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Calculus of Variations

Global Existence, Regularity and Boundedness in a Higher-dimensional Chemotaxis-Navier-Stokes System with Nonlinear Diffusion and General Sensitivity

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Abstract
We consider an incompressible chemotaxis-Navier-Stokes system with nonlinear diffusion and rotational flux

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (nS(x, n, c) \cdot \nabla c), \quad x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, \quad x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N (N = 2, 3)$ with smooth boundary $\partial\Omega$, where $\kappa \in \mathbb{R}$. The chemotaxtic sensitivity S is a given tensor-valued function fulfilling $|S(x, n, c)| \leq S_0(c)$ for all $(x, n, c) \in \bar{\Omega} \times [0, \infty) \times [0, \infty)$ with $S_0(c)$ nondecreasing on $[0, \infty)$. By introducing some new methods (see Sect. 4 and Sect. 5), we prove that under the condition $m > 1$ and some other proper regularity hypotheses on initial data, the corresponding initial-boundary problem possesses at least one global weak solution. The present work also shows that the weak solution could be bounded provided that $N = 2$. Since S is tensor-valued, it is easy to see that the restriction on m here is optimal, which answers the left question in Bellomo-Belloquid-Tao-Winkler (Math Models Methods Appl Sci 25:1663–1763, 2015) and Tao-Winkler (Ann Inst H Poincaré Anal Non Linéaire 30:157–178, 2013). And obviously, this work improves previous results of several other authors (see Remark 1.1).

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1 Introduction

In this paper, we consider the following chemotaxis-Navier-Stokes system with nonlinear diffusion and general sensitivity

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (nS(x,n,c) \cdot \nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \\ (\nabla n^m - nS(x,n,c) \cdot \nabla c) \cdot \nu = \partial_\nu c = 0, u = 0, & x \in \partial\Omega, t > 0, \\ n(x,0) = n_0(x), c(x,0) = c_0(x), u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, where $m > 1$, $\kappa \in \mathbb{R}$ and ν denotes the unit outward normal vector field on $\partial\Omega$. The chemotaxis sensitivity $S(x,n,c)$ is a tensor-valued function satisfying

$$S \in C^2(\bar{\Omega} \times [0,\infty)^2; \mathbb{R}^{N \times N}) \tag{1.2}$$

and

$$|S(x,n,c)| \le S_0(c) \quad \text{for all } (x,n,c) \in \Omega \times [0,\infty)^2 \tag{1.3}$$

with some nondecreasing $S_0 : [0,\infty) \to \mathbb{R}$. Here N denotes the space dimension, $N = 2, 3$. Such system, coupling chemotaxis equations with fluid equations, is proposed to describe the populations of bacteria (or cells) suspended in sessile drops of liquid ([3, 4, 9, 36]). It takes into account not only the convection of bacteria and signal, but also the influence of fluid. In this model, $n = n(x,t)$, $c = c(x,t)$, $u = u(x,t)$ and $P = P(x,t)$ represent the population density, the concentration of chemical signals, the fluid velocity field and the associated pressure, respectively. ϕ is the potential of gravitational field and κ denotes the strength of nonlinear fluid convection. Before establishing our main results, we give the following background knownledge.

Keller-Segel model. In 1970, Keller and Segel ([17]) proposed the mathematical system

$$\begin{cases} n_t = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (nS(n)\nabla c), & x \in \Omega, t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega, t > 0, \\ (D(n)\nabla n - nS(n)) \cdot \nu = \nabla c \cdot \nu = 0, \quad x \in \partial\Omega, t > 0, \\ n(x,0) = n_0(x), c(x,0) = c_0(x), \quad x \in \Omega \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, where n and c are defined as before. The model reflects the interaction between the random diffusion and aggregation of bacteria to the high concentration chemical signals. Extensive mathematical literature has grown on this model and its variants, and the results are rather complete. The most important results are around the existence/boundedness, blow-up and large time behavior. For example, it is well-known that, when $D(n) \equiv 1$ and $S(n) \equiv 1$, solutions to this system may blow up for suitably large initial data in the case $N \ge 3$ ([45]) and $N = 2$ ([12]). When $D(n)$ decays exponentially and satisfies $\frac{S(s)}{D(s)} \le Ks^\alpha$ with constant $K > 0$ and $\alpha \in (0,1)$, the solution is globally bounded in a two-dimensional bounded domain ([6]). Horstmann and Winkler also showed that all solutions to the system are global and uniformly bounded in the case $S(n) \le C(1+n)^{-\alpha}$ with $\alpha > 1 - \frac{2}{N}$, while they may blow up under the requirements that $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ is a ball and S fulfills $S(n) > Cn^{-\alpha}$ with $\alpha < 1 - \frac{2}{N}$ ([14]). Readers can refer to [1, 13, 18, 35, 37, 41, 42, 56–61] for more revelent results about this model and its variants.

Chemotaxis-fluid model. There are many more complex situations in the real life. The change of living environment also plays an important role in immigration. For example, bacteria, such as Bacillus subtilis, live in a thin layer of liquid near solid air-water contact. In such a flow environment, the mutual interaction between cell and fluid may be significant. Considering that the motion of fluid is described by the incompressible (Navier-)Stokes equations, such cell-fluid interaction is given by

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (nS(x, n, c) \cdot \nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c + h(n, c), & x \in \Omega, t > 0, \\ u_t + \kappa(u \cdot \nabla)u + \nabla P = \Delta u + n\nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases}$$

where n, c, u, κ and P as well as ϕ are defined the same as before. Results around this model are influenced by the scalar function S, h and ϕ. When $h(n, c) = n-c$, which means the signal is produced by cells, Liu and Wang ([23]) showed that the solution of this model is global in time and bounded for $\kappa \neq 0$ and $N = 3$ or $\kappa = 0$ and $N = 2, 3$ with $S(x, n, c) = \frac{\xi_0}{(1+\mu c)^2}$. On the other hand, in a three-dimensional setup involving the tensor-valued sensitivity $S(x, n, c)$ satisfying $|S(x, n, c)| \leq S_0(1 + n)^{-\alpha}$, global weak solutions have been shown to exist in [25] for $\alpha > \frac{3}{7}$ and global very weak solutions were obtained for $\alpha > \frac{1}{3}$ in [39] (see also [16]), which in light of the known results for the fluid-free system mentioned above is an optimal restriction on α. We next address the case that $m \neq 1$. For $D(n) = mn^{m-1}$ and $S(x, n, c) = 1$, a globally defined weak solution and at least one global bounded solution can be asserted in the case $m > 2$ ([58]) and $\kappa = 0$ and $m > \frac{4}{3}$ ([59]), respectively. Black [2] showed existence of global (very) weak solutions in the system with $m \neq 1$ and tensor-valued sensitivity under some largeness condition for m. When $h(n, c) = -ng(c)$, cells consume the signal only, where $g(c)$ models the per capita consumption rate. One well-known result is that the system possesses a unique global classical solution converging to the spatially homogeneous equilibrium $(\bar{n}_0, 0, 0)$ with $\bar{n}_0 = \frac{1}{|\Omega|}\int_\Omega n_0$ as $t \to \infty$ in two-dimensional space ([44, 46]). In the case $N = 3$, a globally defined weak solution exists under the requirements that $S(x, n, c) = 1$, $D(n) = 1$ and $\kappa \neq 0$ ([49]). After this, it was shown by Zhang and Li that the same result held in the case $m > \frac{2}{3}$ and $D(n) = n^{m-1}$ ([54]). For more literature, readers can refer to [5, 7, 8, 22, 26, 47, 55, 63, 65] and the references therein.

In order to adapt to more realistic modeling assumptions, further simulation shows that the directional migration of cells may not be parallel to the gradient of the chemical substances. Instead, it involves the rotational flux component, which requires S to be a matrix-valued function in the prototype, for example,

$$S = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha > 0, \ \beta \in \mathbb{R}$$

in two-dimensional case. It brings a great mathematical challenge to the proof, since the loss of some energy structure, which is the key to analyze the scalar-valued S. Consequently, new methods should be found. The most difficult part is to deal with the term $\nabla \cdot (nS(x, n, c) \cdot \nabla c)$. In the case of scalar-valued $S = S(c)$, the main estimates on S are based on the following inequality (see [44, 46])

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega n \ln n + \frac{1}{2}\int_\Omega \frac{S(c)|\nabla c|^2}{g(c)}\right\} + \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{C}\int_\Omega \frac{|\nabla c|^4}{c^3} \leq C\int_\Omega |u|^4, \quad t > 0 \tag{1.4}$$

with some constant C. While if S is of tensor-value, the natural energy inequality like (1.4) would not be available. Indeed, if S is of tensor-value, the following strongly coupling term

$$\int_\Omega n^{p-1}|S(x,n,c)||\nabla n||\nabla c| \quad (p>1)$$

is indispensable. For example, in [48], Winkler constructs a generalized solution to the system

$$\begin{cases} n_t = \Delta n - \nabla\cdot(nS(x,n,c)\cdot\nabla c), & x\in\Omega, t>0,\\ c_t = \Delta c - nc, & x\in\Omega, t>0, \end{cases}$$

where S is a tensor-valued sensitivity with $|S(x,n,c)| \le CS_0(c)$ with S_0 nondecreasing on $[0,\infty)$. And in two-dimensional situations, for a chemotaxis-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(nS(x,n,c)\cdot\nabla c), & x\in\Omega, t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega, t>0,\\ u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega, t>0,\\ \nabla\cdot u = 0, & x\in\Omega, t>0 \end{cases} \tag{1.5}$$

with $D(n) = 1$, it is proved that a global mass-preserving generalized solution could be established; besides, there exists $T>0$ such that the solution satisfies

$$(n,c,u) \in C^{2,1}(\bar\Omega\times[T,\infty)) \times C^{2,1}(\bar\Omega\times[T,\infty)) \times C^{2,1}(\bar\Omega\times[T,\infty);\mathbb{R}^2)$$

and

$$(n(\cdot,t), c(\cdot,t), u(\cdot,t)) \to (\bar n_0, 0, 0) \quad \text{in } L^\infty(\Omega) \text{ as } t\to\infty$$

in [52, 53]. When $N=3$, $D(n) = mn^{m-1}$ and S is scalar-value, many authors study the global existence and boundedness of the solution of (1.5) and weaken the restriction on m step by step. In [9], it requires $m \in [\frac{7+\sqrt{217}}{12}, 2]$. In 2013, $m > \frac{8}{7}$ is need for locally bounded solutions ([34]). In [51], the lower bound of m is extended to $\frac{9}{8}$. Without regard to boundedness, the range of m could be extended to cover the whole range $m\in(1,\infty)$ ([8]) and then $m\in(\frac{2}{3},\infty)$ ([54]). In the case of tensor-valued S, Winkler ([47]) obtained uniform-in-time boundedness of global weak solutions in some bounded and convex domain Ω with $m > \frac{7}{6}$. Later, this restriction was improved to $m > \frac{10}{9}$ ([64]) by one of the current authors. For $|S(x,n,c)| \le S_0(1+n)^{-\alpha}$ and non-decreasing S_0, it is proved that $m\ge 1$ and $m+\alpha > \frac{7}{6}$ are required for the global existence of bounded weak solutions ([40]) with $\alpha>0$. The same result could be established under the requirements that $m+\alpha > \frac{10}{9}$ and $m + \frac{5}{4}\alpha > \frac{9}{8}$ by Wang ([39]), and $m+\alpha > \frac{10}{9}$ by Zheng and Ke ([66]). Inspired by the results mentioned above, we create a new method to further weaken the restriction on m, under the circumstance that S is a tensor-valued function.

Notations. Here and below, for given vectors $v\in\mathbb{R}^N$ and $w\in\mathbb{R}^N$, we define the matrix $v\otimes w$ by letting $(v\otimes w)_{ij} := v_i w_j$, for $i,j\in\{1,\cdots,N\}$. We write $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega)\cap L^2_\sigma(\Omega)$ with $L^2_\sigma(\Omega) := \{\varphi\in L^2(\Omega;\mathbb{R}^N) | \nabla\cdot\varphi = 0\}$ (see [30]).

In order to prepare a precise statement of our main results in these respects, let us assume throughout that the initial data satisfy

$$\begin{cases} n_0 \in C^\iota(\bar\Omega) \text{ for certain } \iota>0 \text{ with } n_0\ge 0 \text{ in } \Omega,\\ c_0\in W^{1,\infty}(\Omega) \text{ is nonnegative and such that } \sqrt{c_0}\in W^{1,2}(\Omega),\\ u_0\in D(A^\gamma) \text{ for some } \gamma\in(\frac{N}{4},1), \end{cases} \tag{1.6}$$

where A denotes the Stokes operator with domain $D(A) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega)$. As for the time-independent gravitational potential function ϕ, we assume for simplicity that

$$\phi \in W^{2,\infty}(\Omega). \tag{1.7}$$

Within the above frameworks, our main results concerning global existence of solutions to (1.1) are as follows.

Theorem 1.1 *Let $m > 1$, $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and assume (1.2)–(1.3) and (1.6)–(1.7) hold. Then the problem (1.1) admits a global-in-time weak solution (n, c, u, P), which is uniformly bounded in the sense that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \ \ \textit{for all} \ \ t > 0$$

with some positive constant C.

Theorem 1.2 *Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary and S satisfies the hypotheses (1.2)–(1.3). Assume ϕ satisfies (1.7), and suppose the initial data n_0, c_0, u_0 satisfy (1.6). If $m > 1$, then there exist functions satisfying*

$$\begin{cases} n \in L_{loc}^{\frac{8m-3}{3}}(\bar{\Omega} \times [0,\infty)), \\ n^m \in L_{loc}^{\frac{8m-3}{4m}}([0,\infty); W^{1,\frac{8m-3}{4m}}(\Omega)), \\ c \in L^4_{loc}([0,\infty); W^{1,4}(\Omega)) \cap L^\infty(\Omega \times (0,\infty)), \\ u \in L^2_{loc}([0,\infty); L^2_\sigma(\Omega;\mathbb{R}^3)) \cap L_{loc}^{\frac{10}{3}}(\Omega \times [0,\infty);\mathbb{R}^3) \cap L^2_{loc}([0,\infty); W^{1,2}_{0,\sigma}(\Omega)), \end{cases}$$

such that (n, c, u) is a global weak solution of the problem (1.1) in the sense of Definition 2.1. This solution can be obtained as the pointwise limit a.e. in $\Omega \times (0,\infty)$ of a suitable sequence of classical solutions to the regularized problem (2.5) below.

Remark 1.1

(i) Theorem 1.1 extends the results of Tao and Winkler [33], in which the authors discussed the chemotaxis-Stokes system ($\kappa = 0$) in a 2D domain. As mentioned earlier, we not only extend the results to the chemotaxis-Navier-Stokes system ($\kappa \neq 0$), but also remove the convexity assumption on the domain.
(ii) In the case $\kappa \neq 0$ in system (1.1), it is hard to obtain the boundedness of the solution of system (1.1).
(iii) We should point out that the ideas of [41, 46, 50, 51] can not deal with (1.1). In fact, (1.1) with rotation loses the natural energy structure, so the relevant study is challenging.

2 Preliminaries and Main Results

Our main efforts center on the discussion of the weak solutions, because of the degeneracy of the system (1.1).

Definition 2.1 (**Weak solutions**) By a global weak solution of (1.1) we mean a triple (n, c, u) of functions

$$\begin{cases} n \in L^1_{loc}(\bar{\Omega} \times [0,\infty)), \\ c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \\ u \in L^1_{loc}([0,\infty); W_0^{1,1}(\Omega;\mathbb{R}^N)), \end{cases} \tag{2.1}$$

such that $n \geq 0$ and $c \geq 0$ a.e. in $\Omega \times (0, \infty)$,

$$\begin{aligned} & nc,\ n^m \ \in L^1_{loc}(\bar{\Omega} \times [0, \infty)) \text{ and} \\ & nS(x, n, c) \cdot \nabla c,\ cu \text{ and } nu \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^N), \end{aligned} \tag{2.2}$$

$\nabla \cdot u = 0$ a.e. in $\Omega \times (0, \infty)$, and that

$$\begin{aligned} & -\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0 \varphi(\cdot, 0) \\ & = \int_0^\infty \int_\Omega n^m \Delta\varphi + \int_0^\infty \int_\Omega n(S(x, n, c) \cdot \nabla c) \cdot \nabla\varphi + \int_0^\infty \int_\Omega nu \cdot \nabla\varphi \end{aligned}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$ as well as

$$\begin{aligned} & -\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0 \varphi(\cdot, 0) \\ & = -\int_0^\infty \int_\Omega \nabla c \cdot \nabla\varphi - \int_0^\infty \int_\Omega nc \cdot \varphi + \int_0^\infty \int_\Omega cu \cdot \nabla\varphi \end{aligned}$$

for each $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$ and

$$\begin{aligned} & -\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) - \kappa \int_0^\infty \int_\Omega u \otimes u \cdot \quad \nabla\varphi \\ & = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\varphi - \int_0^\infty \int_\Omega n\nabla\phi \cdot \varphi \end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega \times [0, \infty); \mathbb{R}^N)$ fulfilling $\nabla \cdot \varphi \equiv 0$.

