - \sum_{j=1}^N \iota_j f(Q_{\lambda_j}) \right) \\
f_{\mathbf{q}}(m_\Delta, \vec{\iota}, \vec{\lambda}, g) &:= -\frac{k^2}{r^2}\Big(f\big(\mathcal{Q}(m_\Delta, \vec{\iota}, \vec{\lambda}) + g\big) - f\big(\mathcal{Q}(m_\Delta, \vec{\iota}, \vec{\lambda})\big) - f'\big(\mathcal{Q}(m_\Delta, \vec{\iota}, \vec{\lambda})\big) g \Big).
\end{aligned}$$

The subscript **i** above stands for "interaction" and **q** stands for "quadratic."

We make use of the estimates,

$$\| f_{\mathbf{i}}(m_\Delta, \vec{\iota}, \vec{\lambda}) \|_{L^1} \lesssim \sum_{j=1}^{N-1} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^k, \quad \| f_{\mathbf{q}}(m_\Delta, \vec{\iota}, \vec{\lambda}, g) \|_{L^1} \lesssim \| g \|_{\mathcal{E}}^2 \tag{5.26}$$

For the $f_{\mathbf{i}}$ estimate we expand to obtain the expression,

$$
\begin{aligned}
\frac{r^2}{k^2}\,\mathrm{D}\,E(\mathcal{Q}(m,\vec{\iota},\vec{\lambda})) &= \frac{1}{2}\sin\left(2\sum_{i=2}^{M}\iota_i Q_{\lambda_i} + 2\iota_1 Q_{\lambda_1}\right) - \frac{1}{2}\sum_{i=1}^{M}\iota_i \sin 2Q_{\lambda_i} \\
&= -\sin\left(2\sum_{i=2}^{M}\iota_i Q_{\lambda_i}\right)\sin^2 Q_{\lambda_1} - \iota_1\sin^2\left(\sum_{i=2}^{M}\iota_i Q_{\lambda_i}\right)\sin 2Q_{\lambda_1} \\
&\quad + \frac{1}{2}\sin\left(2\sum_{i=2}^{M}\iota_i Q_{\lambda_i}\right) - \frac{1}{2}\sum_{i=2}^{M}\iota_i \sin 2Q_{\lambda_i}
\end{aligned}
$$

Iterating this expansion in the last line above and using the identity $k \sin Q = \Lambda Q$ we obtain the pointwise estimates,

$$
|\,\mathrm{D}\,E(\mathcal{Q}(m,\vec{\iota},\vec{\lambda}))| \lesssim \frac{1}{r^2}\sum_{i,j,\ell \text{ not all equal}} \Lambda Q_{\lambda_i}\Lambda Q_{\lambda_j}\Lambda Q_{\lambda_\ell} \tag{5.27}
$$

from which the estimate for $f_{\mathbf{i}}$ in (5.26) follows by way of Lemma 2.8. The estimate for $f_{\mathbf{q}}$ in (5.26) is straightforward.

For each $j \in \{1,\dots,N\}$ we pair (5.25) with $\mathcal{Z}_{\underline{\lambda_j}}$ and use (5.24) to obtain the following system

$$
\begin{aligned}
&\iota_j\lambda_j'\Big(\langle\Lambda Q \mid \mathcal{Z}\rangle - \frac{\iota_j}{\lambda_j}\langle\mathcal{Z}_{\underline{\lambda_j}} \mid g\rangle\Big) + \sum_{i\neq j}\iota_i\lambda_i'\langle\Lambda Q_{\underline{\lambda_i}} \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle \\
&\quad = \langle\mathcal{L}_{\mathcal{Q}}g \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle - \langle f_{\mathbf{i}}(m_\Delta,\vec{\iota},\vec{\lambda}) \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle - \langle f_{\mathbf{q}}(m_\Delta,\vec{\iota},\vec{\lambda},g) \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle - \langle\phi(u,\alpha) \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle.
\end{aligned}
$$

