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Lower bounds for eigenvalues of Laplacian operator and the clamped plate problem

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Abstract
We first investigate the lower bound for higher eigenvalues λ_i of the Laplace operator on a bounded domain and obtain a sharp lower bound. Then, we extent this estimate of the eigenvalues to general cases. Finally, we study the eigenvalues Γ_i for the clamped plate problem and deliver a sharp bound for the clamped plate problem for arbitrary dimension.

Mathematics Subject Classification 35P15 · 58G05

1 Introduction

Let Ω be a bounded domain with piecewise smooth boundary $\partial\Omega$ in an n-dimensional Euclidean space $\mathbf{R}^n$. First of all, we focus on the following Dirichlet eigenvalue problem of Laplacian

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

It is well known that the spectrum of eigenvalue problem (1.1) is real and discrete (cf. [2, 6, 12, 15, 21])

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty,$$

where each λ_i has finite multiplicity which is counted by its multiplicity.

Let $V(\Omega)$ be the volume of Ω, and ω_n the volume of the unit ball in R^n. Then the following well-known Weyl's asymptotic formula holds

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \; k \to \infty,$$

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which implies that

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \sim \frac{n}{n+2}\frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}},\ k\to\infty. \tag{1.2}$$

In 1961, Pólya [23] proved that, if $n=2$ and Ω is a *tiling domain* in R^2, then

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}},\quad \text{for } k=1,2,\ldots,$$

Based on the result above, he proposed the famous conjecture:

Conjecture of Pólya. *If Ω is a bounded domain in R^n, then k-th eigenvalue λ_k of the eigenvalue problem* (1.1) *satisfies*

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}},\quad \text{for } k=1,2,\ldots.$$

During the past six decades, many mathematicians have focused on this problem and the related topics, there are a lot of important results on this aspect (cf. [4, 5, 7, 10, 11, 13, 14, 16, 18]) and we suggest that readers refer [25, 29] for more details. In 1983, Li and Yau [17] verified the famous Li-Yau inequality

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}},\ k=1,2,\ldots. \tag{1.3}$$

It's seen from the asymptotic formula (1.2), that Li-Yau's inequality is the best possible in the sense of the average of eigenvalues. From (1.3), one can derive

$$\lambda_k \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}},\quad \text{for } k=1,2,\ldots,$$

which gives a partial solution to the Pólya conjecture with a factor $\frac{n}{n+2}$. This conjecture is still open up to now.

In [20], Melas obtained the following beautiful estimate which improves (1.3) for $n\geq 1$ and $k\geq 1$

$$\sum_{i=1}^{k}\lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{n+2}{n}} + c_n\frac{V(\Omega)}{I(\Omega)}k,\quad \text{for } k=1,2,\ldots, \tag{1.4}$$

where c_n is a positive constant depending only on n and

$$I(\Omega)=\min_{a\in\mathbf{R}^n}\int_{\Omega}|x-a|^2dx$$

is called the moment of *inertia* of Ω. In fact $c_n \leq \frac{1}{24(n+2)}$. Obviously,

$$I(\Omega)\geq \frac{n}{n+2}V(\Omega)\left(\frac{V(\Omega)}{\omega_n}\right)^{\frac{2}{n}}.$$

In the formula (2.27) of [20], Males requires $c\leq \min\{\frac{1}{6}, \frac{(2\pi)^2}{\omega_n^{\frac{4}{n}}}\}$. According to $\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2}\leq\frac{1}{2}$, we get $c\leq\frac{1}{6}$. Putting $c\leq\frac{1}{6}$ into the formula (2.27) of [20], we get $c_n\leq\frac{1}{24(n+2)}$ in (1.4).

Afterwards, Kovařík, Vugalter and Weidl [13] improved this results when $n = 2$. They proved that

$$\sum_{i=1}^{k} \lambda_i \geq \frac{2\pi}{V(\Omega)} k^2 + C(a_0) V(\Omega)^{-\frac{3}{2}} k^{\frac{3}{2}-\varepsilon(k)} + (1 - a_0) \frac{V(\Omega)}{32 I(\Omega)} k, \tag{1.5}$$

where $C(a_0)$ is a positive constant depending on $a_0 \in [0, 1]$ and the length of the smooth parts of $\partial\Omega$, $\varepsilon(k) = \frac{2}{\sqrt{\log_2(\frac{2\pi k}{c})}}$ and $c = \sqrt{\frac{3\pi}{14}} 10^{-11}$.

The first purpose of this paper is to improve Melas's estimate (1.4) by giving a sharper polynomial inequality, see Corollary 2.4. For more general cases, where $n \geq m \geq 2$ and $k \geq 1$, we obtain a lower bound for eigenvalues in Sect. 3, and we should mention that our result gives a sharp lower bounds by comparing Lemma 2.2 with the polynomial inequality in [20]. As a consequence of our result, we prove the Theorem 3.1. An interesting problem is to investigate the similar problem in a Cartan-Hadamard manifold and we recommend readers to refer to [27, 28] for details.

The second purpose of this paper is to estimate eigenvalues of the following clamped plate problem. Let Ω be a bounded domain in R^n. We consider the following clamped plate problem, which describes characteristic vibrations of a clamped plate:

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacian operator and ν denotes the outward unit normal to the boundary $\partial\Omega$. As is known, this problem has a real and discrete spectrum (cf. [1])

$$0 < \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots \to \infty,$$

where each Γ_i has finite multiplicity which is repeated according to its multiplicity.

For the eigenvalues of the clamped plate problem, Agmon [1] and Pleijel [22] gave the following asymptotic formula

$$\Gamma_k \sim \frac{16\pi^2}{(\omega_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \; k \to \infty.$$

This implies that

$$\frac{1}{k} \sum_{i=1}^{k} \Gamma_i \sim \frac{n}{n+4} \frac{16\pi^2}{(\omega_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \; k \to \infty. \tag{1.6}$$

Furthermore, Levine and Protter [16] proved that the eigenvalues of the clamped plate problem satisfy

$$\frac{1}{k} \sum_{i=1}^{k} \Gamma_i \geq \frac{n}{n+4} \frac{16\pi^4}{(\omega_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}.$$

The formula (1.6) shows that the coefficient of $k^{\frac{4}{n}}$ is the best possible in the sense of the average of eigenvalues. Later, Cheng and Wei [8] improved the above estimate as follows:

$$\frac{1}{k} \sum_{i=1}^{k} \Gamma_i \geq \frac{n}{n+4} \frac{16\pi^4}{(\omega_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}$$

$$
+\left(\frac{n+2}{12n(n+4)}-\frac{1}{1152n^2(n+4)}\right)\frac{V(\Omega)}{I(\Omega)}\frac{n}{n+2}\frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}
+\left(\frac{1}{576n(n+4)}-\frac{1}{27648n^2(n+2)(n+4)}\right)\left(\frac{V(\Omega)}{I(\Omega)}\right)^2,
$$

where $n \geq 1$ and $k \geq 1$.

Recently, by using a different method, Cheng and Wei [9] got better lower bounds for eigenvalues of the clamped plate problem and proved that

$$
\frac{1}{k}\sum_{i=1}^{k}\Gamma_i \geq \frac{n}{n+4}\frac{16\pi^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}}+\frac{n+2}{12n(n+4)}\frac{V(\Omega)}{I(\Omega)}\frac{n}{n+2}\frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}}
+\frac{(n+2)^2}{1152n(n+4)^2}\left(\frac{V(\Omega)}{I(\Omega)}\right)^2, \tag{1.7}
$$

where $n \geq 2$ and $k \geq 1$.

Furthermore, they gave upper bounds for the sum of Γ_i,

$$
\frac{1}{k}\sum_{i=1}^{k}\Gamma_i \leq \frac{1+\frac{4(n+4)(n^2+2n+6)}{n+2}\frac{V(\Omega_{r_0})}{V(\Omega)}}{\left(1-\frac{V(\Omega_{r_0})}{V(\Omega)}\right)^{\frac{n+4}{n}}}\frac{n}{n+4}\frac{16\pi^4}{(\omega_n V(\Omega))^{\frac{4}{n}}}k^{\frac{4}{n}},
$$

where $k \geq V(\Omega)r_0^n$, and

$$
\Omega_r = \left\{x \in \Omega \,\middle|\, \mathrm{dist}(x, \partial\Omega) < \frac{1}{r}\right\}.
$$

In [30], Yildirim and Yolcu improved Cheng and Wei's estimates by replacing the last term in the right hand of (1.7) by a positive term of $k^{\frac{1}{n}}$. For any bounded open set $\Omega \subseteq R^n$, where $n \geq 2$ and $k \geq 1$, Yildirim and Yolcu got the following inequality

$$
\sum_{i=1}^{k}\Gamma_i \geq \frac{n}{n+4}(\omega_n)^{-\frac{4}{n}}\alpha^{-\frac{4}{n}}k^{\frac{4+n}{n}}+\frac{1}{3(n+4)}\frac{(\omega_n)^{-\frac{2}{n}}\alpha^{\frac{2n-2}{n}}k^{\frac{n+2}{n}}}{\rho^2}
+\frac{2}{9(n+4)}\frac{(\omega_n)^{-\frac{1}{n}}\alpha^{\frac{3n-1}{n}}k^{\frac{n+1}{n}}}{\rho^3}, \tag{1.8}
$$

where

$$
\alpha = \frac{V(\Omega)}{(2\pi)^n}, \quad \rho = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}. \tag{1.9}
$$

In Sect. 4, we will improve Yildirim and Yolcu's [30] estimate (1.8) by giving a shaper polynomial inequality when $n \geq 3$, see Corollary 4.4.

2 Lower bounds for sums of Dirichlet eigenvalues

In this section we prove the following theorem.

