Calculus of Variations

Rotational symmetry of solutions of mean curvature flow coming out of a double cone II

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Abstract

Given a double cone $\mathcal{C}$ with entropy at most two that is symmetric across some hyperplane, we show that any integral Brakke flow coming out of the cone must inherit the reflection symmetry for all time, provided the flow is smooth for a short time. As a corollary we prove that any such flow coming out of a rotationally symmetric double cone must stay rotationally symmetric for all time. We also show the existence of a non-self-similar flow coming out of a double cone with entropy at most two, and give an example of such a flow with a finite time singularity. Additionally, we show the existence of self-expanders with triple junctions, which are exceptions to our main theorem.

1 Introduction

A family of properly embedded hypersurface $\{\Sigma_t\}_{t \in I}$ satisfies the mean curvature flow (MCF) equation if:

$$\left(\frac{\partial x}{\partial t}\right)^{\perp} = H_{\Sigma_t}(x), \quad x \in \Sigma_t.$$

Here H_{Σ_t} is the mean curvature vector of Σ_t, x is the position vector and $\perp$ denotes the projection onto the normal bundle. In this paper we study mean curvature flows (MCF) that come out of (smooth) cones. Due to the singularity at the origin, there could be multiple distinct mean curvature flows coming out of a given cone $\mathcal{C}$. We are interested in how symmetries of the cone influence these solutions.

The simplest solutions coming out of cones are self-expanders. We say a properly embedded hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is a *self-expander* if

$$H_{\Sigma}(x) = \frac{x^{\perp}}{2}. \tag{1.1}$$

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Equivalently, Σ is an self-expander if and only if $\{\sqrt{t}\Sigma\}_{t\in(0,\infty)}$ is a solution to the MCF. Self-expanders are of particular importance in the study of singularities of MCF as they arise naturally as models of how MCFs flow through conical singularities.

There exist, however, other solutions coming out of the cone that are not self-similar. A class of such solutions was first sketched by Bernstein–Wang [10] as Morse flow lines between unstable and stable self-expanders asymptotic to the same cone. Note that the existence of such solution requires the existence of an unstable self-expander asymptotic to $\mathcal{C}$, which is not guaranteed.

In the present article, we study flows coming out of a double cone with reflection symmetry. Our main result roughly says that any MCF coming out of a suitable cone with a reflection symmetry across some hyperplane must inherit the symmetry for all future times, provided the flow is initially smooth for a short time (see Sect.2 for terminologies).

Theorem 1.1 *Let $\Pi \subset \mathbb{R}^{n+1}$ be a hyperplane passing through the origin and $\mathbb{H}$ an open half-space with $\partial\mathbb{H} = \Pi$. Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a smooth double cone with $\lambda[\mathcal{C}] < 2$ such that $\mathcal{C} \cap \mathbb{H}$ is a Lipschitz graph over Π. Let $\mathcal{M} = \{\mu_t\}_{t\in[0,\infty)}$ be an integral, unit-regular and cyclic Brakke flow coming out of $\mathcal{C}$; that is,*

$$\lim_{t\to 0} \mu_t = \mathcal{H}^n \llcorner \mathcal{C}$$

as Radon measures. Suppose also $\mathcal{M}$ is smooth on $(0, T)$ for some $T > 0$. If $\mathcal{C}$ is symmetric across Π, then so is $\mathcal{M}$ for $t \in [0, \infty)$. Moreover, $\mathcal{M}$ is smooth away from Π.

Applying the above to rotationally symmetric double cones we obtain:

Corollary 1.2 *Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a smooth rotationally symmetric double cone with $\lambda[\mathcal{C}] < 2$ (see Sect.2 for the precise definitions). Let $\mathcal{M} = \{\mu_t\}_{t\in[0,\infty)}$ be an integral, unit-regular and cyclic Brakke flow coming out of $\mathcal{C}$ that is smooth on $(0, T)$. Then $\mathcal{M}$ is rotationally symmetric with the same axis of symmetry as $\mathcal{C}$. Moreover, $\mathcal{M}$ is smooth away from its axis of symmetry.*

Remark In our previous work [16], we only showed that the flow is rotationally symmetric up to the first singular time T, so the point of the current work is to show that the symmetry still holds after any singularity, which, as part of the conclusion, must lie on the axis of symmetry.

Remark The entropy condition $\lambda[\mathcal{C}] < 2$ is likely redundant. See Sect.7.

In fact, if the cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ is of the form

$$x_1^2 = m^2 \left(x_2^2 + \cdots + x_{n+1}^2\right), \quad m > 0, \tag{1.2}$$

where m is the parameter determined by the cone angle, a much stronger conclusion holds:

Corollary 1.3 *Suppose $\mathcal{C}$ is of the form (1.2) and has entropy $\lambda[\mathcal{C}] < 2$. Suppose $\mathcal{M}$ is an integral, unit-regular and cyclic Brakke flow coming out of $\mathcal{C}$. If $\mathcal{M}$ is smooth on $(0, T)$, then $\mathcal{M}$ is rotationally symmetric across the x_1-axis. The only possible singularity model of $\mathcal{M}$ is the round cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$. Moreover, there can be at most one of such singularity, which, if it exists, must occur at the origin.*

We can also apply Theorem 1.1 to cones with $O(p+1) \times O(n-p+1)$ symmetry to prove:

Corollary 1.4 *Suppose $n \geq 2$ and $1 \leq p \leq n - 1$. Let $\mathcal{C} \subset \mathbb{R}^{n+2}$ be a cone invariant under $O(p+1) \times O(n-p+1)$ with $\lambda[\mathcal{C}] < 2$. Let $\mathcal{M} = \{\mu_t\}_{t\in[0,\infty)}$ be an integral, unit-regular and cyclic Brakke flow coming out of $\mathcal{C}$. If there is $T > 0$ such that $\mathcal{M}$ is smooth on $(0, T)$, then $\mathcal{M}$ inherits the $O(p+1) \times O(n-p+1)$ symmetry (with the same axes of symmetry).*

There are many other related works on the rotational symmetry of self-expanders. Observe that if there is a unique self-expander asymptotic to a rotationally symmetric cone, then it must inherit the rotational symmetry. The first nontrivial result for double cones is obtained by Fong–McGrath [25]. They proved that a mean-convex self-expander (i.e. $H_\Sigma > 0$) asymptotic to a rotationally symmetric double cone is rotationally symmetric. This is later generalized by Bernstein–Wang (Lemma 8.3 in [8]) to weakly stable self-expanders. In our previous work [16], we showed in full generality that any smooth self-expander asymptotic to a rotationally symmetric double cone is rotationally symmetric. Rotational symmetry of self-expanding solitons of other geometric flows has been studied by Chodosh [19] and Chodosh–Fong [14].

Next we briefly comment on some of the assumptions made in Theorem 1.1. First of all, the only extra assumption over the smooth case is the entropy bound $\lambda[\mathcal{C}] < 2$. This is due to the complicated nature of singularities of higher multiplicities arising from Brakke flows. In particular, the entropy condition is essential to the maximum principle Theorem B.1 (in which the entropy controls the Gaussian density) and the proof of Proposition 3.2.

Secondly, the cyclicity of $\mathcal{M}$ is needed to ensure singularities modeled on triple junctions do not appear along the flow. Standard Schauder estimates and pseudolocality arguments can only guarantee smoothness outside of a large ball (see for example Proposition A.2) but do not rule out formations of triple junctions. This condition is explicitly used in Proposition 3.2. In fact, we will show that there exists a self-expander with triple junctions in Sect.6. Our moving plane method will not work for such self-expanders. However, if we assume that $\mathcal{C}$ is also symmetric across the hyperplane perpendicular to its axis of symmetry, then there is still some hope that all self-expanders with triple junctions are rotationally symmetric (See Sect.7).

Finally, it is not immediately clear that our theorem contains more flows than [16], which includes *smooth* self-expanders and low entropy flow lines of Bernstein–Wang [10]. For this reason we will establish an existence result of a non-self-similar flow coming out of a cone $\mathcal{C}$ with $\lambda[\mathcal{C}] < 2$. We warn the readers that, while the flow constructed below agrees with the construction from [10] on the smooth part, they are not necessarily the same flow. This is due to the lack of understanding how expander-mean-convexity is preserved after the singularity.

Theorem 1.5 *Let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a smooth cone with $\lambda[\mathcal{C}] < 2$. Let Σ be a self-expander $C^{2,\alpha}$-asymptotic to $\mathcal{C}$. If Σ is unstable, there exists a non-self-similar immortal integral Brakke flow $\mathcal{M} = \{\mu_t\}_{t \in (0,\infty)}$ such that*

$$\lim_{t \to 0} \mu_t = \mathcal{H}^n \llcorner \mathcal{C}.$$

Moreover, there is $T > 0$ such that $\mathcal{M}$ is smooth on $(0, T)$.

Remark This existence theorem is weaker than the one obtained in [2] in which no entropy assumption is made, but is enough for our purposes. Thanks to the entropy bound, it is much simpler to analyze our situation as the flows we produce are automatically matching. A similar construction for self-shrinkers is carried out by Chodosh–Choi–Mantoulidis–Schulze [13].

Remark Without the more restrictive entropy bound, it is not enough to produce the flow line as a smooth MCF — Some unstable connected self-expanders under the flow will necessarily disconnect in order to reach the stable double-disk solution. See Corollary 1.3 and Proposition 5.7.

Let us now briefly discuss the proof of Theorem 1.1. As in [16], the proof is based on the moving plane method, first used by Alexandrov to prove that embedded, compact CMC

hypersurfaces are round spheres. The method was then employed by Gidas–Ni–Nirenberg [26] who proved radial symmetry of positive solutions to the Yamabe problem, and by Schoen [38] who proved the uniqueness of catenoids. Recently, there are a number of other results on MCF that utilize the moving plane method, including [37] and [17]. Unfortunately since we are dealing with potentially singular flows, these methods for smooth flows are no longer sufficient. Instead we will use a novel variant without smoothness, recently developed by [18] (for parabolic equations) and [28] (for elliptic equations). The key ingredient in the proof is the Hopf lemma without smoothness Theorem B.2, which allows us to upgrade regularity of the flow concurrently with the symmetry.

In order to apply the (geometric) moving plane method to noncompact objects, it is mandatory to have a well-controlled asymptote at infinity. In the asymptotically cylindrical case [18] (see also the preceding [17]), a fine neck analysis is carried out in order to determine the asymptotic expansion near infinity. However, to our advantage, in the asymptotically conical case, the asymptote near infinity is entirely determined by the given cone $\mathcal{C}$, and it is an immediate consequence of the pseudolocality theorem that our flows are nice $C^{2,\alpha}$ normal graphs over $\mathcal{C}$ outside of a large ball with appropriate decay rates (see Appendix A or the works of Bernstein–Wang [9] or [7]). Knowing this, we can then carry out the moving plane method with the usual maximum principle and Hopf lemma replaced by Theorem B.1 and Theorem B.2 respectively (barring some technicality, e.g. the tameness assumption Proposition 3.2).

We will prove Theorem 1.1 in Sect.3. In Sect.4 we apply Theorem 1.1 to prove the various corollaries mentioned above. In Sect.5 we give a construction for Theorem 1.5. In Sect.6 we will prove an ODE existence result of self-expanders with triple junctions, which shows that the cyclic assumption in Theorem 1.1 is indeed necessary. In Appendix A we review the construction of smooth Morse flow line following [10]. Finally in Appendix B we recall the key maximum principle and Hopf lemma from Section 3 of [18].

2 Preliminaries

2.1 Notations

Throughout the paper lower case letters such as x denote points in $\mathbb{R}^{n+1}$, while upper case letters such as X denote points in the spacetime $\mathbb{R}^{n+1} \times [0, \infty)$. $B_r(x)$ denotes the Euclidean ball of radius r centered at x, and $P(X, r)$ denotes the parabolic ball centered at $X = (x, t)$ of radius r, i.e.

$$P(X, r) = B_r(x) \times (t - r^2, t].$$

$\mathcal{T}_r(A)$ denotes the tubular neighborhood of $A \subset \mathbb{R}^{n+1}$ of radius r. Finally, for $x = (x', x_{n+1})$, let

$$B_r^n(x) = \{(y', y_{n+1}) \in \mathbb{R}^{n+1} \mid |y' - x'| < r, \, y_{n+1} = x_{n+1}\},$$

and $C_r(x)$ be the open cylinder of height r over $B_r^n(x)$, i.e.

$$C_r(x) = \{(y', y_{n+1}) \in \mathbb{R}^{n+1} \mid |x' - y'| < r, \, |x_{n+1} - y_{n+1}| < r\}.$$

By a *(hyper)cone* we mean a set $\mathcal{C} \subset \mathbb{R}^{n+1}$ that is invariant under dilation, i.e. $\rho\mathcal{C} = \mathcal{C}$ for all $\rho > 0$. The *link* of $\mathcal{C}$ is $\mathcal{L}(\mathcal{C}) = \mathcal{C} \cap \mathbb{S}^n$. We say $\mathcal{C}$ is smooth if $\mathcal{L}(\mathcal{C})$ is a smooth hypersurface of $\mathbb{S}^n$. A double cone is a cone $\mathcal{C}$ whose link $\mathcal{L}(\mathcal{C})$ has two connected components lying in

opposite hemispheres of $\mathbb{S}^n$. A hypersurface Σ is $C^{k,\alpha}$-*asymptotically conical* to $\mathcal{C}$ if

$$\lim_{\rho \to 0^+} \rho \Sigma = \mathcal{C} \text{ in } C^{k,\alpha}_{loc} \left(\mathbb{R}^{n+1} \setminus \{0\} \right).$$

We will simply say Σ is asymptotically conical to $\mathcal{C}$ if it is smoothly asymptotically conical. In our applications, the cone $\mathcal{C}$ is almost always assumed to be smooth and the asymptotically conical hypersurfaces will come from solutions to MCF, which are in fact smoothly asymptotic to $\mathcal{C}$ by standard Schauder theory.

Given a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, the *Gaussian surface area* of Σ is

$$F[\Sigma] = \int_{\Sigma} e^{-\frac{|x|^2}{4}} d\mathcal{H}^n.$$

Following Colding–Minicozzi [20], the *entropy* of Σ is

$$\lambda[\Sigma] = \sup_{\rho \in \mathbb{R}^+, x_0 \in \mathbb{R}^{n+1}} F[\rho \Sigma + x_0] = \sup_{\rho, x_0} \frac{1}{(4\pi\rho)^{n/2}} \int_{\Sigma} e^{-\frac{|x-x_0|^2}{4\rho}} d\mathcal{H}^n.$$

By Huisken's monotonicity formula, the entropy is nonincreasing along a MCF.

2.2 Integral Brakke flows

For the rest of the article we will be using notations from geometric measure theory. We refer to [39] and [30] for the relevant definitions.

