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Calculus of Variations

Total positive curvature and the equality case in the relative isoperimetric inequality outside convex domains

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Abstract
We settle the case of equality for the relative isoperimetric inequality outside any arbitrary convex set with not empty interior.

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1 Introduction

In [5] Choe, Ghomi and Ritoré proved the following relative isoperimetric inequality outside convex sets, see also [14] for an alternative proof and [12] for a generalization to higher codimension.

Theorem 1.1 [5] *Let* $\mathbf{C} \subset \mathbb{R}^N$ *be a closed convex set with nonempty interior. For any set of finite perimeter and finite measure* $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ *we have*

$$P(\Omega; \mathbb{R}^N \setminus \mathbf{C}) \geq N\left(\frac{\omega_N}{2}\right)^{\frac{1}{N}} |\Omega|^{\frac{N-1}{N}} . \tag{1.1}$$

Moreover, if $\mathbf{C}$ *has a* C^2 *boundary and* Ω *is a bounded set for which the equality in* (1.1) *holds, then* Ω *is a half ball.*

Here and in what follows $P(\Omega; \mathbb{R}^N \setminus \mathbf{C})$ denotes the perimeter of a set Ω in $\mathbb{R}^N \setminus \mathbf{C}$ in the sense of De Giorgi. As observed by the authors in [5] the equality case for general, possibly nonsmooth, convex sets does not follow from their methods as it cannot be handled by a simple approximation argument. However there are many situations in which nonsmooth convex

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sets naturally appear. For instance, in models of vapor-liquid-solid-grown nanowires the nanotube is often described as a semi-infinite convex cylinder with sharp edges and possibly nonsmooth cross sections. In these models super-saturated liquid droplets correspond to isoperimetric regions for the relative perimeter outside the cylinder or more in general for the capillarity energy, see [11, 17]. Experimentally it is observed that in some regimes preferred configurations are given by spherical caps lying on the top facet of the cylinder. Understanding these phenomena from a mathematical point of view was our first motivation to study the equality cases in (1.1) also for nonsmooth convex obstacles, beside the intrinsic geometric interest of the problem.

The main result of this paper reads as follows.

Theorem 1.2 *(The equality case) Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior and let $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ be a set of finite perimeter such that equality holds in (1.1). Then Ω is a half ball supported on a facet of $\mathbf{C}$.*[1]

Observe that, compared to the last part of Theorem 1.1, here we don't have any restriction on the convex set $\mathbf{C}$ and we allow for possibly unbounded competitors. As in [5] the starting point in order the get the characterization of the equality case in (1.1) is an estimate of the positive total curvature $\mathcal{K}^+(\Sigma)$ of a hypersurface $\Sigma \subset \overline{\mathbb{R}^N \setminus \mathbf{C}}$ when the contact angle between $\partial\mathbf{C}$ and Σ is larger than or equal to a fixed $\theta \in (0, \pi)$. Here $\mathcal{K}^+(\Sigma)$ denotes, roughly speaking, the measure of the image of the Gauss map restricted to those points where there exists a support hyperplane, see Definition 1.3 below. To state more precisely our result we need to introduce some notation: Given $\theta \in (0, \pi)$ we denote by S_θ the spherical cap

$$S_\theta := \left\{ y \in \mathbb{S}^{N-1} : \ y \cdot e_N \geq \cos\theta \right\}.$$

Moreover, given $\Sigma \subset \overline{\mathbb{R}^N \setminus \mathbf{C}}$ and a point $x \in \Sigma$ we denote by $N_x\Sigma$ the *normal cone*

$$N_x\Sigma = \left\{ \nu \in \mathbb{S}^{N-1} : \ (y - x) \cdot \nu \leq 0 \text{ for all } y \in \Sigma \right\},$$

that is the set of (exterior) normals to support hyperplanes to Σ. We can now recall the definition of total positive curvature.

Definition 1.3 Let $\mathbf{C}$ be a closed convex set with not empty interior, $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ a bounded open set and $\Sigma := \overline{\partial\Omega \setminus \mathbf{C}}$. The *total positive curvature* of Σ is given by

$$\mathcal{K}^+(\Sigma) := \mathcal{H}^{N-1}\left(\bigcup_{x \in \Sigma \setminus \mathbf{C}} N_x\Sigma \right).$$

The aforementioned estimate on the total positive curvature is provided by the following theorem, which will be proved in Sect. 3.

Theorem 1.4 *Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set of class C^1, $\Omega \subset \mathbb{R}^N \setminus \mathbf{C}$ a bounded open set and $\Sigma := \overline{\partial\Omega \setminus \mathbf{C}}$. Let $\theta_0 \in (0, \pi)$ such that*

$$\nu \cdot \nu_{\mathbf{C}}(x) \leq \cos\theta_0 \quad \textit{whenever } x \in \Sigma \cap \mathbf{C}, \ \ \nu \in N_x\Sigma, \tag{1.2}$$

[1] Theorem 1.1 shows that the isoperimetric profile of $\mathbb{R}^N \setminus \mathbf{C}$ is always greater than or equal to the isoperimetric profile of a half space while Theorem 1.2 characterizes the equality case. Note that in general the isoperimetric profile of $\mathbb{R}^N \setminus \mathbf{C}$ is also greater than or equal to the one of $\mathbf{C}$, see [13, Theorem 6.18], where the equality case is also characterized.

where $\nu_{\mathbf{C}}(x)$ *stands for the outer unit normal to* $\mathbf{C}$ *at* x. *Then,*

$$\mathcal{K}^+(\Sigma) \geq \mathcal{H}^{N-1}(S_{\theta_0})\,. \tag{1.3}$$

Moreover, let $r > 0$ *be such that* $\Sigma \cap \mathbf{C} \subset B_r(0)$. *For any* $\varepsilon > 0$ *there exists* δ, *depending on* ε, θ_0 *and* r, *but not on* $\mathbf{C}$ *or* Ω, *such that if*

$$\nu \cdot \nu_{\mathbf{C}}(x) \leq \cos\theta_0 + \delta \quad \textit{whenever } x \in \Sigma \cap \mathbf{C},\ \ \nu \in N_x\Sigma, \tag{1.4}$$

and

$$\mathcal{K}^+(\Sigma) \leq \mathcal{H}^{N-1}(S_{\theta_0}) + \delta\,, \tag{1.5}$$

then $\Sigma \cap \mathbf{C}$ *is not empty,* $\mathrm{width}(\Sigma \cap \mathbf{C}) \leq \varepsilon$ *and more precisely* $\Sigma \cap \mathbf{C}$ *lies between two parallel* ε*-distant hyperplanes orthogonal to* $\nu_{\mathbf{C}}(x)$ *for some* $x \in \Sigma \cap \mathbf{C}$. *In particular, if* (1.2) *is satisfied and the equality in* (1.3) *holds, then* $\Sigma \cap \mathbf{C}$ *is not empty and lies on a support hyperplane to* $\mathbf{C}$.

Note that in the previous statement $\mathrm{width}(\Sigma \cap \mathbf{C})$ denotes the distance between the closest pair of parallel hyperplanes which contains $\Sigma \cap \mathbf{C}$ in between them, see (3.5). Even though the proof of this theorem follows the general strategy of [4] we are able to improve their result in three directions: (1) we consider a general contact angle $\theta_0 \in (0, \pi)$, whereas in [4] only the case $\theta_0 = \pi/2$ is considered; (2) we do not assume any regularity on Σ and the contact angle condition can be replaced by the weaker condition (1.2); (3) we get a stability estimate on the 'contact part' $\Sigma \cap \mathbf{C}$ which is *independent of the shape of the convex set* $\mathbf{C}$. As we will explain below (2) and (3) are crucial in the proof of Theorem 1.2.

As a consequence of independent interest of the previous theorem we prove a sharp inequality for the Willmore energy, see Theorem 3.9.

Before outlining our strategy of the proof of Theorem 1.2 we briefly recall how in [5] it is proven that a bounded set Ω_0 satisfying the equality in (1.1) is a half ball, when $\mathbf{C}$ is sufficiently smooth. There the idea is to consider the isoperimetric profile

$$I(m) = \inf\left\{P\left(E; \mathbb{R}^N \setminus \mathbf{C}\right) :\ E \subset \mathbb{R}^N \setminus \mathbf{C},\ |E| = m\right\},$$

defined for all $m \in (0, |\Omega_0|]$, and to show that $I(m) = N\left(\frac{\omega_N}{2}\right)^{\frac{1}{N}} m^{\frac{N-1}{N}}$, that is $I(m)$ coincides with the isoperimetric profile $I_{\mathscr{H}}(m)$ of the half space. Moreover, since $I'(|\Omega_0|) = H_\Sigma$, where H_Σ is the mean curvature of $\Sigma = \overline{\partial\Omega_0 \setminus \mathbf{C}}$,

$$\begin{aligned} I(|\Omega_0|)\big(I'(|\Omega_0|)\big)^{N-1} &= \int_{\Sigma\setminus\mathbf{C}} H_\Sigma^{N-1}\, d\mathcal{H}^{N-1} \geq (N-1)^{N-1}\mathcal{K}^+(\Sigma) \\ &\geq (N-1)^{N-1}\mathcal{H}^{N-1}(S_{\pi/2}) = I_{\mathscr{H}}(|\Omega_0|)\big(I'_{\mathscr{H}}(|\Omega_0|)\big)^{N-1}\,, \end{aligned} \tag{1.6}$$

where the first inequality follows from an application of coarea formula and the geometric-arithmetic mean inequality, see for instance the proof of Theorem 3.9, and the second one follows from the estimate of the total curvature proved in [5, Lemma 3.1]. Now, since $I(m) = I_{\mathscr{H}}(m)$ for all $m \in [0, |\Omega_0|]$, all the inequalities in (1.6) are equalities. In particular this implies that $\mathcal{K}^+(\Sigma) = \mathcal{H}^{N-1}(S_{\pi/2})$ and that Σ is umbilical. From this information, it is not difficult to see that Σ must be a half ball.

Note that in the proof of [5, Lemma 3.1] it is crucial that the regular part of Σ meets $\partial\mathbf{C}$ orthogonally and in a C^2 fashion. This can be inferred from the boundary regularity theory for perimeter minimizers which can be applied only if $\mathbf{C}$ is sufficiently smooth. Therefore the above argument fails for a general convex set.

In order to deal with this lack of regularity we implement a delicate argument based on the approximation of $\mathbf{C}$ with more regular convex sets.

Let us describe the argument more in detail. Denote by Ω_0 a set of finite perimeter satisfying the equality in (1.1). For $\eta > 0$ sufficiently small we approximate $\mathbf{C}$ with the closed η-neighborhood $\mathbf{C}_\eta = \mathbf{C} + \overline{B_\eta(0)}$, which is of class $C^{1,1}$. Now the idea is to consider the relative isoperimetric problem in $\mathbb{R}^N \setminus \mathbf{C}_\eta$. In order to force the minimizers to converge to Ω_0 when $\eta \to 0$ and the prescribed mass m converges to $|\Omega_0|$, we introduce the following constrained isoperimetric profiles with obstacle Ω_0:

$$I_\eta(m) = \min\left\{P\left(E; \mathbb{R}^N \setminus \mathbf{C}_\eta\right) : E \subset \Omega_0 \setminus \mathbf{C}_\eta,\ |E| = m\right\} \tag{1.7}$$

for all $m \in (0, |\Omega_0 \setminus \mathbf{C}_\eta|]$. Denote by $\Omega_{\eta,m}$ a minimizer of the above problem and set $\Sigma_{\eta,m} := \overline{\partial\Omega_{\eta,m} \setminus \mathbf{C}_\eta}$. Note that in the general N-dimensional case, both the obstacle Ω_0 and the minimizers $\Omega_{\eta,m}$ may have singularities. Thus, despite the fact that $\partial\mathbf{C}_\eta$ is of class $C^{1,1}$, we cannot apply the known boundary regularity results at the points $x \in \partial\Omega_{\eta,m} \cap \partial\mathbf{C}_\eta \cap \partial\Omega_0$.

However, one useful observation is that $\Omega_{\eta,m}$ is a *restricted* Λ*-minimizer*, i.e., a Λ-minimizer with respect to perturbations that do not increase the "wet part" $\partial\Omega_{\eta,m} \cap \mathbf{C}_\eta$ (see Definition 4.1 below), with a $\Lambda > 0$ which can be made uniform with respect to η and locally uniform with respect to m (see Steps 1 and 2 of the proof of Theorem 1.2). Another important observation is that restricted Λ-minimizers satisfy uniform volume density estimates up to the boundary $\partial\mathbf{C}_\eta$. All these facts are combined to show that the constrained isoperimetric profiles (1.7) are Lipschitz continuous and that their derivatives coincide a.e. with the constant mean curvature $H_{\Sigma^*_{\eta,m}}$ of the regular part $\Sigma^*_{\eta,m}$ of $\Sigma_{\eta,m} \setminus \partial\Omega_0$ (see Steps 3 and 4).

As in the argument of [5] another important ingredient is represented by the inequality

$$\mathcal{K}^+(\Sigma_{\eta,m}) \geq \mathcal{H}^{N-1}(S_{\pi/2}) = \frac{1}{2}N\omega_N\,, \tag{1.8}$$

which would hold by [5, Lemma 3.1] if we could show that $\Sigma_{\eta,m}$ meets $\partial\mathbf{C}_\eta$ orthogonally and in a sufficiently smooth fashion. However, as already observed, due the possible presence of boundary singularities at $\partial\Omega_{\eta,m} \cap \partial\mathbf{C}_\eta \cap \partial\Omega_0$ we cannot show that the aforementioned orthogonality condition is attained in a classical sense. An important step of our argument, which allows us to overcome this difficulty, consists in showing that restricted Λ-minimizers satisfy the $\pi/2$ contact angle condition with respect to $\partial\mathbf{C}_\eta$ in a "viscosity" sense, namely that the following weak Young's law holds:

$$\nu \cdot \nu_{\mathbf{C}_\eta}(x) \leq 0 \quad \text{whenever } x \in \Sigma_{\eta,m} \cap \mathbf{C}_\eta,\ \ \nu \in N_x\Sigma_{\eta,m}\,. \tag{1.9}$$

This is achieved in Step 5 by combining a blow-up argument with a variant of the Strong Maximum Principle that we adapted from [9]. In turn, owing to (1.9) we may apply Theorem 1.4 to obtain (1.8). Having established the latter and with some extra work we can show that $I_\eta(m) \to I_{\mathscr{H}}(m)$ as $\eta \to 0$ for every $m \in (0, |\Omega_0|)$, where we recall $I_{\mathscr{H}}(m) = N\big(\frac{\omega_N}{2}\big)^{\frac{1}{N}} m^{\frac{N-1}{N}}$ is the isoperimetric profile of the half space (see Steps 6 and 7).

With the convergence of the isoperimetric profiles I_η at hand and using again (1.8), we can then prove that for a.e. $m \in (0, |\Omega_0|)$

$$\mathcal{K}^+(\Sigma_{\eta,m}) \to \frac{1}{2}N\omega_N\,, \tag{1.10}$$

and thus $\Sigma_{\eta,m}$ almost satisfies the case of equality in (1.3) for η sufficiently small. Thanks to the last part of Theorem 1.4 we may then infer that $\Sigma_{\eta,m} \cap \mathbf{C}_\eta$ is almost flat and with some extra work that the whole wet part $\partial\Omega_{\eta,m} \cap \mathbf{C}_\eta$ has the same property. By showing that for suitable sequences $m_n \nearrow |\Omega_0|$ and $\eta_n \searrow 0$, $\partial\Omega_{\eta,m_n} \cap \mathbf{C}_\eta \to \partial\Omega_0 \cap \mathbf{C}$ in the Hausdorff sense, we may finally conclude that $\partial\Omega_0 \cap \mathbf{C}$ is flat and lies on a facet of $\mathbf{C}$ (see Step 8). We highlight here that in all the above argument it is crucial that the stability estimate on the width of $\Sigma_{\eta,m} \cap \mathbf{C}_\eta$ provided by our version Theorem 1.4 is independent of the shape of the convex set $\mathbf{C}_\eta$.

Having established that the wet part $\partial\Omega_0 \cap \mathbf{C}$ is flat, more work is still needed in the final step of the proof to deduce again from (1.10) that Ω_0 is umbilical and in turn a half ball supported on a facet of $\mathbf{C}$.

The paper is organized as follows: in Sect. 2 we collect a few known results of the regularity theory of perimeter quasi minimizers needed in the paper. In Sect. 3 we prove Theorem 1.4, while the proof of Theorem 1.2 occupies the whole Sect. 4 with some of the most technical steps outsourced to Sect. 6. Section 5 contains further regularity properties of restricted Λ-minimizers that are needed in the proof of the main result and the proof of the version of the Strong Maximum Principle needed here.

2 Preliminaries

Throughout the paper we denote by $B_r(x)$ the ball in $\mathbb{R}^N$ of center x and radius $r > 0$. In the following we shall often deal with sets of finite perimeter. For the definition and the basic properties of sets of (locally) finite perimeter we refer to the books [3, 15]. Here we fix some notation for later use. Given $E \subset \mathbb{R}^N$ of locally finite perimeter and a Borel set G we denote by $P(E; G)$ the perimeter of E in G. The *reduced boundary* of E will be denoted by $\partial^* E$, while $\partial^e E$ will stand for the *essential boundary* defined as

$$\partial^e E := \mathbb{R}^N \setminus \big(E^{(0)} \cup E^{(1)}\big),$$

where $E^{(0)}$ and $E^{(1)}$ are the sets of points where the density of E is 0 and 1, respectively. Moreover, we denote by ν_E the *generalized exterior normal* to E, which is well defined at each point of $\partial^* E$, and by μ_E the *Gauss-Green measure* associated to E

$$\mu_E := \nu_E \, \mathcal{H}^{N-1} \llcorner \partial^* E \,. \tag{2.1}$$

In the following, when dealing with a set of locally finite perimeter E, we shall always tacitly assume that E coincides with a precise representative that satisfies the property $\partial E = \overline{\partial^* E}$, see [15, Remark 16.11]. A possible choice is given by $E^{(1)}$ for which one may easily check that

$$\partial E^{(1)} = \overline{\partial^* E} \,. \tag{2.2}$$

We recall the well known notion of perimeter (Λ, r_0)-*minimizer* and the main properties which will be used here.