Theorem 2.1 *For any bounded domain $\Omega \subseteq R^n$, $n \geq 2$ we have*

$$\sum_{j=1}^{k} \lambda_j(\Omega) \geq \omega_n^{-\frac{2}{n}} \alpha^{-\frac{2}{n}} k^{\frac{n+2}{n}} - \frac{s_3^3 \alpha^2}{(n+2)\rho^2} k + c_1 \omega_n^{\frac{1}{n}} \frac{s_4^4 \alpha^{\frac{3n+1}{n}} k^{\frac{n-1}{n}}}{(n+2)\rho^3},$$

where

$$c_1 \leq \min\left\{1, \max\left\{\frac{4\sqrt{2} n s_3^3 k^{\frac{1}{n}}}{(3n+1)s_4^4}, \frac{4\sqrt{2}(n+2)k^{\frac{3}{n}}}{(3n+1)s_4^4}\right\}\right\},$$
$$s_l^l = (a+1)^l - a^l,$$

α, ρ are defined by (1.9) and a is defined by (2.16).

Firstly, we introduce some notations and definitions. For a bounded domain Ω, *the moment of inertia* of Ω is defined by

$$I(\Omega) = \min_{a \in R^n} \int_\Omega |x-a|^2 dx.$$

By a translation of the origin and a suitable rotation of axes, we can assume that the center of mass is the origin and

$$I(\Omega) = \int_\Omega |x|^2 dx.$$

We now fix a $k \geq 1$ and let $u_1, \ldots, u_k$ denote an orthonormal set of eigenfunctions of (1.1) corresponding to the set of eigenvalues $\lambda_1(\Omega), \ldots, \lambda_k(\Omega)$. We consider the Fourier transform of each eigenfunction

$$f_j(\xi) = \hat{u}_j(\xi) = (2\pi)^{-n/2} \int_\Omega u_j(x) e^{ix\xi} dx.$$

It seems from Plancherel's Theorem that $f_1, \ldots., f_k$ is an orthonormal set in R^n. Since these eigenfunctions $u_1, \ldots, u_k$ are also orthonormal in $L_2(\Omega)$, Bessel's inequality implies that for every $\xi \in R^n$

$$\sum_{j=1}^{k} |f_j(\xi)|^2 \leq (2\pi)^{-n} \int_\Omega |e^{ix\xi}|^2 dx = (2\pi)^{-n} V(\Omega). \tag{2.1}$$

Since

$$\nabla f_j(\xi) = (2\pi)^{-n/2} \int_\Omega ix u_j(x) e^{ix\xi} dx,$$

we have

$$\sum_{j=1}^{k} |\nabla f_j(\xi)|^2 \leq (2\pi)^{-n/2} \int_\Omega |ixe^{ix\xi}|^2 dx = (2\pi)^{-n} I(\Omega).$$

By the boundary condition, we get

$$\int_{R^n} |\xi|^2 |f_j(\xi)|^2 d\xi = \int_\Omega |\nabla u_j(x)|^2 dx = \lambda_j(\Omega)$$

for each $1 \leq j \leq k$. Set

$$F(\xi) = \sum_{j=1}^{k} |f_j(\xi)|^2.$$

From (2.1), we have

$$0 \leq F(\xi) \leq (2\pi)^{-n} V(\Omega), \tag{2.2}$$

$$|\nabla F(\xi)| \leq 2 \left(\sum_{j=1}^{k} |f_j(\xi)|^2 \right)^{1/2} \left(\sum_{j=1}^{k} |\nabla f_j(\xi)|^2 \right)^{1/2} \leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)} \tag{2.3}$$

for each $\xi \in R^n$. We also get

$$\int_{R^n} F(\xi) d\xi = k, \tag{2.4}$$

$$\int_{R^n} |\xi|^2 F(\xi) d\xi = \sum_{j=1}^{k} \lambda_j(\Omega). \tag{2.5}$$

Assume (by approximating F) that the decreasing function $\phi : [0, +\infty) \to [0, (2\pi)^{-n} V(\Omega)]$ is absolutely continuous. Let $F^*(\xi) = \phi(|\xi|)$ denote the decreasing radial rearrangement of F. Put $\mu(t) = |\{F^* > t\}| = |\{F > t\}|$. It follows from the coarea formula that

$$\mu(t) = \int_t^{(2\pi)^{-n} V(\Omega)} \int_{\{F=s\}} \frac{1}{|\nabla F|} d\sigma_s ds.$$

Since F^* is radial, we have $\mu(\phi(s)) = |\{F^* > \phi(s)\}| = \omega_n s^n$. Differentiating both side of the above equality, we have $n\omega_n s^{n-1} = \mu'(\phi(s))\phi'(s)$ for almost all s. This together with (2.3), $\rho = 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)}$ and the isoperimetric inequality implies

$$\begin{aligned} -\mu'(\phi(s)) &= \int_{\{F=\phi(s)\}} |\nabla F|^{-1} d\sigma_{\phi(s)} \\ &\geq \rho^{-1} \mathrm{Vol}_{n-1}(\{F = \phi(s)\}) \\ &\geq \rho^{-1} n\omega_n s^{n-1}. \end{aligned}$$

For almost all s, we have

$$-\rho \leq \phi'(s) \leq 0. \tag{2.6}$$

Since the map $\xi \mapsto |\xi|^2$ is radial and increasing, applying (2.5), we get

$$k = \int_{R^n} F(\xi) d\xi = \int_{R^n} F^*(\xi) d\xi = n\omega_n \int_0^\infty s^{n-1} \phi(s) ds \tag{2.7}$$

and

$$\sum_{j=1}^{k} \lambda_j(\Omega) = \int_{R^n} |\xi|^2 F(\xi) d\xi \geq \int_{R^n} |\xi|^2 F^*(\xi) d\xi = n\omega_n \int_0^\infty s^{n+1} \phi(s) ds. \tag{2.8}$$

The following lemma will be used in the proof of Theorem 2.1.

Lemma 2.2 *Let $n \geq 2$, $\rho > 0$, $A > 0$. If $\psi : [0, +\infty) \to [0, +\infty)$ is a decreasing function (and absolutely continuous) satisfying*

$$-\rho \leq -\psi'(s) \leq 0 \tag{2.9}$$

and

$$\int_0^\infty s^{n-1}\psi(s)ds = A.$$

Then

$$\int_0^\infty s^{n+1}\psi(s)ds \geq \frac{(nA)^{\frac{n+2}{n}}\psi(0)^{-\frac{2}{n}}}{n} - \frac{s_3^3(nA)\psi(0)^2}{n(n+2)\rho^2} + \frac{s_4^4(nA)^{\frac{n-1}{n}}\psi(0)^{\frac{3n+1}{n}}}{n(n+2)\rho^3},$$

where

$$s_l^l = (a+1)^l - a^l \geq 1.$$

Proof We choose the function $\alpha\psi(\beta t)$ for appropriate $\alpha, \beta > 0$, such that $\rho = 1$ and $\psi(0) = 1$. By [20] we can also assume that $B = \int_0^\infty s^{n+1}\psi(s)ds < \infty$. If we let $q(s) = -\psi'(s)$ for $s \geq 0$, we have $0 \leq q(s) \leq 1$ and $\int_0^\infty q(s) = \psi(0) = 1$. Moreover, integration by parts implies that

$$\int_0^\infty s^n q(s)ds = n\int_0^\infty s^{n-1}\psi(s)ds = nA$$

and

$$\int_0^\infty s^{n+2}q(s)ds \leq (n+2)B.$$

Next, let $0 \leq a < +\infty$ satisfies that

$$\int_a^{a+1} s^n ds = \int_0^\infty s^n q(s)ds = nA. \tag{2.10}$$

By the same argument as in Lemma 1 of [17], such real number a exists. From [20], we have

$$(n+2)B \geq \int_0^\infty s^{n+2}q(s)ds \geq \int_a^{a+1} s^{n+2}ds. \tag{2.11}$$

To estimate the last integral we take $\tau > 0$ to be chosen later. Applying (2.11) and integrating the both sides of the following inequality

$$ns^{n+2} - (n+2)\tau^2 s^n + 2\tau^{n+2} \geq 2\tau^n(s-\tau)^2 + 4s\tau^{n-1}(s-\tau)^2, \ s \in [a, a+1], \tag{2.12}$$

we get

$$\begin{aligned}
&n(n+2)B-(n+2)\tau^2 nA+2\tau^{n+2}\\
&\geq 2\tau^n\int_a^{a+1}(s-\tau)^2+4\tau^{n-1}\int_a^{a+1}s(s-\tau)^2ds\\
&\geq 2\tau^n\left(\frac{s^3}{3}-s^2\tau+s\tau^2\right)\Big|_a^{a+1}\\
&\quad+4\tau^{n-1}\left(\frac{s^4}{4}-\frac{2s^3\tau}{3}+\frac{s^2\tau^2}{2}\right)\Big|_a^{a+1}\\
&=2s\tau^{n+2}+2s^2\tau^{n+1}-2s^3\tau^n-2s^2\tau^{n+1}+s^4\tau^{n-1}\Big|_a^{a+1}\\
&=2\tau^{n+2}-2s_3^3\tau^n+s_4^4\tau^{n-1},
\end{aligned} \tag{2.13}$$

where

$$s_l^l=(a+1)^l-a^l\geq 1.$$

Putting, $\tau=(nA)^{1/n}$ we get

$$B\geq\frac{1}{n}(nA)^{\frac{n+2}{n}}-\frac{s_3^3}{n(n+2)}(nA)+\frac{s_4^4}{n(n+2)}(nA)^{\frac{n-1}{n}}.$$

This proves Lemma 2.2.