Since we are dealing with potentially singular MCF, we need to generalize the classes of MCF to make sense of the flow and hence the symmetries past singularities. The measure-theoretic generalization of MCF is the Brakke flow [4] (see also the more recent book of Tonegawa [41]), which is a flow of varifolds. Given an integral n-rectifiable Radon measure μ, let $V(\mu)$ denote the associated integral varifold, and H its generalized mean curvature vector given by the formula:

$$\int \text{div}_{V(\mu)} X d\mu = - \int H \cdot X d\mu.$$

where X is a compactly supported C^1 vector field.

Given an open set $U \subset \mathbb{R}^{n+1}$, by an *integral n-Brakke flow* in U we mean a family of integral n-rectifiable Radon measures $\mathcal{M} = \{\mu_t\}_{t \in I}$ such that:

(a) For a.e. $t \in I$, $V(\mu)$ has locally bounded first variation and its generalized mean curvature vector H is orthogonal to the approximating tangent space of $V(\mu)$ x-a.e.
(b) For any bounded interval $[a, b] \subset I$ and compact set $K \subset U$,

$$\int_a^b \int_K \left(1 + |H|^2 \right) d\mu_t dt < \infty.$$

(c) For $[a, b] \subset I$ and every $\phi \in C^1_c(U \times [a, b], \mathbb{R}^+)$,

$$\int \phi d\mu_b - \int \phi d\mu_a \leq \int_a^b \int \left(-\phi |H|^2 + |H| \cdot \nabla\phi + \frac{\partial\phi}{\partial t} \right) d\mu_t dt.$$

Since we are working in codimension one we will drop the dependence on n in the definition above and simply refer to it as a Brakke flow.

Brakke flow has two main drawbacks. First, due to the inequality in condition (c), a Brakke flow can vanish abruptly (in fact some evolution must involve such vanishing). In

order to avoid this technical difficulty in Sect.5 we will employ the notion of a matching flow which, in some sense, prevents certain sudden loss of mass. Secondly, and as part of our motivation, Brakke flow does not have to be unique. In fact, there could be multiple (smooth) self-expanders coming out of a fixed cone, which is singular at the origin.

A Brakke flow $\mathcal{M}$ is *unit-regular* if, for a spacetime point $X = (x, t)$, $\mathcal{M}$ is smooth and has no sudden mass loss if a tangent flow at X is a multiplicity one hyperplane. We say $\mathcal{M}$ is *cyclic* if the associated mod-2 flat chain $[V(\mu_t)]$ (see eg. [44]) has no boundary. By works of Ilmanen [30], Brakke flows produced by elliptic regularization are unit-regular and cyclic.

Throughout our presentation, given a Brakke flow $\mathcal{M} = \{\mu_t\}_{t \in I}$, we will write $\mathcal{M}_t = \operatorname{supp} \mu_t$ for $t \in I$.

3 Reflection symmetry

In this section we prove Theorem 1.1. Fix a smooth double cone $\mathcal{C}$ with $\lambda[\mathcal{C}] < 2$ and an integral, unit-regular Brakke flow $\mathcal{M} = \{\mu_t\}$ satisfying the assumptions of Theorem 1.1; that is,

$$\lim_{t \to 0} \mu_t = \mathcal{H}^n \llcorner \mathcal{C},$$

and $\mathcal{M}$ is smooth on $(0, T)$ for some $T > 0$. Given a hyperplane Π, if $\mathcal{C}$ is graphical over Π and symmetric across Π, the main theorem in our previous work [16] implies that $\mathcal{M}$ is symmetric across Π until its first singular time, past which the usual moving plane argument stops to work. For this reason, we will prove Theorem 1.1 using a version of the moving plane method without assuming smoothness, recently developed by [18] (see also [28]).

A technical ingredient we need for the moving plane method is the notion of tameness, which we now define.

Definition 3.1 (Definition 3.1 in [18]) For an integral Brakke flow $\mathcal{M}$ in $\mathbb{R}^{n+1}$, we say $X \in \mathcal{M}$ is a tame point of the flow if the -1 time slice of every tangent flow at X is smooth with multiplicity one away from a singular set $\mathcal{S}$ with $\mathcal{H}^{n-1}(\mathcal{S}) = 0$. We say $\mathcal{M}$ is a tame flow if every point $X \in \mathcal{M}$ is a tame point.

For instance, a tame flow should not have a singularity modeled on a triple junction (that is, three hyperplanes meeting at equal angles) or a multiplicity two plane. Tameness is a key assumption to apply the Hopf lemma without smoothness Theorem B.2, which, in turn, is crucial to the moving plane method. The next proposition establishes tameness of $\mathcal{M}$.

Proposition 3.2 *Let $\mathcal{M}$ be as above. Then $\mathcal{M}$ is a tame flow.*

Proof It suffices to check the definition. Let $\mathcal{X} = \{\nu_t\}_{t \in (-\infty, 0]}$ be a tangent flow at (x_0, t_0). Since $\lambda[\mathcal{C}] < 2$, $\mathcal{X}$ has multiplicity 1 (i.e. the Gaussian density is 1 $\mathcal{H}^{n-1}$-a.e. on $\mathcal{X}$, t-a.e.).

If $\mathcal{X}$ is static or quasi-static, then $\nu_{-1} = \mathcal{H}^n \llcorner \Gamma$ for some stationary cone Γ. If Γ splits off $(n - 2)$-lines, then $\nu_{-1} = \mu_{\mathbb{R}^{n-2}} \times \nu'$ where ν' is a one-dimensional stationary cone in $\mathbb{R}^2$. Hence ν' is a union of half-rays. Since $\lambda[\nu'] = \lambda[\nu_{-1}] < 2$, there are at most 3 rays. Since $\mathcal{M}$ is cyclic, ν' cannot be 3 rays meeting at the origin. This is because the triple junction is not cyclic, and a cyclic Brakke flow cannot have a singularity modeled on a non-cyclic singularity by [44]. Therefore ν' consists of 2 lines and in fact ν_{-1} is smooth. So any singular cone ν_{-1} can split off at most $(n - 3)$-lines, and consequently the singular part of ν_{-1} has codimension at least 3.

If $\mathcal{X}$ is a non-flat self-shrinker, then any tangent cone ν' to ν is a stationary cone with entropy at most 2 (here we used the fact that self-shrinkers are minimal surfaces with respect to the metric $g_{ij} = e^{-\frac{|x|^2}{2n}}\delta_{ij}$ which is conformal to the Euclidean metric). It follows from the above discussion that ν' can split off at most $(n-3)$-lines (if ν' splits of $(n-2)$-lines then it is a multiplicity 1 hyperplane, which, by Allard regularity theorem, means that $\mathcal{S}(\mathcal{X}) \subset \{0\}$ and consequently $\mathcal{X}$ is a multiplicity 1 hyperplane), and so ν_{-1} is smooth away from a set of Hausdorff dimension at most $(n-2)$. Thus $\mathcal{M}$ is tame. $\qquad\square$

Remark With a more restrictive entropy bound it is possible to refine the codimension of the singular set even more. See [6] or Section 4 of [18].

Next we establish some properties of $\mathcal{M}$. The next proposition says that the flow $\mathcal{M}$ stays asymptotically conical for all future time.

Proposition 3.3 *Let $\mathcal{M}$ be as above. Then $\mathcal{M}$ is asymptotically conical to $\mathcal{C}$ for all $t \in (0, \infty)$. Consequently, for every $t > 0$ there is $R = R(t)$ such that $\mathcal{M}_t \setminus B_R(0)$ is a smooth MCF.*

Proof This follows from pseudolocality for Brakke flows and parabolic Schauder estimates as in Proposition A.2 (see also Proposition 4.4 of [7]). $\qquad\square$

Fix an open half space $\mathbb{H}$ and let $\Pi = \partial\mathbb{H}$. We now use the pseudolocality theorem to prove that if $\mathcal{C}$ is graphical over Π, then so is $\mathcal{M}$ outside of a large compact set. This will serve as an asymptotic expansion at infinity and be upgraded into interior graphicality via the moving plane method. For the next two lemmas we write $\mathcal{M}_t^+ = \mathcal{M}_t \cap \mathbb{H}$.

Lemma 3.4 *Suppose $\mathcal{C} \cap \mathbb{H}$ is a Lipschitz graph over Π. let $\mathcal{M}$ be as above, then for every $t > 0$ there is $R = R(t)$ and a smooth function u on $\mathcal{C}$ such that*

$$\mathcal{M}_t^+ \setminus B_R(0) \subset \{p + u(p)\nu_{\mathcal{C}}(p) \mid p \in \mathcal{C}\}.$$

and $|u(p)| \leq C |p|^{-1}$ for $p \in \mathcal{C}$ for some constant $C = C(t)$.

Proof Fix a time t_0. Let us first show that $\mathcal{M}_t^+ \setminus B_R(0)$ can be written as a smooth normal graph over $\mathcal{C}$. By Proposition 3.3, there exists $R = R(t) > 0$ such that $\mathcal{M}_t \setminus B_R(0)$ is asymptotically conical to $\mathcal{C}$. Since $\mathcal{M}_0 = \mathcal{C}$, by pseudolocality theorem for Brakke flows (Theorem 1.5 of [31]), given $\eta > 0$ there exists t_1 such that for $0 < t < t_1$ and $x \in \mathcal{C} \setminus B_1(0)$, $\mathcal{M}_t \cap C_{\sqrt{t_1}}(x)$ can be written as a normal graph over $B^n_{\sqrt{t_1}}(x) \cap T_x\mathcal{C}$ with Lipschitz constant bounded by η. By parabolic rescaling, we see that, for $0 < t < 2t_0$ and $x \in \mathcal{C} \setminus B_{\sqrt{2t_0t_1^{-1}}}(0)$, $\mathcal{M}_t \cap C_{\sqrt{2t_0}}(x)$ can be written as a normal graph over $B^n_{\sqrt{2t_0}}(x)$ with Lipschitz constant bounded by η. In particular putting $t = t_0$ gives the desired graphicality. The regularity of u follows from Proposition 3.3.

To see that the function u decays near infinity, by a similar argument as in Proposition A.1, there exists N such that for all $R > 1$ we have

$$\mathcal{M}_{t_0} \setminus B_{NR\sqrt{t_0+1}}(0) \subset \mathcal{T}_{R^{-1}\sqrt{t_0+1}}(\mathcal{C}).$$

Equivalently, for $R > N\sqrt{t_0 + 1}$,

$$\mathcal{M}_{t_0} \setminus B_{R'}(0) \subset \mathcal{T}_{N(t_0+1)(R')^{-1}}(\mathcal{C}).$$

Enlarge R if needed so that $\mathcal{M}_{t_0}^+ \setminus B_R(0)$ is a normal graph over $\mathcal{C}$. We see that u satisfies $|u(p)| \leq C |p|^{-1}$ for $p \in \mathcal{C}$ and $u(p) \in \mathcal{M}_t^+ \setminus B_{2R}(0)$. $\qquad\square$

Lemma 3.5 *Suppose $\mathcal{C} \cap \mathbb{H}$ is a Lipschitz graph over Π. Let $\mathcal{M}$ be as above. For every $t > 0$ there is $R = R(t)$ such that $\mathcal{M}_t^+ \setminus B_R(0)$ can be written as a graph over Π; that is, the projection $\pi : \mathcal{M}_t^+ \setminus B_R(0) \to \Pi$ is injective.*

Proof By Lemma 3.4, for every $\eta > 0$, there is $R = R(t)$ such that $\mathcal{M}_t^+ \setminus B_R(0)$ is a normal graph over $\mathcal{C}$ with Lipschitz constant bounded by η. Since $\mathcal{C} \cap \mathbb{H}$ is a Lipschitz graph over Π, the unit normal vector ν_Π is not contained in any tangent space to $x' \in (\mathcal{C} \cap \mathbb{H}) \setminus \{0\}$. Therefore by taking η sufficiently small we may make sure that ν_Π is also not contained in any tangent space to $x \in \mathcal{M}_t^+ \setminus B_R(0)$ (here $R = R(t, \eta)$, but of course η in turn depends on t). This proves that $\mathcal{M}_t^+ \setminus B_R(0)$ is graphical over Π as well. □

Proposition 3.2 and Lemma 3.5 allow us to use the moving plane method without smoothness, which we now carry out. Let

$$\Pi^s = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = s\} \times [0, \infty) \subset \mathbb{R}^{n+1} \times [0, \infty)$$

be the hyperplane at level s in spacetime. Given a set $A \subset \mathbb{R}^{n+1} \times [0, \infty)$ and $s \in [0, \infty)$ we let

$$A^{s+} = \{(x, x_{n+1}, t) \in A \mid x_{n+1} > s\} \text{ and } A^{s-} = \{(x, x_{n+1}, t) \in A \mid x_{n+1} < s\}$$

be the parts of A lying above Π_s and below Π_s respectively. Finally, the set

$$A^{s*} = \{(x, x_{n+1}, t) \mid (x, 2s - x_{n+1}, t) \in A\}$$

is the reflection of A across Π_s. We say $A > B$ for $A, B \subset \mathbb{R}^{n+1} \times [0, \infty)$ provided for any $(x, s, t) \in A$ and $(x, s', t) \in B$ we have $s > s'$. In contrast, a subscript t will continue to denote the time t slice of a spacetime set.

To set up the proof of Theorem 1.1, WLOG we may assume the hyperplane is $\{x_{n+1} = 0\}$. Fix a time $T_0 > 0$. We consider $\mathcal{M}$ on $[0, T_0)$ as its spacetime track; namely,

$$\mathcal{M} = \bigcup_{t=0}^{T_0} \mathcal{M}_t \times \{t\} \subset \mathbb{R}^{n+1} \times [0, T_0].$$

Finally, let

$$S = \{s \in (0, \infty) \mid (\mathcal{M}^{s+})^* > \mathcal{M}^{s-}, \text{ and } (\mathcal{M}^{s+})_t \text{ is graphical over } (\Pi^s)_t \text{ for } t \in [0, T_0]\}.$$

Here graphicality means that the projection $\pi_s : (\mathcal{M}^{s+})_t \to (\Pi^s)_t$ is injective for $t \in [0, T]$. Since each $(\mathcal{M}^{s+})_t$ is countably n-rectifiable, graphicality is equivalent to that the unit normal $e_{n+1} = (0, \ldots, 0, 1)$ of $(\Pi^s)_t$ is not contained in the approximate tangent space of $(\mathcal{M}^{s+})_t$ for $t \in [0, T_0]$. Observe that $(\mathcal{M}^{s+})^*$ is asymptotically conical to the translated cone $(\mathcal{C} + 2s e_{n+1}) \times [0, \infty)$ (in the sense of Lemma 3.4 — this ensures that a hypersurface cannot be simultaneously asymptotic to two distinct cones).

First we need a lemma about smoothness of the top part of the flow similar to Proposition 7.4 of [18].