Definition 2.1 Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that a set of locally finite perimeter $E \subset \mathbb{R}^N$ is a *perimeter* (Λ, r_0)-*minimizer* in Ω, $\Lambda \geq 0$ and $r_0 > 0$, if for any ball $B_r(x_0) \subset \Omega$, with $0 < r \leq r_0$ and any $F \subset \mathbb{R}^N$ such that $E \Delta F \subset\subset B_r(x_0)$ we have

$$P(E; B_r(x_0)) \leq P(F; B_r(x_0)) + \Lambda |E \Delta F| \,.$$

In order to state a useful compactness theorem for Λ-minimizers we recall that a sequence $\{\mathcal{C}_n\}$ of closed sets converge in the *Kuratoswki sense* to a closed set $\mathcal{C}$ if the following conditions are satisfied:

(i) if $x_n \in \mathcal{C}_n$ for every n, then any limit point of $\{x_n\}$ belongs to $\mathcal{C}$;
(ii) any $x \in \mathcal{C}$ is the limit of a sequence $\{x_n\}$ with $x_n \in \mathcal{C}_n$.

One can easily see that $\mathcal{C}_n \to \mathcal{C}$ in the sense of Kuratowski if and only if $\mathrm{dist}(\cdot, \mathcal{C}_n) \to \mathrm{dist}(\cdot, \mathcal{C})$ locally uniformly in $\mathbb{R}^N$. In particular, by the Arzelà-Ascoli Theorem any sequence of closed sets admits a subsequence which converge in the sense of Kuratowski.

Throughout the paper, with a common abuse of notation, we write $E_h \to E$ in L^1 (L^1_{loc}) instead of $\chi_{E_h} \to \chi_E$ in L^1 (L^1_{loc}). Moreover, given a sequence of Radon measures μ_h in an open set Ω, we say that $\mu_h \overset{*}{\rightharpoonup} \mu$ *weakly* in* Ω *in the sense of measures* if

$$\int_\Omega \varphi \, d\mu_h \to \int_\Omega \varphi \, d\mu \qquad \text{for all } \varphi \in C^0_c(\Omega)\,.$$

Next theorem is a well known result, see for instance [15, Ch. 21].

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\{E_n\}$ a sequence of locally finite perimeter sets contained in Ω satisfying the following property: there exists $r_0 > 0$ such that for every n, E_n is a perimeter (Λ_n, r_0)-minimizer in Ω, with $\Lambda_n \to \Lambda \in [0, +\infty)$. Then there exist $E \subset \Omega$ of locally finite perimeter and a subsequence $\{n_k\}$ such that*

(i) *E is a (Λ, r_0)-minimizer in Ω;*
(ii) *$E_{n_k} \to E$ in $L^1_{loc}(\Omega)$,*
(iii) *$\partial E_{n_k} \to \mathcal{C}$ in the Kuratowski sense for some closed set $\mathcal{C}$ such that $\mathcal{C} \cap \Omega = \partial E \cap \Omega$;*
(iv) *$\mathcal{H}^{N-1} \llcorner (\partial E_{n_k} \cap \Omega) \overset{*}{\rightharpoonup} \mathcal{H}^{N-1} \llcorner (\partial E \cap \Omega)$ weakly* in Ω in the sense of measures.*

Remark 2.3 From the definition of Kuratowski convergence it is not difficult to see that (ii) and (iii) of Theorem 2.2 imply that, up to extracting a further subsequence if needed, $\overline{E_{n_k}} \to K$ in the sense of Kuratowski, with $K \cap \Omega = \overline{E} \cap \Omega$.

Definition 2.4 Given a set of locally finite perimeter E, we say that a function $h \in L^1_{loc}(\partial^* E)$ is the *weak mean curvature* of E if for any vector field $X \in C^1_c(\mathbb{R}^N; \mathbb{R}^N)$ we have

$$\int_{\partial^* E} \mathrm{div}_\tau X \, d\mathcal{H}^{N-1} = \int_{\partial^* E} h \, X \cdot \nu_E, d\mathcal{H}^{N-1}\,,$$

where $\mathrm{div}_\tau X := \mathrm{div} X - (\partial_{\nu_E} X) \cdot \nu_E$ stands for the tangential divergence of X along $\partial^* E$. If such an h exists we will denote it by $H_{\partial E}$.

Note that if ∂E is of class C^2 then $H_{\partial E}$ coincides with the classical mean curvature, or more precisely with the sum of all principal curvatures. In particular, if E coincides locally with the subgraph of a function u of class C^2 then locally

$$H_{\partial E} = -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right).$$

Concerning the above mean curvature operator, we recall the following useful Strong Maximum Principle, see for instance [18, Th. 2.3], which covers a more general class of quasilinear equations.

Theorem 2.5 *Let $\Omega \subset \mathbb{R}^{N-1}$ be a connected open set and let $u, v \in C^2(\Omega)$ such that $u \le v$ and*

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \lambda = \mathrm{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right)$$

for some constant $\lambda \in \mathbb{R}$. If $u(x_0) = v(x_0)$ for some $x_0 \in \Omega$, then $u \equiv v$.

We recall the following classical regularity result for Λ-minimizers.

Theorem 2.6 *Let E be a perimeter (Λ, r_0)-minimizer in some open set $\Omega \subset \mathbb{R}^N$. Then*

(i) *$\partial^* E \cap \Omega$ is a hypersurface of class $C^{1,\alpha}$ for every $\alpha \in (0, 1)$, relatively open in $\partial E \cap \Omega$. Moreover, $\dim_{\mathcal{H}}((\partial E \setminus \partial^* E) \cap \Omega) \leq N - 8$, where $\dim_{\mathcal{H}}$ stands for the Hausdorff dimension;*
(ii) *$H_{\partial E} \in L^\infty(\partial^* E \cap \Omega)$, with $\|H_{\partial E}\|_{L^\infty} \leq \Lambda$, and thus $\partial^* E \cap \Omega$ is of class $W^{2,p}$ for all $p \geq 1$;*
(iii) *if there exists a C^1 hypersurface Σ touching ∂E at $x \in \Omega$ and lying on one side with respect to ∂E in a neighborhood of x, then $x \in \partial^* E$.*

Items (i) and (ii) are classical, see for instance Theorems 21.8 and 28.1 in [15] for (i) and Theorem 4.7.4 in [2] for (ii).

Concerning (iii) one can show that under the assumption on x the minimal cone obtained by blowing up E around x is contained in a half space. For the existence of such a minimal cone see Theorem 28.6 in [15]. Since any minimal cone contained in a half space is a half space, see for instance [8, Lemma 3], it follows that x is a regular point.

The so-called ε-regularity theory for Λ-minimizers underlying the proof of the above theorem yields that sequences of Λ-minimizers E_h converging in L^1 to a smooth set E are regular for h large and in fact converge in a stronger sense. More precisely, we have the following result, which is well known to the experts.

Theorem 2.7 *Let E_n, E be (Λ, r_0)-minimizers in an open set $\Omega \subset \mathbb{R}^N$ such that $E_n \to E$ in $L^1_{loc}(\Omega)$. Let $x \in \partial^* E \cap \Omega$. Then, up to rotations and translations, there exist a $(N-1)$-dimensional open ball $B' \subset \mathbb{R}^{N-1}$, functions $\varphi_n, \varphi \in W^{2,p}(B')$ for all $p \geq 1$, and $r > 0$ such that $x \in B' \times (-r, r)$ and for n large*

$$\begin{aligned} &\partial E_n \cap (B' \times (-r, r)) = \{(x', \varphi_n(x')) : x' \in B'\}, \\ &\partial E \cap (B' \times (-r, r)) = \{(x', \varphi(x')) : x' \in B'\}, \\ &\varphi_n \to \varphi \quad \text{in } C^{1,\alpha}(\overline{B'}) \text{ for some } \alpha \in (0, 1)\,. \end{aligned} \tag{2.3}$$

Moreover, $H_{\partial E_n}(x', \varphi_n(x')) \overset{*}{\rightharpoonup} H_{\partial E}(x', \varphi(x'))$ in $L^\infty(B')$ and thus $\varphi_n \rightharpoonup \varphi$ in $W^{2,p}(B')$ for all $p \geq 1$.

Properties stated in (2.3) follow from the classical ε-regularity theory, see [22, Th. 1.9] (see also the arguments of Lemma 3.6 in [6]). The last part of the statement then easily follows from Theorem 2.6-(ii) combined with the classical Calderón-Zygmund estimates.

3 An estimate of the total positive curvature

This section is mainly devoted to the proof of Theorem 1.4 and to some applications.

We recall that a set $X \subset \mathbb{S}^{N-1}$ is called *spherically convex* (in short *convex*) if it is geodesically convex, that is, for any pair of points $x_1, x_2 \in X$ there exists a distance minimizing geodesic connecting x_1 and x_2 contained in X.

If $x \in \mathbb{S}^{N-1}$ and $\theta \in (0, \pi)$ we denote by $S_{\theta,x}$ the spherical cap

$$S_{\theta,x} := \{y \in \mathbb{S}^{N-1} : x \cdot y \geq \cos\theta\}\,.$$

If $x = e_N$ we shall simply write S_θ instead of S_{θ,e_N}. Note that $S_{\pi-\theta,-x}$ coincides with $(\mathbb{S}^{N-1} \setminus S_{\theta,x}) \cup \partial S_{\theta,x}$, where $\partial S_{\theta,x}$ denotes the relative boundary of $S_{\theta,x}$ in $\mathbb{S}^{N-1}$. We recall that

$$\mathcal{H}^{N-1}(S_\theta) = (N-1)\omega_{N-1}\int_0^\theta \sin^{N-2}\sigma\, d\sigma\,,$$

and $\mathcal{H}^{N-1}(\mathbb{S}^{N-1}) = N\omega_N$, where ω_N is the measure of the unit ball.

The following lemma extends [4, Proposition 3.1] to general angles.

Lemma 3.1 *Let $X \subset \mathbb{S}^{N-1}$ be spherically convex and closed, with $\mathcal{H}^{N-1}(X) > 0$, let $\theta \in (0,\pi)$ and fix $x \in X$. Then we have*

$$\mathcal{H}^{N-1}(X \cap S_{\theta,x}) \geq \frac{\mathcal{H}^{N-1}(S_\theta)}{N\omega_N}\mathcal{H}^{N-1}(X)\,. \tag{3.1}$$

Moreover, the equality holds if and only if $-x \in X$. Finally, given $\theta_0 \in (0,\pi)$, for every $\varepsilon > 0$ there exists $\delta > 0$, independent of X, such that if $\theta \in [\theta_0/2, \theta_0]$, then

$$\mathcal{H}^{N-1}(X \cap S_{\theta,x}) \leq \left(\frac{\mathcal{H}^{N-1}(S_\theta)}{N\omega_N} + \delta\right)\mathcal{H}^{N-1}(X) \quad \textit{implies} \quad \mathrm{dist}(-x, X) \leq \varepsilon\,.$$

Proof We denote by A the subset of $S_{\theta,x}$ obtained by taking the union of all the minimal geodesics connecting x with the points of $X \cap \partial S_{\theta,x}$. Let $B := S_{\theta,x} \setminus A$. Similarly denote by A^- the subset of $S_{\pi-\theta,-x}$ obtained by taking the union of all the minimal geodesics connecting $-x$ with the points of $X \cap \partial S_{\theta,x} = X \cap \partial S_{\pi-\theta,-x}$, and set $B^- := S_{\pi-\theta,-x} \setminus A^-$.

Assume first that $\mathcal{H}^{N-1}(A) > 0$. We note that

$$\frac{\mathcal{H}^{N-1}(A^-)}{\mathcal{H}^{N-1}(A)} = \frac{\mathcal{H}^{N-1}(S_{\pi-\theta})}{\mathcal{H}^{N-1}(S_\theta)}\,.$$

Thus, we have

$$\mathcal{H}^{N-1}(X \cap A) = \mathcal{H}^{N-1}(A) = \frac{\mathcal{H}^{N-1}(S_\theta)}{\mathcal{H}^{N-1}(S_{\pi-\theta})}\mathcal{H}^{N-1}(A^-) \geq \frac{\mathcal{H}^{N-1}(S_\theta)}{\mathcal{H}^{N-1}(S_{\pi-\theta})}\mathcal{H}^{N-1}(X \cap A^-)\,.$$

Note now that $X \cap B^- = \emptyset$. Indeed, if $y \in X \cap B^-$, then the geodesic connecting y to x is contained in X and intersects $\partial S_{\theta,x}$ at a point $z \in X \cap \partial S_{\theta,x}$. It follows in turn that y belongs to the geodesic connecting z with $-x$, and thus $y \in A^-$, which is a contradiction. Therefore,

$$\begin{aligned}\mathcal{H}^{N-1}(X \cap S_{\theta,x}) &= \mathcal{H}^{N-1}(A) + \mathcal{H}^{N-1}(X \cap B) \geq \mathcal{H}^{N-1}(A)\\ &= \frac{\mathcal{H}^{N-1}(S_\theta)}{\mathcal{H}^{N-1}(S_{\pi-\theta})}\mathcal{H}^{N-1}(A^-)\\ &\geq \frac{\mathcal{H}^{N-1}(S_\theta)}{\mathcal{H}^{N-1}(S_{\pi-\theta})}\mathcal{H}^{N-1}(X \cap S_{\pi-\theta,-x})\,.\end{aligned} \tag{3.2}$$

From this inequality (3.1) follows, recalling that $\mathcal{H}^{N-1}(\mathbb{S}^{N-1}) = N\omega_N$.

If instead $\mathcal{H}^{N-1}(A) = 0$, then $\mathcal{H}^{N-1}(X \setminus S_{\theta,x}) = 0$ and thus (3.1) holds trivially.

If (3.1) holds with the equality, then $\mathcal{H}^{N-1}(A) > 0$ and all the inequalities in (3.2) are equalities. In particular, $\mathcal{H}^{N-1}(A^-) = \mathcal{H}^{N-1}(X \cap S_{\pi-\theta,-x}) > 0$. In turn, recalling that $X \cap S_{\pi-\theta,-x} \subset A^-$ and by closedness we deduce that $-x \in X$. Conversely, if $-x \in X$ then by spherical convexity we have $A^- = X \cap S_{\pi-\theta,-x}$ and also $X \cap B = \emptyset$ since otherwise any geodesic connecting a point $y \in X \cap B$ to $-x$ would intersect $\partial S_{\theta,x} \cap X$, thus implying that y belongs to A, a contradiction. Therefore all the inequalities in (3.2) are equalities and the conclusion follows.

To establish the last part, we argue by contradiction assuming that there exist $\varepsilon > 0$, a sequence of closed spherically convex sets $X_n \ni x$ such that $\mathcal{H}^{N-1}(X_n) > 0$ and a sequence $\theta_n \in [\theta_0/2, \theta_0]$ converging to θ' such that

$$\mathcal{H}^{N-1}(X_n \cap S_{\theta_n,x}) \le \left(\frac{\mathcal{H}^{N-1}(S_{\theta_n})}{N\omega_N} + \frac{1}{n}\right)\mathcal{H}^{N-1}(X_n) \quad \text{but} \quad \operatorname{dist}(-x, X_n) \ge \varepsilon\,. \tag{3.3}$$

We denote by A_n and by A_n^- the sets corresponding to X_n and $S_{\theta_n,x}$ defined as above. Note that $X_n = (X_n \cap S_{\theta_n,x}) \cup (X_n \cap A_n^-)$. From (3.3) it follows that $\mathcal{H}^{N-1}(A_n),\ \mathcal{H}^{N-1}(A_n^-) > 0$ for n large and

$$\frac{\mathcal{H}^{N-1}(X_n \cap S_{\theta_n,x})}{\mathcal{H}^{N-1}(X_n \cap A_n^-)} \le \frac{\mathcal{H}^{N-1}(S_{\theta_n})}{\mathcal{H}^{N-1}(S_{\pi-\theta_n})} + O\left(\tfrac{1}{n}\right).$$

Since $\mathcal{H}^{N-1}(X_n \cap S_{\theta_n,x}) \ge \mathcal{H}^{N-1}(A_n) \ge \frac{\mathcal{H}^{N-1}(S_{\theta_n})}{\mathcal{H}^{N-1}(S_{\pi-\theta_n})}\mathcal{H}^{N-1}(X_n \cap A_n^-)$, it follows that

$$\lim_{n\to\infty} \frac{\mathcal{H}^{N-1}(X_n \cap S_{\theta_n,x})}{\mathcal{H}^{N-1}(X_n \cap S_{\pi-\theta_n,-x})} = \lim_{n\to\infty} \frac{\mathcal{H}^{N-1}(A_n)}{\mathcal{H}^{N-1}(X_n \cap A_n^-)} = \frac{\mathcal{H}^{N-1}(S_{\theta'})}{\mathcal{H}^{N-1}(S_{\pi-\theta'})}\,. \tag{3.4}$$

Note that we have

$$\frac{\mathcal{H}^{N-1}(S_{\theta_n})}{\mathcal{H}^{N-1}(S_{\pi-\theta_n})} = \frac{\mathcal{H}^{N-1}(A_n)}{\mathcal{H}^{N-1}(A_n^-)} \to \frac{\mathcal{H}^{N-1}(S_{\theta'})}{\mathcal{H}^{N-1}(S_{\pi-\theta'})}$$

and thus, from (3.4) we get

$$\lim_{n\to\infty} \frac{\mathcal{H}^{N-1}(A_n^-)}{\mathcal{H}^{N-1}(X_n \cap A_n^-)} = 1\,,$$

which clearly contradicts the fact that by the second inequality in (3.3) we easily infer that $\mathcal{H}^{N-1}(A_n^- \setminus X_n) \ge C(\varepsilon)\mathcal{H}^{N-1}(A_n^-)$, for a positive constant $C(\varepsilon)$ depending only on ε. □

Next we adapt to our case [4, Proposition 4.2]. To this aim we recall some preliminary definitions.