To prove (2.12), we need to show that for any $\tau>0$ we have

$$ns^{n+2}-(n+2)\tau^2 s^n+2\tau^{n+2}-2\tau^n(s-\tau)^2-4s\tau^{n-1}(\tau-s)^2\geq 0. \tag{2.14}$$

Taking $t=\frac{s}{\tau}$, we define $f(t)$ (for $t>0$) by

$$f(t)=nt^{n+2}-(n+2)t^n+2-2(t-1)^2-4t(t-1)^2.$$

Differentiating, $f(t)$ we have

$$\begin{aligned}
f'(t)&=n(n+2)t^{n+1}-(n+2)nt^{n-1}-4(t-1)-4(t-1)^2-8t(t-1)\\
&=\left[n(n+2)t^{n-2}(t+1)-12\right]t(t-1).
\end{aligned}$$

It follows from the above formula that if $n\geq 2$, then $t=1$ is the minimum point of f and $f\geq\min\{f(1)=0,\ f(0)=0\}$. This implies

$$f(t)\tau^{n+2}=ns^{n+2}-(n+2)\tau^2 s^n+2\tau^{n+2}-2\tau^n(s-\tau)^2-4s\tau^{n-1}(\tau-s)^2\geq 0.$$

□

Next we will give the proof of Theorem 2.1.

Proof of Theorem 2.1 Applying Lemma 2.2 to the function ϕ with $A=(n\omega_n)^{-1}k$, $\rho=2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$ and submitting it to (2.8), we obtain

$$\begin{aligned}
\sum_{j=1}^k\lambda_j(\Omega)\geq&\omega_n^{-\frac{2}{n}}\psi(0)^{-\frac{2}{n}}k^{\frac{n+2}{n}}-\frac{s_3^3\psi(0)^2}{(n+2)\rho^2}k\\
&+c_1\omega_n^{\frac{1}{n}}\frac{s_4^4\psi(0)^{\frac{3n+1}{n}}k^{\frac{n-1}{n}}}{(n+2)\rho^3},
\end{aligned} \tag{2.15}$$

where $0 < c_1 \leq 1$ is a constant and a is defined by

$$\int_a^{a+1} \xi^n d\xi = \int_0^\infty -\xi^n \phi'(\xi) d\xi. \tag{2.16}$$

We observe the following facts

(i) $0 < \psi(0) \leq (2\pi)^{-n} V(\Omega)$,
(ii) if R is a positive constant such that $\omega_n R^n = V(\Omega)$, then

$$I(\Omega) \geq \int_{B(R)} |x|^2 dx = \frac{n\omega_n R^{n+2}}{n+2}. \tag{2.17}$$

It follows from the above properties

$$\rho \geq (2\pi)^{-n} \omega_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}. \tag{2.18}$$

On the other hand, we consider the following function

$$g(t) = g_1(t) + g_2(t),$$

for $t \in (0, (2\pi)^{-n} V(\Omega)]$, where

$$g_1(t) = \omega_n^{-\frac{2}{n}} t^{-\frac{2}{n}} k^{\frac{n+2}{n}}$$

and

$$g_2(t) = -\frac{s_3^3 t^2}{(n+2)\rho^2} k + c_1 \omega_n^{\frac{1}{n}} \frac{s_4^4 t^{\frac{3n+1}{n}} k^{\frac{n-1}{n}}}{(n+2)\rho^3}.$$

Then we have

$$(n+2)\rho^2 g_2'(t) = -2s_3^3 kt + c_1 \omega_n^{-\frac{1}{n}} \frac{s_4^4 k^{\frac{n-1}{n}}}{\rho} \frac{3n+1}{n} t^{\frac{2n+1}{n}}.$$

By a direct calculation, we see from $\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$ that

$$\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2} \leq \frac{1}{2},$$

where $\Gamma(t)$ is the Gamma function.

Therefore, in view of (2.18), if

$$c_1 \leq \min\left\{1, \frac{4\sqrt{2} n s_3^3 k^{\frac{1}{n}}}{(3n+1)s_4^4}\right\},$$

then $g_2(t)$ is decreasing on $(0, (2\pi)^{-n} V(\Omega)]$. Now we consider another estimate. Setting

$$G(t) = G_1(t) + G_2(t),$$

where

$$G_1(t) = \omega_n^{-\frac{2}{n}} \psi(0)^{-\frac{2}{n}} k^{\frac{n+2}{n}} + c_1 \omega_n^{\frac{1}{n}} \frac{s_4^4 \psi(0)^{\frac{3n+1}{n}} k^{\frac{n-1}{n}}}{(n+2)\rho^3}$$

and

$$G_2(t) = -\frac{s_3^3\psi(0)^2}{(n+2)\rho^2}k,$$

we have

$$G_1'(t)\rho^2 = -\frac{2}{n}\omega_n^{-\frac{2}{n}}t^{-\frac{n+2}{n}}k^{\frac{n+2}{n}} + \frac{c_1(3n+1)\omega_n^{-\frac{1}{n}}}{n}\frac{s_4^4 t^{\frac{2n+1}{n}}}{(n+2)\rho^2}k^{\frac{n-1}{n}}.$$

Therefore, we conclude that if

$$c_1 \le \frac{4\sqrt{2}(n+2)k^{\frac{3}{n}}}{(3n+1)s_4^4},$$

then $G(t)$ is decreasing on $(0, (2\pi)^{-n}V(\Omega)]$. Finally, we obtain

$$\begin{aligned}\sum_{j=1}^{k}\lambda_j(\Omega) \ge& \omega_n^{-\frac{2}{n}}\alpha^{-\frac{2}{n}}k^{\frac{n+2}{n}} - \frac{s_3^3\alpha^2}{(n+2)\rho^2}k \\ &+ c_1\omega_n^{\frac{1}{n}}\frac{s_4^4\alpha^{\frac{3n+1}{n}}k^{\frac{n-1}{n}}}{(n+2)\rho^3},\end{aligned} \tag{2.19}$$

where α, ρ are defined in the (1.9) and

$$c_1 \le \min\left\{1, \max\left\{\frac{4\sqrt{2}ns_3^3k^{\frac{1}{n}}}{(3n+1)s_4^4}, \frac{4\sqrt{2}(n+2)k^{\frac{3}{n}}}{(3n+1)s_4^4}\right\}\right\}.$$

□

Note that $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$. This together with the above lemma implies the following estimate for higher eigenvalues.

Corollary 2.3 *For any bounded domain $\Omega \subseteq R^n$, $n \ge 2$ and any $k \ge 1$ we have*

$$\begin{aligned}\lambda_k(\Omega) \ge& \omega_n^{-\frac{2}{n}}\alpha^{-\frac{2}{n}}k^{\frac{2}{n}} - \frac{s_3^3\alpha^2}{(n+2)\rho^2} \\ &+ c_1\omega_n^{\frac{1}{n}}\frac{s_4^4\alpha^{\frac{3n+1}{n}}k^{\frac{-1}{n}}}{(n+2)\rho^3},\end{aligned}$$

where

$$\begin{aligned}c_1 \le& \min\left\{1, \max\left\{\frac{4\sqrt{2}ns_3^3k^{\frac{1}{n}}}{(3n+1)s_4^4}, \frac{4\sqrt{2}(n+2)k^{\frac{3}{n}}}{(3n+1)s_4^4}\right\}\right\}, \\ s_l^l =& (a+1)^l - a^l,\end{aligned}$$

α, ρ are defined by (1.9).

In fact, if we choose special a in (2.13), we also have the following result.

Corollary 2.4 *For any bounded domain $\Omega \subseteq R^n$, $n \geq 2$ and any $k \geq 1$ we have*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{n\omega_n^{-\frac{2}{n}}(2\pi)^2 V(\Omega)^{-\frac{2}{n}}}{n+2} k^{\frac{n+2}{n}} + \frac{1}{24(n+2)}\left(\frac{V(\Omega)}{I(\Omega)}\right)k + \frac{\omega_n^{-\frac{1}{n}}\alpha^{\frac{3n+1}{n}}}{9(n+2)\rho^3}k^{\frac{n-1}{n}}. \tag{2.20}$$

Proof Combining with the formula (2.25) in [20] and (57) in [30], we have

$$\begin{aligned} n(n+2)B-(n+2)\tau^2 nA+2\tau^{n+2} &\geq 2\tau^n\int_a^{a+1}(s-\tau)^2+4\tau^{n-1}\int_a^{a+1}s(s-\tau)^2 ds \\ &\geq \frac{\tau^n}{6}+\frac{\tau^{n-1}}{9}. \end{aligned}$$

By using similar discussion in the proof of Theorem 2.1, we get

$$\sum_{i=1}^{k}\lambda_i \geq \frac{n\omega_n^{-\frac{2}{n}}\phi(0)^{-\frac{2}{n}}}{n+2}k^{\frac{n+2}{n}} + \frac{C_1\phi(0)^2}{(n+2)\rho^2}k + \frac{C_2\omega_n^{-\frac{1}{n}}\phi(0)^{\frac{3n+1}{n}}}{(n+2)\rho^3}k^{\frac{n-1}{n}}, \tag{2.21}$$

where $0 < C_1 \leq \frac{1}{6}$ and $0 < C_2 \leq \frac{1}{9}$ are two constants which will be determined later. We consider the following function

$$g(t) = \frac{n\omega_n^{-\frac{2}{n}}t^{-\frac{2}{n}}}{n+2}k^{\frac{n+2}{n}} + \frac{C_1 t^2}{(n+2)\rho^2}k + \frac{C_2\omega_n^{-\frac{1}{n}}t^{\frac{3n+1}{n}}}{(n+2)\rho^3}k^{\frac{n-1}{n}},$$

which would be decreasing on $(0, (2\pi)^{-n}V(\Omega)]$ if $g'((2\pi)^{-n}V(\Omega)) \leq 0$. In view of (2.18), the degression of $g(t)$ is equal to the following inequality

$$2k^{\frac{2}{n}} \geq 2C_1\frac{\omega_n^{\frac{2}{n}}}{(2\pi)^2} + C_2\frac{3n+1}{n}\frac{\omega_n^{\frac{2}{n}}}{(2\pi)^3}$$

Since $k \geq 1$ and $\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2} \leq \frac{1}{2}$, we can choose $C_1 = \frac{1}{6}$. Therefore, C_2 satisfies

$$C_2 \leq \min\{\frac{1}{9}, \tilde{C}_2\},$$

where

$$\tilde{C}_2 = \frac{\sqrt{2}(2\pi)^4}{3.5}\left(2-\frac{1}{6}\right).$$

Obviously, $\tilde{C}_2 \geq \frac{1}{9}$. Hence, we complete our proof. □

3 Lower bounds for Dirichlet eigenvalues in higher dimensions

In this section we will give a universal lower bound on the sum of eigenvalues for $n \geq m+1$, where $m \geq 2$.