The above is diagonally dominate for all sufficiently small $\eta > 0$, hence invertible. We note the brutal estimates,

$$
\begin{aligned}
\Big|\langle\mathcal{L}_{\mathcal{Q}}g \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle\Big| &\lesssim \frac{1}{\lambda_j}\|g\|_{\mathcal{E}} \\
\Big|\langle f_{\mathbf{i}}(m_\Delta,\vec{\iota},\vec{\lambda}) \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle\Big| &\lesssim \frac{1}{\lambda_j}\sum_{j=1}^{N-1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^k \\
\Big|\langle f_{\mathbf{q}}(m_\Delta,\vec{\iota},\vec{\lambda},g) \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle\Big| &\lesssim \frac{1}{\lambda_j}\|g\|_{\mathcal{E}}^2 \\
\Big|\langle\phi(u,\alpha) \mid \mathcal{Z}_{\underline{\lambda_j}}\rangle\Big| &= \frac{1}{\lambda_j}o_n(1)
\end{aligned} \tag{5.28}
$$

We remark that to prove the second inequality in (5.28) we may use (5.27) and the definition of $f_{\mathbf{i}}$. The estimates of the remaining estimates are straightforward and we omit the proofs. It follows that,

$$
\Big|\lambda_j'\Big| \lesssim \frac{1}{\lambda_j}\Big(\mathbf{d}(t) + \zeta_{3,n}\Big)
$$

for some sequence $\zeta_{3,n} \to 0$ as $n \to \infty$. Then (5.9) follows by enlarging ϵ_n (note that because of Remark 5.8, the act of "enlarging" ϵ_n does not affect the sequence $\zeta_{3,n}$). This completes the proof. □

6 Conclusion of the proof

For the remainder of the paper, when we write $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$ we we always assume that $\mathbf{d}(a_n) = \epsilon_n$, $\mathbf{d}(b_n) = \eta$ and $\mathbf{d}(t) \in [\epsilon_n, \eta]$ for all $t \in [a_n, b_n]$. This assumption is valid by Remark 5.8.

Lemma 6.1 *If $\eta_0 > 0$ is small enough, then for any $\eta \in (0, \eta_0]$ there exist $\epsilon \in (0, \eta)$ and $C_u > 0$ with the following property. If $[c, d] \subset [a_n, b_n]$, $\mathbf{d}(c) \le \epsilon$ and $\mathbf{d}(d) \ge \eta$, then,*

$$(d - c)^{\frac{1}{2}} \ge C_u^{-1} \lambda_K(c)$$

Proof If not, there exists $\eta > 0$, sequences $\epsilon_n \to 0$, $[c_n, d_n] \subset [a_n, b_n]$, and $C_n \to \infty$ so that $\mathbf{d}(c_n) \le \epsilon_n$, $\mathbf{d}(d_n) \ge \eta$ and

$$(d_n - c_n)^{\frac{1}{2}} \le C_n^{-1} \lambda_K(c_n) \tag{6.1}$$

We show that in this case $[c_n, d_n] \in \mathcal{C}_{K-1}(\epsilon_n, \eta)$, which contradicts the minimality of K.

First, using (5.9) we see for all j,

$$\left|\lambda_j(t)^2 - \lambda_j(c_n)^2\right| \le C_0(t - c_n) \tag{6.2}$$

for all $t \in [c_n, d_n]$. Hence, using the contradiction assumption (6.1) we can ensure that for large enough n,

$$\frac{3}{4} \le \frac{\lambda_j(t)}{\lambda_j(c_n)} \le \frac{5}{4}$$

for all $j = K, \dots, N$ and all $t \in [c_n, d_n]$. Since $\mathbf{d}(c_n) \to 0$, it follows that,

$$\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} \sum_{j=K}^{N} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\right)^k = 0. \tag{6.3}$$

Next, since $\mathbf{d}(c_n) \to 0$ we can find a sequence r_n such that

$$\lambda_{K-1}(c_n) + (d_n - c_n)^{\frac{1}{2}} \ll r_n \ll \lambda_K(c_n) \text{ and } \lim_{n\to\infty} E(u(c_n) - u^*(c_n); \frac{1}{8}r_n, 8r_n) = 0. \tag{6.4}$$

Since $r_n \ll \alpha(t)$ we see that $u(t, r) - u^*(t, r) = \chi_{\alpha(t)} u(t, r) + (1 - \chi_{\alpha(t)}) m_\Delta \pi = u(t, r)$ for all $r \in (1/8 r_n, 8r_n)$. Letting $\phi(r)$ be a smooth bump equal to 1 for $r \in (1/4, 4)$ and supported for $r \in (1/8, 8)$ with $|\phi'(r)| \le 16$, we apply (2.5) with such a ϕ and deduce that for any $t \in [c_n, d_n]$,

$$E(u(t); \frac{1}{4}r_n, 4r_n) \le E(u(c_n); 1/8 r_n, 8r_n) + C_0 \frac{d_n - c_n}{r_n^2}$$

and hence,

$$\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} E(u(t) - u^*(t); \frac{1}{4}r_n, 4r_n) = 0.$$