In order to solve the difficulties caused by the degenerate diffusion, the nonlinear boundary conditions and the convection terms in Navier-Stokes equation, we consider an appropriately regularized problem of (1.1). To this end, we fix a family $(\rho_\varepsilon)_{\varepsilon \in (0,1)} \in C_0^\infty(\Omega)$ of standard cut-off functions satisfying $0 \leq \rho_\varepsilon \leq 1$ in Ω and $\rho_\varepsilon \nearrow 1$ in Ω as $\varepsilon \searrow 0$, and define

$$S_\varepsilon(x, n, c) := \rho_\varepsilon(x) S(x, n, c), \quad x \in \bar{\Omega}, \quad n \geq 0, \quad c \geq 0 \tag{2.3}$$

for $\varepsilon \in (0, 1)$ to approximate the sensitivity tensor S, which ensures that $S_\varepsilon(x, n, c) = 0$ on $\partial\Omega$. Note that if S complies with (1.3), then so does S_ε, that is,

$$|S_\varepsilon(x, n, c)| \leq S_0(c) \quad \text{for all } (x, n, c) \in \Omega \times [0, \infty)^2, \tag{2.4}$$

where S_0 is the same as that in (1.3). Then for any $\varepsilon \in (0, 1)$, the regularized problem of (1.1) is presented as follows

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta(n_\varepsilon + \varepsilon)^m - \nabla \cdot (n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon), & x \in \Omega, t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - n_\varepsilon c_\varepsilon, & x \in \Omega, t > 0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon - \kappa (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega, t > 0, \\ \nabla n_\varepsilon \cdot \nu = \nabla c_\varepsilon \cdot \nu = 0, u_\varepsilon = 0, & x \in \partial\Omega, t > 0, \\ n_\varepsilon(x, 0) = n_0(x), c_\varepsilon(x, 0) = c_0(x), u_\varepsilon(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.5}$$

where

$$F_\varepsilon(s) = \frac{1}{1 + \varepsilon s} \quad \text{for all } s \geq 0 \tag{2.6}$$

as well as

$$Y_\varepsilon w := (1+\varepsilon A)^{-1} w \quad \text{for all } \ w \in L^2_\sigma(\Omega)$$

and $m > 1$.

Let us begin with the following statement on local well-posedness of (2.5), along with a convenient extensibility criterion. The proof is based on a well-established method involving the Schauder fixed point theorem and standard regularity theory of parabolic equations. For more details, we refer to Lemma 2.1 of [31] (see also Lemma 2.1 of [44] and Lemma 2.1 of [21]).

Lemma 2.1 *Assume that $\varepsilon \in (0,1)$. Then there exist $T_{max,\varepsilon} \in (0,\infty]$ and functions*

$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ c_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ P_\varepsilon \in C^{1,0}(\bar{\Omega} \times (0, T_{max,\varepsilon})) \end{cases}$$

such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.5) classically on $\Omega \times [0, T_{max,\varepsilon})$ with $n_\varepsilon \ge 0$ and $c_\varepsilon \ge 0$, and such that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \ \ as \ \ t \to T_{max,\varepsilon},$$

where γ is given by (1.6).

Next, we are going to introduce some elementary properties of the solutions to (2.5).

Lemma 2.2 *The solution of (2.5) satisfies*

$$\|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)} \ \ \text{for all} \ \ t \in (0, T_{max,\varepsilon}) \tag{2.7}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \ \ \text{for all} \ \ t \in (0, T_{max,\varepsilon}). \tag{2.8}$$

Proof (2.7) and (2.8) follow from an integration of the first equation in (2.5) and an application of the maximum principle to the second equation. □

For simplicity, here and hereafter, we denote

$$C_S := S_0(\|c_0\|_{L^\infty(\Omega)}) \tag{2.9}$$

by using (2.8) and the nondecreasing of S.

Now, let us present the following elementary lemma as a preparation for some estimates in the sequel. The proof of this lemma can be found in [20, 28].

Lemma 2.3 (Lemma 2.7 in [20]) *Let $w \in C^2(\bar{\Omega})$ satisfy $\nabla w \cdot \nu = 0$ on $\partial\Omega$.*

(i) Then

$$\frac{\partial |\nabla w|^2}{\partial \nu} \le C_{\partial\Omega} |\nabla w|^2,$$

where $C_{\partial\Omega}$ is an upper bound on the curvature of $\partial\Omega$.

(ii) Furthermore, for any $\eta > 0$ there is $C(\eta) > 0$ such that for every $w \in C^2(\bar{\Omega})$ with $\nabla w \cdot \nu = 0$ on $\partial\Omega$ fulfills

$$\|\nabla w\|_{L^2(\partial\Omega)} \le \eta \|\Delta w\|_{L^2(\Omega)} + C(\eta)\|w\|_{L^2(\Omega)}.$$

(iii) *For any positive* $w \in C^2(\bar{\Omega})$,

$$\|\Delta w^{\frac{1}{2}}\|_{L^2(\Omega)} \le \frac{1}{2}\|w^{\frac{1}{2}}\Delta \ln w\|_{L^2(\Omega)} + \frac{1}{4}\|w^{-\frac{3}{2}}|\nabla w|^2\|_{L^2(\Omega)}.$$

(iv) *There are* $C_0 > 0$ *and* $\mu_0 > 0$ *such that every positive* $w \in C^2(\bar{\Omega})$ *fulfilling* $\nabla w \cdot \nu = 0$ *on* $\partial\Omega$ *satisfies*

$$-2\int_\Omega \frac{|\Delta w|^2}{w} + \int_\Omega \frac{|\nabla w|^2 \Delta w}{w^2} \le -\mu_0 \int_\Omega w|D^2 \ln w|^2 - \mu_0 \int_\Omega \frac{|\nabla w|^4}{w^3} + C_0 \int_\Omega w. \tag{2.10}$$

Now, we display an important auxiliary interpolation lemma by using the idea which comes from the references [47, 62].

Lemma 2.4 (Lemma 3.8 in [47] and Lemma 2.2 in [62]) *Let* $q \ge 1$,

$$\lambda \in [2q+2, 4q+1]$$

and $\Omega \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary. Then there exists* $C > 0$ *such that for all* $\varphi \in C^2(\bar{\Omega})$ *fulfilling* $\varphi \cdot \frac{\partial \varphi}{\partial \nu} = 0$ *on* $\partial\Omega$, *we have*

$$\|\nabla\varphi\|_{L^\lambda(\Omega)} \le C\left\||\nabla\varphi|^{q-1}D^2\varphi\right\|_{L^2(\Omega)}^{\frac{2(\lambda-3)}{(2q-1)\lambda}} \left\|\varphi\right\|_{L^\infty(\Omega)}^{\frac{6q-\lambda}{(2q-1)\lambda}} + C\|\varphi\|_{L^\infty(\Omega)}.$$

Along with (2.8), Lemma 2.4 asserts the following:

Lemma 2.5 *Let* $\beta \in [1, \infty)$. *There exists a positive constant* $\lambda_{0,\beta}$ *such that the solution of (2.5) satisfies*

$$\|\nabla c_\varepsilon\|_{L^{2\beta+2}(\Omega)}^{2\beta+2} \le \lambda_{0,\beta}(\||\nabla c_\varepsilon|^{\beta-1}D^2 c_\varepsilon\|_{L^2(\Omega)}^2 + 1) \text{ for all } t \in (0, T_{max,\varepsilon}).$$

Finally we recall the following elementary inequality (see Lemma 2.3 in [66]).

Lemma 2.6 *Let* $T > 0$, $\tau \in (0, T)$, $A > 0$, $\alpha > 0$ *and* $B > 0$, *and suppose that* $y : [0, T) \to [0, \infty)$ *is absolutely continuous such that*

$$y'(t) + Ay^\alpha(t) \le h(t) \text{ for a.e. } t \in (0, T)$$

with some nonnegative function $h \in L^1_{loc}([0, T))$ *satisfying*

$$\int_t^{t+\tau} h(s)ds \le B \text{ for all } t \in (0, T - \tau).$$

Then

$$y(t) \le \max\left\{y_0 + B, \frac{1}{\tau^{\frac{1}{\alpha}}}(\frac{B}{A})^{\frac{1}{\alpha}} + 2B\right\} \text{ for all } t \in (0, T).$$

Firstly, as a basic step of the a priori estimates, we establish the main inequality by applying standard testing procedures to the first equation in (2.5).

Lemma 2.7 *Let* $p > 1$ *and* $m > 0$. *Then the solution of (2.5) from Lemma 2.1 satisfies*

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|n_\varepsilon + \varepsilon\|_{L^p(\Omega)}^p + \frac{m(p-1)}{2}\int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 \\ &\le \frac{(p-1)C_S^2}{2m}\int_\Omega (n_\varepsilon + \varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2 \end{aligned} \tag{2.11}$$

for all $t > 0$, *where* C_S *is given by (2.9).*

Proof We multiply the first equation in (2.5) by $(n_\varepsilon+\varepsilon)^{p-1}$ and integrate the result by parts, and then use the Young inequality to obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}+m(p-1)\int_\Omega(n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\
&\le(p-1)\int_\Omega(n_\varepsilon+\varepsilon)^{p-2}n_\varepsilon\nabla n_\varepsilon\cdot(F_\varepsilon(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\\
&\le(p-1)C_S\int_\Omega(n_\varepsilon+\varepsilon)^{p-1}|\nabla n_\varepsilon||\nabla c_\varepsilon|\\
&\le\frac{m(p-1)}{2}\int_\Omega n_\varepsilon(n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2+\frac{(p-1)C_S^2}{2m}\int_\Omega(n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2,
\end{aligned}$$

where we use the fact that $S_\varepsilon(x,n_\varepsilon,c_\varepsilon)=0$ on $\partial\Omega$ and $\nabla\cdot u_\varepsilon=0$, as well as (2.9) and $|F_\varepsilon|\le 1$ (see (2.6)). □

Now we are in the position to show that the solution of the approximate problem (2.5) is actually global in time. That is, $T_{max,\varepsilon}=\infty$ for all $\varepsilon\in(0,1)$.

Lemma 2.8 *Let $m\ge 1$ and $N=2,3$. Then for all $\varepsilon\in(0,1)$, the solution of (2.5) is global in time.*

Proof Multiplying the first equation in (2.5) by $(n_\varepsilon+\varepsilon)^m$, using $\nabla\cdot u_\varepsilon=0$ and the Young inequality, we obtain

$$\begin{aligned}
&\frac{1}{m+1}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^{m+1}_{L^{m+1}(\Omega)}+m^2\int_\Omega(n_\varepsilon+\varepsilon)^{2m-2}|\nabla n_\varepsilon|^2\\
&=-\int_\Omega(n_\varepsilon+\varepsilon)^m\nabla\cdot(n_\varepsilon F_\varepsilon(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\\
&=-\int_\Omega(n_\varepsilon+\varepsilon)^m\nabla\cdot(n_\varepsilon\frac{1}{(1+\varepsilon n_\varepsilon)}S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\\
&\le m\int_\Omega(n_\varepsilon+\varepsilon)^{m-1}n_\varepsilon\frac{1}{(1+\varepsilon n_\varepsilon)}|S_\varepsilon(x,n_\varepsilon,c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon|\\
&\le m\frac{1}{\varepsilon}C_S\int_\Omega(n_\varepsilon+\varepsilon)^{m-1}|\nabla n_\varepsilon||\nabla c_\varepsilon|\\
&\le\frac{m^2}{2}\int_\Omega(n_\varepsilon+\varepsilon)^{2m-2}|\nabla n_\varepsilon|^2+C_1(\varepsilon)\int_\Omega|\nabla c_\varepsilon|^2\quad\text{for all }\ t\in(0,T_{max,\varepsilon}),
\end{aligned}\tag{2.12}$$

by using (1.3) and (2.9), where $C_1(\varepsilon)$ is a positive constant possibly depending on ε. Next, multiplying the second equation with c_ε in (2.5), integrating by parts over Ω and using $\nabla\cdot u_\varepsilon=0$, we have

$$\frac{1}{2}\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)}+\int_\Omega|\nabla c_\varepsilon|^2=-\int_\Omega n_\varepsilon c_\varepsilon^2,\tag{2.13}$$

which combined with the Poincaré inequality, $n_\varepsilon\ge 0$ and $c_\varepsilon\ge 0$ implies that there exists $C_2(\varepsilon)>0$ such that

$$\int_\Omega c_\varepsilon^2\le C_2(\varepsilon)\quad\text{for all }\ t\in(0,T_{max,\varepsilon}).$$

Then integrating (2.13), it yields that for any $\varsigma\in(0,T_{max,\varepsilon})$, there is

$$\int_t^{t+\varsigma}\int_\Omega|\nabla c_\varepsilon|^2\le C_3(\varepsilon)\quad\text{for all }\ t\in(0,T_{max,\varepsilon}-\varsigma)\tag{2.14}$$

with some positive constant $C_3(\varepsilon)$. Recalling (2.7), we derive from the Gagliardo–Nirenberg inequality that for some positive constants $C_4(\varepsilon)$ and $C_5(\varepsilon)$

$$\begin{aligned}
&\int_\Omega (n_\varepsilon+\varepsilon)^{m+1}\\
&=\|(n_\varepsilon+\varepsilon)^m\|^{\frac{m+1}{m}}_{L^{\frac{m+1}{m}}(\Omega)}\\
&\le C_4(\varepsilon)\|\nabla(n_\varepsilon+\varepsilon)^m\|^{\frac{2Nm}{2Nm-N+2}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^m\|^{\frac{m+1}{m}-\frac{2Nm}{2Nm-N+2}}_{L^{\frac{1}{m}}(\Omega)}+C_4(\varepsilon)\|(n_\varepsilon+\varepsilon)^m\|^{\frac{m+1}{m}}_{L^{\frac{1}{m}}(\Omega)}\\
&\le C_5(\varepsilon)(\|\nabla(n_\varepsilon+\varepsilon)^m\|^{\frac{2Nm}{2Nm-N+2}}_{L^2(\Omega)}+1)\ \text{ for all } t\in(0,T_{max,\varepsilon}).
\end{aligned} \tag{2.15}$$

Combining (2.12), (2.15) and the Young inequality, we obtain some positive constant $C_6(\varepsilon)$ satisfying

$$\begin{aligned}
&\frac{1}{m+1}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^{m+1}_{L^{m+1}(\Omega)}+\int_\Omega (n_\varepsilon+\varepsilon)^{m+1}+\frac{m^2}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-2}|\nabla n_\varepsilon|^2\\
&\quad\le C_1(\varepsilon)\int_\Omega |\nabla c_\varepsilon|^2+C_6(\varepsilon)\ \text{ for all } t\in(0,T_{max,\varepsilon}).
\end{aligned} \tag{2.16}$$

Since $\int_t^{t+\varsigma}[\int_\Omega |\nabla c_\varepsilon|^2+C_6(\varepsilon)]$ is bounded (by (2.14)), we infer from (2.16) and Lemma 2.6 that

$$\int_\Omega (n_\varepsilon+\varepsilon)^{m+1}\le C_7(\varepsilon)\quad\text{for all } t\in(0,T_{max,\varepsilon}) \tag{2.17}$$

with some positive constant $C_7(\varepsilon)$.

Testing the third equation of (2.5) against u_ε, integrating by parts and using $\nabla\cdot u_\varepsilon=0$ and $\nabla\cdot(1+\varepsilon A)^{-1}u_\varepsilon\equiv 0$ (see also Lemma 3.5 in [49]), we have

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega |u_\varepsilon|^2+\int_\Omega |\nabla u_\varepsilon|^2\\
&\quad=\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi\ \text{ for all } t\in(0,T_{max,\varepsilon}),
\end{aligned}$$

which in light of (1.6), (1.7) and (2.17) implies that there is $C_8(\varepsilon)$ such that

$$\int_\Omega |u_\varepsilon|^2\le C_8(\varepsilon)\quad\text{for all } t\in(0,T_{max,\varepsilon})$$

and

$$\int_t^{t+\varsigma}\int_\Omega |\nabla u_\varepsilon|^2\le C_8(\varepsilon)\quad\text{for all } t\in(0,T_{max,\varepsilon}-\varsigma). \tag{2.18}$$

Therefore, based on the properties of the Yosida approximation ([27]) of Y_ε, there is $C_9(\varepsilon)>0$ such that

$$\|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)}\le C_9(\varepsilon)\ \text{ for all } t\in(0,T_{max,\varepsilon}). \tag{2.19}$$

Testing the projected Navier-Stokes equation $u_{\varepsilon t}+Au_\varepsilon=\mathcal{P}[-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon+n_\varepsilon\nabla\phi]$ against Au_ε, we derive from $m>1$ as well as (2.19) and (2.17) that for some positive constant $C_{10}(\varepsilon)$, there is

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u_\varepsilon\|^2_{L^2(\Omega)}+\int_\Omega|Au_\varepsilon|^2\\
&=\int_\Omega Au_\varepsilon\mathcal{P}(-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon)+\int_\Omega\mathcal{P}[n_\varepsilon\nabla\phi]Au_\varepsilon\\
&\le\frac{1}{2}\int_\Omega|Au_\varepsilon|^2+\kappa^2\int_\Omega|(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon|^2+\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2\\
&\le\frac{1}{2}\int_\Omega|Au_\varepsilon|^2+C_{10}(\varepsilon)\int_\Omega|\nabla u_\varepsilon|^2+C_{10}(\varepsilon)\ \text{ for all }\ t\in(0,T_{max,\varepsilon}).
\end{aligned}$$

Hence, applying (2.18) and Lemma 2.6 also implies that for some positive constant $C_{11}(\varepsilon)$,

$$\int_\Omega|\nabla u_\varepsilon|^2\le C_{11}(\varepsilon)\quad\text{for all }\ t\in(0,T_{max,\varepsilon}). \tag{2.20}$$

Let $h_\varepsilon(x,t)=\mathcal{P}[n_\varepsilon\nabla\phi-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon]$. Then employing $m>1$, (2.17) as well as (1.7) and (2.19)–(2.20), we obtain

$$\|h_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\le C_{12}(\varepsilon)\quad\text{for all }\ t\in(0,T_{max,\varepsilon})$$

with some positive constant $C_{12}(\varepsilon)$. Due to the regularizing actions of Yosida approximation in the third equation, we can obtain the bounds for $A^\gamma u_\varepsilon(\cdot,t)$ in $L^2(\Omega)$ (see e.g. Lemma 3.9 of [49]) with $\gamma\in(\frac{N}{4},1)$. Since $D(A^\gamma)$ is continuously embedded into $L^\infty(\Omega)$ with $\gamma>\frac{N}{4}$, thus, there is $C_{13}(\varepsilon)>0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C_{13}(\varepsilon)\ \text{ for all }\ t\in(0,T_{max,\varepsilon}). \tag{2.21}$$