Theorem 3.1 *For any bounded domain $\Omega \subseteq R^n$, $n \geq m+1 \geq 3$ and $k \geq 1$, we have*

$$\sum_{i=1}^{k} \lambda_i \geq \omega_n^{-\frac{2}{n}} \alpha^{-\frac{2}{n}} k^{\frac{n+2}{n}} - \frac{2\omega_n^{\frac{m-1}{n}} S_{m+2} \alpha^{\frac{(m+1)n+m-1}{n}}}{(n+2)\rho^{m+1}} k^{\frac{n-m+1}{n}} + c_2 \frac{2\omega_n^{\frac{m}{n}} (m+1) S_{m+3} \alpha^{\frac{(m+2)n+m}{n}}}{(n+2)(m+3)\rho^{m+2}} k^{\frac{n-m}{n}},$$

where

$$c_2 \leq \min\left\{1, \frac{(m+1)n+m-1}{(m+2)n+m} \frac{\sqrt{2} S_{m+2}}{S_{m+3}} \frac{m+3}{m+1} k^{\frac{1}{n}}\right\},$$
$$S_l = (a+1)^l - a^l,$$

$\alpha = \frac{V(\Omega)}{(2\pi)^n}$, $\rho = 2(2\pi)^{-n}\sqrt{V(\Omega) I(\Omega)}$ *and a is defined by (2.16).*

The following lemma will be used in the proof of Theorem 3.1.

Lemma 3.2 *For an integer $n \geq m+1 \geq 0$ and positive real numbers s and τ we have the following inequality:*

$$ns^{n+2} - (n+2)\tau^2 s^n + 2\tau^{n+2} - \sum_{k=1}^{m+1} 2ks^{k-1}\tau^{n-k+1}(\tau - s)^2 \geq 0.$$

Proof Setting $t = \frac{s}{\tau}$, and putting

$$f(t) = nt^{n+2} - (n+2)t^n + 2 - \sum_{k=1}^{m+1} 2kt^{k-1}(t-1)^2,$$

for $t \geq 0$, we get

$$\begin{aligned}
f'(t) =& n(n+2)t^{n+1} - n(n+2)t^{n-1} \\
& - \left[4(t-1) + \sum_{k=1}^{m} (2k(k+1)t^{k-1}(t-1)^2 + 4(k+1)t^k(t-1))\right] \\
=& n(n+2)t^{n+1} - n(n+2)t^{n-1} \\
& - (t-1)\left[4 + \sum_{k=1}^{m} [2k(k+1)t^{k-1}(t-1) + 4(k+1)t^k]\right] \\
=& n(n+2)t^{n+1} - n(n+2)t^{n-1} \\
& - (t-1)\left[2(m+2)(m+1)t^m + \sum_{k=1}^{m-1} (2k(k+1)t^k - 2k(k+1)t^{k-1} + 4(k+1)t^k)\right] \\
=& n(n+2)t^{n+1} - n(n+2)t^{n-1} - 2(m+2)(m+1)t^m(t-1) \\
=& t^m(t-1)\left[n(n+2)t^{n-m-1}(t+1) - 2(m+2)(m+1)\right].
\end{aligned} \tag{3.1}$$

It follows from the above formula that if $n \geq m+1$, then $t=1$ is the minimum point of $f(t)$ and $f \geq \min\{f(1)=0, f(0)=0\}$. So, we get

$$\tau^{n+2} f(t) = ns^{n+2} - (n+2)\tau^2 s^n - \sum_{k=1}^{m+1} 2ks^{k-1}\tau^{n-k+1}(\tau - s)^2 \geq 0.$$

□

Next we will give the proof of Theorem 3.1.

Proof of Theorem 3.1 For $l \geq 0$, $\tau \geq \frac{1}{2}$ and $a \geq 0$, we have

$$\begin{aligned}\int_a^{a+1} s^l(\tau - s)^2 ds &= \frac{s^{l+3}}{l+3} - \frac{2s^{l+2}}{l+2}\tau + \frac{s^{l+1}}{l+1}\tau^2 \Big|_a^{a+1} \\ &= \frac{S_{l+3}}{l+3} - \frac{2S_{l+2}}{l+2}\tau + \frac{S_{l+1}}{l+1}\tau^2,\end{aligned} \tag{3.2}$$

where

$$S_j = (a+1)^j - a^j \geq 1.$$

Therefore, we get

$$n(n+2)B - (n+2)\tau^2 nA + 2\tau^{n+2} \geq \sum_{k=1}^{m+1} 2k\tau^{n-k+1}\left(\frac{S_{k+2}}{k+2} - \frac{2S_{k+1}}{k+1}\tau + \frac{S_k}{k}\tau^2\right).$$

From

$$\begin{aligned}
&\sum_{k=1}^{m+1} 2k\tau^{n-k+1}\left(\frac{S_k}{k}\tau^2 - \frac{2S_{k+1}}{k+1}\tau + \frac{S_{k+2}}{k+2}\right) \\
=&2\tau^{n+2} + 2\sum_{k=1}^{m} S_{k+1}\tau^{n-k+2} - 2\sum_{k=1}^{m+1}\frac{2kS_{k+1}}{k+1}\tau^{n-k+2} + 2\sum_{k=1}^{m+1}\frac{kS_{k+2}}{k+2}\tau^{n-k+1} \\
=&2\tau^{n+2} + 2S_2\tau^{n+1} + \frac{2mS_{m+2}}{m+2}\tau^{n-m+1} + \frac{2(m+1)S_{m+3}}{m+3}\tau^{n-m} \\
&- 2S_2\tau^{n+1} - \frac{4(m+1)S_{m+2}}{m+2}\tau^{n-m+1} \\
&+ 2\sum_{k=2}^{m}\left(1 + \frac{k-1}{k+1} - \frac{2k}{k+1}\right)S_{k+1}\tau^{n-k+2} \\
=&2\tau^{n+2} + 2S_2\tau^{n+1} + \frac{2mS_{m+2}}{m+2}\tau^{n-m+1} + \frac{2(m+1)S_{m+3}}{m+3}\tau^{n-m} \\
&- 2S_2\tau^{n+1} - \frac{4(m+1)S_{m+2}}{m+2}\tau^{n-m+1} \\
=&2\tau^{n+2} - 2S_{m+2}\tau^{n-m+1} + \frac{2(m+1)S_{m+3}}{m+3}\tau^{n-m},
\end{aligned}$$

and

$$\sum_{k=2}^{m}\left(1 + \frac{k-1}{k+1} - \frac{2k}{k+1}\right)S_{k+1}\tau^{n-k+2} = 0,$$

we obtain

$$n(n+2)B-(n+2)\tau^2 nA+2\tau^{n+2} \geq 2\tau^{n+2}-2S_{m+2}\tau^{n-m+1} + \frac{2(m+1)S_{m+3}}{m+3}\tau^{n-m}.$$

Choosing $\tau=(nA)^{\frac{1}{n}}$, we get

$$B \geq \frac{(nA)^{\frac{n+2}{n}}}{n} - \frac{2S_{m+2}(nA)^{\frac{n-m+1}{n}}}{n(n+2)} + \frac{2(m+1)S_{m+3}(nA)^{\frac{n-m}{n}}}{n(n+2)(m+3)}. \tag{3.3}$$

It follows from (3.3) that

$$\int_0^\infty s^{n+1}\psi(s)ds \geq \frac{(nA)^{\frac{n+2}{n}}\psi(0)^{-\frac{2}{n}}}{n} - \frac{2S_{m+2}(nA)^{\frac{n-m+1}{n}}\psi(0)^{\frac{(m+1)n+m-1}{n}}}{n(n+2)\rho^{m+1}} + \frac{2(m+1)S_{m+3}(nA)^{\frac{n-m}{n}}\psi(0)^{\frac{(m+2)n+m}{n}}}{n(n+2)(m+3)\rho^{m+2}}. \tag{3.4}$$

From (2.8), we know

$$\begin{aligned}\sum_{i=1}^k \lambda_i \geq & n\omega_n \int_0^\infty s^{n+1}\psi(s)ds \\ \geq & \omega_n (nA)^{\frac{n+2}{n}}\psi(0)^{-\frac{2}{n}} - \frac{2\omega_n S_{m+2}(nA)^{\frac{n-m+1}{n}}\psi(0)^{\frac{(m+1)n+m-1}{n}}}{(n+2)\rho^{m+1}} \\ & + \frac{2\omega_n(m+1)S_{m+3}(nA)^{\frac{n-m}{n}}\psi(0)^{\frac{(m+2)n+m}{n}}}{(n+2)(m+3)\rho^{m+2}}.\end{aligned}$$

In view of $A=\frac{k}{n\omega_n}$, we have

$$\begin{aligned}\sum_{i=1}^k \lambda_i \geq & \omega_n^{-\frac{2}{n}}\psi(0)^{-\frac{2}{n}}k^{\frac{n+2}{n}} - \frac{2\omega_n^{\frac{m-1}{n}}S_{m+2}\psi(0)^{\frac{(m+1)n+m-1}{n}}}{(n+2)\rho^{m+1}}k^{\frac{n-m+1}{n}} \\ & + c_2\frac{2\omega_n^{\frac{m}{n}}(m+1)S_{m+3}\psi(0)^{\frac{(m+2)n+m}{n}}}{(n+2)(m+3)\rho^{m+2}}k^{\frac{n-m}{n}},\end{aligned} \tag{3.5}$$

where $0<c_2\leq 1$ is a constant.

