Calculus of Variations

Almost-Riemannian manifolds do not satisfy the curvature-dimension condition

Mattia Magnabosco[1] · Tommaso Rossi[1]

Abstract

The Lott–Sturm–Villani curvature-dimension condition $CD(K, N)$ provides a synthetic notion for a metric measure space to have curvature bounded from below by K and dimension bounded from above by N. It was proved by Juillet (Rev Mat Iberoam 37(1), 177–188, 2021) that a large class of sub-Riemannian manifolds do not satisfy the $CD(K, N)$ condition, for any $K \in \mathbb{R}$ and $N \in (1, \infty)$. However, his result does not cover the case of almost-Riemannian manifolds. In this paper, we address the problem of disproving the CD condition in this setting, providing a new strategy which allows us to contradict the one-dimensional version of the CD condition. In particular, we prove that 2-dimensional almost-Riemannian manifolds and strongly regular almost-Riemannian manifolds do not satisfy the $CD(K, N)$ condition for any $K \in \mathbb{R}$ and $N \in (1, \infty)$.

Mathematics Subject Classification 53C17 · 53C22 · 49J52 · 53C23

Contents

Communicated by Andrea Mondino.

✉ Mattia Magnabosco
  magnabosco@iam.uni-bonn.de

  Tommaso Rossi
  rossi@iam.uni-bonn.de

[1] Institut für Angewandte Mathematik, Universität Bonn, Bonn, Germany

1 Introduction

In their seminal works Lott–Villani [20] and Sturm [28, 29] introduced a synthetic notion of curvature-dimension bounds, which is heavily based on the theory of Optimal Transport. They noticed that, in a Riemannian manifold, a uniform lower bound on the Ricci curvature, together with an upper bound on the dimension, is equivalent to a convexity property of the Rényi entropy functionals in the Wasserstein space. This allowed them to define a consistent notion of curvature-dimension bounds for metric measure spaces, known as CD condition. While in the Riemannian setting, the CD condition is equivalent to having bounded geometry, an analogue result does not hold in the sub-Riemannian setting. Sub-Riemannian geometry is a far-reaching generalization of Riemannian geometry: given a smooth manifold M, we define a smoothly varying scalar product only on a subset of *horizontal* directions $\mathcal{D}_p \subset T_p M$ (called distribution) at each point $p \in M$. Under the so-called Hörmander condition, M is horizontally-path connected, and the usual length-minimization procedure yields a well-defined distance d. In particular, differently from what happens in Riemannian geometry, the rank of the distribution $r(p) = \dim \mathcal{D}_p$ may be strictly less than the dimension of the manifold and may vary with the point. In general, we can not expect the CD condition to hold for *truly* sub-Riemannian manifolds. This statement is confirmed by the following result by Juillet.

Theorem 1.1 [19, Corrollary 1.2] *Let M be a complete sub-Riemannian manifold with* $\dim M \geq 3$, *equipped with a smooth positive (i.e. with strictly positive density) measure* $\mathfrak{m}$. *Assume that the possibly varying rank of the distribution is smaller than* $\dim M - 1$. *Then,* $(M, \mathsf{d}, \mathfrak{m})$ *does not satisfy the* $CD(K, N)$ *condition for any* $K \in \mathbb{R}$ *and* $N \in (1, \infty)$.

While this result is quite general, it does not include many cases of interest, such as *almost-Riemannian geometry*. Roughly speaking, an almost-Riemannian manifold is a sub-Riemannian manifold where the rank of the distribution coincides with the dimension of M, at almost every point.[1] For this reason, the technique used to prove Theorem 1.1 can not be adapted to this setting. Indeed, it relies on the construction of two Borel subsets for which the Brunn–Minkowski inequality does not hold, namely, for all $R, \varepsilon > 0$, one can find $A, B \subset M$ such that $\operatorname{diam}(A \cup B) < R$, $\mathfrak{m}(A) \approx \mathfrak{m}(B)$, and such that there exists $t \in (0, 1)$ for which

$$\mathfrak{m}(Z_t(A, B)) \leq \frac{1}{2^{\mathcal{N}-\dim M}} \mathfrak{m}(B)(1 + \varepsilon), \tag{1}$$

where $Z_t(A, B)$ denotes the t-intermediate set between A and B and $\mathcal{N}$ is the so-called *geodesic dimension* of M, see [26] or [2, Definition 5.47] for a precise definition. The inequality (1) allows to contradict the Brunn-Minkowski inequality if and only if the geodesic

---

[1] But not at every point, otherwise the structure would be Riemannian.

dimension $\mathcal{N}$ is strictly greater than $\dim M$. However, in the almost-Riemannian setting, $\mathcal{N} = \dim M$ almost everywhere, making this construction inconclusive. We mention that Juillet [18] disproved the CD condition in the simple example of the standard Grushin plane (cf. Example 2.6) equipped with the Lebesgue measure, by direct computations. Heuristically, disproving the CD condition in almost-Riemannian manifolds is a more challenging task, since they behave in some sense like non-complete Riemannian manifolds. Thus, a new strategy is needed.

Our idea is to exploit the one-dimensional characterization of the CD condition:

$$CD(K, N) \quad \Rightarrow \quad CD^1(K, N), \tag{2}$$

proven by Cavalletti and Mondino [12], and contradict the $CD^1(K, N)$ condition. For any 1-Lipschitz function u, the latter relies on a disintegration of the reference measure, associated with u, in one-dimensional transport rays and requires the $CD(K, N)$ condition to hold along them. The main advantage in dealing with one-dimensional CD spaces is related to a differential characterization of the CD densities, (cf. Lemma 3.2), which is easier to disprove compared with the convexity of the Rényi entropy. In Sect. 3.2, we present a local version of the one-dimensional characterization (2) (cf. Proposition 3.7), which permits to exploit the local structure of sub-Riemannian manifolds. Then, in the case of an almost-Riemannian manifold, equipped with a smooth positive measure $\mathfrak{m}$, we are able to explicitly compute the disintegration and verify that the *one-dimensional $CD(K, N)$* condition along the rays does not hold for any $K \in \mathbb{R}$ and $N \in (1, \infty)$. Our main result is the following, cf. Theorems 5.4 and 6.3. We refer to Sects. 2 and 6 for precise definitions.

Theorem 1.2 *Let M be a complete almost-Riemannian manifold and let $\mathfrak{m}$ be any smooth positive (i.e. with strictly positive density) measure on M. Assume M is either of dimension 2 or strongly regular. Then, the metric measure space $(M, d, \mathfrak{m})$ does not satisfy the $CD(K, N)$ condition for any $K \in \mathbb{R}$ and $N \in (1, +\infty)$.*

Remarkably, for 2-dimensional almost-Riemannian manifolds, we do not require any additional assumption on the structure of the singular region $\mathcal{Z}$, see (4) for the precise definition. However, as soon as the dimension of the manifold increases, the complexity of the computations prevents us to treat the general case and we need an auxiliary control on the behavior of the distribution. Nonetheless, we stress that our procedure is algorithmic and can be applied to any explicit example of almost-Riemannian manifold. This algorithmic procedure has been implemented in the software *Mathematica*, see [23].

A crucial tool for proving Theorem 1.2 will be a truly sub-Riemannian phenomenon, namely the existence of *characteristic points*. For an embedded hypersurface $\Sigma \subset M$, a characteristic point is a point where the distribution is tangent to Σ. Of course, such points do not exist in Riemannian geometry, but as soon as the rank of the distribution $r(p) < \dim M$ for some $p \in M$, they can appear. Usually, characteristic points are source of subtle technical problems, mostly related to the low regularity of the (signed) distance δ_Σ from Σ. Indeed, although being 1-Lipschitz with respect to d, δ_Σ is not smooth around characteristic points (and not even Lipschitz in coordinates). In the proof of Theorem 1.2, we choose a suitable hypersurface Σ, we build the disintegration of $\mathfrak{m}$ associated with a localized version of δ_Σ and we exploit its singular behavior to contradict the differential characterization of the one-dimensional $CD(K, N)$ condition. In particular, Σ is chosen to be *transverse* to the singular region of M in such a way $\Sigma \cap \mathcal{Z}$ exhibits characteristic points; we can then exploit the Riemannian structure at points of $\Sigma \setminus \mathcal{Z}$ to describe the degeneration of δ_Σ in the disintegration of $\mathfrak{m}$. For example, in the standard Grushin plane, where the singular region is $\mathcal{Z} = \{x = 0\}$, a suitable transverse hypersurface is $\Sigma = \{y = 0\}$.

It is worth mentioning that there exists a weaker synthetic notion of curvature bounds, introduced by Ohta [24], called *measure contraction property* or MCP condition. This property seems to be more suited to sub-Riemannian geometry, see for example [7–10]. Finally, we refer to Milman [21] for a relaxation of the CD condition, called *quasi-curvature-dimension* condition, which holds for a certain class of sub-Riemannian manifolds. However, it is not known whether these weaker conditions hold for a general almost-Riemannian manifold.

After submitting this work, Rizzi and Stefani proved in [27] that every sub-Riemannian manifold does not satisfy the $CD(K, \infty)$ condition, using different techniques.

2 Preliminaries

2.1 Almost-Riemannian geometry

We recall some basic facts about almost-Riemannian geometry, following [1].

Definition 2.1 Let M be a smooth, connected manifold. A *sub-Riemannian structure* on M is a triple $(\mathbb{U}, \xi, (\cdot|\cdot))$ satisfying the following conditions:

(i) $\pi_{\mathbb{U}} \colon \mathbb{U} \to M$ is a Euclidean bundle of rank k with base M, namely for all $p \in M$, the fiber $\mathbb{U}_p$ is a vector space equipped with a scalar product $(\cdot|\cdot)_p$, which depends smoothly on p;

(ii) The map $\xi \colon \mathbb{U} \to TM$ is a morphism of vector bundles, i.e. ξ is smooth and such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{\;\xi\;} & TM \\ & {\scriptstyle \pi_{\mathbb{U}}} \searrow & \downarrow {\scriptstyle \pi_M} \\ & & M \end{array}$$

where $\pi_M \colon TM \to M$ denotes the canonical projection of the tangent bundle.

(iii) The distribution $\mathcal{D} = \{\xi(\sigma) \mid \sigma \colon M \to \mathbb{U} \text{ smooth section}\} \subset TM$ satisfies the *Hörmander condition* (also known as bracket-generating condition), namely

$$\mathrm{Lie}_p(\mathcal{D}) = T_p M, \quad \forall\, p \in M.$$

With a slight abuse of notation, we say that M is a sub-Riemannian manifold.

Let $(\mathbb{U}, \xi, (\cdot|\cdot))$ be a sub-Riemannian structure on M. We can define the sub-Riemannian norm on $\mathcal{D}$ as

$$\|v\|_p^2 = \inf\{(u|u)_p \mid u \in \mathbb{U}_p, \ \xi(u) = v\}, \quad \forall\, v \in \mathcal{D}_p, \ p \in M. \tag{3}$$

The norm (3) is well-defined since the infimum is actually a minimum and it induces a scalar product g_p on $\mathcal{D}_p$ by polarization. Notice that different sub-Riemannian structures on M may define the same distributions and induced norms. This is the case for equivalent sub-Riemannian structures.

Definition 2.2 Let $(\mathbb{U}_1, \xi_1, (\cdot|\cdot)_1)$, $(\mathbb{U}_2, \xi_2, (\cdot|\cdot)_2)$ be two sub-Riemannian structures on M. These are said to be *equivalent* if the following conditions hold:

(i) There exists a Euclidean bundle $(\mathbb{V}, (\cdot|\cdot)_{\mathbb{V}})$ and two surjective bundle morphisms $p_i : \mathbb{V} \to \mathbb{U}_i$ such that the following diagram is commutative

$$
\begin{array}{ccc}
\mathbb{V} & \xrightarrow{p_1} & \mathbb{U}_1 \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \xi_1} \\
\mathbb{U}_2 & \xrightarrow[\xi_2]{} & TM
\end{array}
$$
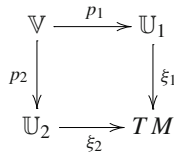
(ii) The projections p_i's are compatible with the scalar products defined on $\mathbb{U}_i$, namely

$$(u|u)_i = \min\{(v|v)_{\mathbb{V}} \mid p_i(v) = u\}, \quad \forall u \in \mathbb{U}_i, \quad i = 1, 2.$$

Definition 2.3 Let M be a sub-Riemannian manifold. The *minimal bundle rank* is the infimum of the rank of Euclidean bundles inducing equivalent structures on M. For $p \in M$, the *local minimal bundle rank* of M at p is the minimal bundle rank of the structure when restricted to a sufficiently small neighborhood $\mathcal{U}_p$.