Definition 3.2 Given a set $X \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$ the *unit normal cone* of X at x is the (possibly empty) set defined as

$$N_x X := \{\nu \in \mathbb{S}^{N-1} : (y - x)\cdot \nu \le 0 \text{ for all } y \in X\}\,.$$

Any hyperplane passing through x and orthogonal to a direction $\nu \in N_x X$ is called a *support hyperplane for X with outward normal ν*. In turn, we define the corresponding *normal bundle* of X as

$$NX := \bigcup_{x\in X} N_x X\,.$$

Given a map $\sigma : X \to \mathbb{S}^{N-1}$ and $\theta \in (0, \pi)$ we introduce the following *restricted normal cone* and *restricted normal bundle* respectively as

$$N_x^{\sigma,\theta} X := N_x X \cap S_{\theta,\sigma(x)} \qquad \text{and} \qquad N^{\sigma,\theta} X := \bigcup_{x\in X} N_x^{\sigma,\theta} X\,.$$

Moreover, we say that a point $x \in X$ is *exposed* if there exists a support hyperplane Π passing through x such that $X \cap \Pi = \{x\}$. Finally, we denote by $\operatorname{width}(X)$ the distance between the closest pair of parallel hyperplanes which contains X in between them, i.e.,

$$\mathrm{width}(X) = \inf_{\nu \in \mathbb{S}^{N-1}} \big(\sup\{x \cdot \nu : x \in X\} - \inf\{x \cdot \nu : x \in X\} \big) . \tag{3.5}$$

Lemma 3.3 *Let $r > 0$ and let $X = \{x_1, \ldots, x_k\} \subset B_r(0)$. Let $\sigma : X \to \mathbb{S}^{N-1}$ be such that $\sigma(x_i) \in N_{x_i}X$ whenever $N_{x_i}X$ is nonempty. Then*

$$\mathcal{H}^{N-1}(N^{\sigma,\theta}X) \geq \mathcal{H}^{N-1}(S_\theta) . \tag{3.6}$$

Moreover, equality holds in (3.6) if and only if X lies in a hyperplane Π such that $\sigma(x_i) \perp \Pi$ whenever x_i is exposed. Finally, given $\theta_0 \in (0, \pi)$, for every $\varepsilon > 0$ there exists $\delta > 0$ (depending also on $r > 0$ and θ_0, but not on σ and not on X) such that if $\theta \in [\theta_0/2, \theta_0]$, then

$$\mathcal{H}^{N-1}(N^{\sigma,\theta}X) \leq \mathcal{H}^{N-1}(S_\theta) + \delta \quad \textit{implies} \quad \mathrm{width}(X) \leq \varepsilon \tag{3.7}$$

and more precisely there exist an exposed point $x \in X$ and two parallel hyperplanes orthogonal to $\sigma(x)$ with mutual distance equal to ε such that X lies between them.

Proof The proof is essentially the same as for [4, Proposition 4.2], using Lemma 3.1 in place of [4, Proposition 3.1]. We give the argument for the sake of completeness. Owing to the compactness of X, for every $\nu \in \mathbb{S}^{N-1}$ there exists a support hyperplane to X with outward normal equal to ν. Thus, $NX = \mathbb{S}^{N-1}$. Observe also that $\nu \in \mathrm{int}_{\mathbb{S}^{N-1}}(N_{x_i}X)$ if and only if the hyperplane orthogonal to ν and passing through x_i is a support hyperplane intersecting X only at x_i (and thus x_i is exposed). In turn, if $i \neq j$ we have

$$\mathrm{int}_{\mathbb{S}^{N-1}}(N_{x_i}X) \cap \mathrm{int}_{\mathbb{S}^{N-1}}(N_{x_j}X) = \emptyset .$$

Since by [4, Lemma 4.1] every $N_{x_i}X$ with nonvanishing $\mathcal{H}^{N-1}$-measure is spherically convex, we may invoke Lemma 3.1 to conclude that

$$\mathcal{H}^{N-1}(N^{\sigma,\theta}X) = \sum_{i=1}^{k} \mathcal{H}^{N-1}(N^{\sigma,\theta}_{x_i}X) \geq \frac{\mathcal{H}^{N-1}(S_\theta)}{N\omega_N} \sum_{i=1}^{k} \mathcal{H}^{N-1}(N_{x_i}X) = \mathcal{H}^{N-1}(S_\theta) , \tag{3.8}$$

thus establishing (3.6).

If equality holds in (3.6), then the above inequality is an equality and in particular

$$\mathcal{H}^{N-1}(N^{\sigma,\theta}_{x_i}X) = \frac{\mathcal{H}^{N-1}(S_\theta)}{N\omega_N}\mathcal{H}^{N-1}(N_{x_i}X) \tag{3.9}$$

whenever $\mathcal{H}^{N-1}(N_{x_i}X) > 0$, that is whenever x_i is exposed. Therefore, by Lemma 3.1 $N^{\sigma,\theta}_{x_i}X$ contains both $\sigma(x_i)$ and $-\sigma(x_i)$ and thus X lies in the hyperplane orthogonal to $\sigma(x_i)$ and passing through x_i. Conversely, if X lies in a hyperplane orthogonal to $\sigma(x_i)$, for every x_i exposed, then also $-\sigma(x_i) \in N_{x_i}X$ and thus by Lemma 3.1 (3.9) holds for all x_i exposed. And thus equality holds also in (3.8).

To prove (3.7) and the last part of the lemma, let $X_n = \{x_1^n, \ldots, x_{k_n}^n\} \subset B_r(0)$ and let $\sigma_n : X_n \to \mathbb{S}^{N-1}$, with $\sigma_n(x_i^n) \in N_{x_i^n}X_n$ whenever x_i^n is exposed, $\theta_n \in [\theta_0/2, \theta_0]$ be such that

$$\mathcal{H}^{N-1}(N^{\sigma_n,\theta_n}X_n) - \mathcal{H}^{N-1}(S_{\theta_n}) \to 0 .$$

Arguing as for (3.8) we then have, in particular, that for every $n \in \mathbb{N}$ there exists $i_n \in \{1, \dots, k_n\}$ such that

$$\frac{\mathcal{H}^{N-1}\left(N^{\sigma_n,\theta_n}_{x^n_{i_n}} X_n\right)}{\mathcal{H}^{N-1}\left(N_{x^n_{i_n}} X_n\right)} - \frac{\mathcal{H}^{N-1}(S_{\theta_n})}{N\omega_N} \to 0\,.$$

By Lemma 3.1 this implies that $\mathrm{dist}(-\sigma_n(x^n_{i_n}), N_{x^n_{i_n}} X_n) \to 0$. From this, owing to the equiboundedness of the X_n's it follows that for every $k \in \mathbb{N}$ and for n large enough X_n lies between the two parallel hyperplanes orthogonal to $\sigma_n(x^n_{i_n})$ and passing through the points $x^n_{i_n}$ and $x^n_{i_n} - \frac{1}{k}\sigma_n(x^n_{i_n})$. In particular, $\mathrm{width}(X_n) \to 0$. □

Next proposition extends the previous lemma to the case of a general compact set X and a continuous map σ.

Proposition 3.4 *Let $X \subset B_r(0)$ be a compact set.*

Let $\sigma : X \to \mathbb{S}^{N-1}$ be a continuous map such that $\sigma(x) \in N_x X$ for all $x \in X$ such that $N_x X \neq \emptyset$. Then,

$$\mathcal{H}^{N-1}(N^{\sigma,\theta} X) \geq \mathcal{H}^{N-1}(S_\theta) \tag{3.10}$$

and if equality holds, then X lies in a hyperplane Π which is orthogonal to $\sigma(x)$ for some $x \in X$. Moreover, given $\theta_0 \in (0, \pi)$ and $\varepsilon > 0$ there exists $\delta_0 > 0$ (depending also on r and θ_0, but not on σ and not on X) such that if $\theta \in [\theta_0/2, \theta_0]$, then

$$\mathcal{H}^{N-1}(N^{\sigma,\theta} X) \leq \mathcal{H}^{N-1}(S_\theta) + \delta_0 \quad \textit{implies} \quad \mathrm{width}(X) \leq \varepsilon \tag{3.11}$$

and more precisely there exist $x \in X$ and two parallel hyperplanes orthogonal to $\sigma(x)$, with mutual distance equal to ε such that X lies between them.

Proof Let $\{X_i\}_{i\in\mathbb{N}}$ be an increasing sequence of discrete subsets of X such that $X_i \to X$ in the Hausdorff sense. We claim that

$$\chi_{N^{\sigma,\theta} X} \geq \limsup_i \chi_{N^{\sigma,\theta} X_i} \quad \text{pointwise in } \mathbb{S}^{N-1}\,. \tag{3.12}$$

To this aim let $\nu \notin N^{\sigma,\theta} X$ and assume by contradiction that (3.12) does not hold at ν and thus that there exist a subsequence $\{i_n\}$ and points $x_n \in X_{i_n}$ such that $\nu \in N^{\sigma,\theta}_{x_n} X_{i_n}$. Passing to a further (not relabelled) subsequence if needed, we may assume that $x_n \to \bar{x} \in X$. Observe that by the continuity of $\sigma(\cdot)$, $\nu \in S_{\theta,\sigma(\bar{x})}$. Fix now any $x \in X$ and due to the Hausdorff convergence find $y_n \in X_{i_n}$ such that $y_n \to x$. Since for every n, $(y_n - x_n)\cdot\nu \leq 0$ passing to the limit we get $(x - \bar{x})\cdot\nu \leq 0$. Due to the arbitrariness of x, we have shown that $\nu \in N_{\bar{x}} X$ and thus $\nu \in N^{\sigma,\theta}_{\bar{x}} X$, a contradiction.

Using the first part of Lemma 3.3 (with X replaced by X_i), (3.12) and Fatou's Lemma we get

$$\mathcal{H}^{N-1}(N^{\sigma,\theta} X) \geq \limsup_i \mathcal{H}^{N-1}(N^{\sigma,\theta} X_i) \geq \liminf_i \mathcal{H}^{N-1}(N^{\sigma,\theta} X_i) \geq \mathcal{H}^{N-1}(S_\theta)\,.$$

Assume now that the first inequality (3.11) holds for some $\theta \in [\theta_0/2, \theta_0]$, with $\delta_0 = \frac{\delta}{2}$, where δ is the constant provided by Lemma 3.3. Then the previous inequality yields for i sufficiently large, depending on θ,

$$\mathcal{H}^{N-1}(N^{\sigma,\theta} X_i) \leq \mathcal{H}^{N-1}(S_\theta) + \delta$$

and thus, thanks to second part of Lemma 3.3 we infer that there exists $x_i \in X_i$ and two parallel hyperplanes orthogonal to $\sigma(x_i)$ with mutual distance equal to ε such that X_i lies between them. By a compactness argument and the continuity of σ, letting $i \to \infty$ we get that there exist $x \in X$ and two parallel hyperplanes orthogonal to $\sigma(x)$ with mutual distance equal to ε such that X lies between them. Thus, in particular width$(X) \le \varepsilon$. This establishes (3.11), which in turn, again by a compactness argument and the continuity of σ, yields the conclusion in the equality case. □

Next we prove a result in the spirit of [4, Theorem 1.1]. In the following $\mathbf{C}$, Ω and Σ will be as in Definition 1.3. Moreover if $x \in \Sigma$ is a point where the tangent hyperplane to Σ exists we denote by $\nu_\Sigma(x)$ the normal to this hyperplane pointing outward with respect to Ω. We give the following definition.

Definition 3.5 We denote by Σ^+ the set of points in $\Sigma \setminus \mathbf{C}$ such that there exists a support hyperplane Π_x with the property that $\Pi_x \cap \Sigma = \{x\}$.

We recall the following result, see [20, Theorem 2.2.9]:

Theorem 3.6 *Let $K \subset \mathbb{R}^N$ be a compact convex set. Then for $\mathcal{H}^{N-1}$-almost every $\nu \in \mathbb{S}^{N-1}$ the support hyperplane for K orthogonal to ν intersects K at a single point.*

Corollary 3.7 *Let $\mathbf{C}$ and $\Sigma \subset \mathbb{R}^N$ be as in Definition 1.3. With the notation above, we have that*

$$\mathcal{K}^+(\Sigma) = \mathcal{H}^{N-1}\left(\bigcup_{x\in\Sigma^+} N_x\Sigma\right),$$

where $\mathcal{K}^+(\Sigma)$ is the total positive curvature defined in Definition 1.3.

Proof Let K denote the closed convex hull of Σ. By Theorem 3.6 we have that for $\mathcal{H}^{N-1}$-a.e. direction $\nu \in \bigcup_{x\in\Sigma\setminus\mathbf{C}} N_x\Sigma$ the corresponding support plane for K intersects K at a single point that necessarily belongs to $\Sigma \setminus \mathbf{C}$ and thus to Σ^+. □

Proof of Theorem 1.4 Observe that if $\Sigma \cap \mathbf{C} = \emptyset$ then

$$\bigcup_{x\in\Sigma\setminus\mathbf{C}} N_x\Sigma = \mathbb{S}^{N-1},$$

hence (1.3) trivially holds.

Hence in the following we may assume that $\Sigma \cap \mathbf{C} \neq \emptyset$.
We denote by $\nu_{\mathbf{C}}$ the outward normal to $\mathbf{C}$. We start by proving (1.3). Let us define $\sigma : \Sigma \cap \mathbf{C} \to \mathbb{S}^{N-1}$ as $\sigma(x) := \nu_{\mathbf{C}}(x)$. Note that since $\mathbf{C}$ is convex the direction $\sigma(x)$ belongs to $N_x\mathbf{C}$ and thus to $N_x(\Sigma \cap \mathbf{C})$ for every $x \in \Sigma \cap \mathbf{C}$.

Given $\nu \in \mathbb{S}^{N-1}$, we denote by $\nu^\perp$ the hyperplane orthogonal to ν and passing through the origin and we set

$$\bar{t} := \max\{t \in \mathbb{R} : (t\nu + \nu^\perp) \cap \Sigma \neq \emptyset\}.$$

Clearly, by definition for every $\nu \in \mathbb{S}^{N-1}$ the hyperplane $\bar{t}\nu + \nu^\perp$ is a support hyperplane for Σ. Fix $\theta \in (0, \theta_0)$. We claim that for every $x \in \Sigma \cap \mathbf{C}$

$$\nu \in N_x(\Sigma \cap \mathbf{C}) \cap S_{\theta,\sigma(x)} \quad \text{implies} \quad \bar{t}\nu + \nu^\perp \cap \Sigma \subset \Sigma \setminus \mathbf{C}. \tag{3.13}$$

Let $t_0 \in \mathbb{R}$ be such that $x + \nu^\perp = t_0\nu + \nu^\perp$ and observe that since $\nu \cdot \sigma(x) \ge \cos\theta > \cos\theta_0$ then by assumption (1.2) $\nu \notin N_x\Sigma$, hence the hyperplane $t_0\nu + \nu^\perp$ enters Ω. Thus it easily

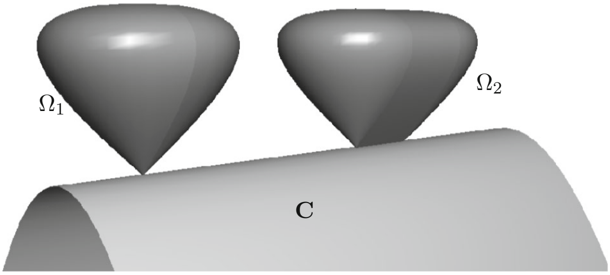

Fig. 1 Both $\Sigma_1 = \overline{\partial\Omega_1 \setminus \mathbf{C}}$ and $\Sigma_2 = \overline{\partial\Omega_2 \setminus \mathbf{C}}$ meet $\mathbf{C}$ with contact angle $\pi/4$ and satisfy the equality in (1.3) with $\theta_0 = 3\pi/4$. Note that $\partial\Omega_1 \cap \mathbf{C}$ is a point and $\partial\Omega_2 \cap \mathbf{C}$ is a segment

follows that $\bar{t} > t_0$. Let $y \in \bar{t}\nu + \nu^\perp \cap \Sigma$. Then $y \notin \Sigma \cap \mathbf{C}$, since otherwise this would contradict the fact that $t_0\nu + \nu^\perp$ is a support hyperplane for $\Sigma \cap \mathbf{C}$. This establishes (3.13). From (3.13) it follows that

$$N^{\sigma,\theta}(\Sigma \cap \mathbf{C}) \subset \bigcup_{x \in \Sigma \setminus \mathbf{C}} N_x \Sigma \,. \tag{3.14}$$

Recall that by Definition 1.3

$$\mathcal{H}^{N-1}\left(\bigcup_{x \in \Sigma \setminus \mathbf{C}} N_x \Sigma\right) = \mathcal{K}^+(\Sigma)\,.$$

Combining the equality above with (3.14), the inequality (1.3) follows from (3.10) with $X = \Sigma \cap \mathbf{C}$, letting $\theta \to \theta_0^-$.

Given $\varepsilon > 0$, let δ_0 be the constant provided by Proposition 3.4 and let $\theta \in [\theta_0/2, \theta_0)$ such that

$$\mathcal{H}^{N-1}(S_{\theta_0}) \leq \mathcal{H}^{N-1}(S_\theta) + \frac{\delta_0}{2}\,. \tag{3.15}$$

Assume that (1.4) and (1.5) hold for some $\delta \in (0, \delta_0/2)$ such that $\cos\theta_0 + \delta < \cos\theta$. Then, using the assumption (1.4), the same argument as before yields (3.13), hence (3.14). Thus, from (1.5) and (3.15) we have in particular

$$\mathcal{H}^{N-1}(N^{\sigma,\theta}(\Sigma \cap \mathbf{C})) \leq \mathcal{K}^+(\Sigma) \leq \mathcal{H}^{N-1}(S_{\theta_0}) + \delta \leq \mathcal{H}^{N-1}(S_\theta) + \delta_0\,.$$

The conclusion follows from Proposition 3.4. □

Remark 3.8 Observe that the equality case in (1.3) does not imply $\partial\Omega \cap \mathbf{C}$ lies on a facet of $\mathbf{C}$. In fact it may happen that $\partial\Omega \cap \mathbf{C}$ is contained in a convex set of Hausdorff dimension strictly less than $N - 1$, see Fig. 1.