When $m=1$, we complete the proof of Theorem 2.1 in Sect. 2. We assume that $m\geq 2$. Putting

$$g(t)=g_1(t)+g_2(t),$$

where

$$g_1(t)=\omega_n^{-\frac{2}{n}}t^{-\frac{2}{n}}k^{\frac{n+2}{n}}$$

and

$$\begin{aligned}g_2(t)=&-\frac{2\omega_n^{\frac{m-1}{n}}S_{m+2}t^{\frac{(m+1)n+m-1}{n}}}{(n+2)\rho^{m+1}}k^{\frac{n-m+1}{n}} \\ & + c_2\frac{2\omega_n^{\frac{m}{n}}(m+1)S_{m+3}t^{\frac{(m+2)n+m}{n}}}{(n+2)(m+3)\rho^{m+2}}k^{\frac{n-m}{n}},\end{aligned}$$

we have

$$\frac{(n+2)\rho^{m+1}\omega_n^{\frac{m}{n}} g_2'(t)}{2k^{\frac{n-m}{n}}} = -\frac{(m+1)n+m-1}{n}\omega_n^{-\frac{1}{n}} S_{m+2} t^{\frac{mn+m-1}{n}} k^{\frac{1}{n}} + c_2 \frac{(m+2)n+m}{n}\frac{(m+1)S_{m+3}}{(m+3)\rho} t^{\frac{(m+1)n+m}{n}}.$$

When

$$c_2 \le \frac{(m+1)n+m-1}{(m+2)n+m}\frac{\sqrt{2}S_{m+2}}{S_{m+3}}\frac{m+3}{m+1}k^{\frac{1}{n}}, \tag{3.6}$$

we get that $g_2(t)$ is decreasing on $(0, (2\pi)^{-n}V(\Omega)]$ by using the following formulas

$$nA = \frac{k}{\omega_n},$$
$$\rho \ge (2\pi)^{-n}\omega_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}.$$

Hence $g(t)$ is also decreasing on $(0, (2\pi)^{-n}V(\Omega)]$. This implies

$$\sum_{i=1}^{k}\lambda_i \ge \omega_n^{-\frac{2}{n}}\psi(0)^{-\frac{2}{n}}k^{\frac{n+2}{n}} - \frac{2\omega_n^{\frac{m-1}{n}} S_{m+2}\psi(0)^{\frac{(m+1)n+m-1}{n}}}{(n+2)\rho^{m+1}}k^{\frac{n-m+1}{n}} + c_2\frac{2\omega_n^{\frac{m}{n}}(m+1)S_{m+3}\psi(0)^{\frac{(m+2)n+m}{n}}}{(n+2)(m+3)\rho^{m+2}}k^{\frac{n-m}{n}}, \tag{3.7}$$

where

$$\psi(0) = \frac{V(\Omega)}{(2\pi)^n},$$

and

$$\rho = \frac{V(\Omega)^{\frac{n+1}{n}}}{(2\pi)^n\omega_n^{\frac{1}{n}}}.$$

□

From the above lemma, we have the following universal lower bounds for higher eigenvalues.

Corollary 3.3 *For any bounded domain $\Omega \subseteq R^n$, $n \ge m+1 \ge 3$ and $k \ge 1$ we have*

$$\lambda_k \ge \omega_n^{-\frac{2}{n}}\alpha^{-\frac{2}{n}}k^{\frac{2}{n}} - \frac{2\omega_n^{\frac{m-1}{n}} S_{m+2}\alpha^{\frac{(m+1)n+m-1}{n}}}{(n+2)\rho^{m+1}}k^{\frac{-m+1}{n}} + c_2\frac{2\omega_n^{\frac{m}{n}}(m+1)S_{m+3}\alpha^{\frac{(m+2)n+m}{n}}}{(n+2)(m+3)\rho^{m+2}}k^{\frac{-m}{n}},$$

where

$$c_2 \le \min\left\{1, \frac{(m+1)n+m-1}{(m+2)n+m}\frac{\sqrt{2}S_{m+2}}{S_{m+3}}\frac{m+3}{m+1}k^{\frac{1}{n}}\right\},$$
$$S_l = (a+1)^l - a^l,$$

$\alpha = \frac{V(\Omega)}{(2\pi)^n}$, $\rho = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$ *and a is defined by (2.16).*

Due to the similar discussion to Corollary 2.4, we have

Corollary 3.4 *For any bounded domain* $\Omega \subseteq R^n$, $n \geq 3$ *and any* $k \geq 1$ *we have*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{n\omega_n^{-\frac{2}{n}}(2\pi)^2 V(\Omega)^{-\frac{2}{n}}}{n+2} k^{\frac{n+2}{n}} + \frac{1}{24(n+2)}\left(\frac{V(\Omega)}{I(\Omega)}\right) k + \frac{\omega_n^{-\frac{1}{n}}\alpha^{\frac{3n+1}{n}}}{9(n+2)\rho^3} k^{\frac{n-1}{n}} + \frac{3\omega_n^{-\frac{1}{n}}\alpha^{\frac{4n+2}{n}}}{80(n+2)\rho^4} k^{\frac{n-2}{n}}.$$

Proof According to Lemma 3.2, we have

$$n(n+2)B - (n+2)\tau^2 nA + 2\tau^{n+2} \geq \frac{\tau^n}{6} + \frac{\tau^{n-1}}{9} + \frac{3\tau^{n-2}}{80}.$$

By using similar discussion in the proof of Theorem 2.1, we get

$$\sum_{i=1}^{k} \lambda_i \geq \frac{n\omega_n^{-\frac{2}{n}}\phi(0)^{-\frac{2}{n}}}{n+2} k^{\frac{n+2}{n}} + \frac{\phi(0)^2}{6(n+2)\rho^2} k + \frac{\omega_n^{-\frac{1}{n}}\phi(0)^{\frac{3n+1}{n}}}{9(n+2)\rho^3} k^{\frac{n-1}{n}} \tag{3.8}$$

$$+ \frac{C_3\omega_n^{-\frac{1}{n}}\phi(0)^{\frac{4n+2}{n}}}{(n+2)\rho^4} k^{\frac{n-2}{n}}, \tag{3.9}$$

where $0 < C_3 \leq \frac{3}{80}$ is a constant which will be chosen. By using the similar discussion in the proof of Corollary 4.4, one can choose $C_3 = \frac{3}{80}$. Hence, we complete our proof. □

4 A universal lower bound on eigenvalues of the clamped plate problem

In this section, let $\phi(z)$ be the decreasing radial rearrangement of $h(z)$ where $h(z)$ is defined as (4.9). Then, a is defined by

$$\int_a^{a+1} z^{n+3} dz = \int_0^{\infty} -z^{n+3}\phi'(z) dz. \tag{4.1}$$

We will give a universal lower bounds on the sum of eigenvalues for $n \geq m$, where $m \geq 1$.

Theorem 4.1 *For any bounded domain* $\Omega \subseteq R^n$, $n \geq m \geq 1$ *and* $k \geq 1$ *we have*

(1) When $n = 1$ *and*

$$\frac{2\sqrt{2}S_3}{5} \leq k,$$

we have

$$\sum_{i=1}^{k} \Gamma_i \geq \omega_n^{-\frac{4}{n}} \alpha^{-\frac{4}{n}} k^{1+\frac{4}{n}} - \omega_n^{\frac{m-4}{n}} \frac{4S_{m+2}}{(n+4)\rho^m} \alpha^{\frac{mn+m-4}{n}} k^{\frac{n-m+4}{n}} + \omega_n^{\frac{m-3}{n}} \frac{4mS_{m+2}}{(n+4)(m+2)\rho^{m+1}} \alpha^{\frac{(m+1)n+m-3}{n}} k^{\frac{n-m+3}{n}}, \tag{4.2}$$

where α, ρ are defined by (1.9) and

$$S_l = (a+1)^l - a^l.$$

(2) When $m \geq 2$, we have

$$\sum_{i=1}^{k} \Gamma_i \geq \omega_n^{-\frac{4}{n}} \alpha^{-\frac{4}{n}} k^{1+\frac{4}{n}} - \omega_n^{\frac{m-4}{n}} \frac{4S_{m+2}}{(n+4)\rho^m} \alpha^{\frac{mn+m-4}{n}} k^{\frac{n-m+4}{n}} + c_3 \omega_n^{\frac{m-3}{n}} \frac{4m S_{m+2}}{(n+4)(m+2)\rho^{m+1}} \alpha^{\frac{(m+1)n+m-3}{n}} k^{\frac{n-m+3}{n}},$$

where

$$c_3 \leq \min\left\{1, \frac{2^{\frac{m+1}{2}}(n+2)(m+2)}{S_{m+2}[(m+1)n+m-3]} k^{\frac{m+1}{n}}\right\}.$$

Next, we recall the definition and serval properties of the symmetric decreasing rearrangements. Let $\Omega \subset R^n$ be a bounded domain. Its symmetric rearrangement Ω^* is the open ball with the same volume as Ω,

$$\Omega^* = \left\{x \in R^n | \; |x| < \left(\frac{V(\Omega)}{\omega_n}\right)\right\}.$$

By using a symmetric rearrangement of Ω, we have

$$I(\Omega) = \int_{\Omega} |x|^2 dx \geq \int_{\Omega^*} |x|^2 dx = \frac{n}{n+2} V(\Omega) \left(\frac{V(\Omega)}{\omega_n}\right)^{\frac{2}{n}}. \tag{4.3}$$

Then we have

$$\int_{R^n} |x|^4 F(x) dx \geq \int_{R^n} |x|^4 F^*(x) dx = n\omega_n \int_0^{\infty} s^{n+3} \phi(s) ds. \tag{4.4}$$

The following lemma is useful in the proof of Theorem 4.1.