Lemma 3.6 *Suppose $s > 0$ and $s \in S$. Then $\mathcal{M}^{s+}$ is a smooth MCF asymptotic to $\mathcal{C}$. Moreover, every point on $\mathcal{M} \cap \{x_{n+1} = s\}$ is a regular point of the flow.*

Proof Let

$$I_s = \{s' \geq s \mid \mathcal{M}^{s'+} \text{ is smooth}\}.$$

By Proposition 3.3, there is $R = R(t)$ such that $\mathcal{M}_t \setminus B_R(0)$ is a smooth MCF asymptotic to $\mathcal{C}$. So for sufficiently large s depending on T_0 we see that $\mathcal{M}^{s+}$ is asymptotic to $\mathcal{C} \times [0, T_0]$. This shows I_s is not empty.

Let $s_0 = \inf I_s$. We first argue that $\mathcal{M} \cap \{x_{n+1} = s_0\}$ consists of regular points. Let $(x, t) \in \mathcal{M} \cap \{x_{n+1} = s_0\}$ and let $\mathcal{M}^*$ be the flow reflected across Π_s. We wish to apply the Hopf lemma Theorem B.2 to $\mathcal{M}$, $\mathcal{M}^*$ and $\mathbb{H} = \{x_n < s\}$ to conclude that (x_0, t_0) is a regular point. To this end we must check that the conditions are satisfied. Tameness follows from Proposition 3.2. We may also assume that $\partial \mathbb{H}$ is not a tangent flow to either $\mathcal{M}$ or $\mathcal{M}^*$ at (x_0, t_0), because otherwise the entropy bound together with Brakke regularity theorem implies (x_0, t_0) is a regular point. Finally, we claim reg $\mathcal{M}_t \cap \mathbb{H}$ and reg $\mathcal{M}_t^* \cap \mathbb{H}$ are disjoint for t sufficiently close to t_0. Suppose not, then there must be point of contact in $\mathbb{H}$:

$$(x_1, t) \in \left(\mathcal{M}_t^{s_0-}\right) \cap \left(\mathcal{M}^{s_0+}\right)_t^* \subset \mathbb{H}.$$

By definition of s_0 we know $\left(\mathcal{M}^{s_0+}\right)_t^*$ is in fact a smooth MCF, so maximum principle Theorem B.1 implies $\mathcal{M}_t^{s_0-}$ agrees with $\left(\mathcal{M}^{s_0+}\right)^*$ in some parabolic cylinder. The same reasoning applied to any other point in $\mathcal{M}^{s_0-} \cap \left(\mathcal{M}^{s_0+}\right)^*$ shows that a connected component of $\mathcal{M}^{s_0-}$ agrees with a connected component of $\left(\mathcal{M}^{s_0+}\right)^*$. This implies that $\left(\mathcal{M}^{s_0+}\right)^*$ is simultaneously asymptotic to $\mathcal{C} \times [0, \infty)$ and $(\mathcal{C} + 2s_0 e_{n+1}) \times [0, \infty)$, a contradiction. Hence the last condition in order to apply Theorem B.2 is satisfied and we conclude that $\mathcal{M} \cap \{x_{n+1} = s_0\}$ is regular.

Lastly we show that $s_0 = s$. This is a consequence of the fact that $\mathcal{M} \cap \Pi^{s_0}$ is compact. Using small balls as barriers similar to Proposition A.1, one sees that there exists some constant N_1 such that

$$\mathcal{M}_t \setminus B_{N_1 R \sqrt{t+1}}(0) \subset \mathcal{T}_{R^{-1} \sqrt{t+1}}(\mathcal{C})$$

for $R > 1$. On the other hand, for a fixed t there is a constant N_2 such that

$$\mathcal{M}_t \cap B_{N_1 \sqrt{t+1}}(0) \subset \mathcal{T}_{N_2}(\mathcal{C}).$$

as the first set is clearly compact. These two facts together imply the existence of a constant N_3 such that

$$\mathcal{M}_t \cap \{x_{n+1} = s_0\} \subset \mathcal{T}_{N_3}(\mathcal{C}) \cap \{x_{n+1} = s_0\}.$$

This shows that $\mathcal{M}_t \cap \{x_{n+1} = s_0\}$ is compact (as $\mathcal{C} \cap \{x_{n+1} = s_0\}$ is compact), and since $t \in [0, T_0]$, $\mathcal{M} \cap \Pi^{s_0}$ is compact as well. To finish the proof, note that by the previous paragraph $\mathcal{M} \cap \{x_{n+1} = s_0\}$ consist of regular points only. At each regular point (x_0, t_0) there is some $r = r(x_0, t_0)$ such that $\mathcal{M}$ is smooth in $P((x_0, t_0), r)$. Since $\mathcal{M} \cap \{x_{n+1} = s_0\}$ is compact, r is uniformly bounded below away from 0, and this is a contradiction unless $s_0 = s$. $\qquad \square$

Proof of Theorem 1.1 To finish the proof we must show S is nonempty, S is open, and S is closed.

Again by Proposition 3.3, for sufficiently large s we can make sure that $\mathcal{M}^{s+}$ is a smooth MCF,

$$\left(\mathcal{M}^{s+}\right)^* \cap \left(\left(\mathcal{M}^{s-}\right) \cap \{x_{n+1} \geq 0\}\right) = \emptyset,$$

and that for any $(x, s_1, t) \in \left(\mathcal{M}^{s+}\right)^*$ and $(x, s_2, t) \in \mathcal{M}^{0-}$ it holds that $s_1 - s_2 \geq 2s - 1$. These two facts imply that for sufficiently large s the inequality $\left(\mathcal{M}^{s+}\right)^* > \mathcal{M}^{s-}$ is valid. On

the other hand, by Lemma 3.5, there is $R = R(T_0)$ such that $(\mathcal{M}^{0+})_{T_0} \setminus B_R(0)$ is graphical over $(\Pi^0)_{T_0}$. So for $s > R$ we have $(\mathcal{M}^{s+})_t$ is graphical over $(\Pi^s)_t$ for all $t \in [0, T_0)$. This shows S is not empty.

It is clear that $(\mathcal{M}^{s+})^* > \mathcal{M}^{s-}$ is an open condition. To see that the graphicality condition is also an open condition, let $\theta_t(x)$ be the angle between the unit normal to the approximate tangent space at a point $x \in \mathcal{M}_t$ and e_{n+1}. Suppose that $s \in S$, then graphicality is equivalent to $\theta_t(x) < \frac{\pi}{2}$ for all $t \in [0, T_0)$ and $x \in (\mathcal{M}^{s+})_t$. Since the flow $\mathcal{M}$ is $C^{2,\alpha}$-asymptotically conical, for given t there exists $\varepsilon > 0$ such that $\theta_t(x) < \pi/2$ for all $x \in (\mathcal{M}^{s'+})_t$ where $|s' - s| < \varepsilon$. Since the time interval is compact, there is a universal ε such that the above holds for all $t \in [0, T_0)$. This shows openness of S.

Finally we show S is closed. Obviously if $s \in S$ then $[s, \infty) \subset S$. So we assume $(s, \infty) \subset S$ and suppose for a contradiction that $s \notin S$. At level s, either $(\mathcal{M}^{s+})^* \cap \mathcal{M}^{s-} \neq \emptyset$ or there is some $t_0 \in [0, T_0)$ such that $(\mathcal{M}^{s+})_{t_0}$ fails to be graphical over $(\Pi^s)_{t_0}$.

In the first case, s is necessarily the first level of contact. By choosing r small enough we can ensure $(\mathcal{M}^{s+})^*$ and $\mathcal{M}^{s-}$ are graphical in $P(X, r)$ where $X \in (\mathcal{M}^{s+})^* \cap \mathcal{M}^{s-}$. Moreover, by Lemma 3.6, the reflected part $(\mathcal{M}^{s+})^*$ is a smooth MCF, so all the conditions of the maximum principle Theorem B.1 are satisfied (note that the Gaussian density bound is automatic from the entropy bound). Applying Theorem B.1, we see $(\mathcal{M}^{s+})^*$ and $\mathcal{M}^{s-}$ agree in a neighborhood of X. Now an identical argument as in the proof of Lemma 3.6 shows that $(\mathcal{M}^{s_0+})^*$ is simultaneously asymptotic to $\mathcal{C} \times [0, \infty)$ and $(\mathcal{C} + 2s_0 e_{n+1}) \times [0, \infty)$, which is again a contradiction.

In the second case, WLOG we may assume t_0 is the first time the graphicality condition fails. Then there necessarily exists a point $X = (x, s, t_0) \in \mathcal{M}_{t_0} \cap \{x_{n+1} = s\}$ whose tangent space contains the vector e_{n+1}. We again check the condition to apply Hopf lemma Theorem B.2 to $\mathcal{M}^1 = (\mathcal{M}^{s+})^*$, $\mathcal{M}^2 = \mathcal{M}^{s-}$ and $\mathbb{H} = \{x_{n+1} < s\}$ as in the proof of Lemma 3.6. Tameness follows from Proposition 3.2. Since e_{n+1} is normal to the hyperplane $\{x_{n+1} = s\}$ and X is a regular point of $\mathcal{M}$ by Lemma 3.6, we see that $\partial\mathbb{H}$ is not the tangent flow to either $\mathcal{M}^1$ or $\mathcal{M}^2$ (here we used the fact that the tangent flow at a regular point agrees with the static flow of the tangent plane). The disjointness of the regular parts of $\mathcal{M}^1$ and $\mathcal{M}^2$ in $\mathbb{H}$ follows identically as in the proof of Lemma 3.6. Hence, we may apply Theorem B.2 to conclude that $\mathcal{M}^1$ and $\mathcal{M}^2$ have distinct tangents, which is a contradiction since the tangent spaces agree at X. This concludes the proof that S is closed.

This shows that $S = (0, \infty)$. At $s = 0$, one sees that the graphicality condition is preserved (alternatively one can run the moving plane method from the other side, i.e. $s < 0$), but the strict inequality $(\mathcal{M}^{0+})^* > \mathcal{M}^{0-}$ does not hold anymore, which implies that $(\mathcal{M}^{0+})^* \cap \mathcal{M}^{0-} \neq \emptyset$. Applying the maximum principle Theorem B.1 once again we conclude $(\mathcal{M}^{0+})^* = \mathcal{M}^{0-}$, and this is the required reflection symmetry across $\Pi^0 = \{x_{n+1} = 0\}$.

4 Rotational symmetry

In this section we prove Corollary 1.2 and Corollary 1.4. First let $\mathcal{C}$ be a smooth, rotationally symmetric double cone. A typical example of such a $\mathcal{C}$ is given by (1.2) which has the x_1 axis as its axis of symmetry. Our theorem allows more general cones of the form

$$x_1^2 = \begin{cases} m_1^2(x_2^2 + x_3^2 + \cdots + x_n^2), & x_1 \geq 0 \\ m_2^2(x_2^2 + x_3^2 + \cdots + x_n^2), & x_1 < 0 \end{cases},$$

where m_1, $m_2 > 0$ (i.e. the top and bottom parts of the cone can have different cone angles). A cone $\mathcal{C}$ of the above form should also satisfy $\lambda[\mathcal{C}] < 2$, but this is not explicitly known. See Sect.7 for more on the entropy of cones.

Proof of Corollary 1.2 WLOG we may assume the axis of symmetry is the x_1-axis. Observe that rotational symmetry is equivalent to reflection symmetry across every hyperplane containing the x_1-axis. Up to a rotation it suffices to show that $\mathcal{M}$ is symmetric across the hyperplane $\{x_{n+1} = 0\}$. Since $\mathcal{C}$ is a smooth graph over $\{x_{n+1} = 0\}$, the desired conclusion follows from Theorem 1.1. □

Proof of Corollary 1.3 The rotational symmetry is Corollary 1.2. Since $\mathcal{C}$ is symmetric across the hyperplane $\{x_1 = 0\}$ as well, we can apply Theorem 1.1 to $\mathbb{H} = \{x_1 > 0\}$ to conclude that $\mathcal{M}$ is smooth away from $\{x_1 = 0\}$. Together with Corollary 1.2 we infer that the only possible singularity of $\mathcal{M}$ is at the origin. Moreover, any tangent flow $\mathcal{X}$ at the first singular time must be rotationally symmetric. By the classification of rotationally symmetric self-shrinkers of Kleene and Møller [33], $\mathcal{X}$ has to be one of the following: a round sphere, a round cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ or a smooth embedded $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Since $\mathcal{M}$ is not closed, we conclude that $\mathcal{X}$ has to be a round cylinder. The uniqueness of the cylinder follows from the work of Colding–Minicozzi [21].

Remark The above corollary does not guarantee the existence of a cylindrical singularity, as it is entirely possible that the flow remains smooth for all times. See Proposition 5.7 for sufficient conditions for the flow to have a singularity.

Next we apply the same method to cones with more general symmetry groups. For this part we will work, for convenience, in $\mathbb{R}^{n+2}$ instead. Let $O(p)$ denote the symmetry group of $\mathbb{S}^{p-1} \subset \mathbb{R}^p$. Fix an integer $1 \leq p \leq n - 1$ and suppose $\mathcal{C}$ is a smooth double cone with $\lambda[\mathcal{C}] < 2$ that has symmetry group $O(p+1) \times O(n-p+1)$. Typical examples are cones $\mathcal{C}_{n,p}$ over the families of minimal hypersurfaces in $\mathbb{S}^n$ given by

$$\mathcal{S}_{n,p} = \sqrt{\frac{p}{n}} \mathbb{S}^p \times \sqrt{\frac{n-p}{n}} \mathbb{S}^{n-p} \subset \mathbb{S}^n.$$

The cones $\mathcal{C}_{n,p}$ are known as Simons-type cones. An immediate consequence of Corollary 1.4 is:

Corollary 4.1 *Any smooth self-expander coming out of a Simons-type cone $\mathcal{C}_{n,p}$ inherits the $O(p+1) \times O(n-p+1)$-symmetry of $\mathcal{C}_{n,p}$.*

Remark Similar results for minimal surfaces have been obtained by Mazet [36], using the elliptic moving plane method.

Proof of Corollary 1.4 Write $(x_1, \ldots, x_{p+1}, y_1, \ldots, y_{n-p+1})$ the standard coordinates on $\mathbb{R}^{n+2}$. WLOG we may assume $\mathcal{L}(\mathcal{C}) = \sigma_1 \times \sigma_2$ where σ_1 is rotationally symmetric across the x_1 axis and σ_2 rotationally symmetric across the the y_1-axis. Evidently showing the rotational symmetry in x-coordinates is enough, as the identical argument works for the y-coordinates. It suffices to show that the reflection symmetry is preserved through all hyperplanes of the form

$$\sum_{i=2}^{p+1} c_i x_i = 0,$$

which, up to an ambient rotation in the x-coordinates, we may assume to be $\{x_{p+1} = 0\}$. Note that the a cone with $O(p+1) \times O(n-p+1)$ symmetry takes the form (up to relabeling)

$$\sum_{i=1}^{p+1} a_i x_i^2 = \sum_{j=1}^{n-p+1} b_j y_j^2$$

for suitable choices of coefficients a_i and b_j. It is not hard to see that $\mathcal{C} \cap \{x_{p+1} > 0\}$ is a graph over $\{x_{p+1} = 0\}$ via

$$x_{p+1} = a_{p+1}^{-1/2} \left(\sum_{j=1}^{n-p+1} b_j y_j^2 - \sum_{i=1}^{p} a_i x_i^2 \right)^{1/2}.$$

Hence Theorem 1.1 applies and the desired reflection symmetry follows (of course, we have to put

$$\Pi^s = \{(x, x_{p+1}, y) \in \mathbb{R}^{n+2} \mid x_{p+1} = s\} \times [0, \infty) \subset \mathbb{R}^{n+2} \times [0, \infty).$$

and change the dimensions in the proofs accordingly).