It is well known that for surfaces $\Sigma \subset \mathbb{R}^3$ without boundary the following inequality holds

$$\int_\Sigma |H_\Sigma|^2\, d\mathcal{H}^2 \geq 16\pi\,.$$

with equality achieved if and only if Σ is a sphere. We now apply Theorem 1.4 to extend this inequality to the following extension of the Willmore energy in N-dimensions

$$\int_{\Sigma \setminus \mathbf{C}} |H_\Sigma|^{N-1}\, d\mathcal{H}^{N-1}\,,$$

for $C^{1,1}$ hypersurfaces with boundary supported on convex sets and with contact angle larger than a given $\theta_0 \in (0, \pi)$. Note that in the next theorem we do not assume any regularity on the convex set $\mathbf{C}$.

Theorem 3.9 (A Willmore type inequality.) *Let* $\mathbf{C}$, Ω *and* Σ *be as in Definition 1.3 and let* $\theta_0 \in (0, \pi)$. *Assume that* $\Sigma \setminus \mathbf{C}$ *is of class* $C^{1,1}$. *Set* $H_\Sigma := \mathrm{div}_\Sigma \nu_\Sigma$ *(where* ν_Σ *is the unit normal to* Σ *pointing outward with respect to* Ω*). Assume also*

$$\nu \cdot \nu' \le \cos\theta_0 \quad \text{whenever } x \in \Sigma \cap \mathbf{C},\ \ \nu \in N_x\Sigma \ \text{ and } \ \nu' \in N_x\mathbf{C}\,. \tag{3.16}$$

Then,

$$\int_{\Sigma\setminus\mathbf{C}} |H_\Sigma|^{N-1}\, d\mathcal{H}^{N-1} \ge (N-1)^{N-1}\mathcal{H}^{N-1}(S_{\theta_0})\,. \tag{3.17}$$

Moreover, if equality holds in (3.17) and $H_\Sigma \ne 0$ *a.e., then* $\Sigma \setminus \mathbf{C}$ *coincides, up to a rigid motion, with an omothetic of* S_{θ_0} *sitting on a facet of* $\mathbf{C}$.

Proof Without loss of generality we may assume that

$$\int_{\Sigma\setminus\mathbf{C}} |H_\Sigma|^{N-1}\, d\mathcal{H}^{N-1} < \infty\,.$$

Set for any $\eta > 0$ sufficiently small $\mathbf{C}_\eta := \mathbf{C} + \overline{B_\eta(0)}$ and $\Sigma_\eta := \overline{\partial\Omega \setminus \mathbf{C}_\eta}$. Observe that $\mathbf{C}_\eta$ satisfies both a outer and inner uniform ball condition and thus is of class $C^{1,1}$, see [7, 16]. Note also that there exists $\theta_\eta \in (0, \theta_0)$ such that

$$\nu \cdot \nu_{\mathbf{C}_\eta}(x) \le \cos\theta_\eta \quad \text{whenever } x \in \Sigma_\eta \cap \mathbf{C}_\eta,\ \ \nu \in N_x\Sigma_\eta\,, \tag{3.18}$$

with $\theta_\eta \to \theta_0$ as $\eta \to 0^+$. Indeed, if not, there would exist a sequence $\eta_h \to 0$, a sequence of points $x_h \in \Sigma_{\eta_h} \cap \mathbf{C}_{\eta_h}$ and a sequence $\nu_h \in N_{x_h}\Sigma_{\eta_h}$, such that $\nu_h \cdot \nu_{\mathbf{C}_{\eta_h}}(x_h) \ge \cos\theta'$ for some $\theta' \in (0, \theta_0)$. We may assume that $x_h \to x \in \Sigma \cap \mathbf{C}$, $\nu_h \to \nu$ and $\nu_{\mathbf{C}_{\eta_h}}(x_h) \to \nu'$. Clearly $\nu \in N_x\Sigma$, $\nu' \in N_x\mathbf{C}$ and $\nu \cdot \nu_{\mathbf{C}}(x) \ge \cos\theta'$, a contradiction to (1.2).

We set

$$\widetilde{\Sigma} = \{x \in \Sigma \setminus \mathbf{C} :\ N_x\Sigma \ne \emptyset\}, \qquad \widetilde{\Sigma}_\eta = \{x \in \Sigma \setminus \mathbf{C}_\eta \,:\, N_x\Sigma_\eta \ne \emptyset\}\,.$$

We claim that

$$\chi_{\widetilde{\Sigma}_\eta} \to \chi_{\widetilde{\Sigma}} \quad \text{pointwise in } \Sigma \setminus \mathbf{C} \text{ as } \eta \to 0\,. \tag{3.19}$$

First of all note that $\widetilde{\Sigma} \setminus \mathbf{C}_\eta \subset \widetilde{\Sigma}_\eta$ for all η, whence

$$\chi_{\widetilde{\Sigma}} = \lim_{\eta\to 0+} \chi_{\widetilde{\Sigma}\setminus\mathbf{C}_\eta} \le \liminf_{\eta\to 0^+} \chi_{\widetilde{\Sigma}_\eta} \quad \text{pointwise in } \Sigma \setminus \mathbf{C}$$

If otherwise $x \notin \widetilde{\Sigma}$, we show that $x \notin \widetilde{\Sigma}_\eta$ for η small. Indeed, assume by contradiction that there exist $\nu_h \in N_x(\Sigma_{\eta_h})$, for a sequence $\eta_h \to 0$. Then, passing to a subsequence, if needed, $\nu_h \to \nu \in N_x\Sigma$, a contradiction. This proves that

$$\chi_{\widetilde{\Sigma}} \ge \limsup_{\eta\to 0^+} \chi_{\widetilde{\Sigma}_\eta} \quad \text{pointwise in } \Sigma \setminus \mathbf{C}$$

and thus (3.19) holds.

Let $(\Sigma_\eta)^+$ the subset of $\widetilde{\Sigma}_\eta$ defined as in Definition 3.5 with Σ replaced by Σ_η. Denote by K_Σ the Gaussian curvature of $\Sigma \setminus \mathbf{C}$ and observe that on $\widetilde{\Sigma}_\eta$ all principal curvatures are nonnegative. By the arithmetic-geometric mean inequality $(N-1)^{N-1}K_\Sigma \le H_\Sigma^{N-1}$ on $\widetilde{\Sigma}_\eta$. Then by Theorem 1.4 we get

$$\begin{aligned}\int_{\Sigma\setminus \mathbf{C}_\eta} |H_\Sigma|^{N-1}\, d\mathcal{H}^{N-1} &\ge \int_{\widetilde{\Sigma}_\eta} H_\Sigma^{N-1}\, d\mathcal{H}^{N-1} \ge (N-1)^{N-1}\int_{\widetilde{\Sigma}_\eta} K_\Sigma\, d\mathcal{H}^{N-1}\\ &\ge (N-1)^{N-1}\int_{(\Sigma_\eta)^+} K_\Sigma\, d\mathcal{H}^{N-1}\\ &= (N-1)^{N-1}\int_{(\Sigma_\eta)^+} \det(D\nu_\Sigma)\, d\mathcal{H}^{N-1}\\ &= (N-1)^{N-1}\mathcal{K}^+(\Sigma_\eta) \ge (N-1)^{N-1}\mathcal{H}^{N-1}(S_{\theta_\eta})\,, \end{aligned}\tag{3.20}$$

where in the second equality we have used Corollary 3.7 and the area formula, since ν_Σ is a Lipschitz map in a neighborhood of Σ^+. Then, letting $\eta \to 0$ and recalling (3.19) and the fact that $\theta_\eta \to \theta_0$, we get

$$\begin{aligned}\int_{\Sigma\setminus \mathbf{C}} |H_\Sigma|^{N-1}\, d\mathcal{H}^{N-1} &\ge \int_{\widetilde{\Sigma}} H_\Sigma^{N-1}\, d\mathcal{H}^{N-1}\\ &\ge (N-1)^{N-1}\int_{\widetilde{\Sigma}} K_\Sigma\, d\mathcal{H}^{N-1} \ge (N-1)^{N-1}\mathcal{H}^{N-1}(S_{\theta_0})\,. \end{aligned}\tag{3.21}$$

In particular (3.17) follows.

If equality holds in (3.17) holds, from (3.20) we have that $\mathcal{K}^+(\Sigma_\eta) - \mathcal{H}^{N-1}(S_{\theta_\eta}) \to 0$. In turn from the second part of Theorem 1.4 we get that $\mathrm{width}(\Sigma_\eta \cap \mathbf{C}_\eta) \to 0$ and more precisely that $\Sigma_\eta \cap \mathbf{C}_\eta$ lies between two parallel hyperplanes orthogonal to $\nu_{\mathbf{C}_\eta}(x_\eta)$ for some $x_\eta \in \Sigma_\eta \cap \mathbf{C}_\eta$ with mutual distance going to zero. Passing to the limit by a simple compactness argument we infer that $\Sigma \cap \mathbf{C}$ lies on a support hyperplane to $\mathbf{C}$.

Note also that in the equality case, if $H_\Sigma \ne 0$ $\mathcal{H}^{N-1}$-a.e., then (3.21) implies that $\Sigma \setminus \mathbf{C} = \widetilde{\Sigma}$. In turn this yields that every $x \in \Sigma \cap \mathbf{C}$ has a support hyperplane to Σ. Moreover, (3.21) yields also that $H_\Sigma^{N-1} = (N-1)^{N-1}K_\Sigma$. In turn this implies that Σ is umbilical and thus, by a classical result, see for instance [19, Th. 3.1], each connected component Σ_i of Σ is contained in a sphere. Since $\Sigma \cap \mathbf{C}$ is contained in a hyperplane Π tangent to $\mathbf{C}$, each Σ_i is either a spherical cap supported on Π and satisfying (3.16) with Σ replaced by Σ_i, or a sphere not intersecting $\mathbf{C}$.

In either case, since (3.16) is satisfied at every point in $\Sigma \cap \mathbf{C}$ (recall that $\Sigma \setminus \mathbf{C} = \widetilde{\Sigma}$), we may apply (3.17) to infer that for every connected component Σ_i we have

$$\int_{\Sigma_i\setminus \mathbf{C}} H_{\Sigma_i}^{N-1}\, d\mathcal{H}^{N-1} \ge (N-1)^{N-1}\mathcal{H}^{N-1}(S_{\theta_0})\,.$$

In particular, since we are in the equality case, there must be only one connected component. Thus Σ is a spherical cap homothetic to S_{θ_0} up to a rigid motion. Finally $\Sigma \cap \mathbf{C}$ by convexity must lie on a facet of $\mathbf{C}$. □

4 The equality case in the relative isoperimetric inequality outside a convex set

In this section we give the proof of Theorem 1.2. Throughout this proof we will denote by $\mathscr{H}$ the half space

$$\mathscr{H} := \{(x', x_N) \in \mathbb{R}^N : x_N > 0\}. \tag{4.1}$$

We will also need the following notions of (Λ, r_0)-minimizer and restricted (Λ, r_0)-minimizer for the relative perimeter, which extend the standard notion of perimeter (Λ, r_0)-minimizer recalled in Definition 2.1.

Definition 4.1 Let $\mathbf{C} \subset \mathbb{R}^N$ be a closed convex set with nonempty interior and let $\Lambda, r_0 > 0$. We say that a set of finite perimeter $E \subset \mathbb{R}^N \setminus \mathbf{C}$ is a (Λ, r_0)*-minimizer of the relative perimeter* $P(\cdot; \mathbb{R}^N \setminus \mathbf{C})$ if for any $F \subset \mathbb{R}^N \setminus \mathbf{C}$ such that $\operatorname{diam}(E \Delta F) \leq r_0$ we have

$$P(E; \mathbb{R}^N \setminus \mathbf{C}) \leq P(F; \mathbb{R}^N \setminus \mathbf{C}) + \Lambda |E \Delta F|.$$

Moreover, we say that $E \subset \mathbb{R}^N \setminus \mathbf{C}$ is a *restricted* (Λ, r_0)*-minimizer* if the above inequality holds for every set $F \subset \mathbb{R}^N \setminus \mathbf{C}$ such that $\operatorname{diam}(E \Delta F) \leq r_0$ and $\partial^* F \cap \mathbf{C} \subset \partial^* E \cap \mathbf{C}$ up to a $\mathcal{H}^{N-1}$-negligible set.

Proof of Theorem 1.2 Let $m_0 > 0$ be a given mass and let Ω_0 be a minimizer of the perimeter outside $\mathbf{C}$ such that $|\Omega_0| = m_0$ and

$$P(\Omega_0; \mathbb{R}^N \setminus \mathbf{C}) = N \Big(\frac{\omega_N}{2}\Big)^{\frac{1}{N}} m_0^{\frac{N-1}{N}}. \tag{4.2}$$

Since Ω_0 solves the isoperimetric problem we have that Ω_0 is a (Λ_0, r_0)-minimizer of the relative perimeter in $\mathbb{R}^N \setminus \mathbf{C}$, see Definition 4.1, for some $\Lambda_0, r_0 > 0$, depending on Ω_0, see for instance the argument of [15, Example 21.3].[2] In turn by Proposition 5.2 Ω_0 satisfies uniform volume density estimates and thus it easily follows that Ω_0 is bounded.

We fix a sufficiently large ball $B_R(0)$ containing $\overline{\Omega_0}$. Note that by standard argument, see also the argument of Step 1 below, Ω_0 solves the following penalized minimum problem

$$\min\{P(E; \mathbb{R}^N \setminus \mathbf{C}) + \Lambda_0 ||E| - m_0| : \ E \subset B_R \setminus \mathbf{C}\},$$

for a possibly larger Λ_0. In particular we have

$$P(\Omega_0; \mathbb{R}^N \setminus \mathbf{C}) \leq P(E; \mathbb{R}^N \setminus \mathbf{C}) + \Lambda_0 |\Omega_0 \Delta E| \qquad \text{for all } E \subset B_R \setminus \mathbf{C}. \tag{4.3}$$

Since in the remaining part of the proof we will always work inside B_R, up to replacing $\mathbf{C}$ with $\mathbf{C} \cap \overline{B_R}$, we may assume without loss of generality that $\mathbf{C}$ is bounded.

Observe that by Theorem 2.6 we may assume that Ω_0 is an open set and that $\partial \Omega_0 \setminus \mathbf{C}$ coincides with the reduced boundary $\partial^* \Omega_0 \setminus \mathbf{C}$ up to an $\mathcal{H}^{N-1}$-negligible set. Let us show that Ω_0 is connected. Indeed, if otherwise $\Omega_0 = \Omega_1 \cup \Omega_2$, with Ω_1 and Ω_2 open, Ω_1 a connected component of Ω_0 with $0 < |\Omega_1| < m_0$, we have by Theorem 1.1

$$\begin{aligned} P(\Omega_0; \mathbb{R}^N \setminus \mathbf{C}) &= P(\Omega_1; \mathbb{R}^N \setminus \mathbf{C}) + P(\Omega_2; \mathbb{R}^N \setminus \mathbf{C}) \\ &\geq N \Big(\frac{\omega_N}{2}\Big)^{\frac{1}{N}} |\Omega_1|^{\frac{N-1}{N}} + N \Big(\frac{\omega_N}{2}\Big)^{\frac{1}{N}} |\Omega_2|^{\frac{N-1}{N}} > N \Big(\frac{\omega_N}{2}\Big)^{\frac{1}{N}} m_0^{\frac{N-1}{N}}, \end{aligned}$$

[2] Note that in [15, Example 21.3] it is proved that a mass constrained minimizer E of the relative perimeter in an open set A is a perimeter (Λ_0, r_0)-minimizer in A according to Definition 2.1. However an inspection of the proof shows that E is also a (Λ_0, r_0)-minimizer of the relative perimeter $P(\cdot, A)$ according to Definition 4.1.

which is a contradiction to (4.2).

For every $\eta \geq 0$ we set $\mathbf{C}_\eta = \mathbf{C} + \overline{B_\eta(0)}$ and, for $\eta \in [0, \bar{\eta}]$ we set $m_\eta := |\Omega_0 \setminus \mathbf{C}_\eta|$, where $\bar{\eta} > 0$ is such that $|\Omega_0 \setminus \mathbf{C}_{\bar{\eta}}| > 0$. Correspondingly, we set for $m \in (0, m_\eta]$

$$I_\eta(m) = \min\{P(E; \mathbb{R}^N \setminus \mathbf{C}_\eta) : \ E \subset \Omega_0 \setminus \mathbf{C}_\eta, \ |E| = m\} \tag{4.4}$$

and denote by $\Omega_{\eta,m}$ any minimizer of the above problem. Note that $\Omega_{0,m_0} = \Omega_0$. Observe also that

$$\sup_{\eta\in[0,\bar{\eta}]} \sup_{m\in(0,m_\eta)} P(\Omega_{\eta,m}) < \infty\,. \tag{4.5}$$

Indeed, given $\eta \in [0, \bar{\eta}]$ and $0 < m \leq m_\eta$ there exists $\eta' \geq \eta$ such that $|\Omega_0 \setminus \mathbf{C}_{\eta'}| = m$. Thus

$$\begin{aligned}P(\Omega_{\eta,m}) &\leq P(\Omega_{\eta,m}; \mathbb{R}^N \setminus \mathbf{C}_\eta) + P(\mathbf{C}_\eta; B_R) \leq P(\Omega_0 \setminus \mathbf{C}_{\eta'}) + P(\mathbf{C}_\eta; B_R)\\ &\leq P(\Omega_0; \mathbb{R}^N \setminus \mathbf{C}) + 2\sup_{s\geq 0} P(\mathbf{C}_s; B_R) \leq P(\Omega_0; \mathbb{R}^N \setminus \mathbf{C}) + 2N\omega_N R^{N-1}\,.\end{aligned}$$

Let us fix $m', m'' \in (0, m_0)$, with $m' < m''$. We claim that there exists $\tilde{\eta} \in (0, \bar{\eta}]$ such that

$$\text{if } \eta \in [0, \tilde{\eta}] \text{ and } U \text{ is a connected component of } \Omega_0 \setminus \mathbf{C}_\eta, \ \text{then} |U| \notin [m', m'']. \tag{4.6}$$

To prove (4.6) we fix $x_0 \in \Omega_0$ and for every η we denote by U_η the connected component of $\Omega_0 \setminus \mathbf{C}_\eta$ containing x_0. Note that U_η increases as η becomes smaller. Given any other point $x \in \Omega_0$ there exists a path connecting x_0 and x contained in Ω_0, thus $x \in U_\eta$ for η small enough. Hence $|U_\eta| \to m_0$, and the claim follows. Note that this argument implies also that for η sufficiently small

$$\partial\Omega_{\eta,m} \cap (\Omega_0 \setminus \mathbf{C}_\eta) \neq \emptyset \qquad \text{for all } \eta \in [0, \tilde{\eta}] \text{ and } m \in [m', m'']. \tag{4.7}$$

We split the remaining part of the proof in several steps. Some of the long technical claims contained in these steps will be proved in "Appendix B" so as not to break the line of reasoning.