Definition 2.4 (*Almost-Riemannian structure*) Let M be a connected, smooth manifold of dimension $n + 1$ and let $(\mathbb{U}, \xi, (\cdot|\cdot))$ be a sub-Riemannian structure on M. We say that M is an *almost-Riemannian manifold* if the local minimal bundle rank of the structure is $n + 1$.

We denote by $\mathcal{Z}$ the set of *singular points*, namely those points where the distribution has not full rank:

$$\mathcal{Z} = \{p \in M \mid \dim(\mathcal{D}_p) < n + 1\}. \tag{4}$$

Notice that $\mathcal{Z}$ is closed, since the rank of the distribution is lower semi-continuous. We say that a point is *Riemannian* if it belongs to $M \setminus \mathcal{Z}$.

Remark 2.5 If the singular set is empty, then the structure on M is Riemannian. Therefore, we will always tacitly assume that $\mathcal{Z} \neq \emptyset$.

A local orthonormal frame for the distribution is the image through ξ of a local orthonormal frame for $\mathbb{U}$. Consequently, by definition of almost-Riemannian manifold of dimension $n+1$, it consists of exactly $n + 1$ vector fields which are linearly independent only at Riemannian points. In particular, local orthonormal frames are standard Riemannian orthonormal frames around Riemannian points.

Example 2.6 (*Grushin plane*) Let $M = \mathbb{R}^2$ and consider the sub-Riemannian structure given by $\mathbb{U} = \mathbb{R}^2 \times \mathbb{R}^2$ with the standard Euclidean scalar product on fibers and

$$\xi : \mathbb{U} \to T\mathbb{R}^2; \quad \xi(x, z, u_1, u_2) = (x, z, u_1, x u_2).$$

As one can check, the resulting distribution is generated by the orthonormal vector fields $X = \partial_x$, $Y = x\partial_z$. The local minimal bundle rank is equal to 2, thus the structure is almost-Riemannian. In this case the singular region is $\mathcal{Z} = \{x = 0\}$ and $\{X, Y\}$ is a (global) orthonormal frame.

Remark 2.7 Any truly sub-Riemannian structure (meaning that is not Riemannian) of rank 2 on a 2-dimensional manifold is always almost-Riemannian, in the sense of Definition 2.4.

2.2 Almost-Riemannian distance

Let $(\mathbb{U}, \xi, (\cdot|\cdot))$ be an almost-Riemannian structure on M. We say that $\gamma : [0, T] \to M$ is a *horizontal curve*, if it is absolutely continuous and

$$\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}, \quad \text{for a.e.} \, t \in [0, T].$$

This implies that there exists a measurable function $u : [0, T] \to \mathbb{U}$, such that

$$\pi_{\mathbb{U}}(u(t)) = \gamma(t), \quad \dot{\gamma}(t) = \xi(u(t)), \quad \text{for a.e.} \, t \in [0, T].$$

Moreover, we have that $u \in L^\infty([0, T], \mathbb{U})$, see [1, Lemma 3.12], therefore the map $t \mapsto \|\dot{\gamma}(t)\|$ is integrable on $[0, T]$. We define the *length* of a horizontal curve as follows:

$$\ell(\gamma) = \int_0^T \|\dot{\gamma}(t)\| \mathrm{d}t.$$

The *almost-Riemannian distance* on M is defined, for any $p, q \in M$, by

$$\mathsf{d}(p, q) = \inf\{\ell(\gamma) \mid \gamma \text{ horizontal curve between } p \text{ and } x\}.$$

By Chow–Rashevskii theorem (see for example [5, Theorem 5.9]), the bracket-generating assumption ensures that the distance $\mathsf{d}\colon M \times M \to \mathbb{R}$ is finite and continuous. Furthermore it induces the same topology as the manifold one. We say that M is *complete*, if the metric space (M, d) is.

2.3 Geodesics and Hamiltonian flow

A *geodesic* is a horizontal curve $\gamma\colon [0, T] \to M$, parameterized with constant speed, such that any sufficiently short segment is length-minimizing. The *almost-Riemannian Hamiltonian* is the function on the cotagent space $H \in C^\infty(T^*M)$ defined by

$$H(\lambda) = \frac{1}{2} \sum_{i=0}^n \langle \lambda, X_i \rangle^2, \quad \lambda \in T^*M, \tag{5}$$

where $\{X_0, \ldots, X_n\}$ is a local orthonormal frame for the almost-Riemannian structure, and $\langle \lambda, \cdot \rangle$ denotes the action of covectors on vectors. The *Hamiltonian vector field* $\vec{H}$ on T^*M is defined by $\varsigma(\cdot, \vec{H}) = dH$, where $\varsigma \in \Lambda^2(T^*M)$ is the canonical symplectic form. Solutions $\lambda\colon [0, T] \to T^*M$ to the *Hamilton equations*

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)), \tag{6}$$

are called *normal extremals*. Their projections $\gamma(t) = \pi(\lambda(t))$ on M, where $\pi : T^*M \to M$ is the bundle projection, are locally length-minimizing horizontal curves parameterized with constant speed, and are called *normal geodesics*. If γ is a normal geodesic with normal extremal λ, then its speed is given by $\|\dot{\gamma}\|_g = \sqrt{2H(\lambda)}$. In particular

$$\ell(\gamma|_{[0,t]}) = t\sqrt{2H(\lambda(0))}, \quad \forall t \in [0, T].$$

There is another class of length-minimizing curves in sub-Riemannian geometry, called *abnormal* or *singular*. As for the normal case, to these curves it corresponds an extremal lift $\lambda(t)$ on T^*M, which however may not follow the Hamiltonian dynamics (6). Here we only observe that an abnormal extremal lift $\lambda(t) \in T^*M$ satisfies

$$\langle \lambda(t), \mathcal{D}_{\pi(\lambda(t))} \rangle = 0 \quad \text{and} \quad \lambda(t) \neq 0, \quad \forall t \in [0, T], \tag{7}$$

that is $H(\lambda(t)) \equiv 0$, therefore abnormal geodesics are always contained in the singular region $\mathcal{Z}$. A geodesic may be abnormal and normal at the same time.

Definition 2.8 Let M be an almost-Riemannian manifold and let $p \in M$. Then, the *almost-Riemannian exponential map* is

$$\exp_p(\lambda) = \pi \circ e^{\vec{H}}(\lambda), \quad \forall \lambda \in T_p^* M, \tag{8}$$

where H denotes the almost-Riemannian Hamiltonian (5) and $e^{\vec{H}}(\lambda)$ is the solution to (6) at time $t = 1$, with initial datum $\lambda \in T_p^* M$.

Note that, in general, $\exp_p$ may not be defined on the whole cotangent space, but if M is complete, then $\vec{H}$ is a complete vector field and (8) is well-posed.

2.4 Length-minimizers to a hypersurface

Let $\Sigma \subset M$ be a smooth hypersurface and fix $q_0 \in \Sigma$. Moreover, let $v \in C^\infty(M)$ be a local defining function for Σ around q_0, namely there exists an open neighborhood $\Sigma_{q_0} \subset \Sigma$ of q_0 such that

$$\Sigma_{q_0} \subset \{v = 0\} \quad \text{and} \quad dv_{|\Sigma_{q_0}} \neq 0. \tag{9}$$

We define the *local signed distance function* from Σ around q_0 as follows:

$$\delta_v := \operatorname{sgn}(v(p)) \cdot \mathsf{d}(p, \{v = 0\}), \quad \forall\, p \in M. \tag{10}$$

Let $\gamma : [0, T] \to M$ be a horizontal curve, parameterized with constant speed, such that $\gamma(0) \in \Sigma$, $\gamma(T) = p \in M \setminus \Sigma$ and assume γ is a minimizer for $\mathsf{d}(\cdot, \Sigma)$, that is $\ell(\gamma) = \mathsf{d}(p, \Sigma)$. In particular, γ is a geodesic and any corresponding normal or abnormal lift, say $\lambda : [0, T] \to T^* M$, must satisfy the transversality conditions, cf. [5, Thm 12.13],

$$\langle \lambda(0), w \rangle = 0, \quad \forall\, w \in T_{\gamma(0)} \Sigma. \tag{11}$$

Equivalently, the initial covector $\lambda(0)$ must belong to the *annihilator bundle* $\mathcal{A}\Sigma$ of Σ with fiber $\mathcal{A}_q \Sigma = \{\lambda \in T_q^* M \mid \langle \lambda, T_q \Sigma \rangle = 0\}$, for any $q \in \Sigma$. The restriction of $\exp_q$ to the annihilator bundle of Σ allows to build (locally) a smooth tubular neighborhood around non-characteristic points. Recall that $q \in \Sigma$ is a *characteristic point*, and we write $q \in C(\Sigma)$, if $\mathcal{D}_q \subset T_q \Sigma$.

Lemma 2.9 *Let $\Sigma \subset M$ be a smooth hypersurface, let $q_0 \in \Sigma \setminus C(\Sigma)$ be a non-characteristic point and $v \in C^\infty(M)$ as in (9). Then, there exist $\varepsilon_{q_0} > 0$ and a neighborhood $\mathcal{O}_{q_0} \subset \Sigma_{q_0}$ of q_0 such that the map*

$$G : (-\varepsilon_{q_0}, \varepsilon_{q_0}) \times \mathcal{O}_{q_0} \to M, \quad G(s, q) = \exp_q(s\lambda(q)), \tag{12}$$

is a diffeomorphism on its image, where $\lambda(q)$ is the unique element (up to a sign) of $\mathcal{A}_q \Sigma$ such that $2H(\lambda(q)) = 1$. Moreover, δ_v is smooth in $G((-\varepsilon_{q_0}, \varepsilon_{q_0}) \times \mathcal{O}_{q_0})$ and[2]

$$G_* \partial_{s\,|(s,q)} = \nabla \delta_v(G(s, q)), \quad \forall (s, q) \in (-\varepsilon_{q_0}, \varepsilon_{q_0}) \times \mathcal{O}_{q_0}. \tag{13}$$

---

[2] The *horizontal gradient* of $f \in C^\infty(M)$ is defined by $g_p(\nabla f, v) = d_p f(v), \forall v \in \mathcal{D}_p$ and $p \in M$.

Remark 2.10 It is known that if Σ has no characteristic points, the signed distance is smooth in a tubular neighborhood of Σ, cf. [17, Proposition 3.1]. This lemma can be regarded as its local version and its proof is a straightforward adaptation of the aforementioned result. Moreover, note that $C(\Sigma) \subset \mathcal{Z}$ and so the Riemannian points of Σ are non-characteristic. Finally, if Σ contains characteristic points, the parameter ε_{q_0}, as well as $\mathcal{O}_{q_0}$, can not be chosen uniformly.

Remark 2.11 By condition (13), for any $q \in \mathcal{O}_{q_0}$, we have

$$(-\varepsilon_{q_0}, \varepsilon_{q_0}) \in s \mapsto G(s, q) \in M$$

is the unique minimizing geodesic (parameterized by unit-speed) from Σ passing through q. Moreover, notice that the initial covector $\lambda(q)$ in (12) is unique up to a sign: the only requirement is to choose this covector in such a way it defines a continuous section of the annihilator bundle.