5 Construction of the matching motion

In this section we use the ideas of Bernstein–Wang in [10] and [7] to produce an immortal Brakke flow starting from a cone $\mathcal{C}$ with $\lambda[\mathcal{C}] < 2$ that is not self-expanding. This demonstrates that Corollary 1.2 is not void.

5.1 Self-expanders

Let us briefly summarize some basic facts about self-expanders. It is often helpful to consider the variational characterization of self-expanders. Formally, self-expanders are critical points of the functional:

$$E[\Sigma] = \int_\Sigma e^{\frac{|x|^2}{4}} d\mathcal{H}^n$$

and equation (1.1) corresponds to the Euler-Lagrange equation of $E[\Sigma]$. We record the second variation of E. For a proof see for example Proposition 4.2 in [12].

Theorem 5.1 *If $\{\phi_t\}_{t \in (-\varepsilon, \varepsilon)}$ is a compactly supported normal variation of Σ with $\frac{d\phi_t}{dt}\big|_{t=0} = f \nu_\Sigma$, where ν_Σ is the outwards unit normal. Then*

$$\frac{d^2}{dt^2}\bigg|_{t=0} E[\phi_t(\Sigma)] = -\int_\Sigma f L_\Sigma f d\mathcal{H}^n$$

where L_Σ is the stability operator of Σ given by

$$L_\Sigma = \Delta_\Sigma + \frac{1}{2} x \cdot \nabla_\Sigma + |A_\Sigma|^2 - \frac{1}{2}.$$

A real number $\mu \in \mathbb{R}$ is an eigenvalue for $-L_\Sigma$ if there is a function $u \in W^1_{\frac{1}{4}}(\Sigma) \setminus \{0\}$ such that $-L_\Sigma u = \mu u$, where

$$W^1_{\frac{1}{4}}(\Sigma) = \{f : \Sigma \to \mathbb{R} \mid \int_\Sigma \left(|f|^2 + |\nabla f|^2 \right) e^{\frac{|x|^2}{4}} d\mathcal{H}^n < \infty\}.$$

The *index* of a self-expander is the number of negative eigenvalues of $-L_\Sigma$, which is equal to

$$\sup\{\dim V \mid V \text{ linear subspace } \subset C_0^2(\Sigma), -\int_\Sigma fL_\Sigma f \leq 0 \ \forall f \in V \setminus \{0\}\}.$$

We say a self-expander is *stable* if it has index zero. By Lemma 4.1 in [8], the operator L_Σ is formally adjoint in $W_{\frac{1}{4}}^0(\Sigma)$, and L_Σ has a discrete spectrum; that is, the eigenvalues of $-L_\Sigma$ can be ordered as $\mu_1 < \mu_2 < \cdots < \mu_n < \cdots$. Moreover, the space of eigenfunctions associated to the lowest eigenvalue μ_1 is 1-dimensional, and any eigenfunction f of μ_1 has a sign.

We need the following basic distance estimates of asymptotically conical self-expanders.

Proposition 5.2 *Let $\mathcal{C}$ be a smooth cone. Suppose Σ is a self-expander $C^{2,\alpha}$-asymptotic to $\mathcal{C}$, then there is $N > 0$ such that $\Sigma \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ for $R > 1$.*

Proof Since Σ is smooth and $\rho\Sigma \to \mathcal{C}$ in $C_{loc}^{2,\alpha}(\mathbb{R}^n \setminus \{0\})$, for sufficiently small ρ we have on $\rho\Sigma \cap (B_1(0) \setminus \{0\})$

$$\left|A_{\rho\Sigma}\right| \leq C.$$

Hence for $x \in \Sigma$ with $|x|$ sufficiently large depending on the above,

$$|A(x)| = \rho \left|A_{\rho\Sigma}(\rho x)\right| \leq C |x|^{-1}$$

if we pick $\rho = \frac{1}{2} |x|^{-1}$ so that $\rho x \in B_1(0)$. This proves that $|A(x)| \leq C |x|^{-1}$ for all $x \in \Sigma$. Together with the self-expander equation, these imply that there is $C > 0$ with

$$\text{dist}(x, \mathcal{C}) < C |x|^{-1}$$

for $x \in \mathcal{C} \setminus B_1(0)$. Finally by scaling it follows that there is N such that for $R \geq 1$.

$$\Sigma \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C}).$$

$\square$

Remark Note that similar to the above we can also estimate the derivatives of A:

$$\left|\nabla^m A\right| \leq C |x|^{-m-1},$$

provided the cone is sufficiently regular.

5.2 Mean curvature flow with boundary

Since we are dealing with noncompact initial hypersurfaces, many existence theorems (in particular the unit density theorem 11.4 in [30]) do not apply directly in our case. To account for this we will utilize White's recent work on MCF with boundary [45], which is a generalization of the Brakke flow in Sect.2.2. For simplicity we will only work in the ambient manifold $\overline{B_R(0)}$.

Given a hypersurface Σ with boundary $\Gamma \subset \partial B_R(0)$ in an open set $U \subset \overline{B_R(0)}$ and an integral n-rectifiable Radon measure μ, the first variation formula becomes:

$$\int \text{div}_{V(\mu)} X d\mu = -\int H \cdot X d\mu + \int \nu_\mu \cdot X d(\mathcal{H}^{n-1} \llcorner \Gamma)$$

for any compactly supported C^1 vector field X, where H is the generalized mean curvature vector and ν the approximating normal vector to Γ. By an *integral n-Brakke flow with boundary* Γ in U we mean a family of integral n-rectifiable Radon measures $\mathcal{M} = \{\mu_t\}_{t \in I}$ satisfying the items (a), (b) and (c) in Sect.2.2 with the extra condition:

(d) For a.e. $t \in I$ the normal vector satisfies $\left| \nu_{\mu_t} \right| \leq 1$ $\mathcal{H}^{n-1}$-a.e on Γ.

For simplicity we will refer to the above as Brakke flow with boundary Γ. This is unambiguous since, by item (d), the boundary Γ stays unchanged under the Brakke flow.

As before, a Brakke flow with boundary Γ is unit-regular if, for a spacetime point $X = (x, t)$, $\mathcal{M}$ is smooth and has no sudden mass loss if a tangent flow at X is a multiplicity one plane. $\mathcal{M}$ is cyclic if the associated mod-2 flat chain $[V(\mu_t)]$ has boundary equal to Γ. By works of White [45], Brakke flows with boundary Γ produced by elliptic regularization are unit-regular and cyclic.

Theorem 5.3 *(Theorem 1.1, Theorem 14.1 of [45]) Let $\Sigma \subset \overline{B_R(0)}$ be a hypersurface with boundary $\Gamma \subset \partial B_R(0)$. There exists a unit-regular and cyclic Brakke flow with boundary Γ, $\mathcal{M} = \{\mu_t\}_{t \in [0,\infty)}$ with $\mu_0 = \mathcal{H}^n \llcorner \Sigma$.*

Similarly Brakke flow with boundary needs not be unique, but White's theorem says that a unit-regular and cyclic one always exists. White proved in addition a strong boundary regularity theorem (Theorem 17.1 of [45]) in the codimension one case, ruling out a scenario where interior singularities could accumulate to a boundary singularity. Hence the boundary Γ truly remains unchanged in the classical sense.

5.3 Level set flows and matching motions

The final ingredient we need is the set-theoretic generalization of MCF, initially developed by [15] and [24] as viscosity solutions to certain PDEs. Given a closed set $\Gamma_0 \subset \mathbb{R}^{n+1}$, we choose any uniformly continuous function u_0 such that $\Gamma_0 = \{x \in \mathbb{R}^{n+1} \mid u_0(x) = 0\}$. There exists a unique $u \in C(\mathbb{R}^{n+1} \times [0, \infty))$ which is a viscosity solution to the problem

$$\begin{cases} u_t = \sum_{i,j=1}^{n+1} \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2} \right) u_{x_i x_j} & \text{on } \mathbb{R}^{n+1} \times [0, \infty) \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^{n+1} \times \{0\}. \end{cases}$$

Let $\Gamma_t = \{x \in \mathbb{R}^{n+1} \mid u(x, t) = 0\}$. We call $\mathcal{K} = \bigcup_{t \in [0,\infty)} \Gamma_t \times \{t\}$ the *level set flow* of Γ_0.

Since the viscosity solution is unique, level set flow is also unique with given initial data. However, level set flows might fatten, i.e. $\mathcal{K}$ might develop a non-empty interior (for example the figure eight fattens immediately). Formally, the level set flow of Γ_0 fattens if $\mathcal{H}^{n+1}(\Gamma_t) > 0$ for some $t > 0$. A theorem of Ilmanen (11.3 in [30]) shows that fattening phenomenon is not generic and can therefore be perturbed away.

Alternatively, it has been observed that the level set flow can be characterized as the "biggest flow" of a closed set satisfying the avoidance principle. There is a rich literature on this more geometrically intuitive way of handling set flows and we refer to [43], [29] and [2] for more information on this approach.

Ilmanen [30] combined ideas from Brakke flows and level set flows and introduced the notion of a matching motion, which turns out to be the suitable notion for our purposes. Let $I_n(U)$ be the set of integral n-current in U.

Definition 5.4 (8.1, 9.1 of Ilmanen [30]) Let $\mathcal{K} \in I_{n+1}(\mathbb{R}^{n+1} \times \mathbb{R}^+)$, $\mathcal{M} = \{\mu_t\}_{t \in [0,\infty)}$ be a Brakke flow and $\Gamma_0 \in I_n(\mathbb{R}^{n+1})$ with finite mass and empty boundary. A pair $(\mathcal{K}, \mathcal{M})$ is an *enhanced motion* with initial data Γ_0 if

(a) $\partial \mathcal{K} = \Gamma_0$ and $\mathcal{K}_t \in I_n(\mathbb{R}^{n+1})$ for a.e. $t \geq 0$;

(b) $\partial \mathcal{K}_t = 0$ and $t \to \mathcal{K}_t$ is continuous in the flat topology for $t \geq 0$;

(c) $\mu_0 = \mu_{\Gamma_0}$, $\mathbb{M}[\mu_t] \leq \mathbb{M}[\mu_0]$ and $\mu_{\mathcal{K}_t} \leq \mu_t$ for a.e. $t \geq 0$.

If the pair $(\mathcal{K}, \mathcal{M})$ further satisfies

(d) $\mu_t = \mu_{\mathcal{K}_t} = \mu_{V(\mu_t)}$ for $t \geq 0$,

then we call it a *matching motion*.

Note that in the above definition we have already abused some notation. Indeed, in our applications $\mathcal{K}$ is going to be the level set flow from Γ_0. A fundamental result of Ilmanen (Section 12 of [30]) shows that a nonfattening level set flow is a matching motion, which justifies our abuse of notation here.

We will use the following result of S. Wang [42] which asserts that limit of low entropy matching motions is a matching motion. Recall that a sequence of Brakke flow $\mathcal{M}_i = \{\mu_t^i\}_{t \in [0,\infty)}$ converges to $\mathcal{M} = \{\mu_t\}_{t \in [0,\infty)}$ if $\mu_t^i \to \mu_t$ as Radon measures and, after possibly passing to a subsequence, $V(\mu_t^i) \to V(\mu_t)$ as varifolds for a.e $t \in [0, \infty)$.

Proposition 5.5 *[Theorem 3.5 of [42]] Let $(\mathcal{K}_i, \mathcal{M}_i)$ be a sequence of matching motions converging to an enhanced motion $(\mathcal{K}, \mathcal{M})$ with $\lambda[\mathcal{M}] < 2$, then $(\mathcal{K}, \mathcal{M})$ is a matching motion.*

Remark The theorem fails without the entropy assumption as the set-theoretic limit of a sequence of grim reapers (which has entropy 2) is two lines but the limit in the sense of currents is empty (as the two lines cancel each other).

5.4 Construction of the smooth flow

Before we construct the weak flow, we briefly summarize the construction of Morse flow line starting from an unstable self-expander [10]. For reader's convenience we have also included a summary of the theorems with some proofs in Appendix A.

Recall that for an unstable self-expander Σ, the stability operator L_Σ has discrete spectrum and eigenfunctions to the lowest eigenvalue $\mu_1 < 0$ have a sign. Let f be the unique positive eigenfunction of μ_1 with $\|f\|_{W_{\frac{1}{4}}^0(\Sigma)} = 1$. For $\varepsilon > 0$ we form the perturbations of Σ by f given by

$$\Sigma^\varepsilon = \Psi^\varepsilon(\Sigma) \text{ where } \Psi^\varepsilon(x) = x + \varepsilon f(x)\nu_\Sigma.$$

By Lemma A.4, there is $N > 0$ depending on ε such that $\Sigma^\varepsilon \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ for all $R > 1$. By Lemma A.5, Σ^ε is expander mean-convex (see Appendix A for the definition) for ε sufficiently small. By the existence theorem Theorem A.3 there is a unique MCF starting from an asymptotically conical, expander mean-convex hypersurface, and the expander-mean-convexity is preserved along the flow until the first singular time. Applying Theorem A.3 to Σ^ε, we get for each ε a unique MCF $\mathcal{M}^\varepsilon = \{\Sigma_t^\varepsilon\}_{t \in [1, T^\varepsilon)}$ with $\Sigma_1^\varepsilon = \Sigma^\varepsilon$. Moreover, by Lemma A.4 and Proposition 5.2, Σ^ε has uniformly bounded curvature, so the interior estimates of Ecker–Huisken [23] implies that the interval of existence of is independent of ε. Moreover, at first singular time T^ε,

$$\lim_{t \to T^\varepsilon} \sup_{\Sigma_t^\varepsilon \cap B_{N'\sqrt{t}}} \left| A_{\Sigma_t^\varepsilon} \right| = \infty$$

for some constant $N' > 0$.

5.5 Construction of the Brakke flow

We now turn to our construction of the weak flow. One of the method to construct a weak flow is via capping off the hypersurfaces $\Sigma^\varepsilon \cap \overline{B_R(0)}$ smoothly and taking a sequence of Brakke flows (or weak set flows) starting from these capped-off hypersurfaces. However, one has to be careful with the above method when working with entropy bounds, since the cap might increase the entropy. On the other hand, the capping method has the advantage that, by a suitable choice of the caps, the expander mean-convexity is preserved through the cap and hence in the limits (here with weak set flows one has to interpret the mean-convexity in the weak sense as well). Such a construction for self-shrinkers can be found in Section 7 of [13] and, for self-expanders, in [2].