Next we claim that

$$\sup_{t\in[c_n,d_n]} E(u(t) - u^*(t); \frac{1}{4}r_n, \infty) \le (N - (K - 1))E(Q) + o_n(1) \tag{6.5}$$

In the case $T_+ < \infty$ we recall that $\alpha(t) = \alpha_n(T_+ - t)^{\frac{1}{2}}$ and we write,

$$E(u(t) - u^*(t); \frac{1}{4}r_n, \infty) = E(u(t) - u^*(t); \frac{1}{4}r_n, \frac{1}{4}\alpha(t)) + E(u(t) - u^*(t); \frac{1}{4}\alpha(t), \infty)$$

Since $\alpha(t) \geq \rho(t)$ we have,

$$\lim_{t\to\infty} E(u(t) - u^*(t); \frac{1}{4}\alpha(t), \infty) = 0$$

Recalling that $u(t,r) - u^*(t,r) = u(t,r)$ for all $r \leq \alpha(t)$ we again apply (2.5) with the cut-off function $\phi(t,r) = (1 - \chi_{4r_n}(r))\chi_{\frac{1}{4}\alpha(t)}(r)$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,r) \leq 0$ we use (2.5) to deduce that for all $t \in [c_n, d_n]$,

$$E\left(u(t) - u^*(t); \frac{1}{4}r_n, \frac{1}{4}\alpha(t)\right) \leq E\left(u(c_n) - u^*(c_n); \frac{1}{8}r_n, \frac{1}{2}\alpha(t)\right) + C_0\frac{d_n - c_n}{r_n^2}$$

and the right hand side tends to zero as $n \to \infty$, proving (6.5) in the case $T_+ < \infty$. If $T_+ = \infty$, we use the same argument, but without the need to truncate at $\alpha(t)$ since we have $u^*(t) := 0$.

Next, using (6.2) with $j = K - 1$ gives,

$$\sup_{t\in[c_n,d_n]} \left|\lambda_{K-1}(t)^2 - \lambda_{K-1}(c_n)^2\right| \lesssim d_n - c_n,$$

and hence

$$\sup_{t\in[c_n,d_n]} \frac{\lambda_{K-1}(t)}{r_n} \lesssim \frac{\lambda_{K-1}(c_n)}{r_n} + \frac{(d_n - c_n)^{\frac{1}{2}}}{r_n} \to 0 \text{ as } n \to \infty$$

given our choice of r_n in (6.4). Using all of the above, we can find $m_n \in \mathbb{Z}$ so that defining,

$$v(t) := (1 - \chi_{r_n})(u(t) - u^*(t)) + \chi_{r_n} m_n \pi$$

we have $v(t) \in \mathcal{E}_{m_n, m_\Delta}$ for $t \in [c_n, d_n]$ and such that

$$\|v(t) - \mathcal{Q}(m_\Delta, \iota_K, \dots, \iota_N, \lambda_K(t), \dots \lambda_N(t))\|_{\mathcal{E}} + \sum_{j=K}^{N-1} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\right)^{\frac{k}{2}} \lesssim \eta$$

It follows that $\mathbf{d}_{m_\Delta, N-K+1}(v(t)) \lesssim \eta$ and we can apply Lemma 2.12 to find modulation parameters $\vec{\iota} \in \{-1, 1\}^{N-K+1}$, $\widetilde{\lambda}_K(t), \dots, \widetilde{\lambda}_N(t)$ and $h(t)$ defined by

$$h(t) = v(t) - \mathcal{Q}(m_\Delta, \iota_K, \dots, \iota_N, \widetilde{\lambda}_K(t), \dots, \widetilde{\lambda}_N(t))$$

so that

$$0 = \left\langle \mathcal{Z}_{\underline{\widetilde{\lambda}_j(t)}} \mid h(t) \right\rangle, \quad \text{and} \quad \|h(t)\|_{\mathcal{E}} + \sum_{j=K}^{N-1} \left(\frac{\widetilde{\lambda}_j(t)}{\widetilde{\lambda}_{j+1}(t)}\right)^{\frac{k}{2}} \lesssim \eta$$