3 The curvature-dimension condition

A triple $(X, \mathsf{d}, \mathfrak{m})$ is called metric measure space if (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a locally finite Borel measure on it. In the following $C([0, 1], X)$ will stand for the space of continuous curves from $[0, 1]$ to X. A curve $\gamma \in C([0, 1], X)$ is called *minimizing geodesic* if

$$\mathsf{d}(\gamma_s, \gamma_t) = |t - s| \cdot \mathsf{d}(\gamma_0, \gamma_1) \quad \text{for every } s, t \in [0, 1],$$

we denote by $\mathrm{Geo}(X)$ the space of minimizing geodesics on X. The metric space (X, d) is said to be geodesic if every pair of points $x, y \in X$ can be connected with a curve $\gamma \in \mathrm{Geo}(X)$. For any $t \in [0, 1]$ we define the evaluation map $e_t \colon C([0, 1], X) \to X$ by setting $e_t(\gamma) := \gamma_t$ and the stretching/restriction operator restr_r^s in $C([0, 1], X)$, defined, for all $0 \leq r < s \leq 1$, by

$$[\mathrm{restr}_r^s(\gamma)]_t := \gamma_{r+t(s-r)}, \quad t \in [0, 1].$$

We denote by $\mathscr{P}(X)$ the set of Borel probability measures on X and by $\mathscr{P}_2(X) \subset \mathscr{P}(X)$ the set of those having finite second moment. We endow the space $\mathscr{P}_2(X)$ with the Wasserstein distance W_2, defined by

$$W_2^2(\mu_0, \mu_1) := \inf_{\pi \in \mathsf{Adm}(\mu_0, \mu_1)} \mathsf{d}^2(x, y) \, \mathrm{d}\pi(x, y),$$

where $\mathsf{Adm}(\mu_0, \mu_1)$ is the set of all the admissible transport plans between μ_0 and μ_1, namely all the measures in $\mathscr{P}(X^2)$ such that $(\mathrm{p}_1)_\sharp \pi = \mu_0$ and $(\mathrm{p}_2)_\sharp \pi = \mu_1$. The metric space $(\mathscr{P}_2(X), W_2)$ is itself complete and separable, moreover, if (X, d) is geodesic, then $(\mathscr{P}_2(X), W_2)$ is geodesic as well. In particular, every geodesic $(\mu_t)_{t \in [0,1]}$ in $(\mathscr{P}_2(X), W_2)$ can be represented with a measure $\eta \in \mathscr{P}(\mathrm{Geo}(X))$, meaning that $\mu_t = (e_t)_\# \eta$. A subset $G \subset \mathrm{Geo}(X)$ is called non-branching if for any pair $\gamma_1, \gamma_2 \in G$ such that $\gamma_1 \neq \gamma_2$, it holds that

$$\mathrm{restr}_0^t(\gamma_1) \neq \mathrm{restr}_0^t(\gamma_2) \quad \text{for every } t \in (0, 1).$$

A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be essentially non-branching if for every two measures $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ which are absolutely continuous with respect to the reference measure $\mathfrak{m}$ ($\mu_0, \mu_1 \ll \mathfrak{m}$), every W_2-geodesic connecting them is concentrated on a non-branching set of geodesics.

3.1 *CD* spaces

In this subsection we introduce the CD condition, pioneered by Sturm and Lott–Villani [20, 28, 29]. This condition aims to generalize, to the context metric measure spaces, the notion of having Ricci curvature bounded from below by K and dimension less than or equal to N. In particular, in the Riemannian setting it is possible to characterize this two bounds in terms of a property whose definition involves only the distance and the (volume) measure. This property, which is stated in Definition 3.1, is given in terms of the following distortion coefficients: for every $K \in \mathbb{R}$ and $N \in (1, +\infty)$

$$\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} \left[\sigma_{K,N-1}^{(t)}(\theta) \right]^{1-\frac{1}{N}},$$

where

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } N\pi^2 > K\theta^2 > 0, \\ t & \text{if } K = 0, \\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K < 0. \end{cases}$$

Definition 3.1 A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be a $CD(K, N)$ space (or to satisfy the $CD(K, N)$ condition) if for every pair of measures $\mu_0 = \rho_0 \mathfrak{m}, \mu_1 = \rho_1 \mathfrak{m} \in \mathscr{P}_2(X)$, absolutely continuous with respect to $\mathfrak{m}$, there exists a W_2-geodesic $(\mu_t)_{t \in [0,1]}$ connecting them and induced by $\eta \in \mathscr{P}(\mathrm{Geo}(X))$, such that for every $t \in [0, 1]$ $\mu_t = \rho_t \mathfrak{m} \ll \mathfrak{m}$ and the following inequality holds for every $N' \geq N$ and every $t \in [0, 1]$

$$\int_X \rho_t^{1-\frac{1}{N'}} \,\mathrm{d}\mathfrak{m} \geq \int_{X \times X} \left[\tau_{K,N'}^{(1-t)}\big(\mathsf{d}(x, y)\big)\rho_0(x)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}\big(\mathsf{d}(x, y)\big)\rho_1(y)^{-\frac{1}{N'}} \right] \mathrm{d}\pi(x, y), \tag{14}$$

where $\pi = (e_0, e_1)_\# \eta$.

In general, the CD condition is not very easy to disprove, however when the reference space is an interval $I \subseteq \mathbb{R}$ the following lemma, whose proof can be find in [14, Lemma A.5], provides a nice strategy.

Lemma 3.2 *Let $I \subset \mathbb{R}$ be an interval and let $h : I \to \mathbb{R}$ be a measurable function such that $(I, |\cdot|, h\mathscr{L}^1)$ is a $CD(K, N)$ space. Then at any point x in the interior of I where h is twice differentiable it holds that*

$$(\log h)''(x) + \frac{1}{N-1}\big((\log h)'(x)\big)^2 \leq -K. \tag{15}$$

Remark 3.3 This lemma holds also for $N = +\infty$, where now the left-hand side of (15) has to be intended as $(\log h)''(x)$ (for the definition of $CD(K, \infty)$ space, see [28]).

In fact, in order to disprove that the space $(I, |\cdot|, h\mathscr{L}^1)$ satisfies $CD(K, N)$ is sufficient to find a point x in the interior of I such that h is twice differentiable in x and

$$(\log h)''(x) + \frac{1}{N-1}\big((\log h)'(x)\big)^2 > -K.$$

Notice also that, if we manage to prove that

$$(\log h)''(x) > -K, \tag{16}$$

we automatically show that $(I, |\cdot|, h\mathcal{L}^1)$ does not satisfy $CD(K, N)$ for every $N \in (1, +\infty]$. This observation will be fundamental in the following, especially in combination with the one-dimensional localization results we are now going to present.

3.2 One-dimensional localization

In this subsection we present a suitable adaptation of the one-dimensional characterization of the CD condition. This property, called $CD^1(K, N)$ condition, has been studied in the general framework of essentially non-branching metric measure spaces with a curvature-dimension bound in [11–14]. We provide a local version of such characterization, that allows us to take advantage of the local structure of almost-Riemannian manifolds.

We recall a general result regarding disintegration of measures. Given a measurable space $(R, \mathscr{R})$, and a function $\mathfrak{Q} : R \to Q$ to a general set Q, we endow Q with the push forward σ-algebra $\mathscr{Q}$ of $\mathscr{R}$, i.e. the biggest σ-algebra on Q such that $\mathfrak{Q}$ is measurable. Moreover, given a finite (non-null) measure ρ on $(R, \mathscr{R})$, consider the measure $\mathfrak{q} := \mathfrak{Q}_{\#}\rho$ on $(Q, \mathscr{Q})$.

Definition 3.4 A disintegration of ρ consistent with $\mathfrak{Q}$ is a map $Q \ni q \mapsto \rho_q \in \mathscr{P}(R)$ such that the following hold:

(1) for all $B \in \mathscr{R}$, $\rho_{\cdot}(B)$ is $\mathfrak{q}$-measurable,
(2) for all $B \in \mathscr{R}$, $C \in \mathscr{Q}$, we have

$$\rho\left(B \cap \mathfrak{Q}^{-1}(C)\right) = \int_C \rho_q(B) \, d\mathfrak{q}(q).$$

A disintegration is called *strongly consistent* with respect to $\mathfrak{Q}$ if, in addition, for all $q \in Q$ it holds that $\rho_q(\mathfrak{Q}^{-1}(q)) = 1$.

Theorem 3.5 [12, Theorem 2.8] *Let $(R, \mathscr{R})$ be a countably generated measurable space and ρ be a finite measure on it. Assume there exists a partition of R as*

$$R = \bigcup_{q \in Q} R_q,$$

denote by $\mathfrak{Q} : R \to Q$ the quotient map and by $(Q, \mathscr{Q}, \mathfrak{q})$ the quotient measure space. If $(Q, \mathscr{Q}) = (X, \mathscr{B}(X))$ where X is a Polish space and $\mathscr{B}(X)$ denotes its Borel σ-algebra, then there exists a unique strongly consistent disintegration $q \mapsto \rho_q$ with respect to $\mathfrak{Q}$.

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space and fix an open subset $\Omega \subset X$ with $0 < \mathfrak{m}(\Omega) = \mathfrak{m}(\bar{\Omega}) < \infty$. Let $u : \bar{\Omega} \to \mathbb{R}$ be a 1-Lipschitz function, define

$$\Gamma_u := \{(x, y) \in \bar{\Omega} \times \bar{\Omega} \mid u(x) - u(y) = \mathsf{d}(x, y)\}$$

and its transpose $\Gamma_u^{-1} := \{(x, y) \in \bar{\Omega} \times \bar{\Omega} \mid (y, x) \in \Gamma_u\}$. Consequently, we introduce the transport relation R_u and the transport set T_u as

$$R_u := \Gamma_u \cup \Gamma_u^{-1} \quad \text{and} \quad T_u := \mathrm{p}_1\left(R_u \setminus \{(x, y) \in \bar{\Omega} \times \bar{\Omega} \mid x = y\}\right),$$

where p_1 denotes the projection on the first factor. Although this is not always the case, if we assume that R_u is an equivalence relation, we may partition the set $\bar{\Omega}$. Letting Q be the set of equivalence classes and $\mathfrak{Q} : \bar{\Omega} \to Q$ the quotient map, we can write

$$\bar{\Omega} = \bigcup_{q \in Q} \gamma_q,$$

where $\gamma_q := \{x \in \bar{\Omega} \mid \mathfrak{Q}(x) = q\}$ for every $q \in Q$. With the quotient map we can endow Q with the quotient σ-algebra $\mathcal{Q}$, that is the finest σ-algebra on Q for which $\mathfrak{Q}$ is measurable. We introduce the following definition to obtain a local version of the one-dimensional localization of Cavalletti and Mondino [12], which better fits the setting of almost-Riemannian geometry, where we have a good local description of geodesics. In [12], the authors define a global partition starting from a globally defined 1-Lipschitz function, see Remark 3.8.

Definition 3.6 We say that a 1-Lipschitz function $u : \bar{\Omega} \to \mathbb{R}$ induces a *one-dimensional partition* of $\bar{\Omega}$ if

(1) R_u is an equivalence relation and $\mathfrak{m}(\bar{\Omega} \setminus T_u) = 0$,
(2) for every $q \in Q$, the set $\gamma_q \subset \bar{\Omega}$ is the image of a geodesic of (X, d),
(3) for every $q \in Q$ there exists $x \in \bar{\Omega}$ such that $\mathfrak{Q}(x) = q$ and $u(x) = 0$.

If u induces a one-dimensional partition, then, in particular, we can choose $Q = \{u = 0\}$. Indeed, the point $x \in \bar{\Omega}$ satisfying (3) of Definition 3.6 is unique. Define the ray map

$$g : \mathrm{Dom}(g) \subset Q \times \mathbb{R} \to \bar{\Omega}$$

by imposing that

$$\mathrm{graph}(g) := \{(q, t, x) \in Q \times \mathbb{R} \times \bar{\Omega} \mid \mathfrak{Q}(x) = q, u(x) = t\}.$$

The ray map g is Borel and bijective, its inverse is

$$\bar{\Omega} \ni x \mapsto g^{-1}(x) := (\mathfrak{Q}(x), u(x)).$$

Moreover, for every $q \in Q$ the map $t \mapsto g(q, \cdot)$ is an isometry, and consequently $\mathcal{H}^1 \llcorner \gamma_q = \mathcal{H}^1 \llcorner \{g(q, t) \mid t \in I_q\} = g(q, \cdot)_\# \mathcal{L}^1$, where $I_q := \mathrm{Dom}(g(q, \cdot))$. Theorem 3.5 ensures that there exists a unique strongly consistent disintegration of $\mathfrak{m} \llcorner \Omega$:

$$\mathfrak{m} \llcorner \Omega = \int_Q \mathfrak{m}_q \, d\mathfrak{q}(q),$$

where $\mathfrak{m}_q$ is a measure concentrated on γ_q and we recall that $\mathfrak{q} := \mathfrak{Q}_\#(\mathfrak{m} \llcorner \Omega)$.