Here we use an alternative approach of Brakke flow with boundary. This construction is less technical and respects the entropy well. However, it is also less clear how expander mean-convexity is preserved through the flows and hence in the limit. This is not needed in our case because we are only concerned with existence. It is an interesting question to determine whether the flows constructed from the two methods above agree (they are, of course, the same notion when smooth, so the point is to determine how each of them flows past singularities). We believe this should be the case with the entropy bound $\lambda[\mathcal{C}] < 2$, but the general picture might be less clear.

The following proposition is the weak flow analogy of the smooth flow produced in Theorem A.3.

Proposition 5.6 *Let $\mathcal{C}$, Σ be as in Theorem 1.5, and Σ^ε be as above. There exists $\varepsilon_0 > 0$ such that, for $|\varepsilon| < \varepsilon_0$, there exists an immortal matching motion $(\mathcal{K}^\varepsilon, \mathcal{M}^\varepsilon)$ where $\mathcal{K}^\varepsilon = \{\Gamma_t^\varepsilon\}_{t\in[1,\infty)}$ and $\mathcal{M}^\varepsilon = \{\mu_t^\varepsilon\}_{t\in[1,\infty)}$ such that $\Gamma_1^\varepsilon = \Sigma^\varepsilon$ and $\mu_1^\varepsilon = \mathcal{H}^n \llcorner \Sigma^\varepsilon$. Moreover the flow $(\mathcal{K}^\varepsilon, \mathcal{M}^\varepsilon)$ agrees with the smooth flow starting from Σ^ε for $t \in [1, T^\varepsilon)$.*

Proof Suppose $\varepsilon > 0$, as the argument for $\varepsilon < 0$ is identical. Let $\Sigma^{\varepsilon,R} = \Sigma^\varepsilon \cap \overline{B_R(0)}$ be the hypersurface in $\overline{B_R(0)}$ with boundary $\Sigma^\varepsilon \cap \partial B_R(0)$. By Theorem 5.3, there exists an unit-regular and cyclic Brakke flow with boundary $\mathcal{M}^{\varepsilon,R} = \{\mu_t^{\varepsilon,R}\}_{t\in[0,\infty)}$ starting from $\Sigma^{\varepsilon,R}$. The flow $\mathcal{M}^{\varepsilon,R} \llcorner B_{R/2}(0)$ is therefore a (usual) Brakke flow inside $B_{R/2}(0)$. Since nonfattening is generic, we may choose a sequence $R_i \to \infty$ such that the associated level set flow of $\mathcal{M}^{\varepsilon,R_i} \llcorner B_{R_i/2}(0)$ is nonfattening. This produces a sequence of matching motions

$$(\mathcal{K}^{\varepsilon,R_i}, \mathcal{M}^{\varepsilon,R_i} \llcorner B_{R_i/2}(0)).$$

By compactness of Brakke flow we may now pass to a subsequence $R_i \to \infty$ to obtain a limiting enhanced motion $(\mathcal{K}^\varepsilon, \mathcal{M}^\varepsilon)$ in $\mathbb{R}^{n+1}$ starting from Σ^ε.

By Lemma 3.5 of [9], $\lambda[\Sigma] = \lambda[\mathcal{C}] < 2$, and by Lemma 6.2 of [10], for every $\delta > 0$ there exists ε_0 such that $|\lambda[\Sigma^\varepsilon] - \lambda[\Sigma]| < \delta$ for $\varepsilon < \varepsilon_0$. Choosing δ small enough so that $\lambda[\mathcal{C}] + \delta < 2$ and ε_0 small according to δ ensures that $\lambda[\mathcal{M}^\varepsilon] = \lambda[\Sigma^\varepsilon] < 2$, and so, in view of Proposition 5.5, $(\mathcal{K}^\varepsilon, \mathcal{M}^\varepsilon)$ is matching.

Finally, using the argument in Proposition A.2 with pseudolocality of MCF replaced by that of Brakke flow there exist $\delta > 0$ and $N' > 0$ such that supp $\mu_t^\varepsilon \setminus B_{N'\sqrt{t}}(0) = \Sigma_t^\varepsilon \setminus B_{N'\sqrt{t}}(0)$ for $t \in [1, 1 + \delta]$. Since $(\mathcal{K}^\varepsilon, \mathcal{M}^\varepsilon)$ is matching, $\Gamma_t^\varepsilon \setminus B_{N'\sqrt{t}}(0) = \Sigma_t^\varepsilon \setminus B_{N'\sqrt{t}}(0)$ as well. It follows from uniqueness of level set flow that Γ_t^ε agrees with Σ_t^ε. Using the matching property again we infer that supp $\mu_t^\varepsilon = \Sigma_t^\varepsilon$. It is easy to see that these flow agree up to the first singular time of Σ_t^ε (i.e. T^ε). $\qquad\square$

Let $(\mathcal{K}^\varepsilon, \mathcal{M}^\varepsilon)$ be the matching motions constructed as above. We can once again take a limit as $\varepsilon \to 0^+$ to obtain a limiting enhanced motion $(\mathcal{K}, \mathcal{M})$ where $\mathcal{K} = \{\Gamma_t\}_{t\in[1,\infty)}$ and

$\mathcal{M} = \{\mu_t\}_{t \in [1,\infty)}$ such that $\Gamma_0 = \Sigma$ and $\mu_0 = \mathcal{H}^n \llcorner \Sigma$. However, this limit is not enough to prove Theorem 1.5 as we will only recover the flow of the self-expander Σ (as in the case of the smooth flow). Moreover, the limit does not attain the cone as its initial data. We need to translate the flow properly so that the starting time can be extended back to 0, and argue that we do not get the original flow of the self-expander back in the process.

Proof of Theorem 1.5 Again WLOG suppose $\varepsilon > 0$. Let $(\mathcal{K}^\varepsilon, \mathcal{M}^\varepsilon)$ be the matching motions from Proposition 5.6. It is convenient to work with the following rescaled MCF:

$$\tilde{\mu}_s^\varepsilon = \mu_t^\varepsilon \circ t^{-\frac{1}{2}} \text{ and } \tilde{\Gamma}_s^\varepsilon = \Gamma_t^\varepsilon, \quad s = \log t.$$

and let $(\tilde{\mathcal{K}}^\varepsilon, \tilde{\mathcal{M}}^\varepsilon)$ denote the rescaled flow. Under the rescaling, a smooth MCF will satisfy the rescaled MCF equation

$$\left(\frac{\partial x}{\partial s}\right)^\perp = H_{\Sigma_s} - \frac{x^\perp}{2},$$

which has self-expanders as stable solutions. The flow $\tilde{\mathcal{M}}^\varepsilon$ is defined on the time interval $[0, \infty)$ with supp $\tilde{\mu}_0^\varepsilon = \Sigma^\varepsilon$. Since Σ is unstable, the lowest eigenvalue λ_1 of $-L_\Sigma$ satisfies $\lambda_1 < \frac{1}{2}$, and consequently the $\mathcal{M}^\varepsilon$ "flows faster" than the parabolic rescaling $\sqrt{t}$. To be precise, for the the flow of the self-expander Σ we have

$$\lim_{\lambda \to 0^+} \text{dist}(\Sigma, \lambda \Sigma_{\lambda^{-2}}) = \lim_{\lambda \to 0^+} \text{dist}\left(\Sigma, \lambda\sqrt{\lambda^{-2}}\Sigma\right) = 0$$

but since we are perturbing Σ by its eigenfunction whose eigenvalue is below $\frac{1}{2}$, we must have

$$\lim_{\lambda \to 0^+} \text{dist}\left(\Sigma^\varepsilon, \lambda \text{ supp } \mu_{\lambda^{-2}}^\varepsilon\right) = \infty.$$

In the rescaled setting, the above means that $\tilde{M}^\varepsilon$ moves out exponentially (the exact rate of which depends on λ_1). As such, we can find a sequence of time s_ε with

$$d\left(\text{supp } \tilde{\mu}_{s_\varepsilon}^\varepsilon, x_0\right) = \gamma$$

for some fixed point $x_0 \in \Sigma$ and positive constant γ. On the other hand $\varepsilon \to 0^+$, the rescaled flows $\tilde{\mathcal{M}}^\varepsilon$ converge to the static rescaled flow of Σ, so in fact $s_\varepsilon \to \infty$ as $\varepsilon \to 0^+$, i.e. one has to go further in time to reach a distance γ away from $x_0 \in \Sigma$. This fact allows us to time translate $\tilde{\mathcal{M}}$ by $-s_\varepsilon$ to obtain a sequence of rescaled MCFs $\tilde{\mathcal{M}}^{\varepsilon,s_\varepsilon} = \{\tilde{\mu}_s^{\varepsilon,s_\varepsilon}\}_{s \in [-s_\varepsilon, \infty)}$ such that

$$\text{supp } \tilde{\mu}_{-s_\varepsilon}^{\varepsilon,s_\varepsilon} = \Sigma^\varepsilon \text{ and } d(\text{supp } \tilde{\mu}_0^{s,s_\varepsilon}, x_0) = \gamma.$$

By compactness of Brakke flows we can take $\varepsilon \to 0^+$ to obtain a limiting flow $\tilde{\mathcal{M}}$ defined on $(-\infty, \infty)$. It is easy to see that $\tilde{\mathcal{M}}$ is not the flow of Σ as $d(\text{supp } \tilde{\mu}_0, \Sigma) \geq \gamma$. Moreover, since Σ is asymptotically conical and $\Sigma^\varepsilon \to \Sigma$ smoothly as $\varepsilon \to 0$, it follows that

$$\lim_{s \to -\infty} \text{supp } \tilde{\mu}_s = \lim_{\varepsilon \to 0} \text{supp } \tilde{\mu}_{-s_\varepsilon}^{\varepsilon,s_\varepsilon} = \lim_{\varepsilon \to 0} \Sigma^\varepsilon = \Sigma.$$

In view of the rescaling, this proves that the flow achieves $\mathcal{C}$ as the initial data.

Since $\tilde{\mu}_s$ is integral, it follows from strong maximum principle for varifolds [40] that $\lim_{s \to -\infty} \tilde{\mu}_s = k\mathcal{H}^n \llcorner \Sigma$. Since $\lambda[\mathcal{C}] < 2$, $k = 1$, so the desired regularity follows from Brakke regularity theorem. $\qquad \square$

5.6 An example of singularity formation

To conclude the section, we give a sufficient condition when there exists a flow coming out of $\mathcal{C}$ that has a singularity. The proof makes heavy use of the structure theory of self-expanders developed by Bernstein and Wang in a series of papers starting from [12]. It would be interesting to see if a simpler proof exists.

Proposition 5.7 *Suppose $\mathcal{C} \subset \mathbb{R}^{n+1}$, $2 \leq n \leq 6$, is a smooth double cone with $\lambda[\mathcal{C}] < 2$ and that the two connected components of $\mathcal{L}(\mathcal{C})$ are graphs over some (fixed) hyperplane. Suppose that there is a connected self-expander asymptotic to $\mathcal{C}$. Then there exists an integral Brakke flow coming out of $\mathcal{C}$ that has a singularity in finite time.*

Remark We note that the above is consistent with the topological uniqueness result of [10]. Indeed, by Proposition 5.6 of [10], if $\lambda[\mathcal{C}] < \lambda[\mathbb{S}^{n-1} \times \mathbb{R}]$, the flow produced by Theorem 1.5 is smooth for all time. On the other hand, any such double cone will not have a connected self-expander.

Proof Let $\sigma = \mathcal{L}(\mathcal{C})$, and let W be the connected component of $\mathbb{S}^n$ lying between the two connected components of σ. By Corollary 1.2 of [12], the set of generic cones (in the sense that there is no C^2-asymptotically self-expander with nontrivial Jacobi fields that fix the infinity) whose link lie in W, is dense near $\mathcal{C}$. These facts allow us to take a sequence of $C^{2,\alpha}$-hypersurfaces σ_i in $\mathbb{S}^2$ such that

- $\sigma_i \to \sigma$ in $C^{2,\alpha}(\mathbb{S}^n)$ as $i \to \infty$;
- $\sigma_i \subset W$ is a smooth double cone for all i.
- $\mathcal{C}_i$ is generic for all i, where $\mathcal{C}_i$ is the cone over σ_i;
- $\lambda[\mathcal{C}_i] < 2$ for sufficiently large i, by Lemma 6.2 of [10].

From the above we immediately see that there exists a unique disconnected, stable self-expander Γ_i $C^{2,\alpha}$-asymptotic to $\mathcal{C}_i$ (by evolution of entire graph [22]). We also see that $\mathcal{C}_i \subset \Omega$, where Ω is the connected component of $\mathbb{R}^{n+1} \setminus \mathcal{C}$ that contains W. Denote by Σ_0 the connected self-expander asymptotic to $\mathcal{C}$. Using a direct method with Σ_0 as the barrier, similar to Lemma 8.2 of [8], we can find a connected self-expander asymptotic to $\mathcal{C}_i$ in Ω', where Ω' is the connected component of $\mathbb{R}^{n+1} \setminus \Sigma_0$ such that the outward unit normal of $\mathcal{C}$ points into Ω'.

Since there exists a connected self-expander for each $\mathcal{C}_i$, by the partial ordering of self-expanders asymptotic to a fixed cone (Theorem 4.1 of [10]), we can pick an innermost connected self-expander Σ_i $C^{2,\alpha}$-asymptotic to $\mathcal{C}_i$ (note that Σ_i might not be unique). We claim that Σ_i is unstable. If not, the mountain pass theorem (Corollary 1.2 [5], this requires $2 \leq n \leq 6$) and the genericity of $\mathcal{C}_i$ imply the existence of an unstable self-expander Σ' lying between Σ_i and Γ_i. Σ' must then be connected, but this contradicts the partial ordering.

Since Σ_i is unstable and $\lambda[\mathcal{C}_i] < 2$, we can produce using Theorem 1.5 an integral Brakke flow $\mathcal{M}^i = \{\mu_t^i\}_{t \in (0,\infty)}$ that moves inwards initially (by expander-mean-convexity) and satisfies $\lim_{t \to 0} \mu_t^i = \mathcal{H}^n \llcorner \mathcal{C}_i$. Suppose for a contradiction that $\mathcal{M}^i$ is smooth for all t, then the flow is expander-mean-convex (in the classical sense) and moves inwards for all time. Moreover, using an almost identical argument as in Proposition 5.1(3) of [10], the rescaled flow (rescaling as in the proof of Theorem 1.5) $\tilde{\mathcal{M}}^i$ converges as $s \to \infty$ to a smooth, stable self-expander asymptotic to $\mathcal{C}_i$, which must lie inside Σ_i. Since Σ_i is an innermost connected self-expander, the stable limit must be Γ_i which is disconnected, a contradiction.