We multiply the second equation in (2.5) by $-\Delta c_\varepsilon$ and use the Young inequality to derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)}+\int_\Omega|\Delta c_\varepsilon|^2\\
&=\int_\Omega c_\varepsilon n_\varepsilon\Delta c_\varepsilon+\int_\Omega\Delta c_\varepsilon u_\varepsilon\cdot\nabla c_\varepsilon\\
&\le\frac{1}{2}\int_\Omega|\Delta c_\varepsilon|^2+\|c_\varepsilon\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2+\|u_\varepsilon\|^2_{L^\infty(\Omega)}\int_\Omega|\nabla c_\varepsilon|^2\ \text{ for all }\ t\in(0,T_{max,\varepsilon}),
\end{aligned}$$

which together with (2.17) as well as (2.21) and (2.14) yields that

$$\int_\Omega|\nabla c_\varepsilon(\cdot,t)|^2\le C_{14}(\varepsilon)\ \text{ for all }\ t\in(0,T_{max,\varepsilon}) \tag{2.22}$$

with some positive constant $C_{14}(\varepsilon)$. An application of the variation of constants formula to c_ε leads to

$$\begin{aligned}
&\|\nabla c_\varepsilon(\cdot,t)\|_{L^4(\Omega)}\\
&\quad\le\|\nabla e^{t(\Delta-1)}c_0\|_{L^4(\Omega)}+\int_0^t\|\nabla e^{(t-s)(\Delta-1)}(c_\varepsilon(\cdot,s)-n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s))\|_{L^4(\Omega)}ds\\
&\qquad+\int_0^t\|\nabla e^{(t-s)(\Delta-1)}\nabla\cdot(u_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s))\|_{L^4(\Omega)}ds\ \text{ for all }\ t\in(0,T_{max,\varepsilon}).
\end{aligned}$$

Now, in view of (1.6), (2.17) as well as (2.21) and (2.22), empolying the L^p-L^q estimates associated heat semigroup, we have some $C_{15}(\varepsilon)>0$ such that

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^4(\Omega)}\le C_{15}(\varepsilon)\text{ for all }\ t\in(0,T_{max,\varepsilon}). \tag{2.23}$$

Let $p > 1+m$. Taking $(n_\varepsilon+\varepsilon)^{p-1}$ as the test function for the first equation of (2.5) and using Lemma 2.7, the Hölder inequality and (2.23), there exists a positive constant $C_{16}(\varepsilon)$ such that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}+\frac{m(p-1)}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\ &\le \frac{(p-1)C_S^2}{2m}\left(\int_\Omega (n_\varepsilon+\varepsilon)^{2(p+1-m)}\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla c_\varepsilon|^4\right)^{\frac{1}{2}}\\ &\le C_{16}(\varepsilon)\left(\int_\Omega (n_\varepsilon+\varepsilon)^{2(p+1-m)}\right)^{\frac{1}{2}} \quad \text{for all } t\in(0,T_{max,\varepsilon}).\end{aligned} \tag{2.24}$$

On the other hand, in view of $m > 1$ and $p > 1+m$, and applying the Gagliardo-Nirenberg inequality and the Young inequality, we derive that there exist positive constants $C_{17}(\varepsilon)$ and $C_{18}(\varepsilon)$ such that

$$\begin{aligned}&C_{16}(\varepsilon)\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|^{\frac{2(p+1-m)}{p+m-1}}_{L^{\frac{4(p+1-m)}{p+m-1}}(\Omega)}\\ &\le C_{17}(\varepsilon)\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|^{\frac{N(2p-2m+1)}{N(m+p-2)+2}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|^{\frac{2(p+1-m)}{p+m-1}-\frac{N(2p-2m+1)}{N(m+p-2)+2}}_{L^{\frac{2}{p+m-1}}(\Omega)}\\ &\quad+C_{17}(\varepsilon)\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|^{\frac{2(p+1-m)}{p+m-1}}_{L^{\frac{2}{p+m-1}}(\Omega)}\\ &\le C_{18}(\varepsilon)(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\|^{\frac{N(2p-2m+1)}{N(m+p-2)+2}}_{L^2(\Omega)}+1)\\ &\le \frac{m(p-1)}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2+C_{18}(\varepsilon) \quad \text{for all } t\in(0,T_{max,\varepsilon}),\end{aligned}$$

which together with (2.24) and an ODE comparison argument entails that

$$\|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}\le C_{19}(\varepsilon) \quad \text{for all } t\in(0,T_{max,\varepsilon}) \quad \text{and} \quad p>1+m, \tag{2.25}$$

where $C_{19}(\varepsilon)$ is a positive constant.

In light of Lemma 2.1 of [15] and the Hölder inequality, we derive that there are $C_{20}(\varepsilon) > 0$ and $C_{21}(\varepsilon) > 0$ such that

$$\begin{aligned}&\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\\ &\le C_{20}(\varepsilon)(1+\sup_{s\in(0,T_{max,\varepsilon})}\|-n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)-u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s)\|_{L^4(\Omega)})\\ &\le C_{20}(\varepsilon)(1+\|c_0(\cdot,s)\|_{L^\infty(\Omega)}\sup_{s\in(0,T_{max,\varepsilon})}\|n_\varepsilon(\cdot,s)\|_{L^4(\Omega)}\\ &\quad+\sup_{s\in(0,T_{max,\varepsilon})}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\sup_{s\in(0,T_{max,\varepsilon})}\|\nabla c_\varepsilon(\cdot,s)\|_{L^4(\Omega)})\\ &\le C_{21}(\varepsilon) \quad \text{for all } t\in(0,T_{max,\varepsilon}).\end{aligned} \tag{2.26}$$

In view of (2.26) and using the outcome of (2.24) with suitably large p as a starting point, we employ a Moser-type iteration (see e.g. Lemma A.1 of [32]) applied to the first equation of (2.5) and obtain some $C_{22}(\varepsilon) > 0$ such that

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C_{22}(\varepsilon) \quad \text{for all } t\in(\tau,T_{max,\varepsilon}) \tag{2.27}$$

with $\tau\in(0,T_{max,\varepsilon})$.

Assume that $T_{max,\varepsilon} < \infty$. In view of (2.21), (2.26) and (2.27), we apply Lemma 2.1 to reach a contradiction. □

3 A Quasi-energy Functional

In this section we establish some suitable ε-independent bounds for solutions to (2.5), which will be a starting point of a series of arguments. Next, in consequence of the space-time L^∞ estimate for c_ε contained in the latter, recalling (iv) of Lemma 2.3, we directly obtain the following result.

Lemma 3.1 *Let $m > 1$. There exists $\kappa_1 > 0$ such that for every $\varepsilon \in (0, 1)$*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \mu_0 \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 + \frac{3\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega \frac{n_\varepsilon |\nabla c_\varepsilon|^2}{c_\varepsilon} \\ &\quad \le -2\int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon + \frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)} \int_\Omega |\nabla u_\varepsilon|^2 + \kappa_1 \quad \textit{for all } \ t > 0, \end{aligned} \tag{3.1}$$

where μ_0 is the same as in (2.10).

Proof From the second equation in (2.5) we see

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} &= 2\int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla c_{\varepsilon t}}{c_\varepsilon} - \int_\Omega \frac{|\nabla c_\varepsilon|^2 c_{\varepsilon t}}{c_\varepsilon^2} \\ &= -2\int_\Omega \frac{\Delta c_\varepsilon c_{\varepsilon t}}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^2 c_{\varepsilon t}}{c_\varepsilon^2} \\ &= -2\int_\Omega \frac{|\Delta c_\varepsilon|^2}{c_\varepsilon} + 2\int_\Omega \Delta c_\varepsilon n_\varepsilon + 2\int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon \cdot \nabla c_\varepsilon) \\ &\quad + \int_\Omega \frac{|\nabla c_\varepsilon|^2 \Delta c_\varepsilon}{c_\varepsilon^2} - \int_\Omega \frac{|\nabla c_\varepsilon|^2 n_\varepsilon}{c_\varepsilon} - \int_\Omega \frac{|\nabla c_\varepsilon|^2 (u_\varepsilon \cdot \nabla c_\varepsilon)}{c_\varepsilon^2}. \end{aligned} \tag{3.2}$$

Together with (2.8), an application of (iv) in Lemma 2.3 yields

$$\begin{aligned} &-2\int_\Omega \frac{|\Delta c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^2 \Delta c_\varepsilon}{c_\varepsilon^2} \\ &\le -\mu_0 \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 - \mu_0 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C(\mu_0)\int_\Omega c_\varepsilon \\ &\le -\mu_0 \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 - \mu_0 \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C(\mu_0)\|c_\varepsilon\|_{L^\infty(\Omega)}|\Omega| \quad \text{for all } \ t > 0 \end{aligned} \tag{3.3}$$

with some positive constant $\mu_0 > 0$ and $C(\mu_0) > 0$. As to the terms containing u_ε, we note that for all $\varepsilon > 0$

$$\begin{aligned} &2\int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon \cdot \nabla c_\varepsilon) \\ &\quad = 2\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon \cdot \nabla c_\varepsilon) - 2\int_\Omega \frac{1}{c_\varepsilon}\nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \\ &\quad\quad - 2\int_\Omega \frac{1}{c_\varepsilon} u_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \quad \text{for all } \ t > 0 \end{aligned}$$

and by writing $\frac{\nabla c_\varepsilon}{c_\varepsilon^2} = \nabla(\frac{1}{c_\varepsilon})$ also

$$\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon \cdot \nabla c_\varepsilon) = 2\int_\Omega \frac{1}{c_\varepsilon} u_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \quad \text{for all } \ t > 0.$$

So that, due to the Young inequality and Lemma 2.2, we conclude that

$$
\begin{aligned}
&2\int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon)-\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)\\
&=-2\int_\Omega \frac{1}{c_\varepsilon}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon)\\
&\le \frac{\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{4}{\mu_0}\int_\Omega c_\varepsilon|\nabla u_\varepsilon|^2\\
&\le \frac{\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega|\nabla u_\varepsilon|^2 \ \text{ for all } \ t>0,
\end{aligned}
\tag{3.4}
$$

where $\mu_0>0$ is the same as in (2.10). Integrating by parts, we have

$$
2\int_\Omega \Delta c_\varepsilon n_\varepsilon=-2\int_\Omega \nabla n_\varepsilon\cdot\nabla c_\varepsilon. \tag{3.5}
$$

Finally, in light of (3.2)–(3.5), we can derive that (3.1) holds. □

In order to absorb the second integral on the right side hand of (3.1), it is necessary to gain the time evolution of $\int_\Omega |u_\varepsilon|^2$, which is the same as most-studied on the chemotaxis-fluid system (see e.g. [64]).

Lemma 3.2 *Let $m>1$. There exists $\kappa_2>0$ such that for any $\varepsilon\in(0,1)$, the solution of (2.5) satisfies*

$$
\frac{d}{dt}\int_\Omega |u_\varepsilon|^2+\int_\Omega|\nabla u_\varepsilon|^2\le\kappa_2\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{2N}{N+2}}(\Omega)} \ \textit{for all} \ t>0. \tag{3.6}
$$

Proof Firstly, multiplying the third equation in (2.5) by u_ε, integrating by parts and using $\nabla\cdot u_\varepsilon=0$, we have

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2+\int_\Omega|\nabla u_\varepsilon|^2=\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi \ \text{ for all } \ t>0. \tag{3.7}
$$

Here we use the Hölder inequality, (1.6) and the continuity of the embedding $W^{1,2}(\Omega)\hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ and find $C_1>0$ such that

$$
\begin{aligned}
\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\phi &\le \|\nabla\phi\|_{L^\infty(\Omega)}\|n_\varepsilon\|_{L^{\frac{2N}{N+2}}(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)}\\
&\le C_1\|n_\varepsilon\|_{L^{\frac{2N}{N+2}}(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)}\\
&\le C_1\|n_\varepsilon+\varepsilon\|_{L^{\frac{2N}{N+2}}(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)}\\
&\le \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}+\frac{1}{2}C_1^2\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{2N}{N+2}}(\Omega)} \ \text{ for all } \ t>0,
\end{aligned}
$$

which together with (3.7) entails (3.6) by choosing $\kappa_2=C_1^2$. □

Lemma 3.3 *Let $m > 1$ and S satisfy (1.2)–(1.3). Suppose that (1.6)–(1.7) hold. Then the solution of (2.5) satisfies*

$$\frac{d}{dt}\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2\right) + \frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 + \mu_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2 + \int_\Omega \frac{n_\varepsilon|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \le 2\int_\Omega |\nabla n_\varepsilon||\nabla c_\varepsilon| + \frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\kappa_2\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{2N}{N+2}}(\Omega)} + \kappa_1 \quad \text{for all } t > 0, \tag{3.8}$$

where μ_0 is as in (2.10).

Proof Taking an evident linear combination of the inequalities provided by Lemmas 3.1–3.2, and using the fact that $-2\int_\Omega \nabla n_\varepsilon \cdot \nabla c_\varepsilon \le 2\int_\Omega |\nabla n_\varepsilon||\nabla c_\varepsilon|$, it implies that (3.8) holds. □

Lemma 3.4 *For $T > 0$, there is $C > 0$ such that for each $\varepsilon \in (0, 1)$, the solution of (2.5) satisfies*

$$\int_\Omega c_\varepsilon^2 \le C \quad \text{for all } t > 0 \tag{3.9}$$

and

$$\int_0^T\int_\Omega |\nabla c_\varepsilon|^2 \le C(T+1) \quad \text{for all } T > 0. \tag{3.10}$$

Proof We multiply the second equation in (2.5) by c_ε to see that

$$\frac{1}{2}\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\nabla c_\varepsilon|^2 = -\int_\Omega n_\varepsilon c_\varepsilon^2 \le 0 \quad \text{for all } t > 0$$

by using $\nabla \cdot u_\varepsilon = 0$ and $n_\varepsilon c_\varepsilon^2 \ge 0$. From the above inequality, (3.9)–(3.10) immediately follows by integrating with respect to time. □

Next we can estimate the integrals on the right-hand sides of (3.8) by taking a totally different approach from [64]. In fact, different from [64], in this paper, we try to use the terms $\int_\Omega \frac{n_\varepsilon|\nabla c_\varepsilon|^2}{c_\varepsilon}$ and $\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2$ by using some careful analysis and a clever choose of $p > 1$, which will be a **new step and method** to solve the chemotaxis system.

Lemma 3.5 *Let $1 < m < 2$ and $N = 2, 3$. Moreover, assume that S satisfy (1.2)–(1.3). Suppose that (1.6)–(1.7) hold. Then for any $p \in (1, \min\{m, 3-m\})$, there exists $C > 0$ independent of ε such that the solution of (2.5) satisfies*

$$\int_\Omega (n_\varepsilon+\varepsilon)^p + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega |u_\varepsilon|^2 \le C \quad \text{for all } t > 0. \tag{3.11}$$

Moreover, for each $T > 0$, one can find a constant $C > 0$ independent of ε such that

$$\int_0^T\int_\Omega \left[\frac{n_\varepsilon+\varepsilon}{\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2 + (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\right] \le C(T+1), \tag{3.12}$$

$$\int_0^T\int_\Omega \left[|\nabla u_\varepsilon|^2 + \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}\right] \le C(T+1) \tag{3.13}$$

and

$$\int_0^T\int_\Omega |\nabla c_\varepsilon|^4 \le C(T+1) \tag{3.14}$$

as well as

$$\int_0^T \int_\Omega \left[c_\varepsilon |D^2 \ln c_\varepsilon|^2 + (n_\varepsilon + \varepsilon)^{m+p-1+\frac{2}{N}} \right] \le C(T+1). \tag{3.15}$$

Proof Since $1 < m < 2$ ensures that

$$1 < \min\{m, 3-m\},$$

one can fix

$$p \in (1, \min\{m, 3-m\}).$$

Therefore, (2.11) entails

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|n_\varepsilon + \varepsilon\|^p_{L^p(\Omega)} + \frac{m(p-1)}{2}\int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 \\ &\quad \le \frac{(p-1)C_S^2}{2m}\int_\Omega (n_\varepsilon + \varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2 \quad \text{for all } t > 0, \end{aligned} \tag{3.16}$$

which in light of (3.8) yields that

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2 + \frac{1}{p}\|n_\varepsilon + \varepsilon\|^p_{L^p(\Omega)}\right) \\ &\quad + \frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 + \mu_0 \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 + \int_\Omega \frac{n_\varepsilon |\nabla c_\varepsilon|^2}{c_\varepsilon} \\ &\quad + \frac{\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{m(p-1)}{2}\int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 \\ &\le \frac{(p-1)C_S^2}{2m}\int_\Omega (n_\varepsilon + \varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2 + 2\int_\Omega |\nabla n_\varepsilon||\nabla c_\varepsilon| \\ &\quad + \frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\kappa_2 \|n_\varepsilon + \varepsilon\|^2_{L^{\frac{2N}{N+2}}(\Omega)} \quad \text{for all } t > 0. \end{aligned} \tag{3.17}$$

In the following, we derive the estimates on the right-hand sides in (3.17) underlying an appropriate interpolation type inequality and basic estimates established in Sect. 2. Indeed, in view of the Young inequality, we have

$$\begin{aligned} &2\int_\Omega |\nabla n_\varepsilon||\nabla c_\varepsilon| \\ &\le \frac{m(p-1)}{4}\int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 + \frac{4}{m(p-1)}\int_\Omega (n_\varepsilon + \varepsilon)^{3-m-p}|\nabla c_\varepsilon|^2 \end{aligned} \tag{3.18}$$

for all $t > 0$. Inserting (3.18) into (3.17), we derive that

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2+\frac{1}{p}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}\right)\\
&\quad+\frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2+\mu_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2+\int_\Omega \frac{n_\varepsilon|\nabla c_\varepsilon|^2}{c_\varepsilon}\\
&\quad+\frac{\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{m(p-1)}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 \qquad (3.19)\\
&\le \frac{(p-1)C_S^2}{2m}\int_\Omega (n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2+\frac{4}{m(p-1)}\int_\Omega (n_\varepsilon+\varepsilon)^{3-m-p}|\nabla c_\varepsilon|^2\\
&\quad+\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\kappa_2\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{2N}{N+2}}(\Omega)} \quad \text{for all } t>0.
\end{aligned}
$$