Proposition 3.7 *Let $(X, \mathsf{d}, \mathfrak{m})$ be a essentially non-branching metric measure space satisfying the $CD(K, N)$ condition, for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Let $\Omega \subset X$ be open and such that $0 < \mathfrak{m}(\Omega) = \mathfrak{m}(\bar{\Omega}) < \infty$ and let $u : \bar{\Omega} \to \mathbb{R}$ be a 1-Lipschitz function providing a one-dimensional partition. Then:*

(1) *for $\mathfrak{q}$-a.e. $q \in Q$, the measure $\mathfrak{m}_q$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \gamma_q$, namely there exists $h_q : I_q \to [0, \infty]$ such that $\mathfrak{m}_q = g(q, \cdot)_\# (h_q \cdot \mathcal{L}^1)$,*
(2) *for $\mathfrak{q}$-a.e. $q \in Q$, $(I_q, |\cdot|, h_q \mathcal{L}^1)$ is a $CD(K, N)$ space.*

Proof The proof of this proposition can be done by adapting the classical global approach to the space $(\bar{\Omega}, \mathsf{d}, \mathfrak{m} \llcorner \Omega)$, see in particular [11, Sect. 6] for point (1) and [12, Theorem 4.2] for point (2). We point out that this space is not necessarily geodesic, hence we can not conclude that it satisfies the $CD(K, N)$ condition and simply apply the known results. However, observe that, in order to deduce properties of the disintegration induced by u, it is enough to study Wasserstein geodesics that follow its transport rays. Since u induces a one-dimensional partition in $\bar{\Omega}$ in the sense of Definition 3.6, all the transport rays are contained in $\bar{\Omega}$ and the CD condition (14) holds along such Wasserstein geodesics. For this reason, we can repeat the standard arguments verbatim, obtaining the result. $\square$

Remark 3.8 In the classical theory of [12], starting from a *globally defined* 1-Lipschitz function on a $CD(K, N)$ essentially non-branching metric measure space, the authors build a one-dimensional partition of the whole space, up to a negligible set, and disintegrate the measure accordingly. Then, the densities in the disintegration satisfy the $CD(K, N)$ condition, providing the one-dimensional characterization $CD^1(K, N)$. In this setting, there is no need for the additional properties of Definition 3.6 on the 1-Lipschitz function u.

In particular, given a 1-Lipschitz function $u : X \to \mathbb{R}$, we can introduce the transport relation R_u and the transport set T_u as before (with X in place of $\bar{\Omega}$) and denote by $\Gamma_u(x)$ the section of Γ_u through x in the first coordinate. Then, we define the set of forward and backward branching points as

$$A^+ := \{x \in T_u \mid \exists y, z \in \Gamma_u(x), (y, z) \notin R_u\}$$

$$A^- := \{x \in T_u \mid \exists y, z \in \Gamma_u^{-1}(x), (y, z) \notin R_u\}.$$

Finally, we define the non-branched transport set and the non-branched transport relation as

$$T_u^{nb} := T_u \setminus (A^+ \cup A^-), \quad \text{and} \quad R_u^{nb} := R_u \cap (T_u^{nb} \times T_u^{nb}).$$

On the one hand, as shown in [11], the essentially non-branching assumption ensures that R_u^{nb} is an equivalence relation on T_u^{nb} and for q-a.e. $q \in Q$, γ_q is isometric to a closed interval of $\mathbb{R}$. On the other hand, if $(X, \mathsf{d}, \mathfrak{m})$ also satisfies the $CD(K, N)$ condition, the set $T_u \setminus T_u^{nb}$ is $\mathfrak{m}$-negligible, cf. [12, Theorem 3.4]. It is then possible to obtain a global result, analogous to Proposition 3.7.

Remark 3.9 Note that showing (16) for $K \in \mathbb{R}$ actually implies that h_q can not be a $CD(K, \infty)$ density. However, for a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, it is not known whether the $CD(K, \infty)$ condition can be characterized with one-dimensional disintegrations.

4 A general strategy for disproving the *CD* condition

The 1-Lipschitz function whose disintegration allows us to disprove the CD condition will be a localized version of the (signed) distance function from a hypersurface Σ. Indeed with this choice we are able to compute explicitly the one-dimensional marginals and to exploit the existence of characteristic points.

4.1 Existence of normal coordinates

Let M be an almost-Riemannian manifold of dimension $n + 1$. We build a convenient set of coordinates around a point in the singular region. This result can be regarded as a generalization of Agrachev et al. [1, Proposition 9.8].

Lemma 4.1 (Normal coordinates) *Let M be an almost-Riemannian manifold of dimension $n + 1$, let $\mathcal{Z} \subset M$ the set of singular points and let $p_0 \in \mathcal{Z}$. Then, there exists a set of coordinates $\varphi = (x, z) \colon \mathcal{U} \to M$ centered at p_0 and a local orthonormal frame $X_0, \ldots, X_n$ for the almost-Riemannian structure on $\mathcal{U}$ such that:*

$$X_0 = \partial_x, \quad X_i = \sum_{j=1}^{n} a_{ij}(x, z)\partial_{z_j}, \quad \forall i = 1, \ldots, n, \tag{17}$$

where a_{ij} are smooth functions in $\mathcal{U}$. Moreover, denoting by $A(x, z) = (a_{ij}(x, z))_{i,j}$, we have $\det A(0, \mathbf{0}) = 0$.

The proof of Lemma 4.1 follows from the existence of a tubular neighborhood around non-characteristic points. We require a preliminary result.

Lemma 4.2 *Let M be an almost-Riemannian manifold and let $p_0 \in M$. Then, there exists a hypersurface $W \subset M$ such that $p_0 \in W \setminus C(W)$.*

Proof We assume by contradiction that $p_0 \in M$ is a characteristic point for any hypersurface $W \subset M$ passing through p_0. By definition of characteristic point, this means that $\mathcal{D}_{p_0} \subset T_{p_0} W$ for any such W. In turn, this implies that $\mathcal{D}_{p_0} = \{0\}$, contradicting the bracket-generating assumption. $\qquad\square$

Proof of Lemma 4.1 Using Lemma 4.2, we find an embedded hypersurface $W \subset M$ such that $p_0 \in W \setminus C(W)$. We use the almost-Riemannian normal exponential map to define the desired coordinates. Indeed, let $v \in C^{\infty}(M)$ be a local defining function as in (9). Then, by Lemma 2.9, there exist $\varepsilon_{p_0} > 0$ and a neighborhood $\mathcal{O}_{p_0} \subset W$ of p_0 such that

$$G \colon (-\varepsilon_{p_0}, \varepsilon_{p_0}) \times \mathcal{O}_{p_0} \to M, \quad G(s, p) = \exp_p(s\lambda(p)),$$

is a diffeomorphism on its image, where $\lambda(p)$ satisfies (11) with $2H(\lambda(p)) = 1$. Moreover, the local signed distance function δ_v is smooth in $G((-\varepsilon_{p_0}, \varepsilon_{p_0}) \times \mathcal{O}_{p_0})$ and

$$G_* \partial_{s|(s,p)} = \nabla \delta_v(G(s, p)), \quad \forall (s, p) \in (-\varepsilon_{p_0}, \varepsilon_{p_0}) \times \mathcal{O}_{p_0}.$$

Thus, fixing any set of coordinates $(z_1, \ldots, z_n)$ for $\mathcal{O}_{p_0}$, and relabelling $s = x$, the coordinates (x, z) satisfies (17). Finally, since $p_0 \in \mathcal{Z}$, the vector fields $\{X_0, \ldots, X_n\}$ are linearly dependent at p_0, meaning that the matrix $A(x, z)$ has zero determinant at $(0, \mathbf{0})$. $\qquad\square$

Remark 4.3 From now on, without loss of generality, whenever we fix a set of coordinates, we will assume that the domain of the chart is the whole $\mathbb{R} \times \mathbb{R}^n$.

4.2 Assumptions on the almost-Riemannian structure

Let M be an almost-Riemannian manifold of dimension $n + 1$, and let $\mathcal{Z} \subset M$ be the set of singular points. Let $p_0 \in \mathcal{Z}$ and let $\Sigma \subset M$ be a hypersurface. To proceed with our general construction, we need two assumptions on the almost-Riemannian structure: in coordinates $(x, z) \in \mathbb{R} \times \mathbb{R}^n$ centered at p_0 given by Lemma 4.1, we require that

(i) the hypersurface Σ consists of Riemannian points except when $x = 0$ and has a characteristic point at the origin, i.e.

$$\Sigma \cap \mathcal{Z} \subset \{x = 0\} \quad \text{and} \quad (0, \mathbf{0}) \in C(\Sigma); \tag{H1}$$

(ii) let $\mathfrak{m}$ be any smooth positive measure on M, then

$$\mathfrak{m}(\mathcal{Z}) = 0. \tag{H2}$$

Here by smooth positive measure, we mean a measure with strictly positive and smooth density with respect to the Lebesgue measure in coordinates.

Remark 4.4 Let us comment on why we need these assumptions. The first one is necessary to have a good local description of the marginals *outside* the set $\{x = 0\}$, in order to exploit the presence of a characteristic point *only* at the origin, cf. Lemma 2.9. The second one is necessary in order to ensure the essentially non-branching property, cf. Lemma 4.7, and to characterize the marginals in the disintegration, cf. (21).

Remark 4.5 We remark that assumption (H2) is not always guaranteed as the next example shows. Consider $C \subset [0, 1]$ a closed subset with positive Lebesgue measure and empty interior and let $f \in C^\infty(\mathbb{R})$ such that $f_{|C} \equiv 0$. Then, define the structure on $\mathbb{R}^4$ with global orthonormal frame:

$$X_0 = \partial_x, \quad X_1 = \partial_{z_1} - \frac{z_2}{2}\partial_{z_3}, \quad X_2 = \partial_{z_2} + \frac{z_1}{2}\partial_{z_3}, \quad X_3 = f(x)\partial_{z_3}.$$

As one can check, the Hörmander condition is verified and the local minimal bundle rank is always 4, since C has empty interior. Thus, the structure is almost-Riemannian. Now fix $\mathfrak{m} = \mathscr{L}^4$, then the singular set has infinite measure indeed, by (18)

$$\mathcal{Z} = \{(x, z) \in \mathbb{R}^4 \mid \det A(x, z) = 0\} = \{(x, z) \in \mathbb{R}^4 \mid f(x) = 0\} = C \times \mathbb{R}^3.$$

Here, the matrix A is given by

$$A(x, z) = \begin{pmatrix} 1 & & \\ & 1 & \\ -\frac{z_2}{2} & \frac{z_1}{2} & f \end{pmatrix}.$$

In an analogue way, one can build an example where (H1) is not verified. Indeed, in the construction above, it is enough to consider a closed set C with empty interior and with an accumulation point at the origin. Then, the hypersurface $\Sigma = \{z_3 = 0\}$ has a characteristic point at the origin, however it intersects the singular region in $C \times \mathbb{R}^2$.

Notice that, in coordinates (17), the singular set can be described by the matrix $A = (a_{ij})_{i,j}$, indeed

$$(x, z) \in \mathcal{Z} \quad \text{if and only if} \quad \det A(x, z) = 0. \tag{18}$$

In particular, along the hypersurface Σ, by (H1), we have

$$x \neq 0 \quad \Rightarrow \quad \det A(x, z_1, \dots, z_{n-1}, 0) \neq 0, \tag{19}$$

since the set $\Sigma \cap \{x \neq 0\}$ consists of Riemannian points. As a consequence, since a Riemannian point is never a characteristic one, $C(\Sigma) \subset \{x = 0\}$. Actually, it is always possible to ensure that $\Sigma = \{z_n = 0\}$ satisfies $(0, \mathbf{0}) \in C(\Sigma)$, so the only condition one should check is (19).