Now let s^i denote the first singular time of the rescaled flows $\tilde{\mathcal{M}}^i$, and time translate $\tilde{\mathcal{M}}^i$ by $-s^i$ to obtain rescaled flows $\tilde{\mathcal{M}}^{i,s^i}$ with a singularity at time 0. By compactness of MCF

we obtain a rescaled flow $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{M}}^{i,s^i} \to \tilde{\mathcal{M}}$ subsequentially. Rescaling back we see that, by upper semicontinuity of the Gaussian density, $\mathcal{M}$ has its first singularity at time $t = 1$. Finally, we claim that $\mathcal{M}$ indeed comes out of the cone $\mathcal{C}$. As $\mathcal{M}$ is smooth on $(0, 1)$, it is enough to show that

$$\lim_{t \to 0} \operatorname{supp} \mu_t \cap \mathbb{S}^n = \sigma \text{ in } C^{2,\alpha}(\mathbb{S}^n). \tag{5.1}$$

Since each $\mathcal{M}^i$ attains $\mathcal{C}_i$ as initial data and $\mathcal{C}_i$ is $C^{2,\alpha}$-regular, we have

$$\lim_{t \to 0} \operatorname{supp} \mu_t^i \cap \mathbb{S}^n = \sigma_i \text{ in } C^{2,\alpha}(\mathbb{S}^n).$$

As $\sigma_i \to \sigma$ in $C^{2,\alpha}(\mathbb{S}^n)$, by a diagonalization argument, we see that (5.1) holds. This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Assuming Conjecture 7.1, the assumptions in Proposition 5.7 are satisfied by cones of the type (1.2), given that the parameter m is sufficiently small. In fact, numerical computations do confirm that the cones $x_1^2 = m^2(x_2^2 + x_3^2)$ for $m \leq 1$ have entropy less than 2. Moreover, by [1], there exists a connected self-expander for sufficiently small m. In these cases, combining the above with Corollary 1.3, we have the much stronger conclusion that any such cone has a potential evolution which disconnects at a cylindrical singularity.

6 Self expanders with triple junctions

An important question in the study of self-expander is to determine the number of self-expanders coming out of a given cone. A classical result of Ecker–Huisken [22] shows that there exists a unique self-expander coming out of a graphical cone. In general, however, Angenent–Ilmanen–Chopp [1] showed numerically that uniqueness fails for double cones. It is proved rigorously by Helmensdorfer [27] that there are at least three distinct smooth self-expanders asymptotic to a rotationally symmetric double cone of the form (1.2) provided the cone angle is sufficiently large (when the cone angle is small, a barrier argument shows that uniqueness indeed holds — see Lemma 8.1 of [8]).

In this section we prove a simple ODE result on the existence of two self-expanders with triple junctions for rotationally symmetric double cones with sufficiently large cone angle. The proof roughly follows the setup of Helmensdorfer [27], although we do not need the clearing out lemma for MCF in the following analysis. This provides an example of a singular self-expander, and also illustrates that the cyclicity assumption in Theorem 1.1 is essential as tameness (Proposition 3.2) clearly fails for self-expanders with triple junction singularities. However, the examples constructed below are in fact still rotationally symmetric.

We consider cones $\mathcal{C}_m$ of the form (1.2), where m is the parameter therein. Observe that $\mathcal{C}_m$ has a rotational symmetry across the x_1-axis as well as a reflection symmetry across the $\{x_1 = 0\}$ hyperplane.

Assume the expander Σ has a triple junction singularity at a point $(0, x_0)$. Imposing rotational symmetry on Σ across the x_1-axis we may assume $x_0 = (a, 0, \ldots, 0)$. Note also at a triple junction singularity, the tangent cone is stationary and is therefore the union of three half-lines meeting at an angle of $2\pi/3$. These observations reduce the problem to finding a function $u : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following ODE (written in spherical coordinates):

$$\frac{u_{rr}}{1 + u_r^2} - \frac{n-1}{u} + \frac{1}{2}ru_r - \frac{1}{2}u = 0. \tag{6.1}$$

with initial data $u(0) = a$ and $u'(0) = \frac{\sqrt{3}}{3}$. The solution is asymptotic to $\mathcal{C}_m$ if

$$\lim_{r \to \infty} \frac{u(r)}{r} = m. \tag{6.2}$$

Of course, by the usual ODE existence and uniqueness theorem, the solution to the problem (6.1) is unique with a given initial data a. First we show that any solution to (6.1) is asymptotically conical.

Proposition 6.1 *For any $a > 0$, there is a (unique) $m = m(a)$ such that (6.2) holds.*

Proof First we prove $u > 0$ for all $r > 0$. Suppose for a contradiction that there is r_0 such that $u(r_0) < 0$. Since initially u and u_r are both positive, by the mean value theorem there must be a local maximum $r_1 \in (0, r_0)$ with $u(r_1) > 0$, but at such an r_1 we can use (6.1) to get

$$u_{rr}(r_1) = \frac{n-1}{u(r_1)} + \frac{1}{2}u(r_1) > 0,$$

a contradiction. So u is indeed positive. By the above calculation, this implies that all critical points of u are local minima.

To continue, let $\alpha(r) = \arctan(u/r)$. Differentiating, we obtain

$$\alpha'(r) = \frac{ru_r - u}{u^2 + r^2} \text{ and } \alpha''(r) = \frac{ru_{rr}}{u^2 + r^2} - \frac{(ru_r - u)(2uu_r + 2r)}{(u^2 + r^2)^2}.$$

At a critical point r_0 of $\alpha(r)$, we have $r_0 u_r(r_0) - u(r_0) = 0$ and (6.1) implies that $u_{rr}(r_0) = \frac{n-1}{u(r_0)} > 0$. Hence $\alpha''(r_0) = \frac{ru_{rr}}{u^2 + r^2} > 0$. Therefore all critical points of $\alpha(r)$ are local minima as well. Since $\alpha(r)$ is bounded and only has local minima, monotone convergence theorem shows that $\lim_{r \to \infty} \alpha(r)$ exists. This limit is clearly positive and concludes the existence of m. $\square$

Knowing the above, our problem becomes essentially a shooting problem: Given m, we wish to find the appropriate initial condition a so that the solution to (6.1) satisfies (6.2). Let $u^a(r)$ be the solution to (6.1) with initial condition $u^a(0) = a$. Consider the asymptotic cone angle parameter m as a function of a:

$$m(a) = \lim_{r \to \infty} \frac{u^a(r)}{r} = \lim_{r \to \infty} u_r^a(r) \in (0, \infty).$$

Proposition 6.2 *$m(a)$ is a continuous function on $(0, \infty)$.*

Proof First we record that

$$u_{rr}(0) = \frac{4}{3}\left(\frac{n-1}{a} + \frac{1}{2}a\right) > 0.$$

From the proof of Proposition 6.1, we see that a critical point of u must be a local minimum, but since u is initially increasing and smooth, there cannot be any critical point at all. So u is strictly increasing and we deduce that $u_r > 0$ for all $r > 0$. On the other hand, l'Hôpital's rule on (6.2) yields $\lim_{r \to \infty} u_r = m$. Going back to (6.1) and taking the limit as $r \to \infty$ yield also $\lim_{r \to \infty} u_{rr} = 0$.

Fix an $a \in (0, \infty)$. Clearly $u^a \geq a$ and u_r^a is bounded above. Therefore we may fix a constant N with $\frac{1}{N} < a$ such that $|a' - a| < \frac{1}{N}$ implies that

$$\frac{1}{u^{a'}} \leq c \tag{6.3}$$

for some constant c depending on a and N. We compute

$$\left(\frac{u}{r}\right)_r = \frac{1}{r}\left(u_r - \frac{u}{r}\right) = \frac{2}{r^2}\left(\frac{n-1}{u} - \frac{u_{rr}}{1 + (u_r)^2}\right). \tag{6.4}$$

For $|a - a'| < \frac{1}{N}$ there are two cases. If $u_{rr}^{a'}$ is never zero, then it is always positive. This immediately gives the bound:

$$\left(\frac{u^{a'}}{r}\right)_r \leq \frac{c}{r^2}. \tag{6.5}$$

If $u_{rr}^{a'}(r_0) = 0$ at some point r_0, differentiating (6.1) once we get

$$\frac{1}{1 + u_r^2}\left(u_{rrr} - \frac{2u_r(u_{rr})^2}{1 + u_r^2}\right) + \frac{n-1}{u^2}u_r + \frac{1}{2}ru_{rr} = 0. \tag{6.6}$$

From this we immediately see that when $u_{rr}^{a'}(r_0) = 0$,

$$u_{rrr}^a(r_0) = -(1 + u_r^2)\frac{n-1}{u^2} < 0,$$

so every critical point of $u_r^{a'}$ is a local maximum, for which there can be at most one of them. Therefore r_0 is the only zero of $u_{rr}^{a'}$. In this case, we see from (6.6) that, at any negative local minimum of $u_{rr}^{a'}$, we have

$$\frac{2u_r^{a'}}{(1 + (u_r^{a'})^2)^2}(u_{rr}^{a'})^2 \leq \frac{(n-1)u_r^{a'}}{(u^{a'})^2} \implies \frac{(u_{rr}^{a'})^2}{(1 + (u_r^{a'})^2)^2} \leq \frac{n-1}{2(u^{a'})^2} \leq c,$$

where we used (6.3). Going back to (6.4), this yields the same type of uniform upper bound as (6.5) (up to increase c). Therefore we conclude (6.5) holds for all $|a - a'| < \frac{1}{N}$.

Integrating (6.5) from r to ∞, we obtain the estimate:

$$m(a') - \frac{u^{a'}(r)}{r} < \frac{c}{r}, \quad |a' - a| < \frac{1}{N}.$$

Now given $\varepsilon > 0$ we can pick $r_0 > 0$ such that $c/r_0 < \varepsilon/3$ and δ so small that $|a' - a| < \delta$ implies that $|u^{a'} - u^a| < \varepsilon/3$ on $(0, r_0]$ (this follows from continuous dependence on initial data as we are now in a compact set). Using the triangle inequality we get that

$$|m(a) - m(a')| \leq \left|m(a) - \frac{u^a(r_0)}{r_0}\right| + \left|\frac{u^a(r_0)}{r_0} - \frac{u^{a'}(r_0)}{r_0}\right| + \left|m(a') - \frac{u^{a'}(r_0)}{r_0}\right| < \varepsilon.$$

This finishes the proof of continuity. $\square$

We are now in the position to prove the existence theorem.

Theorem 6.3 *There is an $M_0 > 0$ such that for all $M > M_0$, there exists at least two distinct values $a_1, a_2 \in (0, \infty)$ depending on M such that $m(a_1) = m(a_2) = M$.*

Proof We will show that $m(a) \to \infty$ both as $a \to 0$ and as $a \to \infty$. In view of Proposition 6.2 this will prove the theorem.

Let us show that $m(a) \to \infty$ as $a \to \infty$. First of all, we show that u_r^a cannot be uniformly bounded above. Indeed if u_r^a is uniformly bounded above by some constant C, then $u - ru_r \geq a - C$ for $r \in [0, 1]$ and so $u_{rr} \geq \frac{1}{2}(a - C)$ on $[0, 1]$ by (6.1). But then

$$u_r^a(1) = \frac{\sqrt{3}}{3} + \int_0^1 u_{rr}^a(r)dr \geq \frac{\sqrt{3}}{3} + \frac{1}{2}(a - C) \to \infty$$

as $a \to \infty$, a contradiction.

Recall from the proof of Proposition 6.2 that u_{rr}^a can have at most one zero. If there is a sequence $a_i \to \infty$ such that $u_{rr}^{a_i}$ has one zero r_i, then (6.4) immediately implies that

$$\left(\frac{u^{a_i}}{r}\right)_r > 0, r > r_i.$$

Integrating the above from r_i to ∞ we get

$$m(a_i) > \frac{u^{a_i}(r_i)}{r_i}. \tag{6.7}$$

On the other hand, at a zero of u_{rr}, (6.1) gives that

$$\frac{u^{a_i}(r_i)}{r_i} = u_r^{a_i}(r_i) - \frac{2(n-1)}{u^{a_i}(r_i)r_i} \tag{6.8}$$

Observe that $u_r^{a_i}(r_i) = \sup_r u_r^{a_i}(r)$ because r_i is a local maximum of $u_r^{a_i}$ and $u_{rr}^{a_i}(r) < 0$ for all $r > r_i$. Since $u_r^{a_i}$ is not uniformly bounded, we have that

$$\lim_{i \to \infty} \frac{u^{a_i}(r_i)}{r_i} = \lim_{i \to \infty} u_r^{a_i}(r_i) - \frac{2(n-1)}{u^{a_i}(r_i)r_i} = \infty,$$

where we also used $u^{a_i}(r_i) > a_i$. Recalling (6.7), we see that $\lim_{i \to \infty} m(a_i) = \infty$.

Otherwise there is $a_0 > 0$ such that u_{rr}^a has no zero for all $a > a_0$. This means that u_r^a is strictly increasing for all $a > a_0$. Since u_r^a is not uniformly bounded we can find a sequence $a_i \to \infty$ and $\{r_i\} \subset [0, \infty)$ such that $u_r^{a_i}(r_i) > i$. Of course monotonicity implies that $m(a_i) \geq u_r^{a_i}(r_i) > i$. This shows that $m(a) \to \infty$ as $a \to \infty$.

Next we will show that $m(a) \to \infty$ as $a \to 0$ as well. Again we will first argue that u_r^a cannot be uniformly bounded near 0. Suppose for a contradiction that there is $a_0 > 0$ such that $|u_r^a| \leq C$ for all $0 < a < a_0$, then since $u_r^a > 0$ we get

$$u^a(r) = a + \int_0^r u_r^a(t)dt \leq a + Cr, \quad a < a_0. \tag{6.9}$$

On the other hand, (6.1) gives that

$$u_{rr}^a \geq \frac{n-1}{u^a} + \frac{1}{2}(u^a - ru_r^a) \geq \frac{n-1}{2u^a} + \sqrt{n-1} - Cr, \quad a < a_0.$$

In particular for sufficiently small a we can ensure $\sqrt{n-1} \geq Cr$ and so that $u_{rr}^a(r) \geq \frac{n-1}{2u^a(r)}$ for $r < \sqrt{a}$. Hence, using (6.9), we may estimate

$$u_r^a(\sqrt{a}) \geq \frac{\sqrt{3}}{3} + \int_0^{\sqrt{a}} \frac{n-1}{2u^a(r)}dr$$

$$\geq \frac{\sqrt{3}}{3} + \frac{n-1}{2} \int_0^{\sqrt{a}} \frac{1}{a+Cr}dr = \frac{\sqrt{3}}{3} + \frac{n-1}{2C} \log(1 + Ca^{-\frac{1}{2}}) \to \infty$$

as $a \to 0$, a contradiction. This shows that u_r^a is not uniformly bounded near 0.