Lemma 4.2 *For integers $n \geq m \geq 1$ and positive real numbers s and τ, we have the following inequality:*

$$ns^{n+4} - (n+4)\tau^4 s^n + 4\tau^{n+4} - \sum_{k=1}^{m} 4ks^{k-1}\tau^{n-k+3}(\tau - s)^2 \geq 0. \tag{4.5}$$

Proof Taking $t = \frac{s}{\tau}$, and putting $f(t)$

$$f(t) = nt^{n+4} - (n+4)t^n + 4 - 4(t-1)^2 - \sum_{k=2}^{m} 4kt^{k-1}(t-1)^2,$$

for $t \geq 0$, we get

$$\begin{aligned}
f'(t) =& n(n+4)t^{n+3} - n(n+4)t^{n-1} \\
& - \left[8(t-1) + \sum_{k=2}^{m} 4k(k-1)t^{k-2}(t-1)^2 + \sum_{k=2}^{m} 8kt^{k-1}(t-1) \right] \\
=& n(n+4)t^{n+3} - n(n+4)t^{n-1} \\
& - (t-1)\left[8 + \sum_{k=2}^{m} 4k(k-1)t^{k-2}(t-1) + \sum_{k=2}^{m} 8kt^{k-1} \right] \\
=& n(n+4)t^{n+3} - n(n+4)t^{n-1} \\
& - (t-1)\left[8 + \sum_{k=2}^{m} 4k(k-1)t^{k-1} - \sum_{k=2}^{m} 4k(k-1)t^{k-2} + \sum_{k=2}^{m} 8kt^{k-1} \right] \\
=& n(n+4)t^{n+3} - n(n+4)t^{n-1} \\
& - (t-1)\left[4m(m+1)t^{m-1} + \sum_{k=3}^{m} (4(k-1)(k-2) - 4k(k-1) + 8(k-1))t^{k-2} \right] \\
=& n(n+4)t^{n+3} - n(n+4)t^{n-1} - 4m(m+1)t^{m-1}(t-1) \\
=& \left[n(n+4)t^{n-m}(t^2+1)(t+1) - 4m(m+1) \right] t^{n-m}(t-1).
\end{aligned}$$

From the above formula, it is clear that when $n \geq m$, we have $t = 1$ is the minimum point of $f(t)$ and then $f \geq \min\{f(1) = 0, f(0) = 0\}$. We get

$$\tau^{n+4} f(t) = ns^{n+4} - (n+4)\tau^4 s^n - \sum_{k=1}^{m} 4ks^{k-1}\tau^{n-k+3}(\tau - s)^2 \geq 0.$$

□

Now, we will give the proof of Theorem 4.1.

Proof of Theorem 4.1 Let $\{u_j\}_{j=1}^{\infty}$ be the eigenfunction corresponding to the eigenvalue Γ_j, $j = 1, 2\dots$ which satisfy

$$\begin{cases} \Delta^2 u_j = \Gamma_j u_j, & \text{in } \Omega, \\ u_j = \frac{\partial u_j}{\partial \nu} = 0, & \text{on } \partial\Omega, \\ \int_\Omega u_i(x)u_j(x)dx = \delta_{ij}, & \text{for any } i, j. \end{cases}$$

Thus, $\{u_j\}_{j=1}^{\infty}$ forms an orthonormal basis of $L^2(\Omega)$. We define a function φ_j by

$$\varphi_j(x) = \begin{cases} u_j(x), & x \in \Omega, \\ 0, & x \in \mathbf{R}^n \backslash \Omega. \end{cases}$$

Denote by $\widehat{\varphi}_j(z)$ the Fourier transform of $\varphi_j(x)$. For any $z \in \mathbf{R}^n$, we have

$$\widehat{\varphi}_j(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \varphi_j(x)e^{i\langle x,z\rangle}dx = (2\pi)^{-\frac{n}{2}} \int_\Omega u_j(x)e^{i\langle x,z\rangle}dx.$$

By the Plancherel formula, we have

$$\int_{\mathbf{R}^n} \widehat{\varphi}_i(z)\widehat{\varphi}_j(z) = \delta_{ij} \tag{4.6}$$

for any i, j. Since $\{u_j\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(\Omega)$, the Bessel inequality implies that

$$\sum_{j=1}^{k} |\widehat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_{\Omega} |e^{i\langle x,z\rangle}|^2 dx = (2\pi)^{-n} V(\Omega).$$

For each $j = 1, \ldots, k$, we deduce from the divergence theorem and $u_j|_{\partial\Omega} = \frac{\partial u_j}{\partial \nu}|_{\partial\Omega} = 0$ that

$$\begin{aligned} z_p^2 \widehat{\varphi}_j(z) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \varphi_j(x)(-i)^2 \frac{\partial^2 e^{i\langle x,z\rangle}}{\partial x_p^2} dx \\ &= -(2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \frac{\partial^2 \varphi_j(x)}{\partial x_p^2} e^{i\langle x,z\rangle} dx \\ &= -\widehat{\frac{\partial^2 \varphi_j}{\partial x_p^2}}(z). \end{aligned}$$

It follows from the Parseval's identity that

$$\begin{aligned} \int_{\mathbf{R}^n} |z|^4 |\widehat{\varphi}_j(z)|^2 dz &= \int_{\mathbf{R}^n} (|z|^2 |\widehat{\varphi}_j(z)|)^2 dz \\ &= \int_{\Omega} |\Delta u_j(x)|^2 dx \\ &= \Gamma_j. \end{aligned} \tag{4.7}$$

Since

$$\nabla \widehat{\varphi}_j(z) = (2\pi)^{-\frac{n}{2}} \int_{\Omega} i x u_j(x) e^{i\langle x,z\rangle} dx,$$

we obtain

$$\sum_{j=1}^{k} |\nabla \widehat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_{\Omega} |i x e^{i\langle x,z\rangle}|^2 dx = (2\pi)^{-n} I(\Omega). \tag{4.8}$$

Putting

$$h(z) := \sum_{j=1}^{k} |\widehat{\varphi}_j(z)|^2, \tag{4.9}$$

one derives from (4.6) that $0 \leq h(z) \leq (2\pi)^{-n} V(\Omega)$. It follows from (4.8) and the Cauchy-Schwarz inequality that

$$\begin{aligned} |\nabla h(z)| &\leq 2 \left(\sum_{j=1}^{k} |\widehat{\varphi}_j(z)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{k} |\nabla \widehat{\varphi}_j(z)|^2 \right)^{\frac{1}{2}} \\ &\leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)} \end{aligned}$$

for every $z \in \mathbf{R}^n$. From the Parseval's identity, we derive

$$\int_{\mathbf{R}^n} h(z) dz = \sum_{j=1}^{k} \int_{\Omega} |u_j(x)|^2 dx = k. \tag{4.10}$$

Applying the symmetric decreasing rearrangement to $h(z)$ and noting that $\zeta = \sup|\nabla h| \leq 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)} := \eta$, we see from (2.6)

$$-\eta \leq -\zeta \leq \phi'(s) \leq 0$$

for almost every s. According to (4.4) and (4.7), we infer

$$\begin{aligned}\sum_{i=1}^{k}\Gamma_i &= \int_{\mathbf{R}^n}|z|^4 h(z)dz\\ &\geq \int_{\mathbf{R}^n}|z|^4 h^*(z)dz\\ &= n\omega_n\int_0^\infty s^{n+3}\phi(s)ds.\end{aligned} \tag{4.11}$$

In order to apply Lemma 4.2, from (4.4) and the definition of A, we take

$$\psi(s) = \phi(s),\;\; A = \frac{k}{n\omega_n},\;\; \eta = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}. \tag{4.12}$$

From (4.3), we deduce that

$$\rho \geq 2(2\pi)^{-n}\left(\frac{n}{n+2}\right)^{\frac{1}{2}}\omega_n^{-\frac{1}{n}}V(\Omega)^{\frac{n+1}{n}}. \tag{4.13}$$

On the other hand, $0 < \phi(0) \leq \sup h^*(z) = \sup h(z) \leq (2\pi)^{-n}V(\Omega)$.