Next, with the help of the Gagliardo–Nirenberg inequality and (2.7), we derive that there are $C_1>0$ and $C_2>0$ such that

$$
\begin{aligned}
&\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{2N}{N+2}}(\Omega)}\\
&=\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{4}{(m+p-1)}}_{L^{\frac{4N}{(N+2)(m+p-1)}}(\Omega)}\\
&\le C_1\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{2\frac{N-2}{N(m+p-2)+2}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{4}{m+p-1}-2\frac{N-2}{N(m+p-2)+2}}_{L^{\frac{2}{m+p-1}}(\Omega)}\\
&\quad+C_1\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{4}{(m+p-1)}}_{L^{\frac{2}{m+p-1}}(\Omega)}\\
&\le C_2(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{2\frac{N-2}{N(m+p-2)+2}}_{L^2(\Omega)}+1) \quad \text{for all } t>0.
\end{aligned}
$$

This combined with $m>1$ and $N=2,3$ implies that there exists a positive constant C_3 such that

$$
\begin{aligned}
&\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\kappa_2\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{2N}{N+2}}(\Omega)}\\
&\quad\le \frac{m(p-1)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2+C_3 \quad \text{for all } t>0
\end{aligned} \qquad (3.20)
$$

by using the Young inequality. Next, inserting (3.20) into (3.19) yields that

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2+\frac{1}{p}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}\right)\\
&\quad+\frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2+\mu_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2+\int_\Omega \frac{n_\varepsilon|\nabla c_\varepsilon|^2}{c_\varepsilon}\\
&\quad+\frac{\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{m(p-1)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\
&\le \int_\Omega \left[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p+1-m}+\frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{3-m-p}\right]|\nabla c_\varepsilon|^2\\
&\quad+C_3 \quad \text{for all } t>0,
\end{aligned}
$$

which together with (2.8) implies that

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2 + \frac{1}{p}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}\right)\\
&\quad + \frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 + \mu_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2\\
&\quad + \frac{\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{m(p-1)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\
&\le \int_\Omega \left[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p+1-m} + \frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{3-m-p} - \frac{n_\varepsilon}{\|c_0\|_{L^\infty(\Omega)}}\right]|\nabla c_\varepsilon|^2 + C_3\\
&\le \int_\Omega \left[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p+1-m} + \frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{3-m-p} - \frac{n_\varepsilon+\varepsilon}{\|c_0\|_{L^\infty(\Omega)}}\right]|\nabla c_\varepsilon|^2\\
&\quad + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{\|c_0\|_{L^\infty(\Omega)}} + C_3 \quad \text{for all } t>0
\end{aligned} \tag{3.21}$$

by using $\varepsilon \in (0,1)$. Therefore,

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2 + \frac{1}{p}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}\right)\\
&\quad + \frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 + \mu_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2\\
&\quad + \frac{\mu_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{m(p-1)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\
&\le \int_\Omega (n_\varepsilon+\varepsilon)\left[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p-m} + \frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{2-m-p} - \frac{1}{\|c_0\|_{L^\infty(\Omega)}}\right]|\nabla c_\varepsilon|^2\\
&\quad + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{\|c_0\|_{L^\infty(\Omega)}} + C_3 \quad \text{for all } t>0.
\end{aligned} \tag{3.22}$$

On the other hand, recalling $m \in (1,2)$ and $p \in (1, \min\{m, 3-m\})$, a direct computation shows

$$\lim_{s\to+\infty}\left[\frac{(p-1)C_S^2}{2m}(s+\varepsilon)^{p-m} + \frac{4}{m(p-1)}(s+\varepsilon)^{2-m-p}\right] = 0.$$

So that, there exists $\eta_0 > 0$, such that for any $s > \eta_0$,

$$\left[\frac{(p-1)C_S^2}{2m}(s+\varepsilon)^{p-m} + \frac{4}{m(p-1)}(s+\varepsilon)^{2-m-p}\right] < \frac{1}{2\|c_0\|_{L^\infty(\Omega)}}.$$

Therefore, by some basic calculations, we have

$$
\begin{aligned}
&\int_\Omega (n_\varepsilon+\varepsilon)[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p-m}+\frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{2-m-p}]|\nabla c_\varepsilon|^2\\
&\le \int_{n_\varepsilon>\eta_0}(n_\varepsilon+\varepsilon)[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p-m}+\frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{2-m-p}]|\nabla c_\varepsilon|^2\\
&\quad+\int_{n_\varepsilon\le\eta_0}(n_\varepsilon+\varepsilon)[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p-m}+\frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{2-m-p}]|\nabla c_\varepsilon|^2\\
&\le \int_{n_\varepsilon>\eta_0}\frac{n_\varepsilon+\varepsilon}{2\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2\\
&\quad+\int_{n_\varepsilon\le\eta_0}(n_\varepsilon+\varepsilon)[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p-m}+\frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{2-m-p}]|\nabla c_\varepsilon|^2\\
&\le \int_\Omega\frac{n_\varepsilon+\varepsilon}{2\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2\\
&\quad+\int_{n_\varepsilon\le\eta_0}[\frac{(p-1)C_S^2}{2m}(n_\varepsilon+\varepsilon)^{p+1-m}+\frac{4}{m(p-1)}(n_\varepsilon+\varepsilon)^{3-m-p}]|\nabla c_\varepsilon|^2\\
&\le \int_\Omega\frac{n_\varepsilon+\varepsilon}{2\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2+\gamma_0\int_{n_\varepsilon\le\eta_0}|\nabla c_\varepsilon|^2\\
&\le \int_\Omega\frac{n_\varepsilon+\varepsilon}{2\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2+\gamma_0\int_\Omega|\nabla c_\varepsilon|^2
\end{aligned}
\tag{3.23}
$$

with

$$
\gamma_0=\frac{(p-1)C_S^2}{2m}(\eta_0+1)^{p+1-m}+\frac{4}{m(p-1)}(\eta_0+1)^{3-m-p}
$$

by using $\varepsilon\in(0,1)$ as well as $m\in(1,2)$ and $p\in(1,\min\{m,3-m\})$. Substituting (3.23) into (3.22), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega|u_\varepsilon|^2+\frac{1}{p}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}\right)\\
&\quad+\frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega|\nabla u_\varepsilon|^2+\mu_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2\\
&\quad+\frac{\mu_0}{4}\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{m(p-1)}{8}\int_\Omega(n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2+\int_\Omega\frac{n_\varepsilon+\varepsilon}{2\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2\\
&\le(\gamma_0+\frac{1}{\|c_0\|_{L^\infty(\Omega)}})\int_\Omega|\nabla c_\varepsilon|^2+C_3\quad\text{for all }\ t>0.
\end{aligned}
\tag{3.24}
$$

Recalling (2.7), we derive from the Gagliardo–Nirenberg inequality that for some positive constants C_4 and C_5 such that

$$
\begin{aligned}
&\int_\Omega(n_\varepsilon+\varepsilon)^{m+p-1+\frac{2}{N}}\\
&=\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{(m+p-1+\frac{2}{N})\frac{2}{m+p-1}}_{L^{(m+p-1+\frac{2}{N})\frac{2}{m+p-1}}(\Omega)}\\
&\le C_4\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^2_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{(m+p-1+\frac{2}{N})\frac{2}{m+p-1}-2}_{L^{\frac{2}{m+p-1}}(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&+C_4\|(n_\varepsilon+\varepsilon)_\varepsilon^{\frac{m+p-1}{2}}\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{(m+p-1+\frac{2}{N})\frac{2}{m+p-1}}\\
&\le C_5(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2+1),
\end{aligned}
$$

which implies that there exist positive constants C_6 and C_7 such that

$$
\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|_{L^2(\Omega)}^2 \ge \frac{1}{C_6}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-1+\frac{2}{N}}-1 \ge \frac{1}{C_6}\int_\Omega (n_\varepsilon+\varepsilon)^p - C_7 \tag{3.25}
$$

by using $m>1$, $N=2,3$ and the Young inequality.

According to the Young inequality and the Poincaré inequality, (3.25) and (2.8), we conclude that with some $C_8>0$ and $C_9>0$, it follows

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2+\frac{1}{p}\|n_\varepsilon+\varepsilon\|_{L^p(\Omega)}^p\\
&\quad+\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-1+\frac{2}{N}}+\int_\Omega |\nabla c_\varepsilon|^4\\
&\quad\le C_8(\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2+\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\int_\Omega |\nabla u_\varepsilon|^2)+C_9 \quad \text{for all } t>0.
\end{aligned} \tag{3.26}
$$

Thus, we infer from (3.24) and (3.26) that there exist $C_{10}>0$ and $C_{11}>0$ such that for all $\varepsilon\in(0,1)$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2+\frac{1}{p}\|n_\varepsilon+\varepsilon\|_{L^p(\Omega)}^p\right)\\
&\quad+C_{10}\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2+\frac{1}{p}\|n_\varepsilon+\varepsilon\|_{L^p(\Omega)}^p\right)\\
&\quad+C_{11}\left(\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-1+\frac{2}{N}}+\int_\Omega |\nabla c_\varepsilon|^4\right)\\
&\quad+\frac{2}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2+\mu_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2\\
&\quad+\frac{\mu_0}{8}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{m(p-1)}{16}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2+\int_\Omega \frac{n_\varepsilon+\varepsilon}{2\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2\\
&\quad\le (\gamma_0+\frac{1}{\|c_0\|_{L^\infty(\Omega)}})\int_\Omega |\nabla c_\varepsilon|^2+C_{11} \quad \text{for all } t>0.
\end{aligned} \tag{3.27}
$$

Now, we define

$$
y_\varepsilon(t):=\left(\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2+\frac{2}{m+1}\|n_\varepsilon+\varepsilon\|_{L^p(\Omega)}^p\right)(\cdot,t) \quad \text{for all } t>0
$$

and

$$
\begin{aligned}
&h_\varepsilon(t)\\
&:= C_{11}\left(\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-1+\frac{2}{N}} + \int_\Omega |\nabla c_\varepsilon|^4\right)(\cdot,t)\\
&\quad+\left(\frac{2}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 + \mu_0\int_\Omega c_\varepsilon |D^2\ln c_\varepsilon|^2 + \int_\Omega \frac{n_\varepsilon|\nabla c_\varepsilon|^2}{c_\varepsilon}\right)(\cdot,t)\\
&\quad+\left(\frac{\mu_0}{8}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{m(p-1)}{16}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 + \int_\Omega \frac{n_\varepsilon+\varepsilon}{2\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2\right)(\cdot,t).
\end{aligned}
$$

For all $t>0$, (3.27) implies that y_ε satisfies

$$
y_\varepsilon'(t) + C_{10}y_\varepsilon(t) + h_\varepsilon(t) \le (\gamma_0 + \frac{1}{\|c_0\|_{L^\infty(\Omega)}})\int_\Omega |\nabla c_\varepsilon|^2 + C_{12} \text{ for all } t>0.
$$

Since $h_\varepsilon(t)\ge 0$ and $\int_t^{t+1}\left[(\gamma_0+\frac{1}{\|c_0\|_{L^\infty(\Omega)}})\int_\Omega |\nabla c_\varepsilon|^2 + C_{12}\right]$ is bounded, from (3.10) as well as (1.6) and Lemma 2.6, we firstly achieve

$$
\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{8}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega |u_\varepsilon|^2 + \frac{2}{m+1}\|n_\varepsilon+\varepsilon\|_{L^p(\Omega)}^p \le C_{13} \quad \text{for all } t>0,
$$

and thus proves (3.11) by using the fact that

$$
|\nabla c_\varepsilon|^2 \le \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}.
$$

Then another integration of (3.27) thereupon shows that (3.12)–(3.15) hold. □

When the nonlinear diffusion is strong enough, the energy type inequality is relatively easy. Actually, for the case of $m>2$, we have the following energy-type inequality by using the Young inequality and the second equation in (2.5).

Lemma 3.6 *Let $m>2$ and $N=2,3$. Then there exists $C>0$ independent of ε such that the solution of (2.5) satisfies*

$$
\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} \le C \quad \textit{for all } t>0. \tag{3.28}
$$

In addition, for any $T>0$, one can find a constant $C>0$ independent of ε such that

$$
\int_0^T\int_\Omega \left[(n_\varepsilon+\varepsilon)^{2m-2+\frac{2}{N}} + (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2\right] \le C(T+1) \quad \textit{for all } T>0. \tag{3.29}
$$

Proof Firstly, picking p as $m-1$ in (2.11), we arrive that

$$
\begin{aligned}
&\frac{1}{m-1}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|_{L^{m-1}(\Omega)}^{m-1} + \frac{m(m-2)}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2\\
&\quad\le \frac{(m-2)C_S^2}{2m}\int_\Omega |\nabla c_\varepsilon|^2 \quad \text{for all } t>0.
\end{aligned} \tag{3.30}
$$

Due to (2.7), one has for any $\varepsilon \in (0,1)$, $\|n_\varepsilon+\varepsilon\|_{L^1(\Omega)} \le \|n_0\|_{L^1(\Omega)}+|\Omega|$. Since $m>2$, by using the Gagliardo-Nirenberg inequality, we can find some positive C_2, C_3, C_4 and C_5 such that

$$\begin{aligned}
&\int_\Omega (n_\varepsilon+\varepsilon)^{m-1}\\
&= \|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{m-1}{m-1}}(\Omega)}^{\frac{m-1}{m-1}}\\
&\le C_2\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{N(m-2)}{1-\frac{N}{2}+N(m-1)}}\|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{m-1}{m-1}-\frac{N(m-2)}{1-\frac{N}{2}+N(m-1)}}+C_2\|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{m-1}{m-1}}\\
&\le C_3(\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|_{L^2(\Omega)}^{\frac{N(m-2)}{1-\frac{N}{2}+N(m-1)}}+1) \text{ for all } t>0
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega (n_\varepsilon+\varepsilon)^{2m-2+\frac{2}{N}}\\
&= \|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{2m-2+\frac{2}{N}}{m-1}}(\Omega)}^{\frac{2m-2+\frac{2}{N}}{m-1}}\\
&\le C_4\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|_{L^2(\Omega)}^2\|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{2m-2+\frac{2}{N}}{m-1}-2}\\
&\quad+C_4\|(n_\varepsilon+\varepsilon)^{m-1}\|_{L^{\frac{1}{m-1}}(\Omega)}^{\frac{2m-2+\frac{2}{N}}{m-1}}\\
&\le C_5(\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|_{L^2(\Omega)}^2+1) \text{ for all } t>0.
\end{aligned} \tag{3.31}$$

The fact $\frac{N(m-2)}{1-\frac{N}{2}+N(m-1)}<2$ enables us to use the Young inequality to deduce that there exists a positive constant C_6 such that

$$\begin{aligned}
\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} &\le \frac{m(m-2)}{8}\times\frac{1}{(m-1)^2}\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\|_{L^2(\Omega)}^2+C_6\\
&= \frac{m(m-2)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2+C_6 \text{ for all } t>0.
\end{aligned}$$

Substituting the above inequality and (3.31) into (3.30), we derive

$$\begin{aligned}
&\frac{1}{m-1}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|_{L^{m-1}(\Omega)}^{m-1}+\int_\Omega (n_\varepsilon+\varepsilon)^{m-1}\\
&\quad+\frac{m(m-2)}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2+C_7\int_\Omega (n_\varepsilon+\varepsilon)^{2m-2+\frac{2}{N}}\\
&\le \frac{(m-2)C_S^2}{2m}\int_\Omega |\nabla c_\varepsilon|^2+C_8 \text{ for all } t>0
\end{aligned} \tag{3.32}$$

with some positive constants C_7 and C_8. Recalling (3.10), in light of a basic calculation, this firstly entails (3.28). And thereafter, an integration of (3.32) yields (3.29). □

Lemma 3.7 *Let $m=2$ and $N=2,3$. Then there exists $C>0$ independent of ε such that the solution of (2.5) satisfies*

$$\int_\Omega n_\varepsilon \ln n_\varepsilon \le C \quad \text{for all } t>0. \tag{3.33}$$

In addition, for any $T > 0$, one can find a constant $C > 0$ independent of ε such that

$$\int_0^T \int_\Omega \left[n_\varepsilon^{2m-2+\frac{2}{N}} + |\nabla n_\varepsilon|^2 \right] \le C(T+1) \text{ for all } T > 0. \tag{3.34}$$