Lemma 4.6 *Let M be an almost-Riemannian manifold and let $p_0 \in \mathcal{Z}$. Then, there exists an hypersurface $\Sigma \subset M$ such that $p_0 \in C(\Sigma)$. Moreover, in coordinates (x, z) as in (17), up to a rotation, we can choose $\Sigma = \{z_n = 0\}$.*

Proof Assume by contradiction that $p_0 \in M$ is not a characteristic point for every hypersurface $W \subset M$ passing through p_0. Then, by definition of characteristic point, we deduce that $\mathcal{D}_{p_0}$ must be transversal to $T_{p_0}W$, for every such W, or equivalently

$$\mathcal{D}_{p_0} + T_{p_0}W = T_{p_0}M,$$

for every $W \subset M$ passing trough p_0. As a consequence, $\mathcal{D}_{p_0} = T_{p_0}M$ and thus $r(p_0) = n + 1$. This gives a contradiction, since $r(p_0) < n + 1$, being $p_0 \in \mathcal{Z}$. Let us show that in coordinates Σ can be chosen as $\{z_n = 0\}$: since $\det A(p_0) = 0$, there exists an invertible matrix $M \in GL(n+1, \mathbb{R})$ such that the last column of the matrix $A(p_0)M$ consists of zeroes. Then, we introduce the following change of coordinates

$$\psi : (x, z) \mapsto (x, \tilde{z}) = (x, M^\mathsf{T} z).$$

In the new coordinates, the generating family for the distribution has the following expression:

$$X_0 = \partial_x, \quad X_i = \sum_{j=1}^{n} a_{ij}(x, z)\partial_{z_j} = \sum_{k,j=1}^{n} a_{ij}(x, \psi^{-1}(z))m_{jk}\partial_{\tilde{z}_k},$$

having denoted by $M = (m_{ij})_{i,j}$. Thus, (17) is still valid and, when evaluated at p_0, the matrix describing the generating family has the last column consisting of zeroes. Finally, this implies that the hypersurface $\Sigma = \{\tilde{z}_n = 0\}$ has a characteristic point at p_0. Indeed,

$$\nabla \tilde{z}_n(p_0) = \sum_{i=1}^{n} X_i(\tilde{z}_n)X_i(p_0) = \sum_{i,j=1}^{n} a_{ij}(0, \mathbf{0})m_{jn}X_i(p_0) = \sum_{i=1}^{n}(A(p_0)M)_{in}X_i(p_0) = 0,$$

since the last column of the matrix $A(p_0)M$ is zero, implying that $\mathcal{D}_{p_0} \subset T_{p_0}\Sigma$. □

Lemma 4.7 *Let M be an almost-Riemannian manifold, equipped with a smooth positive measure $\mathfrak{m}$ and satisfying assumptions* (H2). *Then $(M, d, \mathfrak{m})$ is essentially non-branching.*

Proof Let $\gamma : [0, 1] \to M$ be a minimizing geodesic. Then γ is abnormal if there exists an abnormal extremal lift $\lambda(t) \neq 0$, satisfying (7). But this implies that

$$\gamma(t) \subset \mathcal{Z}, \quad \forall t \in [0, 1].$$

Hence, if γ is a minimizing geodesic with endpoints in the Riemannian region, i.e.

$$\gamma(0), \gamma(1) \in M \setminus \mathcal{Z}, \tag{20}$$

then γ must be strictly normal. As showed in [22, Corollary 6], a strictly normal geodesic $\gamma : [0, 1] \to M$ is branching for some positive time $t \in (0, 1)$ if and only if it contains a non-trivial abnormal subsegment that starts at time 0. Thus, a minimizing geodesic satisfying (20) can not branch for positive times since $\mathcal{Z}$ is closed. Now, let $\eta \in \mathscr{P}(\text{Geo}(X))$ be a W_2-geodesic joining the measures $\mu_0, \mu_1 \in \mathscr{P}_2(X)$, which are absolutely continuous with respect to the reference measure $\mathfrak{m}$ ($\mu_0, \mu_1 \ll \mathfrak{m}$). In particular, notice that $(e_0)_\#\eta = \mu_0$ and $(e_1)_\#\eta = \mu_1$ and therefore, by (H2),

$$\eta(e_0^{-1}(\mathcal{Z})) = \mu_0(\mathcal{Z}) = 0 \quad \text{and} \quad \eta(e_1^{-1}(\mathcal{Z})) = \mu_1(\mathcal{Z}) = 0.$$

Consequently, the measure η is concentrated on $\text{Geo}(X)\setminus\big(e_0^{-1}(\mathcal{Z}) \cup e_1^{-1}(\mathcal{Z})\big)$, which is a non-branching set of geodesics, according to the first part of the proof. □

Remark 4.8 Notice that it is possible to build examples of almost-Riemannian manifolds where (H2) is verified but there exist branching geodesics. Indeed, consider $\mathbb{R}^4$, with the global orthonormal frame given by

$$X_0 = \partial_x, \quad X_1 = \partial_{z_1}, \quad X_2 = \partial_{z_2} + B(z_1, z_2)\partial_{z_3}, \quad X_3 = x\partial_{z_3},$$

where B is a smooth magnetic potential, defined as in [22], namely

$$B(z_1, z_2) = z_1\theta(z_2) + z_2^2\theta(1 - z_2)$$

and $\theta \in C^\infty(\mathbb{R})$ such that $0 \leq \theta \leq 1$, $\theta(r) = 0$ for $r \leq 0$ and $\theta(r) = 1$ for $r \geq 1$. In this situation, we have strictly normal branching geodesic in the singular region. Nevertheless, thanks to Lemma 4.7, $(M, d, \mathfrak{m})$ is essentially non-branching. On the other hand, if the measure of the singular set is positive, it is unclear whether an almost-Riemannian manifold is essentially non-branching.

4.3 Choice of the local disintegration

Let us fix coordinates $(x, z) \in \mathbb{R} \times \mathbb{R}^n$ as in (17). Let $\mathfrak{m}$ be a smooth positive measure on M and let $\Sigma = \{z_n = 0\}$ be defined as Sect. 4.2. Under the assumptions (H1) and (H2), we consider the signed distance function δ_v from Σ, as defined in (10), with $v := z_n$. For ease of notation, we denote δ_v simply by δ. By triangle inequality, δ is always 1-Lipschitz on M with respect to d. However, δ develops singularities at a characteristic point, indeed it is only Hölder (and not Lipschitz) with respect to the Euclidean distance of the chart, see [4, Theorem 4.2]. Roughly speaking, such a singularity is related to the fact that the horizontal gradient of δ, which exists almost everywhere (see, [16, Theorem 8]), becomes tangent to Σ as the base point approaches a characteristic point. Our idea is to exploit this behavior to prove that the disintegration associated with δ does not produce $CD(K, N)$ densities along the transport rays, for any $K \in \mathbb{R}$, $N \geq 1$.

Starting from δ, we build a suitable open and bounded set Ω and we consider the local disintegration of $\mathfrak{m} \llcorner \Omega$ induced by δ, cf. Section 3.2. Set $B := B_r((0, \mathbf{0}))$ for some $r > 0$ and define the open and bounded set

$$\Omega := \bigcup_{q_0 \in (\Sigma \setminus C(\Sigma)) \cap B} \gamma_{q_0}, \quad \text{where} \quad \gamma_{q_0} = G((-f(q_0), f(q_0)) \times \{q_0\}),$$

where G is the map defined in Lemma 2.9 and $f \in C^\infty(\Sigma)$ such that $0 < f(q_0) < \varepsilon_{q_0}$, for every $q_0 \in \Sigma \setminus C(\Sigma)$. Note that G is a local diffeomorphism on Ω and, with this choice of f, $\mathfrak{m}(\Omega) = \mathfrak{m}(\bar{\Omega})$.

Then, δ is a 1-Lipschitz function on $\bar{\Omega}$ inducing a one-dimensional partition in the sense of Definition 3.6. Indeed $Q = \{\delta = 0\} \cap \bar{\Omega} = \Sigma \cap \bar{B}$ and, for every $q \in Q$, the transport ray γ_q of the disintegration coincides with the minimizing geodesics for δ, which exist by completeness. Moreover, $T_\delta = \bar{\Omega} \setminus C(\Sigma)$ and therefore, $\mathfrak{m}(\bar{\Omega} \setminus T_\delta) = 0$. The quotient map $\mathfrak{Q} \colon \bar{\Omega} \to Q$ can be regarded as a projection on the foot of a geodesic,[3] thus $\mathfrak{Q}$ is the inverse of the exponential map, namely

$$G(\delta(p), \mathfrak{Q}(p)) = p, \quad \forall p \in \bar{\Omega},$$

and G is indeed the ray map associated to the partition. Finally, since Ω is defined by the smooth function f, the measure $\mathfrak{q} = \mathfrak{Q}_\#(\mathfrak{m} \llcorner \Omega)$ is smooth on Σ.

4.4 Coordinate expression for the marginals in the disintegration

Using Lemma 2.9 and, in particular, the diffeomorphism (12), we can conveniently represent the one-dimensional densities in the disintegration. Indeed, consider a Riemannian point $q_0 = (\bar{x}, \bar{z}) \in \Sigma$, with $\bar{z}_n = 0$. In particular, thanks to (19), it is enough to assume $\bar{x} \neq 0$. Then, for every Borel set $C \subset \Omega$, on the one hand we have that

$$\int_{\Omega \cap C} \mathrm{d}\mathfrak{m} = \int_\Sigma \int_{-f(q)}^{f(q)} \chi_{G^{-1}(C)} \mathrm{d}(G^*(\mathfrak{m} \llcorner \Omega)),$$

while, on the other hand, making the disintegration explicit and recalling that G is the ray map, we conclude that

---

[3] For any $p \in \bar{\Omega}$, there exists a unique point $\mathsf{f}(p) \in \Sigma \cap \bar{\Omega}$ for which $|\delta(p)| = \mathsf{d}(p, \mathsf{f}(p))$.

$$\int_{\Omega \cap C} d\mathfrak{m} = \int_{\Sigma} \mathfrak{m}_q(\Omega \cap C) dq$$

$$= \int_{\Sigma} \int_{-f(q)}^{f(q)} \chi_C(G(s,q)) d\mathfrak{m}_q dq = \int_{\Sigma} \int_{-f(q)}^{f(q)} \chi_{G^{-1}(C)}(s,q) h_q(s) ds dq.$$

Thus, having fixed a frame for $T\Sigma$, say $\{v_1, \ldots, v_n\}$, the density $h_q(s)$ is given by:

$$h_q(s) = h_q(s) ds(\partial_s) \frac{q(v_1, \ldots, v_n)}{q(v_1, \ldots, v_n)} = \frac{G^* \mathfrak{m}(\partial_s, v_1, \ldots, v_n)}{q(v_1, \ldots, v_n)}$$

$$= \frac{\mathfrak{m}(G_* \partial_s, G_* v_1, \ldots, G_* v_n)}{q(v_1, \ldots, v_n)} = \frac{\mathfrak{m}(\nabla \delta, G_* v_1, \ldots, G_* v_n)}{q(v_1, \ldots, v_n)}, \tag{21}$$

for any $(s,q) \in G^{-1}(\Omega)$, and having used (13) in the last equality.

Remark 4.9 Notice that, from (21), the one-dimensional densities $h_q(s)$ are smooth functions of $(s,q) \in G^{-1}(\Omega)$. Moreover, they do not depend on the choice of coordinates.

We are going to study the second logarithmic derivative of $h_q(s)$, at $s = 0$ and as $q \to 0$, in order to obtain a contradiction with the differential characterization of Lemma 3.2. Firstly, notice that since we are performing derivatives in s, we can disregard constant functions in s. Secondly, by definition $\mathfrak{m}$ is a smooth positive measure, i.e.

$$\mathfrak{m} = m(x,z) dx dz, \quad \text{with} \quad m \in C^\infty(\mathbb{R} \times \mathbb{R}^n), \quad c \le m \le C, \tag{22}$$

for some $C, c > 0$. Moreover, in (21), as a frame for $T\Sigma$, we can choose the vector fields $\{\partial_x, \partial_{z_1}, \ldots, \partial_{z_{n-1}}\}$. In conclusion, we obtain the following expression for the one-dimensional density associated with the disintegration:

$$h_q(s) \propto m(G(s,q)) dx dz(\nabla \delta, G_* \partial_x, \ldots, G_* \partial_{z_{n-1}})_{|(s,q)}. \tag{23}$$

Then, defining the matrix

$$B_q(s) := (\nabla \delta \mid G_* \partial_x \mid \ldots \mid G_* \partial_{z_{n-1}}),$$

where the columns are expressed in coordinates $\{\partial_x, \ldots, \partial_{z_n}\}$, the second logarithm derivative at $s = 0$ is given by:

$$\left(\log(h_q(s))\right)''_{|s=0} = (\log(m(G(s,q)))''_{|s=0} + \left(\log \det(B_q(s))\right)''_{|s=0}$$

$$= (\log(m(G(s,q)))''_{|s=0} + \text{tr}\left(-(B_q^{-1}(0) B_q'(0))^2 + B_q^{-1}(0) B_q''(0)\right), \tag{24}$$

having used Jacobi formula for the determinant of a smooth curve of invertible matrices:

$$\det(B(s))' = \det(B(s)) \text{tr}\left(B^{-1}(s) B'(s)\right).$$

Remark 4.10 We stress that, for any $q \in Q \setminus C(\Sigma)$, $h_q(\cdot)$ is defined on an open interval I_q containing 0. Thus, the derivative in (24) makes sense.