For any $k \geq 1$ and $a \geq 0$, we have

$$\begin{aligned}\int_a^{a+1}s^{k-1}(\tau - s)^2 ds &= \frac{s^{k+2}}{k+2} - \frac{2s^{k+1}}{k+1}\tau + \frac{s^k}{k}\tau^2\Big|_a^{a+1}\\ &= \frac{S_{k+2}}{k+2} - \frac{2S_{k+1}}{k+1}\tau + \frac{S_k}{k}\tau^2,\end{aligned} \tag{4.14}$$

where

$$S_l = (a+1)^l - a^l.$$

Let $D' = \int_a^{a+1}s^{n+4}ds$, from the above lemma, integrating the both sides of (4.5) over $[a, a+1]$, we get

$$n(n+4)D' - (n+4)\tau^4 nA + 4\tau^{n+4} \geq \sum_{k=1}^{m}4k\tau^{n-k+3}\left(\frac{S_k}{k}\tau^2 - \frac{2S_{k+1}}{k+1}\tau + \frac{S_{k+2}}{k+2}\right). \tag{4.15}$$

From

$$\begin{aligned}&\sum_{k=1}^{m}4k\tau^{n-k+3}\left(\frac{S_k}{k}\tau^2 - \frac{2S_{k+1}}{k+1}\tau + \frac{S_{k+2}}{k+2}\right)\\ =&4\tau^{n+4} + 4\sum_{k=1}^{m-1}S_{k+1}\tau^{n-k+4} - 4\sum_{k=1}^{m}\frac{2kS_{k+1}}{k+1}\tau^{n-k+4} + 4\sum_{k=1}^{m}\frac{kS_{k+2}}{k+2}\tau^{n-k+3}\\ =&4\tau^{n+4} + 4S_2\tau^{n+3} + \frac{4mS_{m+2}}{m+2}\tau^{n-m+3} + \frac{4(m-1)S_{m+1}}{m+1}\tau^{n-m+4}\end{aligned}$$

$$
\begin{aligned}
&- 4S_2\tau^{n+3} - \frac{8mS_{m+2}}{m+1}\tau^{n-m+4} \\
&+ 4\sum_{k=2}^{m-1}\left(1 + \frac{k-1}{k+1} - \frac{2k}{k+1}\right)S_{k+1}\tau^{n-k+4} \\
=&4\tau^{n+4} + 4S_2\tau^{n+3} + \frac{4mS_{m+2}}{m+2}\tau^{n-m+3} + \frac{4(m-1)S_{m+1}}{m+1}\tau^{n-m+4} \\
&- 4S_2\tau^{n+3} - \frac{8mS_{m+2}}{m+1}\tau^{n-m+4} \\
=&4\tau^{n+4} - 4S_{m+2}\tau^{n-m+4} + \frac{4mS_{m+2}}{m+2}\tau^{n-m+3},
\end{aligned}
$$

and

$$
4\sum_{k=2}^{m-1}\left(1 + \frac{k-1}{k+1} - \frac{2k}{k+1}\right)S_{k+1}\tau^{n-k+4} = 0,
$$

we get

$$
\begin{aligned}
n(n+4)D' - (n+4)\tau^4 nA + 4\tau^{n+4} \geq& 4\tau^{n+4} - 4S_{m+2}\tau^{n-m+4} \\
&+ \frac{4mS_{m+2}}{m+2}\tau^{n-m+3}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
n(n+4)D' \geq& (n+4)\tau^4(nA) - 4S_{m+2}\tau^{n-m+4} \\
&+ \frac{4mS_{m+2}}{m+2}\tau^{n-m+3}.
\end{aligned}
$$

Taking $\tau = (nA)^{\frac{1}{n}}$, we get

$$
\begin{aligned}
D' \geq& \frac{(nA)}{n}\tau^4 - \frac{4S_{m+2}}{n(n+4)}\tau^{n-m+4} \\
&+ \frac{4mS_{m+2}}{n(n+4)(m+2)}\tau^{n-m+3} \\
\geq& \frac{(nA)^{\frac{n+4}{n}}}{n} - \frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{n(n+4)} \\
&+ \frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{n(n+4)(m+2)}.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
\int_0^{\infty} s^{n+3}\psi(s)ds \geq& \frac{(nA)^{1+\frac{4}{n}}}{n}\psi(0)^{-\frac{4}{n}} - \frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{n(n+4)\rho^m}\psi(0)^{\frac{mn+m-4}{n}} \\
&+ \frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{n(n+4)(m+2)\rho^{m+1}}\rho^{\frac{(m+1)n+m-3}{n}}.
\end{aligned}
$$

According to (4.4), (4.7) and the above inequality, we conclude

$$\begin{aligned}\sum_{i=1}^{k}\Gamma_i =&\int_{\mathbf{R}^n}|z|^4 h(z)dz\\ \geq&\int_{\mathbf{R}^n}|z|^4 h^*(z)dz\\ =&n\omega_n\int_0^{\infty}s^{n+3}\phi(s)ds\\ \geq&n\omega_n\frac{(nA)^{1+\frac{4}{n}}}{n}\psi(0)^{-\frac{4}{n}}-n\omega_n\frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{n(n+4)\rho^m}\psi(0)^{\frac{mn+m-4}{n}}\\ &+n\omega_n\frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{n(n+4)(m+2)\rho^{m+1}}\psi(0)^{\frac{(m+1)n+m-3}{n}}\\ =&\omega_n(nA)^{1+\frac{4}{n}}\psi(0)^{-\frac{4}{n}}-\omega_n\frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{(n+4)\rho^m}\psi(0)^{\frac{mn+m-4}{n}}\\ &+\omega_n\frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{(n+4)(m+2)\rho^{m+1}}\psi(0)^{\frac{(m+1)n+m-3}{n}}.\end{aligned}$$

For $m = 1$ and $n = 1$, we define $f(t)$ as follows

$$f(t) = f_1(t) + f_2(t),$$

on $(0, (2\pi)^{-n}V(\Omega)]$, where

$$f_1(t) = \xi\omega_n(nA)^{1+\frac{4}{n}}t^{-\frac{4}{n}} - \omega_n\frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{(n+4)\rho^m}t^{\frac{-2}{n}},$$

and

$$\begin{aligned}f_2(t) =& (1-\xi)\omega_n(nA)^{1+\frac{4}{n}}t^{-\frac{4}{n}} + \omega_n\frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{(n+4)(m+2)\rho^{m+1}}t^{\frac{2n-2}{n}}\\ =& (1-\xi)\omega_n(nA)^{1+\frac{4}{n}}t^{-\frac{4}{n}} + \omega_n\frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{(n+4)(m+2)\rho^{m+1}}t^{\frac{2n-2}{n}},\end{aligned}$$

for $0 < \xi \leq 1$. Then

$$\frac{nf_1'(t)}{4\omega_n(nA)^{\frac{4}{n}}} = -\xi(nA)t^{-\frac{n+4}{n}} + \frac{2S_{n+2}}{(n+4)\rho}t^{-\frac{n+2}{n}}.$$

When

$$\frac{2\sqrt{2}S_3}{5k} \leq \xi \leq 1,$$

we prove that $f(t)$ decreases on $(0, (2\pi)^{-n}V(\Omega)]$ by using

$$\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2} \leq \frac{1}{2},$$

and

$$\rho \geq (2\pi)^{-n}\omega_n^{-\frac{1}{n}}V(\Omega)^{\frac{n+1}{n}}.$$

Therefore, if

$$\frac{2\sqrt{2}S_3}{5} \leq k,$$

we get

$$\begin{aligned}\sum_{i=1}^{k}\Gamma_i \geq &\omega_n (nA)^{1+\frac{4}{n}}\alpha^{-\frac{4}{n}} - \omega_n \frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{(n+4)\rho^m}\alpha^{\frac{mn+m-4}{n}} \\ &+ \omega_n \frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{(n+4)(m+2)\rho^{m+1}}\alpha^{\frac{(m+1)n+m-3}{n}},\end{aligned}$$

where

$$\alpha = \frac{V(\Omega)}{(2\pi)^n},$$

and

$$\rho = \frac{V(\Omega)^{\frac{n+1}{n}}}{(2\pi)^n \omega_n^{\frac{1}{n}}}.$$

Noting that $A = \frac{k}{n\omega_n}$, we obtain the following inequality

$$\begin{aligned}\sum_{i=1}^{k}\Gamma_i \geq &\omega_n^{-\frac{4}{n}}\alpha^{-\frac{4}{n}}k^{1+\frac{4}{n}} - \omega_n^{\frac{m-4}{n}} \frac{4S_{m+2}}{(n+4)\rho^m}\alpha^{\frac{mn+m-4}{n}}k^{\frac{n-m+4}{n}} \\ &+ \omega_n^{\frac{m-3}{n}} \frac{4mS_{m+2}}{(n+4)(m+2)\rho^{m+1}}\alpha^{\frac{(m+1)n+m-3}{n}}k^{\frac{n-m+3}{n}}.\end{aligned} \tag{4.16}$$

When $m \geq 2$, $F(t)$ is defined by

$$F(t) = F_1(t) + F_2(t)$$

for $t \in (0, (2\pi)^{-n}V(\Omega)]$, where

$$F_1(t) = \omega_n (nA)^{1+\frac{4}{n}} t^{-\frac{4}{n}} + c_3\omega_n \frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{(n+4)(m+2)\rho^{m+1}} t^{\frac{(m+1)n+m-3}{n}},$$

for $0 < c_3 \leq 1$ and

$$F_2(t) = -\omega_n \frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{(n+4)\rho^m} t^{\frac{mn+m-4}{n}}.$$

This implies

$$\begin{aligned}\frac{F_1'(t)}{4\omega_n} = &-\frac{1}{n}(nA)^{1+\frac{4}{n}} t^{-\frac{n+4}{n}} \\ &+ c_3 \frac{(m+1)n+m-3}{n} \frac{mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{(n+2)(m+2)\rho^{m+1}} t^{\frac{mn+m-3}{n}}.\end{aligned}$$

So, if

$$c_3 \leq \frac{2^{\frac{m+1}{2}}(n+2)(m+2)}{S_{m+2}[(m+1)n+m-3]}k^{\frac{m+1}{n}},$$

we obtain that $F(t)$ decreases on $(0, (2\pi)^{-n}V(\Omega)]$, which yields that

$$\sum_{i=1}^{k}\Gamma_i \geq \omega_n^{-\frac{4}{n}}\alpha^{-\frac{4}{n}}k^{1+\frac{4}{n}} - \omega_n^{\frac{m-4}{n}}\frac{4S_{m+2}}{(n+4)\rho^m}\alpha^{\frac{mn+m-4}{n}}k^{\frac{n-m+4}{n}} + c_3\omega_n^{\frac{m-3}{n}}\frac{4mS_{m+2}}{(n+4)(m+2)\rho^{m+1}}\alpha^{\frac{(m+1)n+m-3}{n}}k^{\frac{n-m+3}{n}},$$

where

$$c_3 \leq \min\left\{1, \frac{2^{\frac{m+1}{2}}(n+2)(m+2)}{S_{m+2}[(m+1)n+m-3]}k^{\frac{m+1}{n}}\right\}.$$