Step 1 (*Equivalence with a volume penalized problem*). Fix $0 < m' < m'' < m_0$ and let $0 < \tilde{\eta} \leq \bar{\eta}$ be as in (4.6). We claim that there exists $\Lambda' > 0$ with the following property: for every $\eta \in [0, \tilde{\eta}]$ and $m \in [m', m'']$ we have that $\Omega_{\eta,m}$ is a minimizer of the following problem

$$\min\{P(E; \mathbb{R}^N \setminus \mathbf{C}_\eta) + \Lambda'||E| - m| : \ E \subset \Omega_0 \setminus \mathbf{C}_\eta\}\,. \tag{4.8}$$

The proof of this claim will be given in the "Appendix B".

Step 2 ($\Omega_{\eta,m}$ *is a restricted* Λ-*minimizer*). Fix $0 < m' < m'' < m_0$, let $0 < \tilde{\eta} \leq \bar{\eta}$ be as in (4.6) and set $\Lambda = \max\{\Lambda', \Lambda_0\}$, where Λ' is as in Step 1. We claim that for every $\eta \in [0, \tilde{\eta}]$ and $m \in [m', m'']$, $\Omega_{\eta,m}$ is a restricted Λ-minimizer under the constraint that $\partial^* E \cap \mathbf{C}_\eta \subset \partial^*(\Omega_0 \setminus \mathbf{C}_\eta) \cap \mathbf{C}_\eta$. More precisely, for every set of finite perimeter $E \subset B_R(0) \setminus \mathbf{C}_\eta$ such that $\partial^* E \cap \mathbf{C}_\eta \subset \partial^*(\Omega_0 \setminus \mathbf{C}_\eta) \cap \mathbf{C}_\eta$ up to a $\mathcal{H}^{N-1}$-negligible set

$$P(\Omega_{\eta,m}; \mathbb{R}^N \setminus \mathbf{C}_\eta) \leq P(E; \mathbb{R}^N \setminus \mathbf{C}_\eta) + \Lambda|\Omega_{\eta,m} \Delta E|\,. \tag{4.9}$$

In particular $\Omega_{\eta,m}$ is a restricted (Λ, r_0)-minimizer according to Definition 4.1, choosing for instance $r_0 :=\mathrm{dist}(\Omega_0, \partial B_R(0))$.

Given E as above, from Step 1 we get

$$P\left(\Omega_{\eta,m}; \mathbb{R}^N \setminus \mathbf{C}_\eta\right) \leq P(E \cap \Omega_0; \mathbb{R}^N \setminus \mathbf{C}_\eta) + \Lambda'||E \cap \Omega_0| - |\Omega_{\eta,m}||$$

$$= \mathcal{H}^{N-1}(\partial^* E \cap (\Omega_0 \setminus \mathbf{C}_\eta)) + \mathcal{H}^{N-1}(\partial^* E \cap \partial^* \Omega_0 \cap \{\nu_E = \nu_{\Omega_0}\} \setminus \mathbf{C}_\eta) + \mathcal{H}^{N-1}(\partial^* \Omega_0 \cap E^{(1)}) + \Lambda' \big| |E \cap \Omega_0| - |\Omega_{\eta,m}| \big| . \tag{4.10}$$

Then, using (4.3) and the condition $\partial^* E \cap \mathbf{C}_\eta \subset \partial^*(\Omega_0 \setminus \mathbf{C}_\eta) \cap \mathbf{C}_\eta$, we have

$$\begin{aligned} &\mathcal{H}^{N-1}((\partial^* \Omega_0 \cap \mathbf{C}_\eta) \setminus \mathbf{C}) + \mathcal{H}^{N-1}(\partial^* \Omega_0 \setminus \mathbf{C}_\eta) = P(\Omega_0; \mathbb{R}^N \setminus \mathbf{C}) \\ &\quad \le P(E \cup \Omega_0; \mathbb{R}^N \setminus \mathbf{C}) + \Lambda_0 |E \setminus \Omega_0| \\ &\quad = \mathcal{H}^{N-1}((\partial^* \Omega_0 \cap \mathbf{C}_\eta) \setminus \mathbf{C}) + \mathcal{H}^{N-1}((\partial^* \Omega_0 \cap E^{(0)}) \setminus \mathbf{C}_\eta) \\ &\qquad + \mathcal{H}^{N-1}(\partial^* E \cap \partial^* \Omega_0 \cap \{\nu_E = \nu_{\Omega_0}\} \setminus \mathbf{C}_\eta) + \mathcal{H}^{N-1}(\partial^* E \setminus \overline{\Omega}_0) + \Lambda_0 |E \setminus \Omega_0| . \end{aligned}$$

Simplifying the above inequality, we get

$$\mathcal{H}^{N-1}(\partial^* \Omega_0 \cap E^{(1)}) + \mathcal{H}^{N-1}(\partial^* E \cap \partial^* \Omega_0 \cap \{\nu_E = -\nu_{\Omega_0}\}) \le \mathcal{H}^{N-1}(\partial^* E \setminus \overline{\Omega}_0) + \Lambda_0 |E \setminus \Omega_0| .$$

Combining this inequality with (4.10) we conclude that

$$\begin{aligned} P(\Omega_{\eta,m}; \mathbb{R}^N \setminus \mathbf{C}_\eta) &\le \mathcal{H}^{N-1}(\partial^* E \cap (\Omega_0 \setminus \mathbf{C}_\eta)) + \mathcal{H}^{N-1}(\partial^* E \cap \partial^* \Omega_0 \cap \{\nu_E = \nu_{\Omega_0}\} \setminus \mathbf{C}_\eta) \\ &\quad + \mathcal{H}^{N-1}(\partial^* E \setminus \overline{\Omega}_0) + \Lambda' \big| |E \cap \Omega_0| - |\Omega_{\eta,m}| \big| + \Lambda_0 |E \setminus \Omega_0| \\ &\le P(E; \mathbb{R}^N \setminus \mathbf{C}_\eta) + \max\{\Lambda', \Lambda_0\} |E \Delta \Omega_{\eta,m}| \end{aligned}$$

so that the claim is proven.

Step 3 (*Monotonicity and Lipschitz equicontinuity of the isoperimetric profiles*). We claim that I_η [see (4.4)] is strictly increasing in $[0, m_\eta]$ for all $\eta \in [0, \bar\eta]$. Moreover, for any fixed $0 < m' < m'' < m_0$ and for $0 < \tilde\eta \le \bar\eta$ be as in (4.6), we claim that for $\eta \in [0, \tilde\eta]$, I_η is Λ'-Lipschitz in $[m', m'']$, where Λ' is as in Step 2.

We postpone the proof to "Appendix B".

Step 4 (*A formula for I'_η*). Fix $0 < m' < m'' < m_0$ and let $0 < \tilde\eta \le \bar\eta$ be as in (4.6). For $m \in [m', m'']$ and $\eta \in [0, \tilde\eta]$ we set $\Sigma_{\eta,m} := \overline{\partial \Omega_{\eta,m} \setminus \mathbf{C}_\eta}$ and denote by $\Sigma^*_{\eta,m}$ the regular free part of $\Sigma_{\eta,m}$, that is $\Sigma^*_{\eta,m} := \partial^* \Omega_{\eta,m} \setminus (\partial \Omega_0 \cup \mathbf{C}_\eta)$. Observe that by (4.7) $\Sigma^*_{\eta,m}$ is nonempty. We recall that by a standard first variation argument $\Sigma^*_{\eta,m}$ is a constant mean curvature manifold. We denote by $H_{\Sigma^*_{\eta,m}}$ such a mean curvature.

We claim that at any point $m \in (m', m'')$ of differentiability for I_η, $\eta \in [0, \tilde\eta]$, we have

$$I'_\eta(m) = H_{\Sigma^*_{\eta,m}} . \tag{4.11}$$

To this end we fix $x \in \Sigma^*_{\eta,m}$ and a ball $B_r(x) \subset\subset \Omega_0 \setminus \mathbf{C}_\eta$ such that $\Sigma^*_{\eta,m} \cap B_r(x) = \partial \Omega_{\eta,m} \cap B_r(x)$. Let X be a smooth vector field compactly supported in $B_r(x)$ such that

$$\int_{\Sigma^*_{\eta,m}} X \cdot \nu_{\Omega_{\eta,m}} \, d\mathcal{H}^{N-1} \neq 0 .$$

Consider now the flow associated with X, that is the solution in $\mathbb{R}^N \times \mathbb{R}$ of

$$\begin{cases} \frac{\partial \Phi}{\partial t}(x,t) = X(\Phi(x,t)) \\ \Phi(x,0) = x \end{cases}$$

and set $\Omega_{\eta,m}(t) := \Phi(\Omega_{\eta,m}, t)$. Clearly, $P(\Omega_{\eta,m}(t)); \mathbb{R}^N \setminus \mathbf{C}_\eta) \ge I_\eta(|\Omega_{\eta,m}(t)|)$, with the equality at $t = 0$. Therefore

$$\frac{d}{dt}\Big(P(\Omega_{\eta,m}(t)); \mathbb{R}^N \setminus \mathbf{C}_\eta)\Big)\Big|_{t=0} = \frac{d}{dt}\Big(I_\eta(|\Omega_{\eta,m}(t)|)\Big)\Big|_{t=0} .$$

Note that

$$\frac{d}{dt}\Big(P(\Omega_{\eta,m}(t)); \mathbb{R}^N \setminus \mathbf{C}_\eta\Big)\Big|_{t=0} = H_{\Sigma^*_{\eta,m}} \int_{\Sigma^*_{\eta,m}} X \cdot \nu_{\Omega_{\eta,m}} \, d\mathcal{H}^{N-1} \,,$$
$$\frac{d}{dt}\Big(I_\eta(|\Omega_{\eta,m}(t)|)\Big)\Big|_{t=0} = I'_\eta(m) \frac{d}{dt}\big(|\Omega_{\eta,m}(t)|\big)\Big|_{t=0} = I'_\eta(m) \int_{\Sigma^*_{\eta,m}} X \cdot \nu_{\Omega_{\eta,m}} \, d\mathcal{H}^{N-1} \,,$$

where we have used the well known formulas for the first variation of the perimeter and the volume, see for instance [15, Chap. 17]. Thus (4.11) follows.

Step 5 (*A weak Young's law*). Fix $0 < m' < m'' < m_0$ and let $0 < \tilde{\eta} \le \bar{\eta}$ be as in (4.6). We claim that if $\eta \in [0, \tilde{\eta}]$ and $m \in [m', m'']$, the following weak Young's law holds:

$$\nu \cdot \nu_{\mathbf{C}_\eta}(x) \le 0 \quad \text{whenever } x \in \Sigma_{\eta,m} \cap \mathbf{C}_\eta \text{ and } \nu \in N_x \Sigma_{\eta,m} . \tag{4.12}$$

Let $x \in \Sigma_{\eta,m} \cap \mathbf{C}_\eta$ and $\nu \in N_x \Sigma_{\eta,m}$. Without loss of generality, by rotating the coordinate system if needed, we may assume that $x = 0$, $\nu_{\mathbf{C}_\eta}(0) = e_N$ and $\nu = (\nu_1, 0, \dots, 0, \nu_N)$ with $\nu_1 \le 0$. Note that (4.12) will be proven if we show that

$$\nu_N \ge \ 0 \text{ implies that } \nu_N = 0 \,. \tag{4.13}$$

Set $E_h = h\Omega_{\eta,m}$, $h \in \mathbb{N}$ and $\mathbf{C}_{\eta,h} = h\mathbf{C}_\eta$ and observe that, since $\nu_1 \le 0$ and $\nu_N \ge 0$, $E_h \subset \{x_1 \ge 0\}$. Note also that by (4.9) we have that

$$P(E_h; \mathbb{R}^N \setminus \mathbf{C}_{\eta,h}) \le P(G; \mathbb{R}^N \setminus \mathbf{C}_{\eta,h}) + \frac{1}{h}\Lambda |E_h \Delta G| \tag{4.14}$$

for all sets $G \subset B_{hR}(0) \setminus \mathbf{C}_{\eta,h}$ such that $\partial^* G \cap \mathbf{C}_{\eta,h} \subset \partial^* E_h \cap \mathbf{C}_{\eta,h}$ up to a $\mathcal{H}^{N-1}$-negligible set. Using the density estimate proved in Proposition 5.2 and passing possibly to a not relabelled subsequence we may assume that E_h converge in $L^1_{loc}(\mathbb{R}^N)$ to some set $E \subset \mathscr{H} \cap \{x_1 > 0\}$ [see (4.1)] of locally finite perimeter and that $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$ as Radon measures in $\mathbb{R}^N$, see (2.1) for the definition of μ_E. Finally, given $r > 0$, from the volume density estimate in Proposition 5.2 we get that for h large enough $|E_h \cap B_r(0)| \ge cr^N$ and thus, passing to the limit, we have $|E \cap B_r(0)| \ge cr^N$ for all $r > 0$. This in turn implies that

$$0 \in \partial^e E \subset \partial E \,. \tag{4.15}$$

Since each E_h is a $\frac{\Lambda}{h}$-minimizer, by Theorem 2.2 we have that E is a 0-minimizer that is

$$P(E; B_r(x_0)) \le P(F; B_r(x_0)) \text{ for any } F, B_r(x_0) \text{ s.t. } E\Delta F \subset\subset B_r(x_0) \subset\subset \mathscr{H} \,. \tag{4.16}$$

We claim that also the minimality with respect to inner perturbations passes to the limit. More precisely we want to show that E satisfies the following minimality property: for any cube $Q_r(0) = (-r, r)^N$ and any open set with Lipschitz boundary $V \subset\subset Q_r(0)$[3]

$$\mathcal{H}^{N-1}(\partial E \cap \partial V \cap \mathscr{H}) = 0 \quad \text{implies} \quad P(E; \mathscr{H} \cap Q_r(0)) \le P(E \setminus V; \mathscr{H} \cap Q_r(0)) \,. \tag{4.17}$$

We postpone the proof of this claim to "Appendix B".

[3] Very likely the minimality property of E with respect to inner perturbations holds true also without the condition $\mathcal{H}^{N-1}(\partial E \cap \partial V \cap \mathscr{H}) = 0$. However, this condition is not restrictive for our purposes, while on the other hand it would take some extra technicalities to remove it.

We now denote by $\widehat{E} = E \cup R(E)$ where R denotes the reflection map $R(x', x_N) = (x', -x_N)$. From (4.17) one can easily check that given an open set with Lipschitz boundary $V \subset\subset Q_r(0)$ such that $\mathcal{H}^{N-1}(\partial \widehat{E} \cap \partial V) = 0$ we have

$$P(\widehat{E}; Q_r(0)) \le P(\widehat{E} \setminus V; Q_r(0)).$$

We claim that $\partial \widehat{E}$ contains $\{x_1 = 0\}$. In turn this implies (4.13).

To see this assume first that $\partial \widehat{E}$ intersects $\{x_1 = 0\} \setminus \{x_N = 0\}$ at some point x_0. Then, by Theorem 2.6-(iii) $\partial \widehat{E}$ is a smooth minimal surface in a neighborhood of x_0. In turn, by the Strong Maximum Principle Theorem 2.5 it coincides with the hyperplane $\{x_1 = 0\}$ in a neighborhood of x_0. The same argument shows that $\partial \widehat{E} \cap \{x_1 = 0\}$ is both relatively closed and open in $\{x_1 = 0\}$ and therefore the connected component of $\partial \widehat{E}$ containing x_0 coincides with $\{x_1 = 0\}$. Otherwise, $\partial \widehat{E} \cap \{x_1 = 0\} \subset \{x_1 = 0\} \cap \{x_N = 0\}$ and thus in particular $\mathcal{H}^{N-1}(\partial \widehat{E} \cap \{x_1 = 0\}) = 0$. We may then apply Lemma 5.3 to conclude that $0 \notin \partial \widehat{E}$, thus getting a contradiction to (4.15).

Step 6 (*Convergence of the isoperimetric profiles*). We claim that

$$\lim_{\eta \to 0} I_\eta(m) = I_0(m) \text{ for all } m \in [0, m_0) \quad \text{and} \quad \lim_{\eta \to 0} I_\eta(m_\eta) = I_0(m_0). \tag{4.18}$$

Let η_n be a sequence converging to zero such that $I_{\eta_n}(m) \to \liminf_{\eta \to 0} I_\eta(m)$. Since the perimeters of $\Omega_{\eta_n,m}$ are equibounded, see (4.5), up to a subsequence we may assume that $\Omega_{\eta_n,m}$ converge in L^1 to a set of finite perimeter $E \subset \Omega_0$ with $|E| = m$. Thus, by lower semicontinuity,

$$I_0(m) \le P(E; \mathbb{R}^N \setminus \mathbf{C}) \le \liminf_n P(\Omega_{\eta_n,m}, \mathbb{R}^N \setminus \mathbf{C}_{\eta_n}) = \liminf_{\eta \to 0} I_\eta(m). \tag{4.19}$$

Recall that $\Omega_{0,m}$ denotes a minimizer for the problem defining $I_0(m)$. Since

$$I_\eta(m - |\Omega_{0,m} \cap \mathbf{C}_\eta|) \le P(\Omega_{0,m}; \mathbb{R}^N \setminus \mathbf{C}_\eta) \le I_0(m),$$

using the equilipschitz continuity of I_η proved in Step 3, by letting η tend to 0 in the previous inequality and recalling (4.19) we obtain the first equality in (4.18). The second one follows simply from the fact that $\Omega_{\eta,m_\eta} = \Omega_0 \setminus \mathbf{C}_\eta$.

Note that the above argument shows in particular that if $m \in (0, m_0)$, $\eta_n \to 0$ and $\Omega_{\eta_n,m}$ is a sequence converging in L^1 to a set E, then E is a minimizer for the minimum problem in (4.4) with $\eta = 0$. Recall that any such minimizer is denoted by $\Omega_{0,m}$.