4.5 Computations for the matrix $B_q(s)$

Proceeding with hindsight, we analyze the term of (24) involving $B_q(s)$, as in general it will be more singular than the other one. We expand in s its columns and we deduce an expression for the coefficients of the expansion, using the almost-Riemannian Hamiltonian system.

4.5.1 An expression for the trace term in (24)

We look for an explicit expression for the matrix $B_q(s)$. We may regard $\nabla\delta \in \mathbb{R}^{n+1}$ and $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n+1}$, therefore expanding in s, there exists smooth functions $\mathfrak{f}, \mathfrak{h} \in C^\infty(\Sigma \cap \Omega)$ such that,

$$G(s, q) = q + \nabla\delta(q)s + \frac{1}{2}\mathfrak{f}(q)s^2 + o(s^2),$$

$$\nabla\delta(G(s, q)) = \nabla\delta(q) + \mathfrak{f}(q)s + \frac{1}{2}\mathfrak{h}(q)s^2 + o(s^2), \tag{25}$$

as $s \to 0$. The relation between the two expansions comes from (13). Therefore, we obtain the following formulas for $B_q(s)$ and its derivatives at $s = 0$: for the zero order term, we have

$$B_q(0) = \left(\nabla\delta \mid \partial_x \mid \partial_{z_1} \mid \ldots \mid \partial_{z_{n-1}}\right)_{|q} = \left(\nabla\delta(q) \left|\begin{array}{c}\mathrm{Id}_{n\times n}\\\hline 0 \ \cdots \ 0\end{array}\right.\right).$$

For the first derivative of $B_q(s)$, we differentiate component by component. Notice that we have to take into account the quantities $\partial_{z_i} G(0, q)$, with $i = 0, \ldots, n-1$,[4] therefore we have to differentiate the expansion (25), namely:

$$\partial_{z_i} G(s, q) = \partial_{z_i} + \partial_{z_i}\nabla\delta(q)s + \frac{1}{2}\partial_{z_i}\mathfrak{f}(q)s^2 + o(s^2),$$

as $s \to 0$, for any $i = 0, \ldots, n-1$, where the derivatives have to be interpreted component by component. Therefore, we obtain:

$$B_q'(0) = \left(\mathfrak{f} \mid \partial_x \mid \partial_{z_1} \mid \ldots \mid \partial_{z_{n-1}}\right)_{|q} + \sum_{i=0}^{n-1}\left(\nabla\delta \mid \partial_x \mid \ldots \mid \partial_{z_i}\nabla\delta \mid \ldots \mid \partial_{z_{n-1}}\right)_{|q}.$$

Analogously, we can deduce the expression for the second-order derivative of $B_q(s)$ at $s = 0$:

$$B_q''(0) = \left(\mathfrak{h} \mid \partial_x \mid \partial_{z_1} \mid \ldots \mid \partial_{z_{n-1}}\right)_{|q} + \sum_{i=0}^{n-1}\left(\nabla\delta \mid \partial_x \mid \ldots \mid \partial_{z_i}\mathfrak{f} \mid \ldots \mid \partial_{z_{n-1}}\right)_{|q}$$

$$+ 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\left(\nabla\delta \mid \partial_x \mid \ldots \mid \partial_{z_i}\nabla\delta \mid \ldots \mid \partial_{z_j}\nabla\delta \mid \ldots \mid \partial_{z_{n-1}}\right)_{|q}$$

$$+ 2\sum_{i=0}^{n-1}\left(\mathfrak{f} \mid \partial_x \mid \ldots \mid \partial_{z_i}\nabla\delta \mid \ldots \mid \partial_{z_{n-1}}\right)_{|q}.$$

Inserting the above formulas in the trace term in (24), we obtain the desired expression, in terms of the quantities $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$.

4.5.2 Explicit expression for $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$

In order to obtain an explicit expression for $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$, we study the Hamiltonian system associated with the almost-Riemannian Hamiltonian

$$H(\lambda) = \frac{1}{2}\sum_{i=0}^{n}\langle\lambda, X_i\rangle^2, \quad \forall\lambda \in T^*M,$$

---

[4] Here and below, with a slight abuse of notation, for $i = 0$, we set $z_0 = x$ and $\partial_{z_0} = \partial_x$.

where $\{X_0, \ldots, X_n\}$ is the local orthonormal frame for the distribution defined in (17). In coordinates (x, z), the almost-Riemannian metric g on the Riemannian region is represented by the matrix

$$\left(\begin{array}{c|c} 1 & \\ \hline & (A^\mathsf{T} A)^{-1} \end{array}\right).$$

Therefore the Hamiltonian in canonical coordinates induced by (x, z) is

$$H(p_x, p_z; x, z) = \frac{1}{2} p_x^2 + \frac{1}{2} p_z^\mathsf{T} A^\mathsf{T} A(x, z) p_z,$$

where p_z is a shorthand for $(p_{z_1}, \ldots, p_{z_n})$. The Hamiltonian system then becomes

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p_x} = p_x & \dot{p}_x = -\dfrac{\partial H}{\partial x} = -\dfrac{1}{2} p_z^\mathsf{T} \partial_x \left(A^\mathsf{T} A\right) p_z \\ \dot{z} = \dfrac{\partial H}{\partial p_z} = A^\mathsf{T} A p_z & \dot{p}_z = -\dfrac{\partial H}{\partial z} = -\dfrac{1}{2} p_z^\mathsf{T} \partial_z \left(A^\mathsf{T} A\right) p_z \end{cases} \tag{26}$$

From Lemma 2.9 (cf. also Remark 2.11) we know that the unique minimizing geodesic for δ with initial point $q \in \Sigma \cap \Omega$ has unique (up to a sign) initial covector such that:

$$\langle \lambda(q), T_q \Sigma \rangle = 0 \quad \text{and} \quad 2H(\lambda(q)) = 1. \tag{27}$$

Since $T_q \Sigma = \mathrm{span}\{\partial_x, \partial_{z_1}, \ldots, \partial_{z_{n-1}}\}$, the first condition in (27) implies that $\lambda(q) = p_{z_n} dz_n$. In addition, the second condition in (27) forces $\lambda(q)$ to be of the form:

$$\lambda(q) = \frac{1}{\beta(q)} dz_n, \quad \text{with} \quad \beta(q)^2 = \sum_{k=1}^n a_{kn}(q)^2, \tag{28}$$

where we choose β to be positive. Thus, denoting by $(x^q(s), z^q(s); p_x^q(s), p_z^q(s))$ the solution to (26) with initial datum $(\lambda(q); q)$, the minimizing geodesic for δ starting at q is given by:

$$I_q \ni s \mapsto G(s, q) = (x^q(s), z^q(s)), \tag{29}$$

where I_q is an open interval containing the origin. By (13) and (29), we deduce that

$$\nabla \delta(q) = \partial_s G_{|(0,q)} = \left(\dot{x}^q(0), \dot{z}^q(0)\right).$$

Computing derivatives along s of the equality (13) and recalling the definition of $\mathfrak{f}, \mathfrak{h}$ in (25), we analogously obtain higher-order expression in terms of the solution to (26), precisely:

$$\mathfrak{f}(q) = \left(\ddot{x}^q(0), \ddot{z}^q(0)\right), \quad \mathfrak{h}(q) = \left(\dddot{x}^q(0), \dddot{z}^q(0)\right). \tag{30}$$

We refer to Appendix 1 for the explicit expression of $\mathfrak{f}$ and $\mathfrak{h}$.

4.6 Contradicting the *CD* condition

The idea is to exploit the presence of a characteristic point for Σ at the origin to conclude that

$$\left(\log(h_q(s))\right)''_{|s=0} \xrightarrow{(x,\mathbf{0}) \to (0,\mathbf{0})} +\infty, \tag{31}$$

proving (16) for every $K \in \mathbb{R}$, up to taking x sufficiently small. Keeping in mind (24), we anticipate that the term providing the desired pathology will be

$$\left(\log \det(B_q(s))\right)''_{|s=0}.$$

Observe that, according to Sect. 3, in order to disprove the $CD(K, N)$ condition for every $K \in \mathbb{R}$ and $N \in (1, +\infty)$ we need to show that, given any $K \in \mathbb{R}$, it holds

$$\left(\log(h_q(s))\right)''_{|s=0} > -K,$$

for every q in a q-positive set. However, since the function $(s, q) \to h_q(s)$ is smooth (see Remark 4.9) it is sufficient to prove (31).

Notice that the initial covector for the minimizing geodesic from Σ in (28) is singular at $q = (0, \mathbf{0})$. More precisely, since $(0, \mathbf{0}) \in C(\Sigma)$ and Σ is the level set of $v(x, z) = z_n$, we have that

$$0 = \nabla z_{n|(0,\mathbf{0})} = \sum_{i=1}^{n} X_i(z_n) X_{i|(0,\mathbf{0})} = \sum_{i,j=1}^{n} a_{ij}(0,\mathbf{0}) a_{in}(0,\mathbf{0}) \partial_{z_j},$$

meaning that the function $\beta(q)$ defined in (28) vanishes at q *if and only if* $q \in C(\Sigma)$. In particular, it vanishes at the origin, making the initial covector singular at $q = (0, \mathbf{0})$. Moreover, solving the Hamiltonian system, we deduce that

$$\nabla \delta(q) = (\dot{x}^q(0), \dot{z}^q(0)) = (0, A^\mathsf{T} A \beta^{-1}(q) \partial_{z_n}) = \left(0, \beta^{-1}(q) \sum_{k=1}^{n} a_{ki}(q) a_{kn}(q)\right). \quad (32)$$

Remark 4.11 On the one hand, all the components of $\nabla \delta(q)$, but the first and last, are singular at $q = (0, \mathbf{0})$, as fast as the initial covector (28). On the other hand, since the last component of $\nabla \delta(q)$ is exactly $\beta(q)$ which tends to 0 as $q \to (0, \mathbf{0})$, formally $\nabla \delta(q)$ becomes tangent to Σ at the characteristic point.

In particular, we see that $\nabla \delta(q)$ is singular at the origin and the same goes for the functions $\mathfrak{f}(q), \mathfrak{h}(q)$. Replacing their explicit expressions in (24), we will be able to prove (31).

Remark 4.12 The procedure described in this section for disproving the CD condition is constructive and the algorithm has been implemented in the software *Mathematica*. The code is available online, see [23].

5 2-Dimensional almost-Riemannian manifolds do not satisfy *CD*

In this section, we apply our general strategy to show that 2-dimensional almost-Riemannian manifolds do not satisfy any curvature-dimension condition. The reason why we are able to perform explicit computations is related to the better regularity properties of δ, when $\dim M = 2$, cf. Remark 4.11.

Let M be an almost-Riemannian manifold of dimension 2, with non-empty singular region $\mathcal{Z} \subset M$. We recall the following local description of a general 2-dimensional almost-Riemannian manifold which holds *without* any assumption on the structure of the singular set, see [3, Lemma 17].

Lemma 5.1 *Let M be an almost-Riemannian manifold. Then, for every point $q_0 \in M$, there exists a set of coordinates $\varphi = (x, z) \colon \mathcal{U} \to M$, centered at q_0, such that a local orthonormal frame for the distribution is given by*

$$X = \partial_x, \quad Y = f(x, z)\partial_z.$$

where $f \colon \mathcal{U} \to \mathbb{R}$ is a smooth function. Moreover,

(i) *the integral curves of X are normal extremals, as in* (6)*;*

(ii) *let s be the step of the structure at q_0. If $s = 1$ then $f(0,0) \neq 0$. If $s \geq 2$, we have*

$$f(0,0) = 0, \quad \ldots, \quad \frac{\partial^{s-2} f}{\partial x^{s-2}}(0,0) = 0, \quad \frac{\partial^{s-1} f}{\partial x^{s-1}}(0,0) \neq 0. \tag{33}$$

Remark 5.2 This Lemma improves Lemma 4.1 since we can give additional condition on the function $f(x,z) = \det A(x,z)$, using the Hörmander condition.

In the 2-dimensional case, assumption (H2) is always verified, see [1, Theorem 9.14]. For what concerns assumption (H1), we have the following lemma.