□

For higher eigenvalues, we have the following universal lower bounds

Corollary 4.3 *For any bounded domain $\Omega \subseteq R^n$, $n \geq m \geq 1$ and any $k \geq 1$ we have*

(1) When $n = 1$ and

$$\frac{2\sqrt{2}S_3}{5} \leq k,$$

we have

$$\Gamma_k \geq \omega_n^{-\frac{4}{n}}\alpha^{-\frac{4}{n}}k^{\frac{4}{n}} - \omega_n^{\frac{m-4}{n}}\frac{4S_{m+2}}{(n+4)\rho^m}\alpha^{\frac{mn+m-4}{n}}k^{\frac{-m+4}{n}} + \omega_n^{\frac{m-3}{n}}\frac{4mS_{m+2}}{(n+4)(m+2)\rho^{m+1}}\alpha^{\frac{(m+1)n+m-3}{n}}k^{\frac{-m+3}{n}}, \tag{4.17}$$

where α, ρ are defined by (1.9) and

$$S_l = (a+1)^l - a^l.$$

(2) When $m \geq 2$, we have

$$\Gamma_k \geq \omega_n^{-\frac{4}{n}}\alpha^{-\frac{4}{n}}k^{\frac{4}{n}} - \omega_n^{\frac{m-4}{n}}\frac{4S_{m+2}}{(n+4)\rho^m}\alpha^{\frac{mn+m-4}{n}}k^{\frac{-m+4}{n}} + c_3\omega_n^{\frac{m-3}{n}}\frac{4mS_{m+2}}{(n+4)(m+2)\rho^{m+1}}\alpha^{\frac{(m+1)n+m-3}{n}}k^{\frac{-m+3}{n}}, \tag{4.18}$$

where

$$c_3 \leq \min\left\{1, \frac{2^{\frac{m+1}{2}}(n+2)(m+2)}{S_{m+2}[(m+1)n+m-3]}k^{\frac{m+1}{n}}\right\}.$$

According to Lemma 4.2 and the proof of Theorem 4.1, we also have the following result.

Corollary 4.4 *For any bounded domain $\Omega \subseteq R^n$, $n \geq 3$ and any $k \geq 1$ we have*

$$\sum_{i=1}^{k}\Gamma_i \geq \frac{n}{n+4}(\omega_n)^{-\frac{4}{n}}\alpha^{-\frac{4}{n}}k^{\frac{4+n}{n}} + \frac{1}{3(n+4)}\frac{(\omega_n)^{-\frac{2}{n}}\alpha^{\frac{2n-2}{n}}k^{\frac{n+2}{n}}}{\rho^2} + \frac{2}{9(n+4)}\frac{(\omega_n)^{-\frac{1}{n}}\alpha^{\frac{3n-1}{n}}k^{\frac{n+1}{n}}}{\rho^3} + \frac{3\alpha^4}{40(n+4)\rho^4}k. \tag{4.19}$$

Proof By using Lemma 4.2, we get

$$
\begin{aligned}
ns^{n+4}-(n+4)\tau^4 s^n+4\tau^{n+4} \geq & 4\tau^{n+2}(\tau-s)^2+8s\tau^{n+1}(\tau-s)^2 \\
& +12s^2\tau^n(\tau-s)^2.
\end{aligned}
$$

In view of (4.15) and (57) in [30], integrating the both sides of the above inequality over $[a, a+1]$, we have

$$
\begin{aligned}
n(n+4)D'-(n+4)\tau^4 nA+4\tau^{n+4} \geq & \frac{\tau^{n+2}}{3}+\frac{2\tau^{n+1}}{9}+12\tau^n\int_0^{\frac{1}{2}} s^2(\tau-s)^2 \\
\geq & \frac{\tau^{n+2}}{3}+\frac{2\tau^{n+1}}{9}+12\tau^n \min_{\tau\geq\frac{1}{2}}\int_0^{\frac{1}{2}} s^2(\tau-s)^2 \\
\geq & \frac{\tau^{n+2}}{3}+\frac{2\tau^{n+1}}{9}+\frac{3\tau^n}{40}.
\end{aligned}
$$

By using similar discussion in the proof of Theorem 4.1 and taking $\tau=(nA)^{\frac{1}{n}}$, we get

$$
\int_0^{\infty} s^{n+3}\psi(s)ds \geq \frac{1}{n+4}\tau^{n+4}+\frac{\tau^{n+2}}{3n(n+4)}+\frac{2\tau^{n+1}}{9n(n+4)}+\frac{3\tau^n}{40n(n+4)}.
$$

Hence,we arrive at

$$
\begin{aligned}
\sum_{i=1}^{k}\Gamma_i \geq & \frac{n\omega_n}{n+4}(nA)^{\frac{n+4}{n}}\psi(0)^{-\frac{4}{n}}+\frac{\omega_n(nA)^{\frac{n+2}{n}}}{3(n+4)\rho^2}\psi(0)^{\frac{2n-2}{n}} \\
& +\frac{2\omega_n(nA)^{\frac{n+1}{n}}}{9(n+4)\rho^3}\psi(0)^{\frac{3n-1}{n}}+d_1\frac{3\omega_n(nA)}{40(n+4)\rho^4}\psi(0)^4 \\
= & \frac{n\omega_n}{n+4}\left(\frac{k}{\omega_n}\right)^{\frac{n+4}{n}}\psi(0)^{-\frac{4}{n}}+\frac{\omega_n\left(\frac{k}{\omega_n}\right)^{\frac{n+2}{n}}}{3(n+4)\rho^2}\psi(0)^{\frac{2n-2}{n}} \\
& +\frac{2\omega_n\left(\frac{k}{\omega_n}\right)^{\frac{n+1}{n}}}{9(n+4)\rho^3}\psi(0)^{\frac{3n-1}{n}}+d_1\frac{3\omega_n\left(\frac{k}{\omega_n}\right)}{40(n+4)\rho^4}\psi(0)^4,
\end{aligned}
$$

where $0<d_1\leq 1$ is a constant to be determined. Let $t\in(0,(2\pi)^{-n}V(\Omega)]$, we define

$$
\begin{aligned}
Q(t)= & \frac{n}{n+4}\left(\frac{k}{\omega_n}\right)^{\frac{n+4}{n}}t^{-\frac{4}{n}}+\frac{\left(\frac{k}{\omega_n}\right)^{\frac{n+2}{n}}}{3(n+4)\rho^2}t^{\frac{2n-2}{n}} \\
& +\frac{2\left(\frac{k}{\omega_n}\right)^{\frac{n+1}{n}}}{9(n+4)\rho^3}t^{\frac{3n-1}{n}}+d_1\frac{3\left(\frac{k}{\omega_n}\right)}{40(n+4)\rho^4}t^4,
\end{aligned}
$$

which would be decreasing on $(0,(2\pi)^{-n}V(\Omega)]$ if $Q'((2\pi)^{-n}V(\Omega)) \leq 0$. Obviously, $Q'((2\pi)^{-n}V(\Omega)) \leq 0$ is equal to

$$4\left(\frac{k}{\omega_n}\right)^{\frac{4}{n}}\left(\frac{(2\pi)^n}{V(\Omega)}\right)^{1+\frac{4}{n}} \geq \frac{(2n-2)}{n}\left(\frac{k}{\omega_n}\right)^{\frac{2}{n}}\frac{1}{\rho^2}\left(\frac{V(\Omega)}{(2\pi)^n}\right)^{\frac{n-2}{n}} + \frac{3n-1}{n}\frac{2\left(\frac{k}{\omega_n}\right)^{\frac{1}{n}}}{9(n+4)\rho^3}\left(\frac{V(\Omega)}{(2\pi)^n}\right)^{\frac{2n-1}{n}} + 4d_1\frac{3}{40(n+4)\rho^4}\left(\frac{V(\Omega)}{(2\pi)^n}\right)^3.$$

Due to (2.18) and $\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2} \leq \frac{1}{2}$, if

$$d_1 \leq \min\{1, d_0\},$$

we have $Q'((2\pi)^{-n}V(\Omega)) \leq 0$, where

$$d_0 = \frac{140(2\pi)^2}{3}\left(4\left(\frac{k}{\omega_n}\right)^{\frac{4}{n}} - \frac{1}{\sqrt{2}\pi}\left(\frac{k}{\omega_n}\right)^{\frac{2}{n}} - \frac{2}{21(2\pi)^{\frac{3}{2}}}\left(\frac{1}{2}\right)^{\frac{3}{4}}\left(\frac{k}{\omega_n}\right)^{\frac{1}{n}}\right).$$

By direct computation, one has $d_0 > 1$. Therefore, we obtain the following eigenvalue inequality

$$\sum_{i=1}^{k}\Gamma_i \geq \frac{n\omega_n}{n+4}\left(\frac{k}{\omega_n}\right)^{\frac{n+4}{n}}\alpha^{-\frac{4}{n}} + \frac{\omega_n\left(\frac{k}{\omega_n}\right)^{\frac{n+2}{n}}}{3(n+4)\rho^2}\alpha^{\frac{2n-2}{n}} + \frac{2\omega_n\left(\frac{k}{\omega_n}\right)^{\frac{n+1}{n}}}{9(n+4)\rho^3}\alpha^{\frac{3n-1}{n}} + \frac{3\omega_n\left(\frac{k}{\omega_n}\right)}{20(n+4)\rho^4}\alpha^4.$$

□

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