Step 7 ($I_0 = I_{\mathscr{H}}$ and any minimizer $\Omega_{0,m}$ is a connected open set). We set

$$I_{\mathscr{H}}(m) = N\left(\frac{\omega_N}{2}\right)^{\frac{1}{N}} m^{\frac{N-1}{N}}, \tag{4.20}$$

that is the isoperimetric profile of half spaces. We claim that

$$I_0(m) = I_{\mathscr{H}}(m) \qquad \text{for all } m \in [0, m_0]. \tag{4.21}$$

To this end we fix $0 < m' < m'' < m_0$ and let $0 < \tilde{\eta} \le \bar{\eta}$ be as in (4.6). Recall that by Step 2 for all $\eta \in [0, \tilde{\eta}]$, $\Omega_{\eta,m}$ is a restricted (Λ, r_0)-minimizer for all $m \in [m', m'']$. We claim that for any such η if $x_0 \in \Sigma^+_{\eta,m}$ then $\Sigma_{\eta,m}$ is of class $C^{1,1}$ in a neighborhood of x_0. Here $\Sigma^+_{\eta,m}$ is defined as in Definition 3.5 with Σ and $\mathbf{C}$ replaced by $\Sigma_{\eta,m}$ and $\mathbf{C}_\eta$. Indeed, observe first that if $x_0 \in \Sigma^+_{\eta,m}$ then by Theorem 2.6 $\Sigma_{\eta,m}$ is of class $C^{1,\alpha}$ in a neighborhood of x_0. Moreover, if $x_0 \in \Omega_0$ then, since $H_{\Sigma_{\eta,m}}$ is constant in a neighborhood of x_0, we have that in fact $\Sigma_{\eta,m}$ is analytic in such a neighborhood.

If instead $x_0 \in \partial\Omega_0$, since Ω_0 is a (Λ, r_0)-minimizer and $\partial\Omega_0$ lies on one side with respect to $\Sigma_{\eta,m}$ which is of class $C^{1,\alpha}$ in a neighborhood of x_0, again by Theorem 2.6 we infer that

$\partial\Omega_0$ is of class $C^{1,\alpha}$, hence analytic in a neighborhood of x_0. The claim then follows from Proposition 5.5.

To prove (4.21) observe that the very same argument of (3.20) (with $\widetilde{\Sigma}_\eta$ replaced by $\Sigma^+_{\eta,m}$ and θ_η replaced by $\pi/2$) yields that

$$\int_{\Sigma^+_{\eta,m}\setminus\mathbf{C}_\eta} H^{N-1}_{\Sigma_{\eta,m}}\, d\mathcal{H}^{N-1} \geq (N-1)^{N-1} N \frac{\omega_N}{2}\,. \tag{4.22}$$

Indeed this argument only requires that $\Sigma_{\eta,m}$ is of class $C^{1,1}$ in a neighborhood of $\Sigma^+_{\eta,m}$ and that (3.18) holds. Recall that the latter condition with $\theta_\eta = \pi/2$ is ensured by Step 5. Observe also that if $\Sigma^+_{\eta,m}$ intersects $\partial\Omega_0$ in a set of positive $\mathcal{H}^{N-1}$ measure then for $\mathcal{H}^{N-1}$-a.e. x on such a set

$$H_{\Sigma_{\eta,m}}(x) = H_{\partial\Omega_0} \leq H_{\Sigma^*_{\eta,m}} \tag{4.23}$$

where $\Sigma^*_{\eta,m}$ is the regular free part defined in Step 4 and the inequality follows from Proposition 5.5. Here, with a slight abuse of notation, we denote by $H_{\partial\Omega_0}$ the constant curvature of $\partial^*\Omega_0 \setminus \mathbf{C}$. Therefore the previous inequality, (4.22) and (4.11) imply in particular that for a.e. $m \in (m', m'')$ and for all $\eta \in [0, \tilde{\eta}]$

$$\begin{aligned} I_\eta(m)(I'_\eta(m))^{N-1} &= P(\Omega_{\eta,m}; \mathbb{R}^N \setminus \mathbf{C}_\eta) H^{N-1}_{\Sigma^*_{\eta,m}} \\ &\geq (N-1)^{N-1} N \frac{\omega_N}{2} = I_{\mathscr{H}}(m)(I'_{\mathscr{H}}(m))^{N-1}\,, \end{aligned} \tag{4.24}$$

where the last equality follows from (4.20). Recalling that I_η is Lipschitz in $[m', m'']$ and thus absolutely continuous, raising the above inequality to the power $\frac{1}{N-1}$ and integrating in $[m, m'']$, for any $m \in (m', m'')$ we get

$$I_\eta(m'')^{\frac{N}{N-1}} - I_\eta(m)^{\frac{N}{N-1}} \geq I_{\mathscr{H}}(m'')^{\frac{N}{N-1}} - I_{\mathscr{H}}(m)^{\frac{N}{N-1}}$$

for all $\eta \in [0, \tilde{\eta}]$. Passing to the limit as $\eta \to 0$ and using Step 6 we get

$$I_0(m'')^{\frac{N}{N-1}} - I_0(m)^{\frac{N}{N-1}} \geq I_{\mathscr{H}}(m'')^{\frac{N}{N-1}} - I_{\mathscr{H}}(m)^{\frac{N}{N-1}} \tag{4.25}$$

for all $0 < m < m'' < m_0$. Observe now that $\lim_{m''\to m_0} I_0(m'') = I_0(m_0)$ (this follows by a simple semicontinuity argument and by the fact that I_0 is increasing). Thus, passing to the limit in (4.25) as $m'' \to m_0$, recalling that by assumption $I_0(m_0) = I_{\mathscr{H}}(m_0)$ and that by Theorem 1.1 $I_0(m) \geq I_{\mathscr{H}}(m)$, we get $I_0(m) = I_{\mathscr{H}}(m)$ for all $m \in (0, m_0)$, as claimed.

Finally, since $\Omega_{0,m}$ satisfies the equality case in (1.1), the same argument used for Ω_0 shows that $\Omega_{0,m}$ is a connected open set.

Step 8 ($\partial\Omega_0 \cap \mathbf{C}$ *is flat*). In this step we prove that $\partial\Omega_0 \cap \mathbf{C}$ lies on a hyperplane Π.

To this aim we start by showing that

$$\left(I_\eta^{\frac{N}{N-1}}\right)' \to \left(I_{\mathscr{H}}^{\frac{N}{N-1}}\right)' \qquad \text{in } L^1_{loc}(0, m_0)\,. \tag{4.26}$$

Indeed, given $0 < m' < m'' < m_0$ from (4.24) and the fact that $I_\eta \to I_{\mathscr{H}}$, we have that for a.e. $m \in (m', m'')$ and $\eta \in [0, \tilde{\eta}]$, $\left(I_\eta^{\frac{N}{N-1}}\right)'(m) \geq \left(I_{\mathscr{H}}^{\frac{N}{N-1}}\right)'(m)$ and

$$\int_{m'}^{m''} \left(I_\eta^{\frac{N}{N-1}}\right)'(t)\, dt \to \int_{m'}^{m''} \left(I_{\mathscr{H}}^{\frac{N}{N-1}}\right)'(t)\, dt \quad \text{as } \eta \to 0\,.$$

Hence, (4.26) follows.

Returning to the proof of the flatness of $\partial\Omega_0 \cap \mathbf{C}$, observe that by a simple diagonal argument we can construct two sequences $m_n \to m_0$ and $\eta_n \to 0$ such that Ω_{η_n,m_n} is a Λ_n-minimizer for some $\Lambda_n > 0$ (possibly going to $+\infty$) and

$$I_{\eta_n}(m_n) \to I_0(m_0), \quad \left(I_{\eta_n}^{\frac{N}{N-1}}\right)'(m_n) \to \left(I_{\mathscr{H}}^{\frac{N}{N-1}}\right)'(m_0) = N\left(N\frac{\omega_N}{2}\right)^{\frac{1}{N-1}}.$$

This is possible thanks to Step 2, Step 6 and (4.26). Given $\varepsilon > 0$, let $\delta > 0$ be as in Theorem 1.4 with $\theta_0 = \pi/2$. Recall that δ depends only on ε and on $\mathrm{diam}(\Omega_0)$. Recall also that Σ_{η_n,m_n} is of class $C^{1,1}$ in a neighborhood of $\Sigma^+_{\eta_n,m_n}$, thanks to Step 7. Then from the above convergence, arguing as in the proof of (3.20) with $\widetilde{\Sigma}_\eta$ replaced by $\Sigma^+_{\eta_n,m_n}$, and recalling that the weak Young's inequality (4.12) holds for Σ_{η_n,m_n}, we have that for n large

$$\begin{aligned}\frac{N\omega_N}{2} \le \mathcal{K}^+(\Sigma_{\eta_n,m_n}) &\le (N-1)^{1-N}\int_{\Sigma^+_{\eta_n,m_n}} H^{N-1}_{\Sigma_{\eta_n,m_n}}\, d\mathcal{H}^{N-1} \\ &\le (N-1)^{1-N} P(\Omega_{\eta_n,m_n}; \mathbb{R}^N \setminus \mathbf{C}_{\eta_n}) H^{N-1}_{\Sigma^*_{\eta_n,m_n}} \\ &= \left[\frac{1}{N}\left(I_{\eta_n}^{\frac{N}{N-1}}\right)'(m_n)\right]^{N-1} < \frac{N\omega_N}{2} + \delta\,.\end{aligned}$$

Note that in the third inequality above we have used (4.23). Thus from Theorem 1.4 we get that

$$\mathrm{width}(\Sigma_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}) := \varepsilon_n \to 0\,.$$

More precisely, for n sufficiently large there exists $x_n \in \partial\mathbf{C}_{\eta_n}$ such that

$$\Sigma_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n} \subset \{x : -\varepsilon_n \le (x - x_n)\cdot \nu_{\mathbf{C}_{\eta_n}}(x_n) \le 0\}\,. \tag{4.27}$$

Observe that, up to a not relabelled subsequence,

$$x_n \to \overline{x} \in \mathbf{C}, \qquad \nu_{\mathbf{C}_{\eta_n}}(x_n) \to \overline{\nu} \in N_{\overline{x}}(\mathbf{C})\,. \tag{4.28}$$

Denote by Π the support hyperplane passing through $\overline{x}$ and orthogonal to $\overline{\nu}$ and by $\Pi^\pm$ the half spaces $\{x : (x - \overline{x})\cdot\overline{\nu} \gtrless 0\}$. We claim that $\partial\Omega_0 \cap \mathbf{C} \subset \Pi$ up to a set of $\mathcal{H}^{N-1}$-measure zero.

To prove the claim we first show that, passing possibly to a further subsequence,

$$\partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n} \to K \quad \text{for some } K \subset \partial\Omega_0 \cap \mathbf{C} \text{ s.t. } \mathcal{H}^{N-1}(\partial\Omega_0 \cap \mathbf{C} \setminus K) = 0\,, \tag{4.29}$$

where the convergence is meant in the Kuratowski sense. The existence of a subsequence converging to $K \subset \partial\Omega_0 \cap \mathbf{C}$ follows easily from the compactness properties of Kuratowski convergence, see Sect. 2. To show that $\mathcal{H}^{N-1}(\partial\Omega_0 \cap \mathbf{C} \setminus K) = 0$ observe first that since $\mathbf{C}_{\eta_n} \cap \overline{B_R(0)}$ is a sequence of convex sets converging to the convex set $\mathbf{C} \cap \overline{B_R(0)}$ in the sense of Kuratowski then $P(\mathbf{C}_{\eta_n} \cap \overline{B_R(0)}) \to P(\mathbf{C} \cap \overline{B_R(0)})$. This in turn yields that $\mathcal{H}^{N-1} \llcorner \partial(\mathbf{C}_{\eta_n} \cap \overline{B_R(0)}) \overset{*}{\rightharpoonup} \mathcal{H}^{N-1} \llcorner \partial(\mathbf{C} \cap \overline{B_R(0)})$ and in particular that

$$\mathcal{H}^{N-1} \llcorner \partial\mathbf{C}_{\eta_n} \overset{*}{\rightharpoonup} \mathcal{H}^{N-1} \llcorner \partial\mathbf{C} \quad \text{ in } B_R(0)\,. \tag{4.30}$$

We claim that

$$\limsup_n \mathcal{H}^{N-1}(\partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}) \le \mathcal{H}^{N-1}(K)\,. \tag{4.31}$$

To this aim set $K_\sigma = K + \overline{B_\sigma(0)} \subset B_R(0)$ for $\sigma > 0$ sufficiently small. Then for n sufficiently large $\partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n} \subset K_\sigma \cap \partial\mathbf{C}_{\eta_n}$, hence

$$\mathcal{H}^{N-1}(\partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}) \le \mathcal{H}^{N-1}(K_\sigma \cap \partial\mathbf{C}_{\eta_n}).$$

From this inequality we then have

$$\limsup_n \mathcal{H}^{N-1}(\partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}) \le \limsup_n \mathcal{H}^{N-1}(K_\sigma \cap \partial\mathbf{C}_{\eta_n}) \le \mathcal{H}^{N-1}(K_\sigma \cap \partial\mathbf{C}),$$

where in the last inequality we have used (4.30). Then (4.31) follows letting $\sigma \to 0$. On the other hand, $\Omega_{\eta_n,m_n} \to \Omega_0$ in L^1 and by the lower semicontinuity of perimeter and (4.31)

$$\begin{aligned} P(\Omega_0) = I_0(m_0) + \mathcal{H}^{N-1}(\partial^*\Omega_0 \cap \mathbf{C}) &\le \liminf_n P(\Omega_{\eta_n,m_n}) \\ &= \liminf_n \left[I_{\eta_n}(m_n) + \mathcal{H}^{N-1}(\partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}) \right] \\ &\le I_0(m_0) + \mathcal{H}^{N-1}(K). \end{aligned}$$

Recall that by the volume estimate Proposition 5.2-(ii) $\partial^*\Omega_0 \cap \mathbf{C}$ coincides $\mathcal{H}^{N-1}$-a.e. with $\partial\Omega_0 \cap \mathbf{C}$. Thus the above inequality implies that K coincides $\mathcal{H}^{N-1}$-a.e. with $\partial\Omega_0 \cap \mathbf{C}$. Hence, (4.29) follows.

We finally claim that for n large

$$\partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n} \subset \{x : -\varepsilon_n \le (x - x_n) \cdot \nu_{\mathbf{C}_{\eta_n}}(x_n) \le 0\}. \tag{4.32}$$

To prove this we argue by contradiction assuming that for infinitely many n there exists $y_n \in \partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}$ such that $(y_n - x_n) \cdot \nu_{\mathbf{C}_{\eta_n}}(x_n) < -\varepsilon_n$. Observe that, if this is the case for all such n,

$$F_n := \partial\mathbf{C}_{\eta_n} \cap \{x : (x - x_n) \cdot \nu_{\mathbf{C}_{\eta_n}}(x_n) < -\varepsilon_n\} \subset \partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}. \tag{4.33}$$

Indeed, otherwise there exists $z_n \in F_n \setminus \partial\Omega_{\eta_n,m_n}$ and in turn a continuous path $\gamma \subset F_n$ connecting z_n to y_n (recall that $\mathbf{C}_{\eta_n}$ is bounded). But then this arc must contain a point in $\partial_{\mathbf{C}_{\eta_n}}(\partial\Omega_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}) \subset \Sigma_{\eta_n,m_n} \cap \mathbf{C}_{\eta_n}$, which contradicts (4.27). Therefore, from (4.33), (4.28) and (4.29) we have that

$$\partial\mathbf{C} \cap \{x : (x - \overline{x}) \cdot \overline{\nu} < 0\} = \partial\mathbf{C} \cap \Pi^- \subset \partial\Omega_0 \cap \mathbf{C}.$$

Then, let $\bar{t} := \min\{t \le 0 : \Pi + t\overline{\nu} \cap \mathbf{C} \ne \emptyset\}$ and set for $t \in (\bar{t}, 0)$, $\mathbf{C}^t := \mathbf{C} \cap (\Pi^+ + t\overline{\nu})$. Note that, from the above inclusion, $P(\Omega_0 \cup (\mathbf{C} \setminus \mathbf{C}^t); \mathbb{R}^N \setminus \mathbf{C}^t) = P(\Omega_0; \mathbb{R}^N \setminus \mathbf{C}) = I_{\mathscr{H}}(m_0)$, but this contradicts (1.1) since $|\Omega_0 \cup (\mathbf{C} \setminus \mathbf{C}^t)| > m_0$. Hence (4.32) holds for n large enough.

Finally, from (4.32) and (4.29) we have that $\partial\Omega_0 \cap \mathbf{C} \subset \Pi$ up to a set of vanishing $\mathcal{H}^{N-1}$ measure.

Step 9 (*Conclusion*). In this final step we show that Ω_0 is a half ball.

To this aim we fix $m \in (0, m_0)$ and a sequence $\eta_n \to 0$ such that

$$I_{\eta_n}(m) \to I_0(m) = I_{\mathscr{H}}(m), \quad \left(I_{\eta_n}^{\frac{N}{N-1}}\right)'(m) \to \left(I_{\mathscr{H}}^{\frac{N}{N-1}}\right)'(m) = N\left(N\frac{\omega_N}{2}\right)^{\frac{1}{N-1}} \tag{4.34}$$

Owing to Steps 6-8 we can find such a sequence for a.e. $m \in (0, m_0)$. Thanks to Step 2, we may assume that there exists $\Lambda > 0$ such that $\Omega_{\eta_n,m}$ is a Λ-minimizer for all n. By Theorem 2.6-(ii) this implies in particular that $|H_{\Sigma_{\eta_n,m}}| \le \Lambda$ $\mathcal{H}^{N-1}$-a.e. on $\partial^*\Omega_{\eta_n,m} \setminus \mathbf{C}_{\eta_n}$.