Proof Using the first equation of (2.5), from integration by parts we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon =& \int_\Omega n_{\varepsilon t} \ln n_\varepsilon + \int_\Omega n_{\varepsilon t} \\
=& \int_\Omega \Delta(n_\varepsilon+\varepsilon)^2 \ln n_\varepsilon - \int_\Omega \ln n_\varepsilon \nabla\cdot(n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon) \\
& - \int_\Omega \ln n_\varepsilon u_\varepsilon \cdot \nabla n_\varepsilon \\
\le& -2\int_\Omega \frac{(n_\varepsilon+\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega S_0(c_\varepsilon)|\nabla n_\varepsilon||\nabla c_\varepsilon| \\
\le& -2\int_\Omega |\nabla n_\varepsilon|^2 + C_S\int_\Omega |\nabla n_\varepsilon||\nabla c_\varepsilon| \\
\le& -\int_\Omega |\nabla n_\varepsilon|^2 + \frac{1}{4}C_S^2\int_\Omega |\nabla c_\varepsilon|^2 \text{ for all } t > 0
\end{aligned}$$

by using (2.6). Based on the elementary inequality $z\ln z \le \frac{3}{2}z^{\frac{5}{3}}$ for all $z \ge 0$, and from $\frac{2}{m} < \frac{10}{3m} < \frac{2N}{N-2}$ for $m = 2$, we apply the Gagliardo-Nirenberg inequality to obtain positive constants C_1 and C_2 independent of $\varepsilon \in (0,1)$ such that

$$\begin{aligned}
&\int_\Omega n_\varepsilon \ln n_\varepsilon \\
&\le \frac{3}{2}\int_\Omega n_\varepsilon^{\frac{5}{3}} \\
&= \frac{3}{2}\|n_\varepsilon\|_{L^{\frac{5}{3}}}^{\frac{5}{3}} \\
&\le C_1\|\nabla n_\varepsilon\|_{L^2(\Omega)}^{\frac{5}{3}\frac{N-\frac{3N}{5}}{1-\frac{N}{2}+N}} \|n_\varepsilon\|_{L^1(\Omega)}^{\frac{5}{3}-\frac{5}{3}\frac{N-\frac{3N}{5}}{1-\frac{N}{2}+N}} + C_1\|n_\varepsilon\|_{L^1(\Omega)}^{\frac{5}{3}} \\
&\le C_2(\|\nabla n_\varepsilon\|_{L^2(\Omega)}^{\frac{5}{3}\frac{N-\frac{3N}{5}}{1-\frac{N}{2}+N}} + 1) \text{ for all } t > 0.
\end{aligned}$$

Since $N \le 3$ also indicates that

$$0 < \frac{5}{3}\frac{N-\frac{3N}{5}}{1-\frac{N}{2}+N} < 2,$$

whence by means of the Young inequality we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 \\
&\le \frac{1}{4}C_S^2\int_\Omega |\nabla c_\varepsilon|^2 + C_3 \text{ for all } t > 0
\end{aligned} \tag{3.35}$$

with some positive constant C_3. Recalling $m = 2$, by using the Gagliardo-Nirenberg inequality and (2.7), we can find some positive constants C_4 and C_5 such that

$$\begin{aligned}\int_\Omega n_\varepsilon^{2m-2+\frac{2}{N}} &= \int_\Omega n_\varepsilon^{2+\frac{2}{N}} \\ &\le C_4\|\nabla n_\varepsilon\|_{L^2(\Omega)}^2 \|n_\varepsilon\|_{L^1(\Omega)}^{\frac{2}{N}} + C_4\|n_\varepsilon\|_{L^1(\Omega)}^{2+\frac{2}{N}} \\ &\le C_5(\|\nabla n_\varepsilon\|_{L^2(\Omega)}^2 + 1) \quad \text{for all } t > 0,\end{aligned}$$

which combined with (3.35) implies that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{4C_5}\int_\Omega n_\varepsilon^{2m-2+\frac{2}{N}} + \frac{1}{4}\int_\Omega |\nabla n_\varepsilon|^2 \\ &\le \frac{1}{4}C_S^2 \int_\Omega |\nabla c_\varepsilon|^2 + C_6 \quad \text{for all } t > 0\end{aligned} \tag{3.36}$$

with some $C_6 > 0$. According to Lemma 3.4, there exists $C_7 > 0$ such that $\int_t^{t+1}[\frac{1}{4}C_S^2\int_\Omega |\nabla c_\varepsilon|^2 + C_6] \le C_7$ for all $t > 0$. Thanks to Lemma 2.6, it derives (3.33), and then (3.34) follows by integrating (3.36) in time. This completes the proof of Lemma 3.7. □

Lemma 3.8 *Let $m \ge 2$ and $N = 2, 3$. Then there exists $C > 0$ such that the solution of (2.5) satisfies*

$$\int_\Omega [|u_\varepsilon|^2 + |\nabla c_\varepsilon|^2] \le C \quad \textit{for all } t > 0. \tag{3.37}$$

Moreover, for any $T > 0$, it holds that

$$\int_0^T \int_\Omega \left[|\nabla c_\varepsilon|^4 + |\nabla u_\varepsilon|^2 + |\Delta c_\varepsilon|^2\right] \le C(T+1). \tag{3.38}$$

Proof We multiply the second equation in (2.5) by $-\Delta c_\varepsilon$ and integrate by parts to see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla c_\varepsilon\|_{L^2(\Omega)}^2 + \int_\Omega |\Delta c_\varepsilon|^2 \\ &= \int_\Omega n_\varepsilon c_\varepsilon \Delta c_\varepsilon + \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon \\ &= \int_\Omega n_\varepsilon c_\varepsilon \Delta c_\varepsilon - \int_\Omega \nabla c_\varepsilon \cdot \nabla(u_\varepsilon \cdot \nabla c_\varepsilon) \\ &= \int_\Omega n_\varepsilon c_\varepsilon \Delta c_\varepsilon - \int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon),\end{aligned} \tag{3.39}$$

where we have used the fact that

$$\int_\Omega \nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot u_\varepsilon) = \frac{1}{2}\int_\Omega u_\varepsilon \cdot \nabla|\nabla c_\varepsilon|^2 = 0 \quad \text{for all } t > 0.$$

On the other hand, we make use of Lemma 2.2 and the Young inequality to derive

$$\int_\Omega n_\varepsilon c_\varepsilon \Delta c_\varepsilon \le C_1^2 \int_\Omega n_\varepsilon^2 + \frac{1}{4}\int_\Omega |\Delta c_\varepsilon|^2 \quad \text{for all } t > 0, \tag{3.40}$$

with some positive constant C_1. In the last equation in (3.39), we use the Cauchy-Schwarz inequality to obtain

$$-\int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \le \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^4(\Omega)}^2 \quad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Now thanks to Lemma 2.2 and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\nabla c_\varepsilon\|^2_{L^4(\Omega)} \le & C_2\|\Delta c_\varepsilon\|_{L^2(\Omega)}\|c_\varepsilon\|_{L^\infty(\Omega)} + C_2\|c_\varepsilon\|^2_{L^\infty(\Omega)} \\ \le & C_3\|\Delta c_\varepsilon\|_{L^2(\Omega)} + C_3 \quad \text{for all } t > 0\end{aligned} \tag{3.41}$$

with some positive constants $C_2 > 0$ and $C_3 > 0$. Thus, we use the Young inequality to derive

$$\begin{aligned}& -\int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \\ & \le \|\nabla u_\varepsilon\|_{L^2(\Omega)}[C_3\|\Delta c_\varepsilon\|_{L^2(\Omega)} + C_3] \\ & \le C_4\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{4}\|\Delta c_\varepsilon\|^2_{L^2(\Omega)} + C_4 \quad \text{for all } t > 0\end{aligned} \tag{3.42}$$

with some positive constant C_4. Inserting (3.40) and (3.42) into (3.39), we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2 \le C_4\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_1^2\int_\Omega n_\varepsilon^2 + C_5. \tag{3.43}$$

Apart from that, (3.41) and the Young inequality also guarantee the existence of C_6 such that

$$\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla c_\varepsilon\|^4_{L^4(\Omega)} \le C_6\|\Delta c_\varepsilon\|^2_{L^2(\Omega)} + C_6 \quad \text{for all } t > 0,$$

which along with (3.43) implies that there is $C_7 > 0$ such that

$$\begin{aligned}& \frac{1}{2}\frac{d}{dt}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{4}\int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{4C_6}\|\nabla c_\varepsilon\|^4_{L^4(\Omega)} + \frac{1}{4C_6}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} \\ & \le C_4\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_1^2\int_\Omega n_\varepsilon^2 + C_7 \quad \text{for all } t > 0.\end{aligned} \tag{3.44}$$

Now, we try to analyze the evolution of $\int_\Omega |u_\varepsilon|^2$, which contributes to absorbing $\|\nabla u_\varepsilon\|^2_{L^2(\Omega)}$ on the right-hand side of (3.44). To this end, multiplying the third equation of (2.5) by u_ε, integrating by parts and using $\nabla \cdot u_\varepsilon = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\phi \quad \text{for all } t > 0.$$

Recalling the Poincaré inequality we can find a constant $C_\Omega > 0$ fulfilling

$$\|\psi\|^2_{L^2(\Omega)} \le C_\Omega\|\nabla\psi\|^2_{L^2(\Omega)} \quad \text{for all } \psi \in W_0^{1,2}(\Omega).$$

Then the Young inequality along with the assumed boundedness of $\nabla\phi$ yields

$$\begin{aligned}& \frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 + \frac{1}{4C_\Omega}\int_\Omega |u_\varepsilon|^2 \\ & \le \frac{1}{4C_\Omega}\int_\Omega |u_\varepsilon|^2 + \frac{1}{4C_\Omega}\int_\Omega |u_\varepsilon|^2 + C_\Omega\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \\ & \le \frac{1}{2}\int_\Omega |\nabla u_\varepsilon|^2 + C_\Omega\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \quad \text{for all } t > 0.\end{aligned} \tag{3.45}$$

Noticing that $m \ge 2$ and $N \le 3$ imply $2m - 2 + \frac{2}{N} \ge 2$, then by using Lemma 2.6, Lemma 3.6 and Lemma 3.7, we obtain that there is a positive constant C_8 such that

$$\int_\Omega u_\varepsilon^2 \le C_8 \quad \text{for all } t > 0.$$

Therefore, integration of (3.45) entails that

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \le C_9 \ \text{ for all } \ T > 0$$

with some $C_9 > 0$. This combined with Lemma 3.6 and Lemma 3.7 implies that

$$\int_0^T [C_4 \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_1^2 \int_\Omega n_\varepsilon^2 + C_7] \le C_{10}(T+1) \ \text{ for all } \ T > 0.$$

Thereupon an integration of (3.44) yields for some C_{11}

$$\int_\Omega |\nabla c_\varepsilon|^2 \le C_{11} \quad \text{for all } \ t > 0$$

and

$$\int_0^T \int_\Omega \left[|\Delta c_\varepsilon|^2 + |\nabla c_\varepsilon|^4\right] \le C_{11}(T+1) \ \text{ for all } \ T > 0.$$

□

Now we can obtain the staring point through the following lemma.

Lemma 3.9 *Let $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ be the solution of (2.5) and $m \ge 2$. Then there exists a positive constant C such that*

$$\sup_{t\in(0,\infty)} \int_\Omega (n_\varepsilon + \varepsilon)^{3m-3+\frac{2}{N}} \le C. \tag{3.46}$$

Proof Choosing p as $3m - 3 + \frac{2}{N}$ in (2.11), implies that

$$\begin{aligned} &\frac{1}{3m-3+\frac{2}{N}} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|^{3m-3+\frac{2}{N}}_{L^{3m-3+\frac{2}{N}}(\Omega)} \\ &\quad + \frac{m(3m-4+\frac{2}{N})}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{m+3m-3+\frac{2}{N}-3} |\nabla n_\varepsilon|^2 \\ &\le \frac{(3m-4+\frac{2}{N})C_S^2}{2m} \int_\Omega (n_\varepsilon+\varepsilon)^{3m-3+\frac{2}{N}+1-m} |\nabla c_\varepsilon|^2 \quad \text{for all } \ t > 0. \end{aligned} \tag{3.47}$$

Due to (2.7), one has for any $\varepsilon \in (0, 1)$, $\|n_\varepsilon + \varepsilon\|_{L^1(\Omega)} \le \|n_0\|_{L^1(\Omega)} + |\Omega|$. Therefore, an application of the Gagliardo-Nirenberg inequality yields that there are positive constants $C_i (i = 1, 2, 3, 4)$ such that

$$\begin{aligned} &\int_\Omega (n_\varepsilon + \varepsilon)^{3m-3+\frac{2}{N}} \\ &= \|(n_\varepsilon + \varepsilon)^{2m-2+\frac{1}{N}}\|^{\frac{3m-3+\frac{2}{N}}{2m-2+\frac{1}{N}}}_{L^{\frac{3m-3+\frac{2}{N}}{2m-2+\frac{1}{N}}}(\Omega)} \\ &\le C_1 \|\nabla (n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|^{\frac{N(3m-4+\frac{2}{N})}{1-\frac{N}{2}+N(2m-2+\frac{1}{N})}}_{L^2(\Omega)} \|(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|^{\frac{3m-3+\frac{2}{N}}{2m-2+\frac{1}{N}} - \frac{N(3m-4+\frac{2}{N})}{1-\frac{N}{2}+N(2m-2+\frac{1}{N})}}_{L^{\frac{1}{2m-2+\frac{1}{N}}}(\Omega)} \\ &\quad + C_1 \|(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|^{\frac{3m-3+\frac{2}{N}}{2m-2+\frac{1}{N}}}_{L^{\frac{1}{2m-2+\frac{1}{N}}}(\Omega)} \end{aligned}$$

$$\leq C_2(\|\nabla(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|_{L^2(\Omega)}^{\frac{N(3m-4+\frac{2}{N})}{1-\frac{N}{2}+N(2m-2+\frac{1}{N})}}+1) \text{ for all } t>0$$

and

$$\begin{aligned}
&\int_\Omega (n_\varepsilon+\varepsilon)^{4m-4+\frac{4}{N}}\\
&=\|(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|_{L^{\frac{4m-4+\frac{4}{N}}{2m-2+\frac{1}{N}}}(\Omega)}^{\frac{4m-4+\frac{4}{N}}{2m-2+\frac{1}{N}}}\\
&\leq C_3\|\nabla(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|_{L^2(\Omega)}^2\|(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|_{L^{\frac{1}{2m-2+\frac{1}{N}}}(\Omega)}^{\frac{4m-4+\frac{4}{N}}{2m-2+\frac{1}{N}}-2}\\
&\quad+C_3\|(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|_{L^{\frac{1}{2m-2+\frac{1}{N}}}(\Omega)}^{\frac{4m-4+\frac{4}{N}}{2m-2+\frac{1}{N}}}\\
&\leq C_4(\|\nabla(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|_{L^2(\Omega)}^2+1) \text{ for all } t>0.
\end{aligned} \tag{3.48}$$

Apart from these, (3.48) also implies that

$$\|\nabla(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|_{L^2(\Omega)}^2+1\geq\frac{1}{C_4}\int_\Omega(n_\varepsilon+\varepsilon)^{4m-4+\frac{4}{N}} \text{ for all } t>0,$$

which in view of the Young inequality implies that

$$\begin{aligned}
&\frac{(3m-4+\frac{2}{N})C_S^2}{2m}\int_\Omega(n_\varepsilon+\varepsilon)^{3m-3+\frac{2}{N}+1-m}|\nabla c_\varepsilon|^2\\
&\leq\frac{1}{2C_4}\times\frac{1}{(2m-2+\frac{1}{N})^2}\times\frac{m(3m-4+\frac{2}{N})}{8}\int_\Omega(n_\varepsilon+\varepsilon)^{4m-4+\frac{4}{N}}\\
&\quad+C_5\int_\Omega|\nabla c_\varepsilon|^4\\
&\leq\frac{1}{(2m-2+\frac{1}{N})^2}\times\frac{m(3m-4+\frac{2}{N})}{8}[\|\nabla(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|_{L^2(\Omega)}^2+1]\\
&\quad+C_5\int_\Omega|\nabla c_\varepsilon|^4\\
&=\frac{m(3m-4+\frac{2}{N})}{8}\int_\Omega(n_\varepsilon+\varepsilon)^{m+3m-3+\frac{2}{N}-3}|\nabla n_\varepsilon|^2\\
&\quad+\frac{1}{(2m-2+\frac{1}{N})^2}\times\frac{m(3m-4+\frac{2}{N})}{8}+C_5\int_\Omega|\nabla c_\varepsilon|^4 \text{ for all } t>0
\end{aligned} \tag{3.49}$$

with some $C_5>0$. Due to $m\geq 2$, we observe that

$$\frac{N(3m-4+\frac{2}{N})}{1-\frac{N}{2}+N(2m-2+\frac{1}{N})}<2,$$

and thus by another application of the Young inequality we obtain that there is $C_6 > 0$ such that

$$\begin{aligned}
&\int_\Omega (n_\varepsilon+\varepsilon)^{3m-3+\frac{2}{N}}\\
&\le \frac{m(3m-4+\frac{2}{N})}{8}\times\frac{1}{(2m-2+\frac{1}{N})^2}\|\nabla(n_\varepsilon+\varepsilon)^{2m-2+\frac{1}{N}}\|^2_{L^2(\Omega)}+C_6\\
&=\frac{m(3m-4+\frac{2}{N})}{8}\int_\Omega (n_\varepsilon+\varepsilon)^{m+3m-3+\frac{2}{N}-3}|\nabla n_\varepsilon|^2+C_6 \ \text{ for all } t>0.
\end{aligned}$$

Substituting the above inequality and (3.49) into (3.47), we have

$$\begin{aligned}
&\frac{1}{3m-3+\frac{2}{N}}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^{3m-3+\frac{2}{N}}_{L^{3m-3+\frac{2}{N}}(\Omega)}+\int_\Omega (n_\varepsilon+\varepsilon)^{3m-3+\frac{2}{N}}\\
&+\frac{m(3m-4+\frac{2}{N})}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{m+3m-3+\frac{2}{N}-3}|\nabla n_\varepsilon|^2\\
&\le C_5\int_\Omega |\nabla c_\varepsilon|^4+C_7 \quad \text{for all } t>0
\end{aligned}$$

with some positive constant C_7. This combined with (3.38) and Lemma 2.6 yields that there exists a positive constant C_8 such that

$$\int_\Omega (n_\varepsilon+\varepsilon)^{3m-3+\frac{2}{N}} \le C_8 \quad \text{for all } t>0. \tag{3.50}$$

□

Lemma 3.10 *Let $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ be the solution of (2.5) and $m > 1$. Then there are $C > 0$ and $p_0 > 1$ such that*

$$\sup_{t\in(0,\infty)}\int_\Omega (n_\varepsilon+\varepsilon)^{p_0}\le C. \tag{3.51}$$

Proof Let

$$p_0=\begin{cases} p \ \text{ if } \ 1<m<2\\ 3m-3+\frac{2}{N} \ \text{ if } \ m\ge 2,\end{cases}$$

where $p\in(1,\min\{m,3-m\})$ is the same as that in (3.11). Then an elementary computation shows that $p_0 > 1$, so that, (3.50) and (3.11) entails (3.51). □

Lemma 3.11 *Let $m \ge 2$ and $N = 2, 3$. Then for each $T > 0$, one can find a constant $C > 0$ independent of ε such that the solution of (2.5) satisfies*

$$\int_0^T\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}|\nabla c_\varepsilon|^2\le C(T+1). \tag{3.52}$$

Proof Recalling (3.1), we integrate by parts to derive

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\mu_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2+\frac{3\mu_0}{4}\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\int_\Omega\frac{n_\varepsilon|\nabla c_\varepsilon|^2}{c_\varepsilon}\\
&\le 2\int_\Omega n_\varepsilon\Delta c_\varepsilon+\frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega|\nabla u_\varepsilon|^2+\kappa_1\\
&\le \int_\Omega n_\varepsilon^2+\int_\Omega|\Delta c_\varepsilon|^2+\frac{4}{\mu_0}\|c_0\|_{L^\infty(\Omega)}\int_\Omega|\nabla u_\varepsilon|^2+\kappa_1 \quad \text{for all } t>0
\end{aligned} \tag{3.53}$$

by using the Young inequality. Next, $m \geq 2$ entails that

$$3m - 3 + \frac{2}{N} \geq 2,$$

so that, using (3.46) and (3.38), we derive that

$$\int_0^T \int_\Omega \left[n_\varepsilon^2 + |\Delta c_\varepsilon|^2 + \frac{4}{\mu_0} \|c_0\|_{L^\infty(\Omega)} |\nabla u_\varepsilon|^2 + \kappa_1 \right] \leq C(T+1) \quad \text{for all} \quad T > 0.$$

Then the result of (3.52) can be obtained by an integration of (3.53) and using (1.6). □

4 Boundedness for the Case $N = 2$

In this subsection, we obtain some regularity properties for n_ε, c_ε, and u_ε in the following form on the basis of Lemma 3.5.