Lemma 5.3 *Let M be a 2-dimensional almost-Riemannian manifold and let $q_0 \in \mathcal{Z}$. Consider the curve in normal coordinates $\Sigma = \{z = 0\}$. Then, up to restricting the chart, $\Sigma \cap \mathcal{Z} = C(\Sigma) = \{(0,0)\}$.*

Proof Recall that if $\Sigma = \{v = 0\}$, for $v \in C^\infty$ with never-vanishing differential, then

$$p \in C(\Sigma) \quad \Leftrightarrow \quad \nabla v(p) = 0,$$

where ∇u denotes the horizontal gradient of v. In particular, in the normal coordinates given by Lemma 5.1, the singular region is $\mathcal{Z} \cap \mathcal{U} = \{(x,z) \mid f(x,z) = 0\}$, thus setting $v(x,z) = z$,

$$p = (x,0) \in C(\Sigma) \quad \Leftrightarrow \quad f(x,0) = 0.$$

Since $q_0 \in \mathcal{Z}$, then $\dim(\mathcal{D}_{q_0}) < 2$ and the almost-Riemannian structure has step $s \geq 2$ at q_0. Thus $f(0,0) = 0$ and consequently $p = q_0 \in C(\Sigma)$. On the other hand, if $0 < |x| < \varepsilon$, $f(x,0) \neq 0$. Indeed, by the vanishing condition (33) on f, we can expand $f(x,0)$ as a Taylor series at $x = 0$, obtaining

$$f(x,0) = \frac{\partial^{s-1} f}{\partial x^{s-1}}(0,0)x^{s-1} + o(x^{s-1}), \quad \text{as } x \to 0.$$

where the leading term is not zero. This implies that there exists a smooth function $r \in C^\infty(-\varepsilon, \varepsilon)$, such that $r(x) \neq 0$, for every $x \in (-\varepsilon, \varepsilon)$ and $f(x,0) = r(x)x^{s-1}$, which never vanishes on $\Sigma \cap \mathcal{U} \setminus \{q_0\}$, up to restricting the domain of the chart $\mathcal{U}$. □

Now, thanks to Lemmas 5.1 and 5.3, we can follow the general strategy (cf. Sect. 4) to disprove the CD condition. First of all, notice that the matrix $A = (f(x,z))$ has only one entry, so the Hamiltonian system is greatly simplified. More precisely, the initial covector (28) becomes:

$$\lambda(x) = \frac{1}{f(x,0)} dz, \quad \forall x \neq 0.$$

Thus, as one can check using (26), (30) and (32) we have

$$\nabla \delta(x) = (0, f(x,0)), \quad \mathfrak{f}(x) = \left(-\frac{\partial_x f}{f}, f \partial_z f \right)_{|(x,0)},$$

$$\mathfrak{h}(x) = \left(\star, -\frac{2(\partial_x f)^2}{f} + f(\partial_z f)^2 + f^2 \partial_z^2 f \right)_{|(x,0)}. \tag{34}$$

Here we have omitted the first component of $\mathfrak{h}(x)$, since we will not need it.

Second of all, we can replace the quantities (34) in the matrix $B_q(s)$, defined in (23). After a long but routine computation, we obtain the following expression for the logarithmic second derivative of $\det B_q(s)$ at $s = 0$, namely

$$(\log \det B_x(s))''(0) = \left(f \partial_z^2 f + \frac{(\partial_x f)^2 - f \partial_x^2 f}{f^2} \right)_{|(x,0)}.$$

We are in position to prove the main result of this section.

Theorem 5.4 *Let M be a complete 2-dimensional almost-Riemannian manifold and let $\mathfrak{m}$ be any smooth positive measure on M. Then, the metric measure space $(M, d, \mathfrak{m})$ does not satisfy the $CD(K, N)$ condition for any $K \in \mathbb{R}$ and $N \in (1, +\infty)$.*

Proof As explained in Sect. 4, we have to show that the quantity $(\log h_x(s))''(0)$ diverges at $+\infty$ as $x \to 0$. Recall that, by the proof of Lemma 5.3, there exists a never-vanishing function $r \in C^\infty(-\varepsilon, \varepsilon)$ such that

$$f(x, 0) = r(x)x^{s-1}, \quad \text{with} \quad f, r \in C^\infty(\mathcal{U}).$$

Therefore, using the smoothness of both f and r, we deduce that

$$\begin{aligned}
(\log \det B_x(s))''(0) &= \frac{(\partial_x f(x,0))^2 - f(x,0)\partial_x^2 f(x,0)}{f(x,0)^2} + O(1) \\
&= \left(\frac{s-1}{x^2} + \frac{\partial_x r(x)^2 - r(x)\partial_x^2 r(x)}{r(x)^2} \right) + O(1) \\
&= \frac{s-1}{x^2} + O(1),
\end{aligned} \tag{35}$$

which diverges to $+\infty$ as $x \to 0$, since $(0,0) \in \mathcal{Z}$ and therefore $s > 1$. Moreover, let us remark that the singularity is polynomial of order -2. We are left to take care of the first term in (24): by a direct computation and using (22), one can check that

$$\begin{aligned}
\left| (\log m(G(s, x)))''(0) \right| &\le C_0 \left(|\partial_s G(0, x)|_e^2 + |\partial_s^2 G(0, x)|_e \right) \le C_1 + C_2 \left| \frac{\partial_x f(x, 0)}{f(x, 0)} \right| \\
&= C_1 + C_2 \left| \frac{s-1}{x} + \frac{\partial_x r(x)}{r(x)} \right|,
\end{aligned} \tag{36}$$

where $| \cdot |_e$ denotes the Euclidean norm of $\mathbb{R}^2$. Since the singularity in (36) is polynomial of order 1, it is negligible compared to the one in (35), and we conclude that:

$$(\log \det h_x(s))''(0) \xrightarrow{x \to 0} +\infty,$$

disproving the $CD(K, N)$ condition for any $K \in \mathbb{R}$ and $N \in (0, +\infty)$, as desired. $\quad\square$

Remark 5.5 A similar argument can be carried out for *generic* 3-dimensional almost-Riemannian manifolds, in the sense of Agrachev et al. [3, Definition 2]. Indeed, in this situation we have a convenient description of a local orthonormal frame, and of the matrix A, cf. [6, Theorem 2].

6 Strongly regular almost-Riemannian manifolds do not satisfy *CD*

In this section, we prove that strongly regular almost-Riemannian manifolds do not satisfy any curvature-dimension condition. Strongly regular almost-Riemannian manifolds have been studied in [15, 25]. In this setting, we can deal with the complexity of the computations thanks to a nice local description of the singularities of the structure. We recall the following definition.

Definition 6.1 Let M be a n-dimensional almost-Riemannian manifold. Assume that the singular set $\mathcal{Z} \subset M$ is an embedded hypersurface without characteristic points. Then, for any $q_0 \in \mathcal{Z}$, there exist local coordinates (x, z) centered at q_0 such that $\mathcal{Z} = \{x = 0\}$ in coordinates, and condition (17) is verified, namely a local orthonormal frame for the distribution is given by

$$X_0 = \partial_x, \quad X_i = \sum_{j=1}^{n} a_{ij}(x, z)\partial_{z_j}, \quad \forall\, i = 1, \ldots, n,$$

for some smooth functions a_{ij}, so that, denoting by $A = (a_{ij})_{i,j}$,

$$\det A(x, z) = 0 \quad \text{if and only if} \quad x = 0.$$

We say that M is a *strongly regular* almost-Riemannian manifold, if there exists $l \in \mathbb{N}$ such that

$$a_{ij}(x, z) = x^l \hat{a}_{ij}(x, z) \quad \text{with} \quad \det(\hat{a}_{ij})(0, z) \neq 0, \tag{37}$$

for all $(0, z)$ in the domain of the chart.

Remark 6.2 Although being formulated in coordinates, the notion of a strongly regular almost-Riemannian structure on M is intrinsic. In particular, condition (37), as well as the order l, do not depend neither on the choice of $q_0 \in \mathcal{Z}$ nor on the coordinates (x, z), see [25] for further details.

In order to apply our general strategy, we have to ensure that conditions (H1) and (H2) are verified. The former is a consequence of the very definition of strongly regular almost-Riemannian structure and we pick Σ as in Lemma 4.6 so that also the latter condition is satisfied. We proceed by computing the second logarithmic derivative of the one-dimensional densities,

$$\begin{aligned}
\left(\log \det B_q(s)\right)''_{|s=0} = {} & \frac{1}{\beta}\Bigg[\mathfrak{h}_n + \beta\partial_x\mathfrak{f}_0 + \sum_{i=1}^{n-1}\left(\beta\partial_{z_i}\mathfrak{f}_i - \beta_i\partial_{z_i}\mathfrak{f}_n\right) - 2\mathfrak{f}_0\partial_x\beta \\
& + 2\sum_{i=1}^{n-1}\left(\mathfrak{f}_n\partial_{z_i}\beta_i - \mathfrak{f}_i\partial_{z_i}\beta\right) + 2\sum_{0<i<j<n}\det\begin{pmatrix}\partial_{z_i}\beta_i & \partial_{z_j}\beta_i & \beta_i \\ \partial_{z_i}\beta_j & \partial_{z_j}\beta_j & \beta_j \\ \partial_{z_i}\beta & \partial_{z_j}\beta & \beta\end{pmatrix}\Bigg] \\
& - \frac{1}{\beta^2}\left(\mathfrak{f}_n + \sum_{i=1}^{n-1}\left(\partial_{z_i}\beta_i\beta - \beta_i\partial_{z_i}\beta\right)\right)^2, \tag{38}
\end{aligned}$$

where $\beta_i, \mathfrak{f}_i, \mathfrak{h}_i$ denote the components of $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$ respectively. This computation follows from the trace term in (24), using the property that the first component of $\nabla\delta$ is identically zero, cf. (32).

Theorem 6.3 *Let M be a complete strongly regular almost-Riemannian manifold and let $\mathfrak{m}$ be any smooth positive measure on M. Then, the metric measure space $(M, d, \mathfrak{m})$ does not satisfy the $CD(K, N)$ condition for any $K \in \mathbb{R}$ and $N \in (1, +\infty)$.*

Proof As in the proof of Theorem 5.4, we have to show that the quantity $\left(\log h_q(s)\right)''(0)$ diverges at $+\infty$ as $q \to 0$ along Σ. To do that, the idea is to highlight the most singular terms in x, namely those where a derivative in x appears. Let us discuss the order in x of the quantities in (38), using as well formulas from Appendix 1. Firstly, since M is strongly regular, (37) holds and we have

$$\beta(x, z)^2 = \sum_{i=1}^{n} a_{kn}^2(x, z) = x^{2l} \sum_{i=1}^{n} \hat{a}_{kn}^2(x, z) = x^{2l} \hat{\beta}(x, z)^2, \tag{39}$$

with $\hat{\beta}(0, z) \neq 0$. Thus, β has order l in x. Similarly, the components β_i of $\nabla \delta$ are given by (42),

$$\beta_i(x, z) = \frac{\alpha_i(x, z)}{\beta(x, z)} = \frac{1}{\beta(x, z)} \sum_{k=1}^{n} a_{ki}(x, z) a_{kn}(x, z) = \frac{x^l}{\hat{\beta}(x, z)} \sum_{k=1}^{n} \hat{a}_{ki}(x, z) \hat{a}_{kn}(x, z).$$

Therefore, also β_i's have order l in x. A crucial remark before moving forward is that, thanks to the strongly regular assumption on M, computing derivatives along z-directions does not change the order in x of the quantities. Thus, for example,

$$\text{ord}_x \partial_{z_j} \beta_i(x, z) = l, \quad \forall i, j = 1, \dots, n.$$

Reasoning in this way, for the functions $\mathfrak{f}_i$ defined in (43), we have:

$$\text{ord}_x \mathfrak{f}_i(x, z) = 2l, \quad \forall i = 1, \dots, n,$$

and the same is true for any derivative in z-directions. For what concerns $\mathfrak{f}_0$, recall that

$$\mathfrak{f}_0(x, z) = -\frac{\partial_x \beta(x, z)}{\beta(x, z)} \quad \Rightarrow \quad \text{ord}_x \mathfrak{f}_0(x, z) = -1. \tag{40}$$

From (40), it is clear that derivatives in the x-direction encode all the possible singularities of second logarithmic derivatives of $h_q(s)$. Finally, using (44), we see that

$$\text{ord}_x \mathfrak{h}_n(x, z) = l - 2, \quad \text{and} \quad \mathfrak{h}_n(x, z) = -\frac{2(\partial_x \beta(x, z))^2}{\beta(x, z)} + O(x^{3l}).$$