Arguing as in the previous step, see also the proof of (3.20), we have then

$$\begin{aligned}\frac{N\omega_N}{2} \le \mathcal{K}^+(\Sigma_{\eta_n,m}) &= \int_{\Sigma^+_{\eta_n,m}} K_{\Sigma_{\eta_n,m}}\,d\mathcal{H}^{N-1} \le (N-1)^{1-N}\int_{\Sigma^+_{\eta_n,m}} H^{N-1}_{\Sigma_{\eta_n,m}}\,d\mathcal{H}^{N-1} \\ &\le (N-1)^{1-N}P(\Omega_{\eta_n,m};\mathbb{R}^N\setminus \mathbf{C}_{\eta_n})H^{N-1}_{\Sigma^*_{\eta_n,m}} \\ &= \Big[\frac{1}{N}\Big(I_{\eta_n}^{\frac{N}{N-1}}\Big)'(m)\Big]^{N-1} \to \frac{N\omega_N}{2}\,,\end{aligned} \tag{4.35}$$

where we recall $K_{\Sigma_{\eta_n,m}}$ is the Gaussian curvature of $\Sigma_{\eta_n,m}$ and we used (4.23). We start by observing that, since $H_{\Sigma_{\eta_n,m}}(x) \le H_{\Sigma^*_{\eta_n,m}}$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Sigma^+_{\eta_n,m}$, from the third inequality in (4.35) we have in particular that

$$\lim_n \mathcal{H}^{N-1}(\Sigma^+_{\eta_n,m}) = \lim_n P(\Omega_{\eta_n,m};\mathbb{R}^N\setminus \mathbf{C}_{\eta_n}) = \mathcal{H}^{N-1}(\partial^*\Omega_{0,m}\setminus \mathbf{C})\,. \tag{4.36}$$

Note that $H_{\Sigma_{\eta_n,m}}$ may only take the constant values $H_{\partial\Omega_0}$ or $H_{\Sigma^*_{\eta_n,m}}$. Then, again from (4.35) and from (4.23), it follows that

$$\text{either}\quad H_{\Sigma^*_{\eta_n,m}} \to H_{\partial\Omega_0} \quad\text{or}\quad \mathcal{H}^{N-1}\big((\partial\Omega_{\eta_n,m}\cap\partial\Omega_0)\setminus \mathbf{C}_{\eta_n}\big) \to 0\,. \tag{4.37}$$

Fix now $x \in \partial^*\Omega_{0,m}\setminus \mathbf{C}$. Since $\Omega_{\eta_n,m} \to \Omega_{0,m}$ in L^1 and $P(\Omega_{\eta_n,m};\mathbb{R}^N\setminus \mathbf{C}_{\eta_n}) \to P(\Omega_{0,m};\mathbb{R}^N\setminus \mathbf{C})$ thanks to the first condition in (4.34), we have that $\mathcal{H}^{N-1}\llcorner\partial^*\Omega_{\eta_n,m} \overset{*}{\rightharpoonup} \mathcal{H}^{N-1}\llcorner\partial^*\Omega_{0,m}$ in $\mathbb{R}^N\setminus \mathbf{C}$. In turn, by Theorem 2.7 it follows that, up to rotations and translations, there exist a $(N-1)$-dimensional ball $B' \subset \mathbb{R}^{N-1}$, functions $\varphi_n, \varphi \in W^{2,p}(B')$, and $r > 0$ such that $x \in B'\times(-r,r)$ and

$$\begin{aligned}&\partial\Omega_{\eta_n,m}\cap(B'\times(-r,r)) = \{(x',\varphi_n(x')) : x'\in B'\},\\ &\partial\Omega_{0,m}\cap(B'\times(-r,r)) = \{(x',\varphi(x')) : x'\in B'\},\\ &\varphi_n \rightharpoonup \varphi \quad \text{in } W^{2,p}(B') \text{ for all} p\ge 1,\\ &H_{\Sigma_{\eta_n,m}}(x',\varphi_n(x')) \rightharpoonup H_{\Sigma_{0,m}}(x',\varphi(x')) \quad \text{in } L^p(B') \text{ for all } p\ge 1\,,\end{aligned}$$

Recalling (4.37) the fourth condition above implies that

$$H_{\Sigma_{\eta_n,m}}(x',\varphi_n(x')) \to H_{\Sigma_{0,m}}(x',\varphi(x')) \equiv H_{\Sigma^*_{0,m}}$$

strongly in $L^p(B')$ for all $p \ge 1$. In turn, see for instance [1, Lemma 7.2], this implies

$$\varphi_n \to \varphi \quad \text{strongly in } W^{2,p}(B') \text{ for all } p\ge 1\,. \tag{4.38}$$

Note also that, since from (4.36) $\mathcal{H}^{N-1}(\Sigma_{\eta_n,m}\setminus\Sigma^+_{\eta_n,m}) \to 0$, we have that for every $y \in (B'\times(-r,r))\cap\Sigma_{0,m}$ there exists a sequence $y_n \in (B'\times(-r,r))\cap\Sigma^+_{\eta_n,m}$ such that $y_n \to y$. Therefore, using the L^1 convergence of $\Omega_{\eta_n,m}$ to $\Omega_{0,m}$ we conclude that the tangent hyperplane to $\partial\Omega_{0,m}$ at y is also a support hyperplane. Thus we have shown that all principal curvatures at any point in $(B'\times(-r,r))\cap\Sigma_{0,m}$ are nonnegative. Thus, from the second inequality in (4.35), recalling (4.36) and (4.38) we may conclude that

$$K_{\Sigma_{0,m}} = (N-1)^{1-N}H^{N-1}_{\Sigma_{0,m}} = (N-1)^{1-N}H^{N-1}_{\Sigma^*_{0,m}} \quad \text{on } (B'\times(-r,r))\cap\Sigma_{0,m}\,.$$

The equality above implies that $\Sigma_{0,m}\cap(B'\times(-r,r)))$ is umbilical. Hence $\partial^*\Omega_{0,m}\setminus \mathbf{C}$ is umbilical, thus each connected component of $\partial^*\Omega_{0,m}\setminus \mathbf{C}$ lies on a sphere of radius $R_m = (N-1)/H_{\Sigma^*_{0,m}}$. Consider the unique unbounded connected component of $U := \mathbb{R}^N\setminus\overline{\Omega_{0,m}}$. Then, recalling Step 8 and that $\Omega_{0,m}$ is connected (see Step 7), $\partial U\setminus \mathbf{C}$ is contained in a sphere of radius R_m intersecting $\mathbf{C}$ on Π. Thus $\partial U\setminus \mathbf{C}$ is a spherical cap and $\Omega_{0,m}$ is contained in the region enclosed by $\partial U \setminus \mathbf{C}$ and Π. In particular $\Omega_{0,m}$ is contained in the half space Π^+ determined by Π not containing $\mathbf{C}$. Since $P(\Omega_{0,m};\Pi^+) = P(\Omega_{0,m};\mathbb{R}^N \setminus \mathbf{C}) = N\big(\frac{\omega_N}{2}\big)^{\frac{1}{N}} m^{\frac{N-1}{N}}$, by Theorem 19.21 in [15] for a.e. m we conclude that for such m $\Omega_{0,m}$ is a half ball. Since the argument above can be carried out for a.e. $m \in (0, m_0)$, in particular there exists a sequence $m_n \to m_0$ such that Ω_{0,m_n} is a half ball. Hence Ω_0 is a half ball. □

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5 Appendix A: some auxiliary results

In this section we collect some auxiliary results needed in the proof of Theorem 1.2.

5.1 Density estimates

Density estimates for (Λ, r_0)-minimizers are well known. However for the sake of completeness we give the proof of the proposition below showing that such density estimates are independent of the convex obstacle.

Lemma 5.1 *Let $\mathbf{C}$ be a closed convex set with nonempty interior and $F \subset \mathbb{R}^N \setminus \mathbf{C}$ a bounded set of finite perimeter. Then*

$$P(F;\partial\mathbf{C}) \le P(F;\mathbb{R}^N \setminus \mathbf{C})$$

Proof Assume that $\mathbf{C}$ is bounded and let H_i be a sequence of closed half spaces such that $\mathbf{C} = \bigcap_{i=1}^{\infty} H_i$. Since $\mathbf{C} = (\mathbf{C} \cup F) \cap \bigcap_{i=1}^{\infty} H_i$ we have

$$P(\mathbf{C}) \le \liminf_n P\left((\mathbf{C} \cup F) \cap \bigcap_{i=1}^{n} H_i\right) \le P(\mathbf{C} \cup F)\,,$$

where the last inequality follows by applying repeatedly the inequality $P(G \cap H_i) \le P(G)$ where G is a set of finite perimeter. Since $P(\mathbf{C}\cup F) = \mathcal{H}^{N-1}(\partial\mathbf{C} \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial^* F \setminus \mathbf{C})$, the conclusion follows observing that $P(\mathbf{C}) = \mathcal{H}^{N-1}(\partial\mathbf{C} \cap F^{(0)}) + \mathcal{H}^{N-1}(\partial\mathbf{C} \cap \partial^* F)$.

If $\mathbf{C}$ is not bounded, since $F \subset\subset B$ for a suitable closed ball B, the conclusion follows by the same argument as before, replacing $\mathbf{C}$ with $\mathbf{C} \cap B$. □

Proposition 5.2 *Let $\mathbf{C}$ be a closed convex set with nonempty interior and let $E \subset \mathbb{R}^N \setminus \mathbf{C}$ be a restricted (Λ, r_0)-minimizer of the relative perimeter $P(\cdot;\mathbb{R}^N \setminus \mathbf{C})$ according to Definition 4.1. Then there are positive constants $c_1 = c_1(N)$ and $C_1 = C_1(N)$ independent of $\mathbf{C}$ such that for all $r \in (0, \min\{r_0, N/(4\Lambda)\})$ we have:*

(i) *for all* $x \in \mathbb{R}^N \setminus int(\mathbf{C})$

$$P(E; B_r(x)) \le C_1 r^{N-1} \ ,$$

(ii) *for all* $x \in \partial^* E$

$$|E \cap B_r(x)| \ge c_1 r^N \ .$$

Moreover E is equivalent to an open set Ω *such that* $\partial\Omega = \partial^e\Omega$*, hence* $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^*\Omega) = 0$*, and (ii) holds at any point* $x \in \partial\Omega$*.*

Proof Given $x \in \mathbb{R}^N \setminus \mathrm{int}(\mathbf{C})$ and $r < \min\{r_0, N/(4\Lambda)\}$, we set $m(r) := |E \cap B_r(x)|$. Recall that for a.e. such r we have $m'(r) = \mathcal{H}^{N-1}(E^{(1)} \cap \partial B_r(x))$ and $\mathcal{H}^{N-1}(\partial^* E \cap \partial B_r(x)) = 0$. For any such r we set $F := E \setminus B_r(x)$. Then, using Definition 4.1, we have

$$P(E; B_r(x) \setminus \mathbf{C}) \le \mathcal{H}^{N-1}(\partial B_r(x) \cap E^{(1)}) + \Lambda |E \cap B_r(x)| \le C_1 r^{N-1} \tag{5.1}$$

for a suitable constant C_1. In turn

$$P(E; B_r(x)) \le P(E; B_r(x) \setminus \mathbf{C}) + \mathcal{H}^{N-1}(\partial(\mathbf{C} \cap B_r(x))) \le C_1 r^{N-1} + \mathcal{H}^{N-1}(\partial B_r(x)) \, ,$$

where in the last inequality we estimated the perimeter of $\mathbf{C} \cap B_r(x)$ with the perimeter of the larger convex set $B_r(x)$. Thus (i) follows by taking C_1 larger.

Observe now that by Lemma 5.1

$$P(E \cap B_r(x); \partial\mathbf{C}) \le P(E \cap B_r(x); \mathbb{R}^N \setminus \mathbf{C}) \, .$$

Thus, using also (5.1), we have

$$\begin{aligned} P(E \cap B_r(x)) &= P(E \cap B_r(x); \mathbb{R}^N \setminus \mathbf{C}) + P(E \cap B_r(x); \partial\mathbf{C}) \\ &\le 2P(E \cap B_r(x); \mathbb{R}^N \setminus \mathbf{C}) = 2P(E; B_r(x) \setminus \mathbf{C}) + 2m'(r) \\ &\le 4m'(r) + 2\Lambda m(r) \, . \end{aligned}$$

In turn, using the isoperimetric inequality and the fact that $2\Lambda r < N/2$ we get

$$\begin{aligned} N\omega_N^{\frac{1}{N}} m(r)^{\frac{N-1}{N}} &\le P(E \cap B_r(x)) \le 4m'(r) + 2\Lambda m(r) \\ &\le 4m'(r) + 2\Lambda r \omega_N^{\frac{1}{N}} m(r)^{\frac{N-1}{N}} \le 4m'(r) + \frac{N}{2}\omega_N^{\frac{1}{N}} m(r)^{\frac{N-1}{N}} \, . \end{aligned}$$

Then from the previous inequality we get

$$\frac{N}{2}\omega_N^{\frac{1}{N}} m(r)^{\frac{N-1}{N}} \le 4m'(r) \, .$$

Observe now that if in addition $x \in \partial^* E$, then $m(r) > 0$ for all r as above. Thus, we may divide the previous inequality by $m(r)^{\frac{N-1}{N}}$, and integrate the resulting differential inequality thus getting

$$|E \cap B_r(x)| \ge c_1 r^N \, ,$$

for a suitable positive constant c_1 depending only on N.

We show that $\overline{\partial^* E} \subset \partial^e E$. To this aim note that (ii) holds for every $x \in \overline{\partial^* E}$. Thus, if $x \in \mathbb{R}^N \setminus \mathbf{C}$, since both E and $\mathbb{R}^N \setminus E$ are Λ-minimizers in a neighborhood of x we have that $|B_r(x) \setminus E| \ge c_1 r^N$ for r small. Thus $x \notin (E^{(0)} \cup E^{(1)})$, that is $x \in \partial^e E$. If $x \in \partial\mathbf{C} \cap \overline{\partial^* E}$ then there exists a constant $c_2 > 0$, depending on x such that for r small $|\mathbf{C} \cap B_r(x)| \ge c_2 r^N$. This estimate, together with (ii) again implies that $x \in \partial^e E$.

Hence $\mathcal{H}^{N-1}(\overline{\partial^* E} \setminus \partial^* E) \leq \mathcal{H}^{N-1}(\partial^e E \setminus \partial^* E) = 0$, where the last equality follows from Theorem 16.2 in [15].

Set now $\Omega = E^{(1)} \setminus \partial E^{(1)}$. Recalling that $\partial E^{(1)} = \overline{\partial^* E}$, see (2.2), we have that Ω is an open set equivalent to E such $\partial\Omega = \partial E^{(1)}$. Hence the conclusion follows. □

5.2 A maximum principle

Next result is essentially the strong maximum principle proved in [9, Lemma 2.13]. However, we have to apply it in a slightly different situation and therefore we indicate the changes needed in the proof.

Lemma 5.3 *Let $E \subset \{x_1 > 0\}$ be a set of locally finite perimeter such that*

$$\mathcal{H}^{N-1}((\partial E \setminus \partial^* E) \setminus \{x_1 = 0\}) = 0 \tag{5.2}$$

satisfying the following minimality property: for every $r > 0$ and every open set with Lipschitz boundary $V \subset\subset Q_r(0)$ such that $\mathcal{H}^{N-1}(\partial E \cap \partial V) = 0$ we have

$$P(E; Q_r(0)) \leq P(E \setminus V; Q_r(0)) \,. \tag{5.3}$$

Assume also that $\mathcal{H}^{N-1}(\partial E \cap \{x_1 = 0\}) = 0$. Then $0 \notin \partial E$.[4]

The proof of lemma above is in turn based on the following variant of [9, Lemma 2.12]. To this aim, given $r > 0$ we set $C_r := (0, r) \times D_r$, where $D_r := \{x' \in \mathbb{R}^{N-1} : |x'| < r\}$.

Lemma 5.4 *Let E be as in Lemma 5.3, let $\bar{r} > 0$ and let $u_0 \in C^2(D_{\bar{r}}) \cap Lip(D_{\bar{r}})$ with $0 < u_0 < \bar{r}$ on $\overline{D}_{\bar{r}}$. Assume also that*

$$E^{(1)} \cap [(0, \bar{r}) \times \partial D_{\bar{r}}] \subset \{(x_1, x') \in (0, \bar{r}) \times \partial D_{\bar{r}} : x_1 \geq u_0(x')\}\,,$$

$$\operatorname{div}\left(\frac{\nabla u_0}{\sqrt{1 + |\nabla u_0|^2}}\right) = 0 \qquad \text{in } D_{\bar{r}}$$

and

$$\mathcal{H}^{N-1}\big(\partial E \cap \partial\{(x_1, x') \in C_{\bar{r}} : x_1 < u_0(x')\}\big) = 0\,. \tag{5.4}$$

Then,

$$E^{(1)} \cap C_{\bar{r}} \subset \{(x_1, x') \in C_{\bar{r}} : x_1 \geq u_0(x')\}\,.$$

Proof The proof goes exactly as the one of Lemma 2.12 in [9] as it is based on the comparison with he competitor $F = E \setminus V$, where $V = \{(x_1, x') \in C_{\bar{r}} : x_1 < u_0(x')\}$. Observe that assumption (5.4) guarantees that such a competitor satisfies $\mathcal{H}^{N-1}(\partial E \cap \partial V) = 0$, which is required in order (5.3) to hold. □

Proof of Lemma 5.3 For reader's convenience we reproduce the proof of Lemma 2.13 in [9] with the small changes needed in our case.

We choose $\bar{r} > 0$ so that $\mathcal{H}^{N-1}(\partial E \cap \partial C_{\bar{r}}) = 0$ and $\mathcal{H}^{N-2}(\partial E \cap \partial D_{\bar{r}}) = 0$, where with a slight abuse of notation $\partial D_{\bar{r}}$ stands for the relative boundary of $D_{\bar{r}}$ in $\{x_1 = 0\}$. Note that a.e. $r > 0$ satisfies these conditions thanks to (5.2) and to the assumption $\mathcal{H}^{N-1}(\partial E \cap \{x_1 = 0\}) = 0$. Define now a function $w_E : \overline{D_{\bar{r}}} \to [0, \infty]$ by setting

$$w_E(x') = \inf\{x_1 \in \mathbb{R} : (x_1, x') \in \overline{C_{\bar{r}}} \cap \partial E\}.$$

[4] Here as usual we assume that $\partial E = \overline{\partial^* E}$.