Lemma 4.1 *Let $m > 1$ and $N = 2$. Then there exists $C > 0$ independent of ε such that the solution of (2.5) satisfies*

$$\int_\Omega |\nabla u_\varepsilon|^2 \leq C \text{ for all } t > 0. \tag{4.1}$$

Proof In view of (3.11), from $D(1 + \varepsilon A) := W^{2,2}(\Omega) \cap W_{0,\sigma}^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, there are positive constants C_1 and C_2 such that

$$\|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)} = \|(I + \varepsilon A)^{-1} u_\varepsilon\|_{L^\infty(\Omega)} \leq C_1 \|u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_2 \text{ for all } t > 0. \tag{4.2}$$

Next, testing the projected Navier-Stokes equation $u_{\varepsilon t} + Au_\varepsilon = \mathcal{P}[-\kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon + n_\varepsilon \nabla\phi]$ by Au_ε, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}} u_\varepsilon\|_{L^2(\Omega)}^2 + \int_\Omega |Au_\varepsilon|^2 \\
&= \int_\Omega Au_\varepsilon \mathcal{P}(-\kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon) + \int_\Omega \mathcal{P}(n_\varepsilon \nabla\phi) Au_\varepsilon \\
&\leq \frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + \kappa^2 \int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon|^2 + \|\nabla\phi\|_{L^\infty(\Omega)}^2 \int_\Omega n_\varepsilon^2 \text{ for all } t > 0.
\end{aligned} \tag{4.3}$$

In view of the Young inequality and (4.2), there is $C_3 > 0$ such that

$$\begin{aligned}
\kappa^2 \int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon|^2 &\leq \kappa^2 \|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla u_\varepsilon|^2 \\
&\leq C_3 \int_\Omega |\nabla u_\varepsilon|^2 \text{ for all } t > 0,
\end{aligned}$$

which together with (4.3) and the fact that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$ (see Theorem 2.1.1 of [30]) yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 + \int_\Omega |\Delta u_\varepsilon|^2 \leq C_4 \int_\Omega |\nabla u_\varepsilon|^2 + \|\nabla\phi\|_{L^\infty(\Omega)}^2 \int_\Omega n_\varepsilon^2 \text{ for all } t > 0 \tag{4.4}$$

with some C_4. Let

$$q_0 = \begin{cases} 2m - 1 & \text{if } m \geq 2, \\ m + p & \text{if } 1 < m < 2, \end{cases}$$

where $p \in (1, \min\{m, 3-m\})$ is the same as that in (3.11). For any $m > 1$ ensures that

$$q_0 > 2,$$

therefore, in view of Lemmas 3.5–3.7 and the Young inequality, (4.4) directly leads to (4.1) by performing some basic calculations. □

Lemma 4.2 *Let $m > 1$ and $N = 2$. Then there exists a positive constant C independent of ε such that the solution of (2.5) from Lemma 2.1 satisfies*

$$\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t > 0$$

as well as

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t > 0$$

and

$$\|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t > 0,$$

where γ is the same as that in (1.6).

Proof Now, involving the variation-of-constants formula for c_ε and applying $\nabla \cdot u_\varepsilon = 0$ in $x \in \Omega, t > 0$, we have

$$c_\varepsilon(\cdot, t) = e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-s)(\Delta-1)}(-n_\varepsilon(\cdot, s)c_\varepsilon(\cdot, s) + c_\varepsilon(\cdot, s) + \nabla\cdot(u_\varepsilon(\cdot, s)c_\varepsilon(\cdot, s))ds, \ \ t > 0.$$

So that, for any $2 < q < \min\{\frac{2p_0}{(2-p_0)_+}, 4\}$, where $p_0 > 1$ is the same as (3.51), there is

$$\begin{aligned} &\|\nabla c_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \\ &\leq \|\nabla e^{t(\Delta-1)}c_0\|_{L^q(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)}[-n_\varepsilon(\cdot, s)c_\varepsilon(\cdot, s) + c_\varepsilon(\cdot, s)]\|_{L^q(\Omega)}ds \\ &+ \int_0^t \|\nabla e^{(t-s)(\Delta-1)}\nabla\cdot(u_\varepsilon(\cdot, s)c_\varepsilon(\cdot, s))\|_{L^q(\Omega)}ds. \end{aligned} \tag{4.5}$$

To address the right-hand side of (4.5), in view of (1.6), it can be derived through the standard L^p-L^q estimates on Neumann heat semigroup (see Lemma 1.3 of [43])

$$\|\nabla e^{t(\Delta-1)}c_0\|_{L^q(\Omega)} \leq C_5 \text{ for all } t > 0$$

with some positive constant C_5. Since

$$-\frac{1}{2} - \frac{2}{2}\left(\frac{1}{p_0} - \frac{1}{q}\right) > -1,$$

recalling (3.51) and (2.8), we deduce from the standard L^p-L^q estimates on Neumann heat semigroup that there are $\lambda_1 > 0$, $C_6 > 0$ and $C_7 > 0$ such that

$$\begin{aligned} &\int_0^t \|\nabla e^{(t-s)(\Delta-1)}[-n_\varepsilon(\cdot, s)c_\varepsilon(\cdot, s) + c_\varepsilon(\cdot, s)]\|_{L^q(\Omega)}ds \\ &\leq C_6 \int_0^t [1 + (t-s)^{-\frac{1}{2}-\frac{2}{2}(\frac{1}{p_0}-\frac{1}{q})}]e^{-\lambda_1(t-s)}\|c_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}[\|n_\varepsilon(\cdot, s)\|_{L^{p_0}(\Omega)} + 1]ds \\ &\leq C_6\|c_0\|_{L^\infty(\Omega)} \int_0^t [1 + (t-s)^{-\frac{1}{2}-\frac{2}{2}(\frac{1}{p_0}-\frac{1}{q})}]e^{-\lambda_1(t-s)}[\|n_\varepsilon(\cdot, s)\|_{L^{p_0}(\Omega)} + 1]ds \\ &\leq C_7 \text{ for all } t > 0. \end{aligned}$$

Now, we estimate the third term on the right-hand side of (4.5). In fact, since, $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for any $p > 1$, the boundedness of $\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ (see (4.1)) as well as $\|\nabla c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ (see (3.11) and (3.37)) and $\|c_\varepsilon(\cdot,t)\|_{L^2(\Omega)}$ (see (3.9)) yields that there exists a positive constant C_8 such that

$$\|u_\varepsilon(\cdot,t)\|_{L^{16}(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{L^{16}(\Omega)} \le C_8 \text{ for all } t > 0.$$

Pick $0 < \iota < \frac{1}{2}$ satisfying $\frac{1}{2} + \frac{2}{2}(\frac{1}{8} - \frac{1}{4}) < \iota$ and $\tilde{\kappa} \in (0, \frac{1}{2} - \iota)$. In light of Hölder's inequality, we derive from the standard L^p-L^q estimates on Neumann heat semigroup that there exist constants λ_2, C_9, C_{10}, as well as C_{11} and C_{12} such that

$$\begin{aligned}
&\int_0^t \|\nabla e^{(t-s)(\Delta-1)}\nabla\cdot(u_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s))\|_{L^q(\Omega)}ds \\
&\le C_9 \int_0^t \|(-\Delta+1)^\iota e^{(t-s)(\Delta-1)}\nabla\cdot(u_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s))\|_{L^4(\Omega)}ds \\
&\le C_{10} \int_0^t (t-s)^{-\iota-\frac{1}{2}-\tilde{\kappa}} e^{-\lambda_2(t-s)} \|u_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)\|_{L^8(\Omega)}ds \\
&\le C_{11} \int_0^t (t-s)^{-\iota-\frac{1}{2}-\tilde{\kappa}} e^{-\lambda_2(t-s)} \|u_\varepsilon(\cdot,s)\|_{L^8(\Omega)}\|c_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds \\
&\le C_{11}\|c_0\|_{L^\infty(\Omega)} \int_0^t (t-s)^{-\iota-\frac{1}{2}-\tilde{\kappa}} e^{-\lambda_2(t-s)} \|u_\varepsilon(\cdot,s)\|_{L^8(\Omega)}ds \\
&\le C_{12} \text{ for all } t > 0
\end{aligned}$$

by using (2.8). Combining the above estimates, we obtain a positive constant C_{13} such that

$$\int_\Omega |\nabla c_\varepsilon(x,t)|^q \le C_{13} \text{ for all } t > 0 \text{ and some } q \in \left(2, \min\left\{\frac{2p_0}{(2-p_0)_+}, 4\right\}\right). \tag{4.6}$$

For any $p > 2 + m$, in view of (2.11) and (4.6), we derive from the Hölder inequality that there exists a positive constant C_{14} such that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)} + \frac{m(p-1)}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 + \|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)} \\
&\le \frac{(p-1)C_S^2}{2m}\int_\Omega (n_\varepsilon+\varepsilon)^{p+1-m}|\nabla c_\varepsilon|^2 + \|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)} \\
&\le \frac{(p-1)C_S^2}{2m}\left(\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{q}{q-2}(p+1-m)}\right)^{\frac{q-2}{q}}\left(\int_\Omega |\nabla c_\varepsilon|^q\right)^{\frac{2}{q}} + \|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)} \\
&\le \frac{(p-1)C_S^2C_{14}}{2m}\left(\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{q}{q-2}(p+1-m)}\right)^{\frac{q-2}{q}} + \|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)} \text{ for all } t > 0,
\end{aligned}$$

where C_S is the same as that in (2.9). Recalling (2.7), we employ the Gagliardo-Nirenberg inequality and find positive constants $C_{15} > 0$, $C_{16} > 0$ as well as $C_{17} > 0$ and $C_{18} > 0$

satisfying

$$\begin{aligned}&\frac{(p-1)C_S^2C_{14}}{2m}\left(\int_\Omega (n_\varepsilon+\varepsilon)^{\frac{q}{q-2}(p+1-m)}\right)^{\frac{q-2}{q}}\\&=\frac{(p-1)C_S^2C_{14}}{2m}\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(p+1-m)}{m+p-1}}_{L^{\frac{2q(p+1-m)}{(q-2)(m+p-1)}}(\Omega)}\\&\le C_{15}\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{2\frac{p-m+\frac{2}{q}}{m+p-1}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(p+1-m)}{m+p-1}-2\frac{p-m+\frac{2}{q}}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)}\\&\quad+C_{15}\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{2(p+1-m)}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)}\\&\le C_{16}(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{2\frac{p-m+\frac{2}{q}}{m+p-1}}_{L^2(\Omega)}+1)\ \text{ for all }\ t>0\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega (n_\varepsilon+\varepsilon)^p\\&=\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{2p}{m+p-1}}_{L^{\frac{2p}{m+p-1}}(\Omega)}\\&\le C_{17}\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{2\frac{p-1}{m+p-1}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{2p}{m+p-1}-2\frac{p-1}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)}\\&\quad+C_{17}\|(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{\frac{2p}{m+p-1}}_{L^{\frac{2}{m+p-1}}(\Omega)}\\&\le C_{18}(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+p-1}{2}}\|^{2\frac{p-1}{m+p-1}}_{L^2(\Omega)}+1)\ \text{ for all }\ t>0.\end{aligned}$$

Since $p>2+m$ as well as $q>2$ and $m>1$, we see that

$$2\frac{p-m+\frac{2}{q}}{m+p-1}<2\quad\text{and}\quad 2\frac{p-1}{m+p-1}<2,$$

which allow for an application of the Young inequality to entail some positive constant C_{19} such that

$$\frac{1}{p}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}+\|n_\varepsilon+\varepsilon\|^p_{L^p(\Omega)}\le C_{19}\ \text{ for all }\ t>0.$$

By a comparison argument, this in particular entails that there is $C_{20}>0$ such that

$$\int_\Omega (n_\varepsilon+\varepsilon)^p\le C_{20}\quad\text{for all }\ t>0\quad\text{and}\quad p>2. \tag{4.7}$$

Fix $\gamma\in(\frac{1}{2},1)$ and define

$$M(T):=\sup_{t\in(0,T)}\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\quad\text{for all }\ T>0.$$

Let $t_0:=(t-1)_+$ for any $t\in(0,T)$, then from the variation-of-constants formula of u_ε and the regularization estimates on Stokes semigroup ([11]), one can find $C_{21}>0$ and $\lambda_3>0$

fulfilling

$$\begin{aligned}
&\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \\
&\le \|A^\gamma e^{-(t-t_0)A} u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)} + \int_{t_0}^t \|A^\gamma e^{-(t-\tau)A} h_\varepsilon(\cdot,\tau)d\tau\|_{L^2(\Omega)} d\tau \\
&\le \|A^\gamma u_0\|_{L^2(\Omega)} + C_{21}\int_{t_0}^t (t-\tau)^{-\gamma-\frac{2}{2}(\frac{1}{2}-\frac{1}{2})} e^{-\lambda_3(t-\tau)}\|h_\varepsilon(\cdot,\tau)\|_{L^2(\Omega)} d\tau,
\end{aligned} \tag{4.8}$$

where $h_\varepsilon(\cdot,\tau) = \mathcal{P}[n_\varepsilon(\cdot,\tau)\nabla\phi - \kappa(Y_\varepsilon u_\varepsilon(\cdot,\tau)\cdot\nabla)u_\varepsilon(\cdot,\tau)]$. If $t \in (0,1]$, then by (1.6), there is C_{22} such that

$$\|A^\gamma e^{-(t-t_0)A} u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)} = \|A^\gamma e^{-A} u_0\|_{L^2(\Omega)} \le C_{22}.$$

Whereas if $t > 1$, due to $t - t_0 = 1$ and using the boundedness of $\int_\Omega |\nabla u_\varepsilon|^2$ (see (4.1)), we have

$$\|A^\gamma e^{-(t-t_0)A} u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)} \le C_{23}(t-t_0)^{-\gamma}\|u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)} \le C_{24} \tag{4.9}$$

with $C_{23} > 0$ and $C_{24} > 0$. In the following we will estimate $\|h_\varepsilon(\cdot,\tau)\|_{L^2(\Omega)}$. Choose $\beta \in (\frac{1}{2},\gamma)$. Then we have the embedding $D(A^\beta) \hookrightarrow L^\infty(\Omega)$ (see [11]). Thus, there exist $C_{25} > 0$, $C_{26} > 0$ and $C_{27} > 0$ such that

$$\begin{aligned}
\|h_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le\ & C_{25}\|(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + C_{25}\|n_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \\
\le\ & C_{26}\|u_\varepsilon\|_{L^\infty(\Omega)}\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + C_{26} \\
\le\ & C_{27}\|A^\beta u_\varepsilon\|_{L^2(\Omega)}\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + C_{26} \quad \text{for all } t > 0.
\end{aligned} \tag{4.10}$$

On the other hand, recalling (4.1), then by using the interpolation between $D(A^\gamma)$ and $D(A^{\frac{1}{2}})$ (see [10]), we have $C_{28} > 0$ and $C_{29} > 0$ such that

$$\|A^\beta u_\varepsilon\|_{L^2(\Omega)} \le C_{28}\|A^\gamma u_\varepsilon\|^a_{L^2(\Omega)}\|\nabla u_\varepsilon\|^{1-a}_{L^2(\Omega)} \le C_{29}M^a(T)$$

with $a = \frac{2\beta-1}{2\gamma-1} \in (0,1)$. This together with (4.10) and (4.1) implies that there exists some $C_{30} > 0$ such that

$$\|h_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_{30}M^a(T) + C_{26} \quad \text{for all } t \in (0,T). \tag{4.11}$$