Finally, we can evaluate the order in x of the functions in (38): the lowest order is -2 coming from the terms $\mathfrak{h}_n \beta^{-1}$, $\partial_x \mathfrak{f}_0$ and $\beta^{-1} \mathfrak{f}_0 \partial_x \beta$. Thus, denoting by $z' = (z_1, \dots, z_{n-1}, 0)$, we obtain:

$$(\log \det B_q(s))''_{|s=0} = \left(\frac{\mathfrak{h}_n(x, z') - 2\mathfrak{f}_0 \partial_x \beta(x, z')}{\beta(x, z')} + \partial_x \mathfrak{f}_0(x, z') \right) + O(1)$$

$$= \left(\frac{-\partial_x^2 \beta(x, z') \beta(x, z') + (\partial_x \beta(x, z'))^2}{\beta^2(x, z')} \right) + O(1).$$

Now using (39), we can reason as in the 2-dimensional case, cf. (35), to conclude that

$$\left(\log \det B_q(s)\right)''(0) \xrightarrow{q \to (0, \mathbf{0})} +\infty.$$

Once again, also in this situation, the singularity in x of the quantity $\left(\log \det B_q(s)\right)''(0)$ is polynomial of order -2. Finally, using the same argument used in (36) for the 2-dimensional

case, we can show that the density of the measure $\mathfrak{m}$ produces a polynomial singularity of order -1, which is negligible as $q \to (0, \mathbf{0})$. Finally, we obtain

$$\left(\log \det h_q(s) \right)'' (0) \xrightarrow{q \to (0, \mathbf{0})} +\infty,$$

disproving the $CD(K, N)$ condition for any $K \in \mathbb{R}$ and $N \in (1, +\infty)$, as desired. $\qquad\square$

Remark 6.4 We stress once again that, thanks to the strongly regular assumption on M, the order of the structure (and thus the order of β) controls the orders in x, not only of the functions β_i, $\mathfrak{f}_i$ and $\mathfrak{h}_n$, but also of their derivatives in the z-directions. Below, we provide an example of regular (but not strongly regular) structure where the orders of the derivatives are not controlled by the order of β. Nevertheless our strategy to disprove the CD condition works.

In full generality, it is possible to prove that $(\log \det h_x(s))'' (0)$ actually diverges, however there is no criterion of determining the *sign* of the leading order, without requiring some additional regularity on the structure. On the other hand, as characteristic points encode the truly sub-Riemannian behavior of almost-Riemannian manifolds, we believe that our strategy should always be effective.

Example 6.5 Let $M = \mathbb{R}^4$ and in coordinates (x, z_1, z_2, z_3) consider the almost-Riemannian structure defined by the global vector fields

$$X_0 = \partial_x, \quad X_1 = \partial_{z_1} - \frac{z_2}{2} \partial_{z_3}, \quad X_2 = \partial_{z_2} + \frac{z_1}{2} \partial_{z_3}, \quad X_3 = x \partial_{z_3}.$$

The singular region is given by $\mathcal{Z} = \{x = 0\}$ and is an embedded hypersurface without characteristic points. Notice that M is regular, see [25, Definition 7.10] for the precise definition, but not strongly regular, thus we can not apply Theorem 6.3. Nevertheless, if we consider $\Sigma = \{z_3 = 0\}$, assumptions (H1) and (H2) are verified, therefore, we can apply our general strategy. Setting $\mathfrak{m} = \mathcal{L}^4$, an explicit computation leads to

$$\left(\log \det h_q(s) \right)'' (0) = \frac{8x^2 - 4(z_1^2 + z_2^2)}{(4x^2 + z_1^2 + z_2^2)^2}, \tag{41}$$

which diverges at $+\infty$ along the curve $(x, 0, 0, 0)$ as $x \to 0$, disproving the $CD(K, N)$ condition for any $K \in \mathbb{R}$ and $N \geq 1$. A few remarks are in order: first of all, the function $\beta(x, z_1, z_2) = 4x^2 + z_1^2 + z_2^2$ has order 2 in x but this is not true for its derivatives in the z-directions. Second of all, the numerator of (41) does not have a sign, highlighting the difficulties of the general case of determining the behavior of the leading term.

Data Availability All data generated or analysed during this study are included in this published article.

Appendix A. Explicit expression for $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$

In order to obtain an explicit expression for $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$, we study the Hamiltonian system associated with the sub-Riemannian Hamiltonian. Recall that in canonical coordinates induced by (x, z), given by (17), the Hamiltonian is

$$H(p_x, p_z; x, z) = \frac{1}{2}p_x^2 + \frac{1}{2}p_z^\mathsf{T} A^\mathsf{T} A(x, z) p_z,$$

where p_z is a shorthand for $(p_{z_1}, \ldots, p_{z_n})$. For the hypersurface $\Sigma \subset M$, given by (H1), from Lemma 2.9, we know that the unique minimizing geodesic for δ with initial point $q \in \Sigma \backslash C(\Sigma)$ has unique (up to a sign) initial covector:

$$\lambda(q) = \frac{1}{\beta(q)}dz_n, \quad \text{with} \quad \beta(q)^2 = \sum_{k=1}^{n} a_{kn}(q)^2$$

Thus, if $(x(s), z(s), p_x(s), p_z(s))^5$ is the solution to (26) with initial datum $(\lambda(q); q)$, we deduce that

$$\beta_0(q) = \dot{x}(0) = 0, \quad \beta_i(q) = \dot{z}_i(0) = \frac{1}{\beta(q)}\sum_{k=1}^{n} a_{ki}(q)a_{kn}(q) = \frac{\alpha_i(q)}{\beta(q)}, \quad \forall i = 1, \ldots, n, \quad (42)$$

having denoted $\nabla\delta(q) = (\beta_0(q), \ldots, \beta_n(q))$. Moreover, notice that by definition $\beta_n(q) = \beta(q)$. In an analogous way, we can compute $\mathfrak{f} = (\mathfrak{f}_0, \ldots, \mathfrak{f}_n)$:

$$\mathfrak{f}_0(q) = \ddot{x}(0) = -\frac{\partial_x\beta(q)}{\beta(q)}$$

$$\mathfrak{f}_i(q) = \ddot{z}_i(0) = \frac{1}{\beta^2(q)}\left[\sum_{l=1}^{n}\partial_{z_l}\alpha_i(q)\alpha_l(q) - \frac{1}{2}\sum_{j,k=1}^{n}a_{ki}(q)a_{kj}(q)\partial_{z_j}\beta^2(q)\right]. \quad (43)$$

Finally, taking the third-order derivatives in s of the solution to (26), we obtain $\mathfrak{h}$. Notice, however, that we only need the n-th component of $\mathfrak{h}$ in (38), thus:

$$\mathfrak{h}_n = \dddot{z}_n(0) = \frac{1}{\beta^3}\left[-\frac{(\partial_x\beta^2)^2}{2} + \sum_{j,r,l=1}^{n}\alpha_l\alpha_r\partial_{z_lz_r}^2(\beta^2) + \sum_{j,l=1}^{n}\beta^2\partial_{z_l}(\beta^2)\mathfrak{f}_l\right.$$

$$\left. - \sum_{j,l=1}^{n}\alpha_l\partial_{z_l}\alpha_j\partial_{z_j}(\beta^2) - \frac{1}{2}\sum_{j,l=1}^{n}\alpha_j\left(\alpha_l\partial_{z_jz_l}(\beta^2) - \partial_{z_j}\alpha_l\partial_{z_l}(\beta^2)\right)\right]. \quad (44)$$

References

1. Agrachev, A., Barilari, D., Boscain, U.: A comprehensive introduction to sub-Riemannian geometry. In: Volume 181 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2020). From the Hamiltonian viewpoint, With an appendix by Igor Zelenko

---

5 We drop the superscript q only in this section to ease the notation.

2. Agrachev, A., Barilari, D., Rizzi, L.: Curvature: a variational approach. Mem. Am. Math. Soc. **256**(1225):v+142 (2018)
3. Agrachev, A., Boscain, U., Sigalotti, M.: A Gauss–Bonnet-like formula on two-dimensional almost-Riemannian manifolds. Discrete Contin. Dyn. Syst. **20**(4), 801–822 (2008)
4. Albano, P., Cannarsa, P., Scarinci, T.: Regularity results for the minimum time function with Hörmander vector fields. J. Differ. Equ. **264**(5), 3312–3335 (2018)
5. Agrachev, A., Sachkov, Y.L.: Control theory from the geometric viewpoint. Volume 87 of Encyclopaedia of Mathematical Sciences, Control Theory and Optimization, II. Springer, Berlin (2004)
6. Boscain, U., Charlot, G., Gaye, M., Mason, P.: Local properties of almost-Riemannian structures in dimension 3. Discrete Contin. Dyn. Syst. **35**(9), 4115–4147 (2015)
7. Balogh, Z.M., Kristály, A., Sipos, K.: Jacobian determinant inequality on corank 1 Carnot groups with applications. J. Funct. Anal. **277**(12), 108293, 36, (2019)
8. Barilari, D., Rizzi, L.: Sharp measure contraction property for generalized H-type Carnot groups. Commun. Contemp. Math. **20**(6), 17500, 24,81 (2018)
9. Barilari, D., Rizzi, L.: Sub-Riemannian interpolation inequalities. Invent. Math. **215**(3), 977–1038 (2019)
10. Badreddine, Z., Rifford, L.: Measure contraction properties for two-step analytic sub-Riemannian structures and Lipschitz Carnot groups. Ann. Inst. Four. (Grenoble) **70**(6), 2303–2330 (2020)
11. Cavalletti, F.: Monge problem in metric measure spaces with Riemannian curvature-dimension condition. Nonlinear Anal. **99**, 136–151 (2014)
12. Cavalletti, F., Mondino, A.: Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. Invent. Math. **208**(3), 803–849 (2017)
13. Cavalletti, F., Mondino, A.: New formulas for the Laplacian of distance functions and applications. Anal. PDE **13**(7), 2091–2147 (2020)
14. Cavalletti, F., Milman, E.: The globalization theorem for the curvature-dimension condition. Invent. Math. **226**(1), 1–137 (2021)
15. Chitour, Y., Prandi, D., Rizzi, L.: Weyl's law for singular Riemannian manifolds (2019). arXiv preprint arXiv:1903.05639
16. Franchi, B., Hajłasz, P., Koskela, P.: Definitions of Sobolev classes on metric spaces. Ann. Inst. Four. (Grenoble) **49**(6), 1903–1924 (1999)
17. Franceschi, V., Prandi, D., Rizzi, L.: On the essential self-adjointness of singular sub-Laplacians. Potential Anal. **53**(1), 89–112 (2020)
18. Juillet, N.: On a method to disprove generalized Brunn–Minkowski inequalities. In: Probabilistic Approach to Geometry, Volume 57 of Adv. Stud. Pure Math., pp. 189–198. The Mathematical Society of Japan (2010)
19. Juillet, N.: Sub-Riemannian structures do not satisfy Riemannian Brunn–Minkowski inequalities. Rev. Mat. Iberoam. **37**(1), 177–188 (2021)
20. Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. Math. (2) **169**(3), 903–991 (2009)
21. Milman, E.: The quasi curvature-dimension condition with applications to sub-Riemannian manifolds. Commun. Pure Appl. Math. **74**(12), 2628–2674 (2021)
22. Mietton, T., Rizzi, L.: Branching geodesics in sub-Riemannian geometry. Geom. Funct. Anal. **30**(4), 1139–1151 (2020)
23. Magnabosco, M., Rossi, T.: An algorithmic procedure to disprove the CD condition for almost-Riemannian manifolds (2022). https://github.com/TRenghia/CD-on-AR-manifolds
24. Ohta, S.-I.: On the measure contraction property of metric measure spaces. Comment. Math. Helv. **82**(4), 805–828 (2007)
25. Prandi, D., Rizzi, L., Seri, M.: Quantum confinement on non-complete Riemannian manifolds. J. Spectr. Theory **8**(4), 1221–1280 (2018)
26. Rizzi, L.: Measure contraction properties of Carnot groups. Calc. Var. Part. Differ. Equ. **55**(3), 20 (2016)
27. Rizzi, L., Stefani, G.: Failure of curvature-dimension conditions on sub-Riemannian manifolds via tangent isometries (2023). arXiv preprint arXiv:2301.00735
28. Sturm, K.-T.: On the geometry of metric measure spaces. I. Acta Math. **196**(1), 65–131 (2006)
29. Sturm, K.-T.: On the geometry of metric measure spaces. II. Acta Math. **196**(1), 133–177 (2006)