Suppose there is a sequence $a_i \to 0$ such that $u_{rr}^{a_i}$ has one zero r_i. In view of (6.8), if r_i is bounded away from 0, then taking $i \to \infty$ will give

$$\lim_{i \to \infty} \frac{u^{a_i}(r_i)}{r_i} = \lim_{i \to \infty} u_r^{a_i}(r_i) - \frac{2(n-1)}{u^{a_i}(r_i)r_i} = \infty,$$

where we used the fact that $u^{a_i}(r_i) \geq a_i + \frac{\sqrt{3}}{3}r_i$ and that u_r^a is not uniformly bounded near 0. By (6.7), we can conclude $m(a_i) \to \infty$ as $i \to \infty$ as before. So we henceforth assume that $r_i \to 0$ and assume for a contradiction that there is a constant C such that $r^{-1}u^{a_i}(r) \leq C$ uniformly for $r \geq r_i$. Since u_r is decreasing on $[r_i, \infty)$ we get

$$u^{a_i}(2r_i) \geq \int_{r_i}^{2r_i} u_r^{a_i}(t)dt \geq r_i u_r^{a_i}(2r_i) \implies u_r^{a_i}(2r_i) \leq \frac{u^{a_i}(2r_i)}{r_i} \leq 2C.$$

Since $u_{rr}^{a_i}(2r_i) < 0$, (6.1) gives that

$$0 < \frac{u^{a_i}(2r_i)}{2r_i} \leq \frac{-2(n-1)}{2r_i u^{a_i}(2r_i)} + u_r^{a_i}(2r_i).$$

Rearranging, we deduce that $r_i u^{a_i}(2r_i) \geq (n-1)u_r^{a_i}(2r_i)^{-1} \geq \frac{1}{2}(n-1)C^{-1}$, and so

$$2Cr_i^2 \geq r_i u^{a_i}(2r_i) \geq \frac{1}{2}(n-1)C^{-1} \implies r_i^2 \geq \frac{1}{4}(n-1)C^{-2}$$

a contradiction as $r_i \to 0$. Hence $r^{-1}u^{a_i}(r)$ is not uniformly bounded on (r_i, ∞), so for each i we may find $r_i' \in (r_i, \infty)$ such that $(r_i')^{-1}u^{a_i}(r_i') > i$. (6.7) then implies $m(a_i) \to \infty$ as $i \to \infty$.

Otherwise u_{rr}^a has no zero for sufficiently small a and monotonicity as before implies that $m(a) \to \infty$ as $a \to 0$. This completes the proof. □

7 Further remarks

We conclude our article with some open questions and conjectures, some of which we have already alluded to before.

The most natural question to ask is what happens if $\mathcal{C}$ is rotationally symmetric and the link $\mathcal{L}(\mathcal{C})$ has three (or more) connected components. It is not expected that self-expanders asymptotic to $\mathcal{C}$ will be rotationally symmetric. This should be compared to the case of minimal surfaces - the Costa surface, which is not rotationally symmetric, has two catenoidal ends and one planar end. It is therefore natural to expect that something similar happens if the cone is given by the union of a rotationally symmetric double cone and a hyperplane. We suspect that gluing method by desingularizing the connected expander (asymptotic to the cone) with the hyperplane will produce a counterexample.

Another important problem is to determine the entropy of rotationally symmetric double cones. In fact, we conjecture that the assumption $\lambda[\mathcal{C}] < 2$ in Corollary 1.2 is redundant (where as the same assumption in Theorem 1.1 is essential). More precisely we conjecture:

Conjecture 7.1 *Let $\mathcal{C}$ be a rotationally symmetric double cone of the form:*

$$x_1^2 = \begin{cases} m_1(x_2^2 + x_3^2 + \cdots + x_n^2), & x_1 \geq 0 \\ m_2(x_2^2 + x_3^2 + \cdots + x_n^2), & x_1 < 0 \end{cases},$$

where $m_1, m_2 > 0$. Then $\lambda[\mathcal{C}] < 2$.

When $m_1 = m_2 = m$, observe that when $m \to 0$, $\mathcal{C}$ converges to a multiplicity 2 plane which has entropy 2, and that when $m \to \infty$, $\mathcal{C}$ converges after suitable translations to a cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ which has entropy strictly less than 2. This also explains why the two connected components of $\mathcal{L}(\mathcal{C})$ need to be in different half-spaces, for otherwise when m is small the double cone could be close to a multiplicity two cylinder which has entropy strictly larger than 2.

It is not so hard to calculate the Gaussian area of $\mathcal{C}$ at the origin, but unlike self-shrinkers, the maximum in our case needs not to happen at the origin (the argument for self-shrinkers can be found in eg. Section 7 of [20]). In fact when m is large, the entropy is achieved far away from the origin (in a region where the cone looks more like a cylinder). Although we have strong numerical evidence that the conjecture is true, the analysis of the Gaussian area functional on the cone centered away from the origin is complicated to handle.

Less clear is the entropy of cones with $O(p + 1) \times O(n - p + 1)$ symmetry. Ilmanen and White [32] give an exact formula for the Gaussian density of $\mathcal{C}_{n,p}$ at the origin:

$$\Theta_{\mathcal{C}_{n,p}}(0) = \frac{\sigma_p \sigma_{n-p}}{\sigma_n} \left(\frac{p}{n}\right)^{p/2} \left(\frac{n-p}{n}\right)^{(n-p)/2},$$

where σ_p is the volume of the unit sphere in $\mathbb{R}^{p+1}$. It can be checked this is less than 2 for all n, p (the proof is by rather tedious computation so we omit it here, but one can easily verify by numerics as well). Since $\mathcal{C}_{n,p}$ are minimal, they are also self-shrinkers. It follows from a theorem of Colding–Minicozzi [20] that the entropy of $\mathcal{C}_{n,p}$ is achieved at the origin. Hence, in fact, $\lambda[\mathcal{C}_{n,p}] = \Theta_{\mathcal{C}_{n,p}}(0) < 2$. For example, the cone $\mathcal{C}_{2,1} \subset \mathbb{R}^4$ has entropy $\frac{3}{2}$. It is therefore reasonable to make the following conjecture, partly due to Solomon:

Conjecture 7.2 *[cf. Section 4 of [32]] Any cone with $O(p + 1) \times O(n - p + 1)$ symmetry has entropy at least that of $\mathcal{C}_{n,p}$ and at most 2.*

Finally we discuss the rotational symmetry of singular self-expanders. As we remarked before, the Hopf lemma Theorem B.2 cannot deal with triple junction singularities which are a possibility as we have seen in Sect.6.

Question 7.3 *What can we say about the symmetry of a singular self-expander coming out of a rotationally symmetric double cone?*

In particular, if Σ is a self-expander asymptotic to $\mathcal{C}_m$ (notation as in Sect.6) that is smooth away from $\{x_1 = 0\}$ and only has triple junction singularities on $\{x_1 = 0\}$, is Σ rotationally symmetric?

It is possible that one can use a parabolic variant of the argument of Bernstein–Maggi [3] for singular Plateau surfaces (in particular Lemma 2.4 therein) to prove the desired rotational symmetry.

Appendix A. Construction of the Smooth Flow

Here we give a more detailed overview of the construction of smooth Morse flow lines from Sect.5.4. The construction is an adaptation of the work of Bernstein and Wang in a series of papers, including [5], [11], [10] and [7].

Throughout the section, we assume $\mathcal{C} \subset \mathbb{R}^{n+1}$ is a smooth cone. The first proposition was originally proven for self-shrinkers in [7] and we give a modified proof for self-expanders.

Proposition A.1 *(cf. Lemma 4.3 of [7]) Suppose Σ is an hypersurface $C^{2,\alpha}$-asymptotic to $\mathcal{C}$, and there is $N > 0$ such that $\Sigma \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ for $R > 1$. If $\{\Sigma_t\}_{t \in [1,T]}$ is an integral Brakke flow starting from Σ in $\mathbb{R}^{n+1}$, then there is a constant $N' > 0$ such that*

$$\Sigma_t \setminus B_{N'R\sqrt{t}}(0) \subset \mathcal{T}_{R^{-1}\sqrt{t}}(\mathcal{C})$$

for $R > 1$.

Proof For any $x \in \mathbb{R}^{n+1} \setminus (B_{NR}(0) \cup \mathcal{T}_{R^{-1}}(\mathcal{C}))$ let

$$\rho(x) = \inf\{\rho' \geq 0 \mid B_{\rho'}(x) \cap (B_{NR}(0) \cup \mathcal{T}_{R^{-1}}(\mathcal{C})) \neq \emptyset\}$$

and let

$$\rho_t(x) = \begin{cases} \sqrt{\rho(x)^2 - 2n(t-1)} & \rho(x)^2 \geq 2n(t-1) \\ 0 & \rho(x)^2 < 2n(t-1) \end{cases}$$

be the corresponding MCF starting from $B_\rho(x)$. Let

$$U_t = \bigcup_{\rho_t(x) > 0} B_{\rho_t(x)}(x).$$

be the time t slice of the above MCF. Since initially $\Sigma \cap U_1 = \emptyset$ maximum principle implies that $U_t \cap \Sigma_t = \emptyset$ for all $t \in [1, T)$. This proves that

$$\Sigma_t \setminus B_{NR+\sqrt{2n(t-1)}}(0) \subset \mathcal{T}_{R^{-1}+\sqrt{2n(t-1)}}(\mathcal{C}) \tag{A.1}$$

since $\mathbb{R}^{n+1} \setminus (B_{NR+\sqrt{2n(t-1)}}(0) \cup \mathcal{T}_{R^{-1}+\sqrt{2n(t-1)}}(\mathcal{C})) \subset U_t$.

Next we consider the map $\Phi : \mathcal{C} \setminus \{0\} \times \mathbb{R} \to \mathbb{R}^{n+1}$ given by $\Phi(x, \lambda) = x + \lambda \nu_\mathcal{C}$. Choose $\lambda_0 < 1/2$ depending on $\mathcal{C}$ small enough so that $\Phi|_{\mathcal{L}(\mathcal{C}) \times (-2\lambda_0, 2\lambda_0)}$ is a diffeomorphism onto its image. It follows, since $\mathcal{C}$ is a cone, that Φ is a diffeomorphism on the set $\{(x, \lambda) \mid |\lambda| < 2\lambda_0 |x|\}$.

Consider, for some N' to be chosen later,

$$V_t = \mathbb{R}^{n+1} \setminus (U_t \cup B_{N'R\sqrt{t}}(0)).$$

We first claim that N' can be chosen so that $y \in V_t$ can be written as $x + \lambda' |x| \nu_\mathcal{C}(x)$ for some $|\lambda'| < \lambda_0$. Indeed, since $y \notin U_t$ we have that $\text{dist}(y, \mathcal{C}) \leq R^{-1} + \sqrt{2n(t-1)}$ provided we choose N' large enough so that

$$N'R\sqrt{t} > NR + \sqrt{2n(t-1)}.$$

Let x be the nearest point projection of y onto $\mathcal{C}$, then using $R > 1$ we have that

$$\frac{|y - x|}{|x|} \leq \frac{1 + \sqrt{2nt}}{N'\sqrt{t} - 1 - \sqrt{2nt}} = \frac{(\sqrt{t})^{-1} + \sqrt{2n}}{N' - (\sqrt{t})^{-1} - \sqrt{2n}} < \frac{1 + \sqrt{2n}}{N' - 1 - \sqrt{2n}} < \lambda_0$$

provided we choose N' large depending on n only. Hence the claim holds.

Let $y \in \Sigma_t \setminus B_{N'R\sqrt{t}}(0) \subset V_t$. We claim that, up to further increasing N', $\text{dist}(y, \mathcal{C}) < R^{-1}\sqrt{t}$ and this will finish the proof. To this end consider $y_0 = x + \lambda_0 |x| \nu_\mathcal{C}$. We have that

$$|y_0| = |x + \lambda_0 |x| \nu_\mathcal{C}| = \sqrt{1 + \lambda_0^2} |x| > NR + \lambda_0 |x| - \frac{1}{3}R^{-1},$$

if N' is chosen large enough so that

$$|x| \geq (N'R - \sqrt{2n})\sqrt{t} - R^{-1} > 4NR - R^{-1},$$

where we used the fact that $\sqrt{1 + \lambda_0^2} - \lambda_0 > \frac{1}{3}$. This implies that $B_{\lambda_0 |x| - R^{-1}}(y_0) \cap B_{NR}(0) = \emptyset$. Since $\text{dist}(y_0, \mathcal{C}) = \lambda_0 |x|$ we conclude that

$$\rho(y_0) = \lambda_0 |x| - R^{-1}.$$

Increasing N' if necessary we can also ensure that

$$\lambda_0 |x| - R^{-1} > \sqrt{2n(t-1)}.$$

Since $y \notin B_{\rho_t(y_0)}(y_0) \subset U_t$ we can estimate

$$\text{dist}(y, \mathcal{C}) \leq \text{dist}(y_0, \mathcal{C}) - \rho_t(y_0) = \lambda_0 |x| - \sqrt{(\lambda_0 |x| - R^{-1})^2 - 2n(t-1)}.$$

To finish the proof we compute, for large N',

$$\begin{aligned}
(\lambda_0 |x| - R^{-1}\sqrt{t})^2 &- ((\lambda_0 |x| - R^{-1})^2 - 2n(t-1)) \\
&= 2R^{-1}\lambda_0 |x|(1 - \sqrt{t}) + R^{-2}(t-1) + 2n(t-1) \\
&= R^{-1}(\sqrt{t} - 1)(-2\lambda_0 |x| + (R^{-1} + 2nR)(\sqrt{t} + 1)) \leq 0.
\end{aligned}$$

which is equivalent to

$$\lambda_0 |x| - R^{-1}\sqrt{t} \leq \sqrt{(\lambda_0 |x| - R^{-1})^2 - 2n(t-1)} \implies \text{dist}(y, \mathcal{C}) \leq R^{-1}\sqrt{t}$$

provided N' is chosen large enough so that

$$\lambda_0 |x| \geq (N'R - \sqrt{2n})\sqrt{t} - R^{-1} \geq (R^{-1} + 2nR)\sqrt{t}.$$

$\square$

The next proposition shows the desired regularity for an asymptotically conical MCF. Since the proof is used repeatedly in our presentation, we have also included a proof for the sake of completeness. In the proof we will use the notation $[f]_{\alpha;\Omega}$ to denote the α-Hölder seminorm of f on Ω (note Ω could be a subset of $\mathbb{R}^{n+1}$ or a time interval), i.e.

$$[f]_{\alpha;\Omega} = \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Proposition A.2 *(Lemma 5.3(2) of [10]) Suppose Σ is an hypersurface $C^{2,\alpha}$-asymptotic to $\mathcal{C}$ and let $\{\Sigma_t\}_{t\in[1,T)}$ be a MCF starting from Σ. If there is $N > 0$ such that $\Sigma_1 \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ for all $R > 1$, then there is $N' > 0$ such that $\Sigma_t \setminus B_{N'\sqrt{t}}(0)$ can be written as a (smooth) normal graph over $\mathcal{C} \setminus B_{N'\sqrt{t}}(0)$. In particular we have the uniform curvature bound*

$$\sup_{t\in[1,T)} \sup_{\Sigma_t \setminus B_{N'\sqrt{t}}} |A_{\Sigma_t}| < \infty.$$