Due to $\gamma < 1$, $t - t_0 \le 1$, (4.8)–(4.9) and (4.11), we have

$$\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_{31} + C_{32}M^a(T)$$

for some positive constants C_{31} and C_{32}. Since $t \in (0,T)$ is arbitrary, we further have

$$M(T) \le C_{31} + C_{32}M^a(T).$$

Then a standard ODE comparison argument implies that there is $C_{33} > 0$ such that

$$\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_{33} \quad \text{for all } t > 0,$$

which combined with the fact that $D(A^\gamma)$ is continuously embedded into $L^\infty(\Omega)$ implies that for some positive constant C_{34} such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_{34} \quad \text{for all } t > 0. \tag{4.12}$$

To prove the boundedness of $\|\nabla c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$, we rewrite the variation-of-constants formula for c_ε in the form

$$c_\varepsilon(\cdot,t) = e^{t(\Delta-1)}c_0 + \int_0^t e^{(t-s)(\Delta-1)}[-n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s) + c_\varepsilon(\cdot,s) - u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s)]ds \text{ for all } t>0.$$

Thanks to $q>2$ (by $2<q<\min\{\frac{2p_0}{(2-p_0)_+},4\}$), one has

$$\frac{1}{2}+\frac{2}{2q}<1.$$

So that, one can pick

$$\theta\in(\frac{1}{2}+\frac{2}{2q},1),$$

and by $N=2$, one can derive that $D((-\Delta+1)^\theta)\hookrightarrow W^{1,\infty}(\Omega)$ (see [14]). Therefore, in light of L^p–L^q estimates associated with the heat semigroup, (4.7) as well as (4.12) and (2.8), we derive that there exist positive constants λ_4, C_{35}, C_{36}, C_{37}, and C_{38} such that

$$\begin{aligned}
&\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}\\
&\le C_{35}\|(-\Delta+1)^\theta c_\varepsilon(\cdot,t)\|_{L^q(\Omega)}\\
&\le C_{36}t^{-\theta}e^{-\lambda_4 t}\|c_0\|_{L^q(\Omega)}\\
&\quad + C_{36}\int_0^t (t-s)^{-\theta}e^{-\lambda_4(t-s)}\|-n_\varepsilon(\cdot,s)c_\varepsilon(\cdot,s)+c_\varepsilon(\cdot,s)-u_\varepsilon(\cdot,s)\cdot\nabla c_\varepsilon(\cdot,s)\|_{L^q(\Omega)}ds\\
&\le C_{37}+C_{37}\int_0^t (t-s)^{-\theta}e^{-\lambda_4(t-s)}\|c_0(\cdot,s)\|_{L^\infty(\Omega)}(\|n_\varepsilon(\cdot,s)\|_{L^q(\Omega)}+1)ds\\
&\quad + C_{37}\int_0^t (t-s)^{-\theta}e^{-\lambda_4(t-s)}\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c_\varepsilon(\cdot,s)\|_{L^q(\Omega)}ds\\
&\le C_{38} \text{ for all } t>0.
\end{aligned} \tag{4.13}$$

Next, using the outcome of (4.7) with suitably large p as a starting point, recalling the boundedness of $\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}$ (see (4.13)), we may invoke Lemma A.1 in [32] which by means of a Moser-type iteration applied to the first equation in (2.5) and establish

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C_{39} \quad \text{for all} \quad t>0$$

with some positive constant C_{39}. The proof of Lemma 4.2 is thus completed. □

Now we can establish global existence and boundedness in the approximate problem (2.5) by using Lemma 4.2 and an idea of [60] (see also [24, 47]).

Proof of Theorem 1.1. Firstly, according to the standard parabolic regularity theory (see e.g. Theorem IV.5.3 of [19]) to the second equation and third equation in system (2.5), there exists a positive constant C_ϵ such that

$$\|c_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])}\le C_\epsilon \quad \text{for all} \quad t\in(0,\infty) \tag{4.14}$$

and

$$\|u_\varepsilon(\cdot,t)\|_{C^{\mu,\frac{\mu}{2}}(\Omega\times[t,t+1])}\le C_\epsilon \quad \text{for all} \quad t\in(0,\infty).$$

Next, employing the same arguments as that in the proof of Lemmas 3.22–3.23 in [47], and taking advantages of Lemma 4.2, we conclude that for all $T > 0$ and $\varepsilon \in (0, 1)$, there exists $C(T)$ independent of ε such that

$$\int_0^T \|\partial_t (n_\varepsilon + \varepsilon)^\varsigma(\cdot, t)\|_{(W_0^{2,2}(\Omega))^*} dt \le C(T) \ \text{ for all } \ t \in (0, T)$$

and

$$\int_0^T \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^\varsigma|^2 \le C(T) \ \text{ for all } \ t \in (0, T) \tag{4.15}$$

with $\varsigma > \max\{m, 2(m-1)\}$. Then combined with (4.14)–(4.15) as well as Lemma 4.2, the Aubin-Lions compactness lemma (see e.g. Simon [29]) and the Egorov theorem, one can derive the existence of a sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ such that

$$\begin{aligned} &n_\varepsilon \rightharpoonup n \ \text{ weakly star in } \ L^\infty(\Omega \times (0, \infty)), \\ &n_\varepsilon \to n \ \text{ in } \ C^0_{loc}([0, \infty); (W_0^{2,2}(\Omega))^*), \\ &c_\varepsilon \to c \ \text{ in } \ C^0_{loc}(\bar{\Omega} \times [0, \infty)), \\ &n_\varepsilon \to n \ \text{ a.e. in } \ \Omega \times (0, \infty) \end{aligned}$$

as well as

$$u_\varepsilon \to u \ \text{ in } \ C^0_{loc}(\bar{\Omega} \times [0, \infty))$$

and

$$Du_\varepsilon \rightharpoonup Du \ \text{ weakly star in } \ L^\infty(\Omega \times (0, \infty))$$

hold for some limit $(n, c, u) \in (L^\infty(\Omega \times (0, \infty)))^4$ with nonnegative n and c. Based on the above convergence properties, we can pass to the limit in each term of weak formulation for (2.5) to construct a global weak solution of (1.1). Finally, the boundedness of (n, c, u) may result from the boundedness of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ (see Lemma 4.2) and the Banach-Alaoglu theorem. This completes the proof of Theorem 1.1. □

5 Further ε-independent Estimates on (2.5) in the Case $N = 3$

In order to pass to limits in (2.5) with safety in the case $N = 3$, we need some more ε-independent estimates for the solution. Indeed, by means of the interpolation, the estimates from Lemma 3.5 imply bounds for further spatio-temporal integrals.

Lemma 5.1 *Let $m > 1$ and $N = 3$. Then there exists a positive constant C such that the solution of (2.5) satisfies*

$$\int_\Omega (n_\varepsilon + \varepsilon)^m \le C \quad \text{for all } \ t > 0 \tag{5.1}$$

as well as

$$\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 \le C \quad \text{for all } \ T > 0 \tag{5.2}$$

and

$$\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-1+\frac{2}{N}m} \le C \quad \text{for all } \ T > 0. \tag{5.3}$$

Proof Picking $p = m$ in (2.11), one has

$$\begin{aligned} &\frac{1}{m}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^m_{L^m(\Omega)} + \frac{m(m-1)}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2 \\ &\quad\le \frac{(m-1)C_S^2}{2m}\int_\Omega (n_\varepsilon+\varepsilon)|\nabla c_\varepsilon|^2 \end{aligned} \tag{5.4}$$

for all $t > 0$. In the following, we will estimate the right-side of (5.4). Recalling (2.7), in light of the Gagliardo-Nirenberg inequality, there exist positive constants C_1, C_2 and C_3 independent of $\varepsilon \in (0, 1)$ such that

$$\begin{aligned} &\int_\Omega (n_\varepsilon+\varepsilon)^m \\ &= \|(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^{\frac{2m}{2m-1}}_{L^{\frac{2m}{2m-1}}(\Omega)} \\ &\le C_1(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^{\frac{3m-3}{3m-2}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^{\frac{2m}{2m-1}-\frac{3m-3}{3m-2}}_{L^{\frac{2}{2m-1}}(\Omega)} + \|(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^{\frac{2m}{2m-1}}_{L^{\frac{2}{2m-1}}(\Omega)}) \\ &\le C_2(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^{\frac{3m-3}{3m-2}}_{L^2(\Omega)} + 1) \\ &\le \frac{m(m-1)}{2}\times\frac{2}{(2m-1)^2}\|\nabla(n_\varepsilon+\varepsilon)^{\frac{2m-1}{2}}\|^2_{L^2(\Omega)} + C_3, \end{aligned}$$

where in the last inequality we have used the Young inequality. Inserting the above inequality into (5.4), one has some positive constant C_4 such that

$$\begin{aligned} &\frac{1}{m}\frac{d}{dt}\|n_\varepsilon+\varepsilon\|^m_{L^m(\Omega)} + \frac{m(m-1)}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2 + \int_\Omega (n_\varepsilon+\varepsilon)^m \\ &\quad\le \frac{(m-1)C_S^2}{2m}\int_\Omega (n_\varepsilon+\varepsilon)|\nabla c_\varepsilon|^2 + C_4 \ \text{ for all } \ t > 0. \end{aligned} \tag{5.5}$$

In the case when $m \ge 2$, by virtue of $\varepsilon \in (0, 1)$, (3.10) as well as (3.52) and (2.8), we can find $C_5 > 0$ such that for all $T > 0$,

$$\begin{aligned} &\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)|\nabla c_\varepsilon|^2 \\ &\quad\le \int_0^T\int_\Omega \frac{n_\varepsilon+1}{c_\varepsilon}c_\varepsilon|\nabla c_\varepsilon|^2 \\ &\quad\le \|c_0\|_{L^\infty(\Omega)}\int_0^T\int_\Omega \frac{n_\varepsilon}{c_\varepsilon}|\nabla c_\varepsilon|^2 + \|c_0\|_{L^\infty(\Omega)}\int_0^T\int_\Omega |\nabla c_\varepsilon|^2 \\ &\quad\le C_5(T+1). \end{aligned} \tag{5.6}$$

Whereas for $1 < m < 2$, by means of (3.12) we derive

$$\begin{aligned} \int_0^T\int_\Omega (n_\varepsilon+\varepsilon)|\nabla c_\varepsilon|^2 &\le \|c_0\|_{L^\infty(\Omega)}\int_0^T\int_\Omega \frac{n_\varepsilon+\varepsilon}{\|c_0\|_{L^\infty(\Omega)}}|\nabla c_\varepsilon|^2 \\ &\le C_6(T+1) \quad \text{for all } \ T > 0 \end{aligned}$$

with some $C_6 > 0$. This combining with (5.5)–(5.6) yields (5.1)–(5.2) by means of an ODE comparison argument. In view of (5.1), the Gagliardo-Nirenberg inequality entails that there exist $C_7 > 0$, $C_8 > 0$ and $C_9 > 0$ such that

$$
\begin{aligned}
&\int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{m+m-1+\frac{2}{N}m} ds \\
&= \int_0^T \|(n_\varepsilon+\varepsilon)^{\frac{m+m-1}{2}}\|^{\frac{2(m+m-1+\frac{2}{N}m)}{m+m-1}}_{L^{\frac{2(m+m-1+\frac{2}{N}m)}{m+m-1}}(\Omega)} ds \\
&\le C_7 \int_0^T \|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+m-1}{2}}\|^2_{L^2(\Omega)} \|(n_\varepsilon+\varepsilon)^{\frac{m+m-1}{2}}\|^{\frac{2(m+m-1+\frac{2}{N}m)}{m+m-1}-2}_{L^{\frac{2m}{m+m-1}}(\Omega)} ds \\
&\quad + C_7 \int_0^T \|(n_\varepsilon+\varepsilon)^{\frac{m+m-1}{2}}\|^{\frac{2(m+m-1+\frac{2}{N}m)}{m+m-1}}_{L^{\frac{2m}{m+m-1}}(\Omega)} ds \\
&\le C_8 (\int_0^T \|\nabla(n_\varepsilon+\varepsilon)^{\frac{m+m-1}{2}}\|^2_{L^2(\Omega)} ds + 1) \\
&\le C_9 (T+1) \text{ for all } T > 0.
\end{aligned} \tag{5.7}
$$

The proof is completed. ☐

Next, an application of Lemma 5.1 also enables us to get a higher order regularity of n_ε and u_ε in the case $N = 3$.

Lemma 5.2 *Let $m > 1$ and $N = 3$. Then there exists a positive constant C such that the solution of (2.5) satisfies*

$$
\int_0^T \int_\Omega |\nabla(n_\varepsilon+\varepsilon)^m|^{\frac{8m-3}{4m}} \le C \quad \textit{for all } T > 0
$$

and

$$
\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} \le C(T+1) \ \textit{for all } T > 0. \tag{5.8}
$$

Proof Recalling $N = 3$, then the Young inequality, (5.2) and (5.7) enable us to obtain that

$$
\begin{aligned}
&\int_0^T \int_\Omega |\nabla(n_\varepsilon+\varepsilon)^m|^{\frac{8m-3}{4m}} \\
&= m^{\frac{8m-3}{4m}} \int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{\frac{(m-1)(8m-3)}{4m}} |\nabla(n_\varepsilon+\varepsilon)|^{\frac{8m-3}{4m}} \\
&= m^{\frac{8m-3}{4m}} \int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{\frac{(m-1)(8m-3)}{4m} - \frac{(2m-3)(8m-3)}{8m}} (n_\varepsilon+\varepsilon)^{\frac{(2m-3)(8m-3)}{8m}} |\nabla(n_\varepsilon+\varepsilon)|^{\frac{8m-3}{4m}} \\
&= m^{\frac{8m-3}{4m}} \int_0^T \int_\Omega (n_\varepsilon+\varepsilon)^{\frac{(8m-3)}{8m}} (n_\varepsilon+\varepsilon)^{\frac{(2m-3)(8m-3)}{8m}} |\nabla(n_\varepsilon+\varepsilon)|^{\frac{8m-3}{4m}} \\
&\le C_1 \int_0^T [\int_\Omega (n_\varepsilon+\varepsilon)^{2m-3} |\nabla n_\varepsilon|^2 + \int_\Omega (n_\varepsilon+\varepsilon)^{m+m-1+\frac{2}{N}m}] \\
&\le C_2 (T+1) \text{ for all } T > 0
\end{aligned}
$$

with some positive constants C_1 and C_2. Finally, due to (3.11) and (3.13), employing the Hölder inequality and the Gagliardo-Nirenberg inequality, we conclude that there exist positive constants C_3 and C_4 such that

$$
\begin{aligned}
\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} &= \int_0^T \|u_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}} \\
&\le C_3 \int_0^T \left(\|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 \|u_\varepsilon\|_{L^2(\Omega)}^{\frac{4}{3}} + \|u_\varepsilon\|_{L^2(\Omega)}^{\frac{10}{3}} \right) \\
&\le C_4(T+1) \text{ for all } T > 0.
\end{aligned}
$$

The proof is completed. □

6 Regularity Properties of Time Derivatives

In preparation of an Aubin-Lions type compactness argument, besides the ε-independent estimates derived before (see Lemma 3.5 and Lemma 5.2), the time regularity is also indispensable.

Lemma 6.1 *Let $m > 1$ and $N = 3$. Assume that (1.6) and (1.7) hold. Then for any $T > 0$, one can find $C > 0$ independent of ε such that*

$$
\int_0^T \|\partial_t (n_\varepsilon + \varepsilon)^m (\cdot, t)\|_{(W^{2,q}(\Omega))^*} dt \le C(T+1) \tag{6.1}
$$

as well as

$$
\int_0^T \|\partial_t c_\varepsilon(\cdot, t)\|_{(W^{1,\frac{5}{2}}(\Omega))^*}^{\frac{5}{3}} dt \le C(T+1) \tag{6.2}
$$

and

$$
\int_0^T \|\partial_t u_\varepsilon(\cdot, t)\|_{(W_{0,\sigma}^{1,\frac{5}{2}}(\Omega))^*}^{\frac{5}{3}} dt \le C(T+1). \tag{6.3}
$$

Proof Fix $t > 0$. Multiplying the first equation in (2.5) by $m(n_\varepsilon + \varepsilon)^{m-1}\varphi \in C^\infty(\bar{\Omega})$, it follows

$$
\begin{aligned}
&\left| \int_\Omega [(n_\varepsilon + \varepsilon)^m]_t \varphi \right| \\
&= \left| \int_\Omega \left[\Delta (n_\varepsilon + \varepsilon)^m - \nabla \cdot (n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) - u_\varepsilon \cdot \nabla n_\varepsilon \right] \cdot m (n_\varepsilon + \varepsilon)^{m-1} \varphi \right| \\
&\le \left| -m^2 \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} (n_\varepsilon + \varepsilon)^{m-1} \nabla n_\varepsilon \cdot \nabla \varphi \right| \\
&\quad + \left| -m^2 (m-1) \int_\Omega (n_\varepsilon + \varepsilon)^{m-1} (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 \varphi \right|
\end{aligned}
$$

$$
\begin{aligned}
&+m\left|\int_\Omega F_\varepsilon(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)(m-1)n_\varepsilon(n_\varepsilon+\varepsilon)^{m-2}\nabla n_\varepsilon\cdot\nabla c_\varepsilon\varphi\right|\\
&+m\left|\int_\Omega F_\varepsilon(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)n_\varepsilon(n_\varepsilon+\varepsilon)^{m-1}\nabla c_\varepsilon\cdot\nabla\varphi\right|+\left|\int_\Omega(n_\varepsilon+\varepsilon)^m u_\varepsilon\cdot\nabla\varphi\right|\\
&\le m^2\left\{\int_\Omega(n_\varepsilon+\varepsilon)^{m-1}(n_\varepsilon+\varepsilon)^{m-1}|\nabla n_\varepsilon|\right\}\|\varphi\|_{W^{1,\infty}(\Omega)}\\
&+m^3\left\{\int_\Omega(n_\varepsilon+\varepsilon)^{m-1}(n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2\right\}\|\varphi\|_{W^{1,\infty}(\Omega)}\\
&+\left\{\int_\Omega m^2C_S(n_\varepsilon+\varepsilon)^{m-1}|\nabla n_\varepsilon||\nabla c_\varepsilon|\right\}\|\varphi\|_{W^{1,\infty}(\Omega)}\\
&+\left\{\int_\Omega[C_Sm(n_\varepsilon+\varepsilon)^m|\nabla c_\varepsilon|+(n_\varepsilon+\varepsilon)^m|u_\varepsilon|]\right\}\|\varphi\|_{W^{1,\infty}(\Omega)}.
\end{aligned}
$$