In fact, using (6.3) and the fact that the $\widetilde{\lambda}_j(t)$ satisfy $|\widetilde{\lambda}_j(t)/\lambda_j(t) - 1| \lesssim \eta$, we have,

$$\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} \sum_{j=K}^{N-1} \left(\frac{\widetilde{\lambda}_j(t)}{\widetilde{\lambda}_{j+1}(t)}\right)^{\frac{k}{2}} = 0.$$

And, thus, using (2.13) along with (6.5) we have the bound,

$$\|h(t)\|_{\mathcal{E}} \lesssim \sum_{j=K}^{N-1} \Big(\frac{\widetilde{\lambda}_j(t)}{\widetilde{\lambda}_{j+1}(t)}\Big)^{\frac{k}{2}} + o_n(1)$$

and thus $\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} \|h(t)\|_{\mathcal{E}} = 0$ as well. Letting $\rho_{K-1}(t) := r_n$ for $t \in [c_n, d_n]$ we have proved that

$$\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} \mathbf{d}(t; \rho_{K-1}(t)) = 0$$

which means we can find $\widetilde{\eta} > 0$, $\widetilde{\epsilon}_n \to 0$ such that $[c_n, d_n] \in \mathcal{C}_{K-1}(\widetilde{\epsilon}_n, \widetilde{\eta})$ contradicting the minimality of K. □

Lemma 6.2 *Let $\eta_0 > 0$ be as in Lemma 6.1, $\eta \in (0, \eta_0]$, $\epsilon_n \to 0$ be some sequence, and let $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$. Then, there exist $\epsilon \in (0, \eta)$, $n_0 \in \mathbb{N}$, and $c_n, d_n \in (a_n, b_n)$ such that for all $n \geq n_0$, we have*

$$\mathbf{d}(t) \geq \epsilon, \quad \forall\, t \in [c_n, d_n], \tag{6.6}$$

$$d_n - c_n = \frac{1}{n}\lambda_K(c_n)^2, \tag{6.7}$$

and

$$\frac{1}{2}\lambda_K(c_n) \leq \lambda_K(t) \leq 2\lambda_K(c_n) \quad \forall\, t \in [c_n, d_n]. \tag{6.8}$$

Proof Choose $\epsilon > 0$ so that Lemma 6.1 holds and define $c_n := \sup\{t \in [a_n, b_n] \mid \mathbf{d}(t) \leq \epsilon\}$. Then $\mathbf{d}(c_n) = \epsilon$ and by Lemma 6.1 we have

$$b_n - c_n \geq C_u^{-1}\lambda_K(c_n).$$

We then let $d_n := c_n + \frac{1}{n}\lambda_K(c_n)^2$ and for n sufficiently large we have $d_n < b_n$. Then by (5.9) we have,

$$\left|\frac{\lambda_K(t)^2}{\lambda_K(c_n)^2} - 1\right| \lesssim \frac{d_n - c_n}{\lambda_K(c_n)} = \frac{1}{n}.$$

from which (6.8) follows. □

Lemma 6.3 *There exists $\eta_1 > 0$ with the following property. Let $\eta \in (0, \eta_1]$, $\epsilon_n \to 0$ and let $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$. If $\{s_n\}_n$ and $\{r_n\}_n$ are any sequences such that $s_n \in [a_n, b_n]$ for all n, $1 \ll r_n \ll \lambda_{K+1}(s_n)/\lambda_K(s_n)$, and $\lim_{n\to\infty} \boldsymbol{\delta}_{r_n\lambda_K(s_n)}(u(s_n)) = 0$, then $\lim_{n\to\infty} \mathbf{d}(s_n) = 0$.*

Proof Let R_n be a sequence such that $r_n\lambda_K(s_n) \ll R_n \ll \lambda_{K+1}(s_n)$. Without loss of generality, we can assume $\nu(s_n) \leq R_n \leq \alpha(s_n)$, since it suffices to replace R_n by $\nu(s_n)$ for all n such that $R_n < \nu(s_n)$. If $K = N$ we can similarly ensure that $R_n \leq \alpha(s_n)$. Let $M_n, m_n, \vec{\sigma}_n \in \{-1, 1\}^{M_n}, \vec{\mu}_n \in (0, \infty)^{M_n}$ be parameters such that

$$\|u(t_n) - \mathcal{Q}(m_n, \vec{\sigma}_n, \vec{\mu}_n)\|^2_{H(r\leq r_n\lambda_K(s_n))} + \sum_{j=1}^{M_n}\Big(\frac{\mu_{n,j}}{\mu_{n,j+1}}\Big)^k + \frac{\mu_{n,M_n}}{r_n\lambda_K(s_n)} \to 0, \tag{6.9}$$