Observe that w_E is nonnegative and lower semicontinuous on $\overline{D_{\bar r}}$, with the property that

$$E^{(1)} \cap C_{\bar r} \subset \{(x_1, x') : x' \in D_{\bar r},\, x_1 \geq w_E(x')\}\,.$$

Recalling that $\mathcal{H}^{N-2}(\partial E \cap \partial D_{\bar r}) = 0$, we have that $w_E > 0$ $\mathcal{H}^{N-2}$-a.e. on $\partial D_{\bar r}$. Therefore there exists a family $(\varphi_t)_{t\in(0,1)} \subset C^\infty(\partial D_{\bar r})$ such that

$$0 \leq \varphi_{t_1} \leq \varphi_{t_2} \leq \min\left\{w_E, \frac{\bar r}{2}\right\} \quad \varphi_{t_1} \not\equiv \varphi_{t_2} \quad \text{for all } 0 < t_1 < t_2 < 1\,.$$

By Lemma 2.11 in [9] for every $t \in (0,1)$ there exists $u_t \in C^\infty(D_{\bar r}) \cap \mathrm{Lip}(D_{\bar r})$ such that

$$\begin{cases} \mathrm{div}\left(\dfrac{\nabla u_t}{\sqrt{1+|\nabla u_t|^2}}\right) = 0 & \text{in } D_{\bar r}, \\ u_t = \varphi_t & \text{on } \partial D_{\bar r}\,. \end{cases}$$

Note that by the Strong Maximum Principle Theorem 2.5 we have that $0 < u_{t_1} < u_{t_2} < \bar r/2$ in $D_{\bar r}$ for every $0 < t_1 < t_2 < 1$. Therefore the graphs Γ_t of u_t are mutually disjoint in $C_{\bar r}$ and so $\mathcal{H}^{N-1}(\Gamma_t \cap \partial E) = 0$ for all but countably many $t \in (0,1)$. In particular there exists $\bar t$ such that (5.4) holds with u_0 replaced by $u_{\bar t}$. Therefore we may apply Lemma 5.4 to conclude that $E^{(1)} \cap C_{\bar r} \subset \{(x_1, x') \in C_{\bar r} : x_1 \geq u_{\bar t}(x')\}$ so that in particular $w_E(0) \geq u_{\bar t}(0) > 0$, hence $0 \notin \partial E$. □

5.3 A regularity result

The following proposition is a slight variant of a result contained in [21].

Proposition 5.5 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $x_0 \in \partial\Omega$ be such that $\partial\Omega$ is of class C^2 in a neighborhood U of x_0. Let $E \subset \Omega$ satisfy*

$$P(E) \leq P(F) \quad \text{for all } F \subset \Omega, |F| = |E|, \text{ s.t. } E \Delta F \subset\subset U\,. \tag{5.5}$$

If there exists a support hyperplane Π to E at x_0 such that $\partial E \cap \Pi = \{x_0\}$, then ∂E is of class $C^{1,1}$ in a neighborhood V of x_0. Moreover if $\partial^* E \cap \Omega \cap V \neq \emptyset$, then for $\mathcal{H}^{N-1}$-a.e. $x \in \partial E \cap \partial\Omega \cap V$

$$H_{\partial\Omega}(x) \leq H\,, \tag{5.6}$$

where H denotes the constant curvature of $\partial^* E \cap \Omega \cap V$.

Proof Observe that by a standard argument (5.5), together with the assumption that $\partial\Omega$ of class C^2, implies that E is a (Λ, r_0)-minimizer in a possibly smaller naeighborhood U' of x_0. Hence, since there exists a support hyperplane to ∂E at x_0, by Theorem 2.6 ∂E is of class $C^{1,\alpha}$ in a neighborhood of x_0. Moreover, up to a change of coordinate system, we may assume that the support hyperplane to E at x_0 is the horizontal hyperplane $\{x_N = 0\}$ and $E \subset \{x_N > 0\}$. Since $\{x_N = 0\} \cap \partial E = \{x_0\}$, there exists $\varepsilon > 0$ sufficiently small such that $E \cap \{x_N = \varepsilon\}$ is an $(N-1)$-dimensional relatively open set, denoted by ω, and there exist $\beta \in C^2(\omega)$, $u \in C^{1,\alpha}(\omega)$ whose graphs coincide with $\partial\Omega \cap (\omega \times (-r, r))$ and $\partial E \cap (\omega \times (-r, r))$ respectively, for some $r > 0$, with $u = 0$ on $\partial\omega$ and $\beta \leq u \leq 0$. The $C^{1,1}$ regularity of ∂E then follows arguing exactly as in the proof at p. 658 of [21]. Finally, inequality (5.6) is also a byproduct of the same proof, see (3.5) in [21]. □

6 Appendix B: some steps of the proof of Theorem 1.2

6.1 Proof of the claim of Step 1

We argue by contradiction assuming that there exist a sequence $\Lambda_h \to +\infty$, $\eta_h \in [0, \tilde{\eta}]$, $\eta_h \to \eta_0$, $m_h \in [m', m'']$ converging to some m, and a sequence $E_h \subset \Omega_0 \setminus \mathbf{C}_{\eta_h}$ such that each E_h is a minimizer of (4.8) with Λ', m and η replaced by Λ_h, m_h and η_h respectively, and $|E_h| \neq m_h$. Since $P(E_h; \mathbb{R}^N \setminus \mathbf{C}_{\eta_h}) \leq P(\Omega_{\eta_h, m_h}; \mathbb{R}^N \setminus \mathbf{C}_{\eta_h})$, from (4.5) we have that the perimeters of E_h are equibounded perimeters. Therefore, without loss of generality we may assume that E_h converges in L^1 to some set $F \subset \Omega_0 \setminus \mathbf{C}_{\eta_0}$ such that $|F| = m$. We assume also that $|E_h| < m_h$ for all h, the other case being analogous. Note also that, since $\Lambda_h \to +\infty$ we have $m_h - |E_h| \to 0$.

Observe now that (4.6) implies that there exists a point $x_0 \in \partial^* F \cap (\Omega_0 \setminus \mathbf{C}_{\eta_0})$. Arguing as in Step 1 of Theorem 1.1 in [10], given $\varepsilon > 0$ sufficiently small, we can find nearby x_0 a point x' and $r > 0$ such that $B_r(x') \subset\subset \Omega_0 \setminus \mathbf{C}_{\eta_0}$ and

$$|F \cap B_{r/2}(x')| < \varepsilon r^N , \quad |F \cap B_r(x')| > \frac{\omega_N}{2^{N+2}} r^N .$$

Therefore, for h sufficiently large, we also have

$$|E_h \cap B_{r/2}(x')| < \varepsilon r^N , \quad |E_h \cap B_r(x')| > \frac{\omega_N}{2^{N+2}} r^N .$$

We can now continue as in the proof of [10, Theorem 1]. We recall the main construction for the reader's convenience. For a sequence $0 < \sigma_h < 1/2^N$ to be chosen, we introduce the following bilipschitz maps:

$$\Phi'_x(x) := \begin{cases} x' + (1 - \sigma_h(2^N - 1))(x - x') & \text{if } |x - x'| \leq \frac{r}{2}, \\ x + \sigma_h \left(1 - \dfrac{r^N}{|x - x'|^N}\right)(x - x') & \frac{r}{2} \leq |x - x'| < r, \\ x & |x - x'| \geq r. \end{cases}$$

Setting $\widetilde{E}_h := \Phi_h(E_h)$, arguing as for the proof of [10, formula (14)], we have

$$\mathcal{H}^{N-1}\big(\partial^* E_h \setminus \mathbf{C}_{\eta_h}\big) - \mathcal{H}^{N-1}(\partial^* \widetilde{E}_h \setminus \mathbf{C}_{\eta_h}) \geq -2^N N \sigma_h \mathcal{H}^{N-1}(\partial^* E_h \setminus \mathbf{C}_{\eta_h}) . \quad (6.1)$$

Moreover, arguing exactly as in Step 4 of the proof of [10, Theorem 1] we have

$$|\widetilde{E}_h| - |E_h| \geq \sigma_h r^N (c - \varepsilon C)$$

for suitable universal constants $c, C > 0$. If we fix ε so that the negative term in the brackets does not exceed half the positive one, then we have

$$|\widetilde{E}_h| - |E_h| \geq \frac{c}{2} \sigma_h r^N . \quad (6.2)$$

In particular from this inequality it is clear that we can choose σ_h so that $|\widetilde{E}_h| = m_h$; this implies $\sigma_h \to 0$. With this choice of σ_h, recalling that $\Lambda_h \to +\infty$ and that the perimeters of E_h are equibounded, it follows from (6.1) and (6.2) that

$$\begin{aligned} P(\widetilde{E}_h; \mathbb{R}^N \setminus \mathbf{C}_{\eta_h}) + \Lambda_h ||\widetilde{E}_h| - m_h| &\leq P(E_h; \mathbb{R}^N \setminus \mathbf{C}_{\eta_h}) + \Lambda_h ||E_h| - m_h| \\ &\quad + 2^N N \sigma_h \mathcal{H}^{N-1}(\partial^* E_h \setminus \mathbf{C}_{\eta_h}) - \Lambda_h \frac{c}{2} \sigma_h r^N \\ &< P(E_h; \mathbb{R}^N \setminus \mathbf{C}_{\eta_h}) + \Lambda_h ||E_h| - m_h| \end{aligned}$$

for h large, thus contradicting the minimality of E_h.

6.2 Proof of the claim of Step 3

We start by showing that the functions I_η are strictly increasing in $[0, m_\eta]$ for all $\eta \in [0, \bar{\eta}]$. To this end we fix $m \in (0, m_\eta]$ and a point $x \in \pi_{\mathbf{C}_\eta}(\Omega_{\eta,m})$, where $\pi_{\mathbf{C}_\eta}$ is the orthogonal projection on $\mathbf{C}_\eta$. Let Π be the tangent hyperplane to $\mathbf{C}_\eta$ at x. Define $\Pi_t = \Pi + t\nu_{\mathbf{C}_\eta}(x)$ for $t \in \mathbb{R}$ and set

$$\bar{t} = \max\left\{t \geq 0 :\ \Pi_t \cap \overline{\Omega_{\eta,m}} \neq \emptyset\right\}.$$

Note that $\bar{t} > 0$ and that $\Pi_{\bar{t}}$ is a support hyperplane for $\Omega_{\eta,m}$ with $\mathrm{dist}(\Pi_{\bar{t}}, \mathbf{C}_\eta) = \bar{t}$. For all $t \in (0, \bar{t})$ we denote by $\Omega_{\eta,m,t}$ the intersection of $\Omega_{\eta,m}$ with the half space with boundary Π_t containing $\mathbf{C}_\eta$. Then $I_\eta(|\Omega_{\eta,m,t}|) \leq P(\Omega_{\eta,m,t}; \mathbb{R}^N \setminus \mathbf{C}_\eta) < P(\Omega_{\eta,m}; \mathbb{R}^N \setminus \mathbf{C}_\eta) = I_\eta(m)$. Since the function $t \to |\Omega_{\eta,m,t}|$ is increasing and continuous in a left neighborhood of $\bar{t}$ and $|\Omega_{\eta,m,t}| < |\Omega_{\eta,m}|$ if $t < \bar{t}$, it follows that

$$\text{for every } m \in (0, m_\eta] \text{ there exists } \varepsilon > 0 \text{ s.t. } I_\eta(s) < I_\eta(m) \text{ for all } s \in (m - \varepsilon, m)\,. \tag{6.3}$$

Let $I = \{0 < s < m :\ I_\eta(\sigma) \leq I_\eta(m) \text{ for all } \sigma \in [s, m)\}$. We claim that $I = (0, m)$. Indeed if $\bar{m} = \inf I > 0$, then there exist $m_n \in I$, with $m_n \to \bar{m}^+$. Since the minimizers Ω_{η,m_n} are equibounded sets with equibounded perimeters, see (4.5), up to a subsequence we may assume that Ω_{η,m_n} converge to a set $E \subset \Omega_0 \setminus \mathbf{C}_\eta$ with $|E| = \bar{m}$. Then, by the lower semicontinuity of the perimeter we conclude that $I_\eta(\bar{m}) \leq P(E; \mathbb{R}^N \setminus \mathbf{C}_\eta) \leq \liminf_n I_\eta(m_n) \leq I_\eta(m)$. In turn, (6.3) implies that there exists exists a left neighborhood $(\bar{m} - \varepsilon, \bar{m})$ such that $I_\eta(s) < I_\eta(\bar{m}) \leq I_\eta(m)$ for all $s \in (\bar{m} - \varepsilon, \bar{m})$ which is a contradiction to the fact that $\bar{m} = \inf I$. This contradiction proves that I_η is increasing. The strict monotonicity now follows from (6.3).

Finally if, $m_1, m_2 \in [m', m'']$, from (4.8) we have for $\eta \in [0, \tilde{\eta}]$

$$I_\eta(m_2) = P\left(\Omega_{\eta,m_2}; \mathbb{R}^N \setminus \mathbf{C}_\eta\right) \leq P\left(\Omega_{\eta,m_1}; \mathbb{R}^N \setminus \mathbf{C}_\eta\right) + \Lambda'|m_2 - m_1|\,.$$

This proves the Λ'-Lipschitz continuity of I_η.

6.3 Proof of claim (4.17)

Let us start by assuming also that

$$\mathcal{H}^{N-1}(\partial E_h \cap \partial V \cap \mathscr{H}) = 0 \qquad \text{for all } h \in \mathbb{N}\,. \tag{6.4}$$

To this aim we fix $\delta > 0$ and set $\mathscr{H}_\delta := \{x \in \mathscr{H} :\ x_N > \delta\}$ and $(E)_\delta = E + B_\delta(0)$. Then we denote by $\Phi_h : \overline{Q_r(0)} \to \overline{Q_r(0)}$ a sequence of C^1 diffeomorphisms converging in C^1 to the identity map as $h \to +\infty$ with the property that $\Phi_h(\overline{Q_r(0) \cap \mathscr{H}}) = \overline{Q_r(0) \setminus \mathbf{C}_{\eta,h}}$, $\Phi_h(\partial\mathscr{H} \cap Q_r(0)) = \partial\mathbf{C}_{\eta,h} \cap Q_r(0)$ and $\Phi_h(x) = x$ if $x \in \mathscr{H}_\delta$. Recalling the Λ-minimality property (4.14), we have using (6.4) and observing that $\Phi_h(V) \subset\subset Q_r(0)$ for h sufficiently large

$$\begin{aligned}
P(E_h; Q_r(0) \setminus \mathbf{C}_{\eta,h}) &\leq P(E_h \setminus \Phi_h(V); Q_r(0) \setminus \mathbf{C}_{\eta,h}) + \frac{\Lambda}{h}|\Phi_h(V)| \\
&\leq P(E_h; (Q_r(0) \setminus \mathbf{C}_{\eta,h}) \setminus \overline{\Phi_h(V)}) + P(\Phi_h(V); (Q_r(0) \setminus \mathbf{C}_{\eta,h}) \cap E_h) \\
&\quad + \mathcal{H}^{N-1}(\partial\Phi_h(V) \cap \partial E_h \cap \{x_N \leq \delta\} \cap (Q_r(0) \setminus \mathbf{C}_{\eta,h})) + \frac{\Lambda}{h}|\Phi_h(V)|\,.
\end{aligned}$$

Since

$$P(E_h; Q_r(0) \setminus \mathbf{C}_{\eta,h}) \geq P(E_h; (Q_r(0) \setminus \mathbf{C}_{\eta,h}) \setminus \overline{\Phi_h(V)}) + P(E_h; (Q_r(0) \setminus \mathbf{C}_{\eta,h}) \cap \Phi_h(V)),$$

and using the fact that $\mathscr{H}_\delta \cap V = \mathscr{H}_\delta \cap \Phi_h(V) \subset (Q_r(0) \setminus \mathbf{C}_{\eta,h}) \cap \Phi_h(V)$, the inequality above yields

$$\begin{aligned} P(E_h; \mathscr{H}_\delta \cap V) &\leq P(\Phi_h(V); (Q_r(0) \setminus \mathbf{C}_{\eta,h}) \cap E_h) \\ &\quad + \mathcal{H}^{N-1}(\partial\Phi_h(V) \cap \{x_N \leq \delta\} \cap (Q_r(0) \setminus \mathbf{C}_{\eta,h})) + \frac{\Lambda}{h}|\Phi_h(V)| \\ &\leq P(V; Q_r(0) \cap \mathscr{H}_\delta \cap E_h) \\ &\quad + 2\mathcal{H}^{N-1}(\partial\Phi_h(V) \cap \{x_N \leq \delta\} \cap (Q_r(0) \setminus \mathbf{C}_{\eta,h})) + \frac{\Lambda}{h}|\Phi_h(V)| \\ &\leq P(V; Q_r(0) \cap \mathscr{H}_\delta \cap (E)_\delta) \\ &\quad + 2(\mathrm{Lip}(\Phi_h))^{N-1} P(V; \{0 < x_N \leq \delta\}) + \frac{\Lambda}{h}|\Phi_h(V)|, \end{aligned} \tag{6.5}$$

where in the last inequality we used the fact that $\Phi_h^{-1}((Q_r(0) \setminus \mathbf{C}_{\eta,h}) \cap \{x_N \leq \delta\}) = Q_r(0) \cap \{0 < x_N \leq \delta\}$ and the fact that $\overline{E_h}$ converge in the Kuratowski sense to $\overline{E}$ in $\mathscr{H}_\delta$, see Remark 2.3. By the lower semicontinuity of the perimeter, passing to the limit in (6.5)

$$P(E; \mathscr{H}_\delta \cap V) \leq P(V; Q_r(0) \cap \mathscr{H}_\delta \cap (E)_\delta) + 2P(V; \{0 < x_N \leq \delta\}).$$

In turn, by letting $\delta \to 0$ we have

$$P(E; \mathscr{H} \cap V) \leq P(V; Q_r(0) \cap E), \tag{6.6}$$

which is equivalent to (4.17) thanks to first condition in (4.17). To remove (6.4) it is enough to consider a sequence of smooth sets $V_j \subset\subset Q_r(0)$, $V \subset\subset V_j$, satisfying the first condition in (4.17) and (6.4), and such that $V_j \to V$ in L^1 and $P(V_j; Q_r(0)) \to P(V; Q_r(0))$. The conclusion then follows by applying (4.17) with V replaced by V_j and passing to the limit thanks to the first condition in (4.17).

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