Proof Fix $t \in [1, T)$ and let $\delta > 0$. Since Σ is asymptotically conical, by Proposition 5.2 there is N_1 and $\varepsilon > 0$ depending on δ such that for any $x_0 \in \mathcal{C} \setminus B_{N_1}(0)$, $\Sigma_1 \cap C_\eta(x_0)$ can be written as a graph $\tilde{f}_{x_0}(x)$ over some neighborhood of x_0 in $T_x\mathcal{C}$ containing $B_\eta^n(x_0)$. Here

$\eta = \varepsilon |x|$. Moreover up to increasing N_1 we can ensure Proposition A.1 holds, and that $\tilde{f}_{x_0}$ satisfies the estimates

$$\sum_{i=0}^{2} \eta^{-1+i} \sup_{B_\eta^n(x_0)} |\nabla^i \tilde{f}_{x_0}| + r^{1+\alpha} [\nabla^2 \tilde{f}_{x_0}]_\alpha < \delta \tag{A.2}$$

By pseudolocality of MCF (Theorem 1.5 in [31]), given $\varepsilon > 0$, there is $N_1 > 0$ such that for every $x_0 \in \Sigma_1 \setminus B_{N_1}(0)$ and $s \in [1, t]$, $\Sigma_s \cap C_{\eta/2}(x_0)$ can be written as a normal graph over $\Sigma_1 \cap B_{\eta/2}^n(x_0)$. Combining the above two facts, we see that for sufficiently small δ and ε, $\Sigma_s \cap C_{\eta/2}(x_0)$ can be written as the graph of a function $f_{x_0}(s, x)$ over some neighborhood of $T_{x_0}\mathcal{C}$ for all $s \in [1, t]$ and $x_0 \in \mathcal{C} \setminus B_{N_1}(0)$. Moreover, f_{x_0} satisfies the pointwise estimates

$$(\eta/2)^{-1} \sup_{B_{\eta/2}^n(x_0)} |f_{x_0}(s, \cdot)| + \sup_{B_{\eta/2}^n(x_0)} |\nabla_x f_{x_0}(s, \cdot)| < 1.$$

For the rest of the proof we fix an x_0 and put $f = f_{x_0}$. Since $\{\Sigma_s\}_{s \in [1,t]}$ is a graphical MCF near x_0, f satisfies the evolution equation:

$$\frac{\partial f}{\partial s} = \sqrt{1 + |\nabla_x f|^2} \, \text{div} \left(\frac{\nabla_x f}{\sqrt{1 + |\nabla_x f|^2}} \right).$$

This is a quasilinear parabolic equation, so we may use Hölder estimates (eg. Theorem 1.1 in Chapter 4 of [35]) to get that

$$\sup_{s \in [1,t]} [\nabla_x f(s, \cdot)]_{\alpha; B_{\eta/4}(x_0)} + \sup_{B_{\eta/4}^n(x_0)} [\nabla_x f(\cdot, x)]_{\alpha/2; [1,t]} \le C(\eta/4)^{-\alpha}.$$

for any $\alpha \in (0, 1)$. Standard Schauder estimates (see eg. Chapter 5 of [34] or Theorem 5.1 in Chapter 4 of [35]) yield higher order estimates of the form

$$\sum_{i=0}^{2} (\eta/8)^{i-1} \sup_{B_{\eta/8}^n(x_0)} |\nabla^i f(s, \cdot)| + (\eta/8)^{1+\alpha} [\nabla^2 f(s, \cdot)]_{\alpha; B_{\eta/8}(x_0)} \le C$$

for $s \in [1, t]$ and

$$\sup_{x \in B_{\eta/8}^n(x_0)} [\nabla_x f_{x_0}(s, x)]_{\frac{1}{2}; [1,t]} \le C(\eta/8)^{-1}.$$

From the above we may estimate

$$|f_{x_0}(s, x) - f_{x_0}(1, x_0)| \le C(s-1)(\eta/8)^{-1} + \delta |x - x_0| + C(\eta/8)^{-1} |x - x_0|^2$$

where we also used the evolution equation and the fact that $|\nabla_x f(1, x_0)| < \delta$ from (A.2). These implies that, for $\rho < 1/8$, a fixed $s \in [1, t]$ and $x_0 \in \mathcal{C} \setminus B_{\tilde{N}\sqrt{s}}(0)$,

$$(\rho\eta)^{-1} \sup_{x \in B_{\rho\eta}^n(x_0)} |f(s, x)| \le (\rho\eta)^{-1} N_1 |x_0|^{-1} + C(s-1)\rho^{-1}\eta^{-2} + \delta + C\rho$$

$$\le \frac{(\rho\varepsilon)^{-1} N_1}{\tilde{N}^2 s} + \frac{C(s-1)\rho^{-1}}{\tilde{N}^2 s} + \delta + C\rho$$

where we used that $|f(1, x_0)| < N_1 |x_0|^{-1}$ by Proposition A.1. The right hand side of the above equation can be made arbitrarily small provided we choose δ and ρ small enough and $\tilde{N}$ large enough. Similarly we can estimate the derivative

$$|\nabla_{x_0} f_{x_0}(s, x) - f_{x_0}(1, x_0)| \le C(\eta/8)^{-1} |x - x_0| + C(\eta/8)^{-1}\sqrt{s-1}$$

and

$$\sup_{x \in B^n_{\rho\eta}(x_0)} |\nabla_x f(s,x)| \leq \delta + C\rho + C\frac{\sqrt{s-1}}{\tilde{N}\sqrt{s}}$$

which can be made arbitrarily small as well. These two decay estimates together with Schauder estimates above with $\eta/8$ replaced by $\rho\eta$ give

$$\sum_{i=0}^{2} (\eta/8)^{i-1} \sup_{B^n_{\eta/8}(x_0)} \left|\nabla^i f(s,\cdot)\right| + (\eta/8)^{1+\alpha}[\nabla^2 f(s,\cdot)]_{\alpha;B_{\eta/8}(x_0)} \leq \frac{1}{2} + C(\rho + \rho^{1+\alpha})$$

which can be made to be less than 1 provided ρ is chosen small enough. This proves that $\Sigma_t \cap C_{\rho\eta}(x_0)$ is a graph over (a neighborhood of) $T_{x_0}\mathcal{C}$ for $x_0 \in \mathcal{C} \setminus B_{\tilde{N}\sqrt{t}}(0)$ with derivative bounds up to the second order. The curvature bounds follow easily, and higher order bounds follow similarly using Schauder estimates.   □

Given a MCF $\{\Sigma_t\}_{t\in I}$, the *expander mean curvature* of Σ_t is

$$E_{\Sigma_t}(x) = 2tH_{\Sigma_t} + \langle x, \nu_{\Sigma_t}\rangle.$$

We say $\{\Sigma_t\}$ is *expander-mean-convex* if the $E_{\Sigma_t}(x) > 0$ along the flow. For a fixed time t and a hypersurface Σ, the relative expander mean curvature of Σ is

$$E_{\Sigma}^t(x) = 2tH_{\Sigma} + x^{\perp}.$$

For $\beta > 0$ define the auxiliary function $g_\beta : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g_\beta(s) = s^{-\beta}e^{-\beta s}.$$

We now prove the main existence theorem for expander-mean-convex hypersurfaces without entropy bound.

Theorem A.3 *(Existence Theorem, cf. Proposition 5.1 of [10]) Let Σ be a hypersurface $C^{2,\alpha}$-asymptotic to $\mathcal{C}$ with no closed components. Suppose that there is N such that $\Sigma \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ and that there is $c, \beta > 0$ such that*

$$E_\Sigma(x) \geq cg_\beta(1+|x|^2) > 0, \quad x \in \Sigma.$$

Then there exists a unique MCF $\{\Sigma_t\}_{t\in[1,T)}$ with $\Sigma_1 = \Sigma$, where T is the first singular time (possibly ∞). Moreover the MCF satisfies

(1) Σ_t is $C^{2,\alpha}$-asymptotic to $\mathcal{C}$ for all $t \in [1,T)$.
(2) $E_{\Sigma_t}(x) > cg_\beta(1 + |x|^2 + 2n(t-1))$ for all $t \in [1,T)$ and $x \in \Sigma_t$.
(3) If $T < \infty$, we have

$$\lim_{t \to T} \sup_{\Sigma_t \cap B_{N'\sqrt{t}}} |A_{\Sigma_t}| = \infty.$$

Proof Consider the map $\Phi : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+1}$ given by

$$\Phi(x,\lambda) = x + \lambda\nu_\Sigma.$$

Since Σ is asymptotically conical, we can choose ε sufficiently small so that the above map is a diffeomorphism onto its image for every $\lambda \in (-\varepsilon, \varepsilon)$. Using this parametrization we can invoke standard existence theorem for MCF to conclude that there exists a unique MCF starting from $\Sigma_1 = \Sigma$. For the properties, Item (1) and (3) follow from Proposition A.2, and Item (2) is Lemma 5.4 of [10].

Given an unstable self-expander Σ asymptotic to $\mathcal{C}$, we wish to apply Theorem A.3 to the perturbed self-expander Σ^ε defined by

$$\Sigma^\varepsilon = \Psi^\varepsilon(\Sigma) \text{ where } \Psi^\varepsilon(x) = x + \varepsilon f(x)\nu_\Sigma.$$

It remains to check that Σ^ε satisfies the assumption of Theorem A.3. We need the following C^0-estimate on the first eigenfunction f.

Lemma A.4 *(Proposition 3.2 of [5]) Let Σ be an unstable connected self-expander. Let f be the unique positive eigenfunction of L_Σ with eigenvalue $\mu_1 < 0$ and $\|f\|_{W^0_{1/4}(\Sigma)} = 1$. Then f satisfies the following C^0 estimate:*

$$C^{-1}(1+|x|^2)^{-\frac{1}{2}(n+1-2\mu_1)}e^{-\frac{|x|^2}{4}} \leq f \leq C(1+|x|^2)^{-\frac{1}{2}(n+1-2\mu_1)}e^{-\frac{|x|^2}{4}}.$$

where $C = C(\Sigma)$.

Lemma A.4 and Proposition 5.2 imply that there exists $N > 0$ such that $\Sigma^\varepsilon \setminus B_{NR}(0) \subset \mathcal{T}_{R^{-1}}(\mathcal{C})$ for $R > 1$, so the first condition to apply Theorem A.3 is satisfied. Finally we need to show that perturbing by the first eigenfunction produces a hypersurface with positive expander mean curvature. This relies on the fact that $-L_\Sigma$ is the linearization of the self-expander equation.

Lemma A.5 *Let Σ be a connected self-expander $C^{2,\alpha}$-asymptotic to $\mathcal{C}$. Let f be the unique positive eigenfunction corresponding to μ_1 of L_Σ with $\|f\|_{W^0_{1/4}(\Sigma)} = 1$. Then there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ there is $\beta = \beta(\varepsilon)$ such that*

$$E_{\Sigma^\varepsilon}(x) \geq cg_\beta(1+|x|^2).$$

Here Σ^ε is the image of Σ under the map $\Phi(x) = x + \varepsilon f(x)\nu_\Sigma$.

Proof By Lemma A.2 of [11] we have (the computation is long, but the result should be standard),

$$E_{\Sigma^\varepsilon}(x) = -\varepsilon L_\Sigma f + \varepsilon^2 Q(f, \langle x, \nabla_\Sigma f \rangle, \nabla_\Sigma f, \nabla^2_\Sigma f)$$

for some homogeneous quadratic polynomial Q with bounded coefficients. When $\varepsilon > 0$, it follows from Lemma A.4, that, up to further shrinking ε,

$$E_{\Sigma^\varepsilon}(x) \geq \varepsilon\mu_1 C^{-1}(1+|x|^2)^{-\frac{1}{2}(n+1-2\mu_1)}e^{-\frac{1+|x|^2}{4}} \geq cg_\beta(1+|x|^2)$$

where $\beta = \frac{1}{2}(n+1-2\mu_1) > 0$. The case $\varepsilon < 0$ can be handled similarly. $\quad\square$

Lemma A.5 shows that Σ^ε satisfies the second condition of Theorem A.3 for sufficiently small ε. As such, Theorem A.3 can be applied to conclude the short-time existence of an expander mean-convex MCF starting from Σ^ε.

Appendix B. Maximum principles

Here we record the maximum principles from Section 3 of [18] for Brakke flows. These are the essential tools in applying the moving plane method without smoothness. If $\mathcal{M}$ is an integral Brakke flow and $X = (x_0, t_0) \in \text{supp}\,\mathcal{M}$, the Gaussian density at X is

$$\Theta_\mathcal{M}(X) = \lim_{\rho \to 0} \frac{1}{(4\pi\rho^2)^{n/2}} \int_{\mathcal{M}_{t_0-\rho^2}} e^{-\frac{|x-x_0|^2}{4\rho^2}} d\mathcal{H}^n.$$

$\Theta_{\mathcal{M}}(X)$ is well-defined by Huisken's monotonicity formula. Observe that an entropy upper bound automatically gives upper bounds on all Gaussian densities.

Theorem B.1 *(Maximum principle for Brakke flows, Theorem 3.4 of [18]) Let $\mathcal{M}$ be a smooth MCF defined in a parabolic ball $P(X, r)$, where $X = (x_0, t_0) \in \operatorname{supp} \mathcal{M}$ and $r > 0$ is sufficiently small such that $\operatorname{supp} \mathcal{M}$ separates $P(x, r)$ into two open connected components U and U'. Let $\mathcal{M}'$ be an integral Brakke flow in $P(X, r)$ with $X \in \operatorname{supp} \mathcal{M}'$ and Gaussian density $\Theta_X(\mathcal{M}') < 2$. If $\operatorname{supp} \mathcal{M}' \subset U \cup \operatorname{supp} \mathcal{M}$, then X is a smooth point for $\mathcal{M}'$, and $\mathcal{M}'$ agrees with $\mathcal{M}$ in a small parabolic ball.*

Theorem B.2 *(Hopf lemma for tame Brakke flows, Theorem 3.19 of [18]) Let $\mathcal{M}$ and $\mathcal{M}'$ be two integral Brakke flows defined in a parabolic ball $P(X, r)$ where $X = (x_0, t_0) \in \operatorname{supp} \mathcal{M} \cap \mathcal{M}'$. Suppose X is a tame point (see Definition 3.1) for both $\mathcal{M}$ and $\mathcal{M}'$ and let $\mathbb{H} \subset \mathbb{R}^{n+1}$ be an open half space with $x_0 \in \partial \mathbb{H}$. If in addition $\partial \mathbb{H}$ is not the tangent flow to either $\mathcal{M}$ or $\mathcal{M}'$, and $\operatorname{reg} \mathcal{M}_t \cap \mathbb{H}$ and $\operatorname{reg} \mathcal{M}'_t \cap \mathbb{H}$ are disjoint for $t \in (t_0 - r^2, t_0)$, then $\mathcal{M}$ and $\mathcal{M}'$ are smooth at $(0, 0)$ with distinct tangents.*

We remark that in contrast to the usual smooth maximum principle and Hopf lemma, the smoothness is in fact a conclusion in both of the statements above.

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