which exist by the definition of the localized distance function (3.1). Since $\mathbf{d}(t) \leq \eta$ on $[a_n, b_n]$ we can choose $\eta_1 > 0$ sufficiently small so that,

$$\Big(K - \frac{1}{2}\Big)E(Q) \leq \liminf_{n\to\infty} E(u(s_n); 0, r_n\lambda_K(s_n))$$

$$\leq \limsup_{n\to\infty} E(u(s_n); 0, r_n\lambda_K(s_n)) \leq \Big(K + \frac{1}{2}\Big)E(Q),$$

after noting that the radiation u^* is negligible on the region $r \leq r_n\lambda_K(s_n)$. Hence, $M_n = K$ for n large enough. We set $\mu_{n,j} := \lambda_j(s_n)$ and $\sigma_{n,j} := \iota_j$ for $j > K$. We claim that

$$\lim_{n\to\infty} \left(\|u(s_n) - u^* - \mathcal{Q}(m_\Delta, \vec{\sigma}_n, \vec{\mu}_n)\|_{\mathcal{E}}^2 + \sum_{j=1}^{N} \Big(\frac{\mu_{n,j}}{\mu_{n,j+1}}\Big)^k \right) = 0.$$

By the definition of $\mathbf{d}$, the proof will be finished. First, recall that $\mu_{n,K} \ll r_n\mu(t_n)$, so $\mu_{n,K}/\mu_{n,K+1} \to 0$. In the region $r \leq r_n\lambda_K(s_n)$, convergence follows from (6.9), since the energy of the exterior bubbles asymptotically vanishes there. In the region $r \geq R_n$, the energy of the interior bubbles vanishes, hence it suffices to apply (5.11). In particular, by the above and (5.10),

$$\lim_{n\to\infty} E(u(s_n); 0, r_n\lambda_K(s_n)) = KE(Q),$$
$$\lim_{n\to\infty} E(u(s_n); R_n, \infty) = (N-K)E(Q) + E(u^*),$$

which implies

$$\lim_{n\to\infty} E(u(s_n); r_n\lambda_K(s_n), R_n) = 0,$$

and (2.3) yields convergence of the error also in the region $r_n\lambda_K(s_n) \leq r \leq R_n$. □

Proof of Theorem 1 Assume the theorem is false and let $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$ be a sequence of disjoint collision intervals given by Lemma 5.9, and $\eta > 0$ is sufficiently small so that Lemmas 6.1 and 6.3 hold. Let $\epsilon > 0$, n_0, and $[c_n, d_n]$ be as in Lemma 6.2.

We claim that there exists $c_0 > 0$ such that for every $n \geq n_0$,

$$\inf_{t\in[c_n,d_n]} \lambda_K(t)^2\|\partial_t u(t)\|_{L^2}^2 \geq c_0. \tag{6.10}$$

If not, we could, after passing to a subsequence, find a sequence $s_n \in [c_n, d_n]$ such that

$$\lim_{n\to\infty} \lambda_K(s_n)\|\partial_t u(s_n)\|_{L^2} = 0$$

But then an application of Lemma 3.1 gives a sequence $r_n \to \infty$ such that, after passing to a further subsequence, $\lim_{n\to\infty} \boldsymbol{\delta}_{r_n\lambda_K(s_n)}(u(s_n)) = 0$. But then Lemma 6.3 gives that $\lim_{n\to\infty} \mathbf{d}(s_n) = 0$, which contradicts (6.6). Thus (6.10) holds.

Therefore, using (6.10), (6.8), and (6.7) we have

$$\sum_{n\geq n_0} \int_{c_n}^{d_n} \|\partial_t u(t)\|_{L^2}^2\, \mathrm{d}t \geq \frac{c_0}{4}\sum_{n\geq n_0}\int_{c_n}^{d_n} \lambda_K(c_n)^{-2}\,\mathrm{d}t \geq \frac{c_0}{4}\sum_{n\geq n_0} n^{-1} = \infty.$$

On the other hand, by (2.2) and the fact that the $[c_n, d_n]$ are disjoint, we have,

$$\sum_{n\geq n_0}\int_{c_n}^{d_n} \|\partial_t u(t)\|_{L^2}^2\,\mathrm{d}t \leq \int_0^{T_*} \|\partial_t u(t)\|_{L^2}^2\,\mathrm{d}t < \infty,$$

which is a contradiction. □

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