Due to the embedding $W^{2,q}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ for $q>3$, we deduce from the Young inequality that there exist positive constants C_1 and C_2 such that

$$
\begin{aligned}
&\int_0^T\|\partial_t n_\varepsilon^m(\cdot,t)\|_{(W^{2,q}(\Omega))^*}dt\\
&\le C_1\left\{\int_0^T\int_\Omega(n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2+\int_0^T\int_\Omega(n_\varepsilon+\varepsilon)^{2m-1}+\int_0^T\int_\Omega(n_\varepsilon+\varepsilon)|\nabla c_\varepsilon|^2\right\}\\
&\quad+C_1\left\{\int_0^T\int_\Omega|\nabla c_\varepsilon|^4+\int_0^T\int_\Omega(n_\varepsilon+\varepsilon)^{\frac{4}{3}m}+\int_0^T\int_\Omega(n_\varepsilon+\varepsilon)^{\frac{10}{7}m}+\int_0^T\int_\Omega|u_\varepsilon|^{\frac{10}{3}}\right\}\\
&\le C_2\left\{\int_0^T\int_\Omega(n_\varepsilon+\varepsilon)^{2m-3}|\nabla n_\varepsilon|^2+\int_0^T\int_\Omega|\nabla c_\varepsilon|^4+\int_0^T\int_\Omega(n_\varepsilon+\varepsilon)^{\frac{8m}{3}-1}\right\}\\
&\quad+C_2\left\{\int_0^T\int_\Omega|u_\varepsilon|^{\frac{10}{3}}+\int_0^T\int_\Omega(n_\varepsilon+\varepsilon)|\nabla c_\varepsilon|^2+T\right\}\quad\text{for all }\ T>0.
\end{aligned}
$$

According to the bounds provided by Lemma 3.5 and Lemma 5.2, it readily yields (6.1). For any chosen $\varphi\in C^\infty(\bar{\Omega})$, we use it to test n_ε-equation in (2.5) and use (2.8) to obtain

$$
\begin{aligned}
&\left|\int_\Omega\partial_t c_\varepsilon(\cdot,t)\varphi\right|\\
&=\left|\int_\Omega[\Delta c_\varepsilon-c_\varepsilon+n_\varepsilon-u_\varepsilon\cdot\nabla c_\varepsilon]\cdot\varphi\right|\\
&=\left|-\int_\Omega\nabla c_\varepsilon\cdot\nabla\varphi-\int_\Omega n_\varepsilon c_\varepsilon\varphi+\int_\Omega c_\varepsilon u_\varepsilon\cdot\nabla\varphi\right| \qquad (6.4)\\
&\le\left\{\|\nabla c_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}+\|n_\varepsilon c_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}+\|n_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}+\|c_\varepsilon u_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}\right\}\|\varphi\|_{W^{1,\frac{5}{2}}(\Omega)}\\
&\le\left\{\|\nabla c_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}+\|c_0\|_{L^\infty(\Omega)}\|n_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}+\|c_0\|_{L^\infty(\Omega)}\|u_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}\right\}\|\varphi\|_{W^{1,\frac{5}{2}}(\Omega)}
\end{aligned}
$$

for all $t > 0$. In view of the Young inequality and $m > 1$, (6.4) implies that there exists $C_3 > 0$ fulfilling

$$\int_0^T \|\partial_t c_\varepsilon(\cdot,t)\|^{\frac{5}{3}}_{(W^{1,\frac{5}{2}}(\Omega))^*} dt$$
$$\le C_3 \left(\int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{8m}{3}-1} + \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} + T \right) \quad \text{for all } T > 0,$$

which combined with Lemma 3.5 and Lemma 5.2 implies (6.2).

Finally, for the proof of (6.3), we pick $t > 0$ and multiply the third equation in (2.5) by an arbitrary solenoidal $\varphi \in C^\infty_{0,\sigma}(\Omega;\mathbb{R}^3)$. Then by using the Hölder inequality, we obtain

$$\left| \int_\Omega \partial_t u_\varepsilon(\cdot,t)\varphi \right|$$
$$= \left| - \int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi - \kappa \int_\Omega (Y_\varepsilon u_\varepsilon \otimes u_\varepsilon) \cdot \nabla\varphi + \int_\Omega n_\varepsilon \nabla\phi \cdot \varphi \right|$$
$$\le \left\{ \|\nabla u_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} + |\kappa| \|Y_\varepsilon u_\varepsilon \otimes u_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)} + \|n_\varepsilon \nabla\phi\|_{L^{\frac{5}{3}}(\Omega)} \right\} \|\varphi\|_{W^{1,\frac{5}{2}}(\Omega)} \quad \text{for all } t > 0. \tag{6.5}$$

Since $\|Y_\varepsilon v\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}$ for all $v \in L^2_\sigma(\Omega)$, together with (1.7) and the Young inequality, (6.5) further implies that there exist positive constants C_4 and C_5 such that

$$\int_0^T \|\partial_t u_\varepsilon(\cdot,t)\|^{\frac{5}{3}}_{(W^{1,\frac{5}{2}}_{0,\sigma}(\Omega))^*} dt$$
$$\le C_4 \left(\int_0^T \int_\Omega |\nabla u_\varepsilon|^{\frac{5}{3}} + \int_0^T \int_\Omega |Y_\varepsilon u_\varepsilon \otimes u_\varepsilon|^{\frac{5}{3}} + \int_0^T \int_\Omega n_\varepsilon^{\frac{5}{3}} \right)$$
$$\le C_5 \left(\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega |Y_\varepsilon u_\varepsilon|^2 + \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{8m}{3}-1} + T \right) \quad \text{for all } T > 0.$$

This combined with the outcome of Lemma 3.5 and Lemma 5.2, we immediately obtain (6.3). □

In order to guarantee the pointwise convergence for each component of the approximate solution, some further estimates on $n_\varepsilon u_\varepsilon$, $u_\varepsilon \cdot \nabla c_\varepsilon$ and $n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon$ are also needed.

Lemma 6.2 *Let $m > 1$ and $N = 3$, and suppose that (1.6) and (1.7) hold. Then for any $T > 0$, one can find $C > 0$ independent of ε such that*

$$\int_0^T \int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{10(8m-3)}{3(8m+7)}} \le C(T+1) \tag{6.6}$$

as well as

$$\int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^{\frac{20}{11}} \le C(T+1) \tag{6.7}$$

and

$$\int_0^T \int_\Omega |n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon|^{\frac{4(8m-3)}{9+8m}} \le C(T+1). \tag{6.8}$$

Proof In light of (2.9), (3.14), (5.3), (5.8) and the Young inequality, we derive that there exist positive constants C_6, C_7 and C_8 such that

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon u_\varepsilon|^{\frac{10(8m-3)}{3(8m+7)}} &\le \left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^4\right)^{\frac{5(8m-3)}{6(8m+7)}}\left(\int_0^T\int_\Omega n_\varepsilon^{\frac{8m-3}{3}}\right)^{\frac{12}{8m+9}}\\ &\le C_6(T+1) \ \text{ for all } \ T>0,\\ \int_0^T\int_\Omega |u_\varepsilon\cdot\nabla c_\varepsilon|^{\frac{20}{11}} &\le \left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^4\right)^{\frac{5}{11}}\left(\int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}}\right)^{\frac{6}{11}}\\ &\le C_7(T+1) \ \text{ for all } \ T>0\end{aligned}$$

and

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon|^{\frac{4(8m-3)}{9+8m}} &\le \left(\int_0^T\int_\Omega |\nabla c_\varepsilon|^4\right)^{\frac{8m-3}{9+8m}}\left(\int_0^T\int_\Omega n_\varepsilon^{\frac{8m-3}{3}}\right)^{\frac{12}{9+8m}}\\ &\le C_8(T+1) \ \text{ for all } \ T>0.\end{aligned}$$

These already establish (6.6)–(6.8). □

7 Passing to the Limit. Proof of Theorem 1.2

Now, let us take the limit of the approximate solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)_{\varepsilon\in(0,1)}$, and prove that the limit functions of each component make up the solution of the problem (1.1) in the sense of Definition 2.1.

We are now in the position to extract a suitable sequence of numbers ε along which the respective solutions approach a limit in convenient topologies.

Proof of Theorem 1.2 First, based on Lemma 5.2, we see that $(n_\varepsilon+\varepsilon)^m_{\varepsilon\in(0,1)}$ is bounded in $L_{loc}^{\frac{8m-3}{4m}}([0,\infty); W^{1,\frac{8m-3}{4m}}(\Omega))$, whereas $\partial_t(n_\varepsilon+\varepsilon)^m$ is bounded in $L^1_{loc}([0,\infty);(W^{2,q}(\Omega))^*)$ thanks to Lemma 6.1. Therefore, a variant of the Aubin-Lions lemma ([29]) asserts that $(n_\varepsilon+\varepsilon)^m_{\varepsilon\in(0,1)}$ is relatively compact in $L_{loc}^{\frac{8m-3}{4m}}(\bar{\Omega}\times[0,\infty))$ with respect to the strong topology therein. Thus, one can choose $\varepsilon=\varepsilon_j\subset(0,1)_{j\in\mathbb{N}}$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and $(n_\varepsilon+\varepsilon)^m\to z_1^m$, and hence $n_\varepsilon\to z_1$ a.e. in $\Omega\times(0,\infty)$ for some nonnegative measurable $z_1:\Omega\times(0,\infty)\to\mathbb{R}$. Now, with the help of the Egorov theorem, we conclude that necessarily $z_1=n$, thus

$$n_\varepsilon\to n \ \text{ a.e. in } \ \Omega\times(0,\infty). \tag{7.1}$$

Therefore, due to (5.3)–(5.8), $\frac{8m-3}{4m}>1$ and $\frac{8m-3}{3}>1$, there exists a subsequence $\varepsilon=\varepsilon_j\subset(0,1)_{j\in\mathbb{N}}$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$

$$(n_\varepsilon+\varepsilon)^{m-1}\nabla n_\varepsilon \rightharpoonup n^{m-1}\nabla n \ \text{ in } \ L_{loc}^{\frac{8m-3}{4m}}(\bar{\Omega}\times[0,\infty)) \tag{7.2}$$

and

$$n_\varepsilon\rightharpoonup n \ \text{ in } \ L_{loc}^{\frac{8m-3}{3}}(\bar{\Omega}\times[0,\infty)). \tag{7.3}$$

Likewise, Lemma 3.5, Lemma 5.2 and Lemma 6.1 also imply that there is $C_1>0$ such that

$$\|c_\varepsilon\|_{L^2_{loc}([0,\infty);W^{1,2}(\Omega))}\le C_1(T+1), \qquad \|\partial_t c_\varepsilon\|_{L_{loc}^{\frac{5}{3}}([0,\infty);(W^{1,\frac{5}{2}}(\Omega))^*)}\le C_1(T+1)$$

and

$$\|u_\varepsilon\|_{L^2_{loc}([0,\infty);W^{1,2}(\Omega))} \le C_1(T+1), \qquad \|\partial_t u_\varepsilon\|_{L^{\frac{5}{3}}_{loc}([0,\infty);(W^{1,\frac{5}{2}}_{0,\sigma}(\Omega))^*)} \le C_1(T+1).$$

In view of the embeddings $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (W^{1,\frac{5}{2}}(\Omega)))^*$, we can again infer from the Aubin-Lions lemma ([29]) that there indeed exist $\varepsilon = \varepsilon_j \subset (0,1)_{j\in\mathbb{N}}$ and the limit functions c and u such that

$$c_\varepsilon \to c \text{ in } L^2_{loc}(\bar{\Omega}\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty), \tag{7.4}$$

$$u_\varepsilon \to u \text{ in } L^2_{loc}(\bar{\Omega}\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty) \tag{7.5}$$

as well as

$$\nabla c_\varepsilon \rightharpoonup \nabla c \text{ in } L^2_{loc}(\bar{\Omega}\times[0,\infty)) \tag{7.6}$$

and

$$\nabla u_\varepsilon \rightharpoonup \nabla u \text{ in } L^2_{loc}(\bar{\Omega}\times[0,\infty)). \tag{7.7}$$

For the same u as that in (7.5), in view of (5.8) and (3.11), one can thus pick $\varepsilon = \varepsilon_j \subset (0,1)_{j\in\mathbb{N}}$ such that $\varepsilon_j \searrow 0$ as $j\to\infty$ and

$$u_\varepsilon \rightharpoonup u \text{ in } L^{\frac{10}{3}}_{loc}(\bar{\Omega}\times[0,\infty)),$$

$$u_\varepsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty_{loc}((0,\infty);L^2(\Omega)).$$

Similarly, for the same c as that in (7.4), recalling (2.8) and (3.14), we can also choose $\varepsilon = \varepsilon_j \subset (0,1)_{j\in\mathbb{N}}$ such that $\varepsilon_j \searrow 0$ as $j\to\infty$ and

$$c_\varepsilon \rightharpoonup c \text{ in } L^4_{loc}([0,\infty);W^{1,4}(\Omega)),$$

$$c_\varepsilon \overset{*}{\rightharpoonup} c \text{ in } L^\infty(\Omega\times(0,\infty)).$$

Next, let $g_\varepsilon(x,t) := -n_\varepsilon c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon$. Together with Lemma 2.2 and Lemma 5.2, we make use of the Young inequality and obtain some $C_2>0$ such that

$$\begin{aligned}\int_0^T\int_\Omega |n_\varepsilon c_\varepsilon|^{\frac{8m-3}{3}} &\le \|c_0\|_{L^\infty(\Omega)}\int_0^T\int_\Omega |n_\varepsilon|^{\frac{8m-3}{3}}\\ &\le C_2(T+1) \text{ for all } T>0\end{aligned} \tag{7.8}$$

and therefore, we further deduce from (6.7) that $c_{\varepsilon t} - \Delta c_\varepsilon = g_\varepsilon$ is bounded in $L^{\min\{\frac{20}{11},\frac{8m-3}{3}\}}(\Omega\times(0,T))$ for any $\varepsilon\in(0,1)$. So that, one may invoke the standard parabolic regularity theory to infer that $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^{\min\{\frac{20}{11},\frac{8m-3}{3}\}}((0,T);W^{2,\min\{\frac{20}{11},\frac{8m-3}{3}\}}(\Omega))$. This together with (6.2) and the Aubin-Lions lemma enables us to find a subsequence $\varepsilon = \varepsilon_j \subset (0,1)_{j\in\mathbb{N}}$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and $\nabla c_{\varepsilon_j} \to z_2$ in $L^{\min\{\frac{20}{11},\frac{8m-3}{3}\}}(\Omega\times(0,T))$ for all $T\in(0,\infty)$ and some $z_2\in L^{\min\{\frac{20}{11},\frac{8m-3}{3}\}}(\Omega\times(0,T))$ as $j\to\infty$, hence $\nabla c_{\varepsilon_j}\to z_2$ a.e. in $\Omega\times(0,\infty)$ as $j\to\infty$. In view of (7.6) and the Egorov theorem, we conclude that $z_2 = \nabla c$, and

$$\nabla c_\varepsilon \to \nabla c \quad \text{a.e. in } \Omega\times(0,\infty) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{7.9}$$

Thereupon, in view of (1.2), (2.3) and (7.1), we may further infer that

$$n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon \to nS(x,n,c)\cdot\nabla c \quad \text{a.e. in } \Omega\times(0,\infty) \quad \text{as } \varepsilon:=\varepsilon_j\searrow 0. \tag{7.10}$$

On the other hand, it follows from (6.8) and the Aubin-Lions lemma that there exists a further subsequence $\varepsilon = \varepsilon_j \subset (0,1)_{j\in\mathbb{N}}$ such that

$$n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot\nabla c_\varepsilon \rightharpoonup z_3 \quad \text{in } L_{loc}^{\frac{4(8m-3)}{9+8m}}(\Omega\times(0,\infty)),$$

which together with the Egorov theorem and (7.10) implies that $z_3 = nS(x,n,c)\nabla c$, and therefore, (7.10) can be rewritten as

$$n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot\nabla c_\varepsilon \rightharpoonup nS(x,n,c)\nabla c \\ \text{in } L_{loc}^{\frac{4(8m-3)}{9+8m}}(\Omega\times(0,\infty)) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{7.11}$$

Employing the same arguments as those in the proof of (7.11), and taking advantage of (6.6), (6.7), (7.1), (7.4), (7.5), (7.8) and (7.9), we conclude that

$$n_\varepsilon c_\varepsilon \to nc \ \text{ in } L_{loc}^{\frac{8m-3}{3}}(\bar\Omega\times(0,\infty)) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0, \tag{7.12}$$

$$n_\varepsilon u_\varepsilon \rightharpoonup nu \quad \text{in } L_{loc}^{\frac{10(8m-3)}{3(8m+7)}}(\Omega\times(0,\infty)) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0 \tag{7.13}$$

as well as

$$u_\varepsilon\cdot\nabla c_\varepsilon \rightharpoonup u\cdot\nabla c \quad \text{in } L_{loc}^{\frac{20}{11}}(\Omega\times(0,\infty)) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0$$

and

$$c_\varepsilon u_\varepsilon \rightharpoonup cu \ \text{ in } L_{loc}^{\frac{10}{3}}(\bar\Omega\times(0,\infty)) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{7.14}$$

Here we have used the fact that

$$\int_0^T\int_\Omega |c_\varepsilon u_\varepsilon|^{\frac{10}{3}} \le \|c_0\|_{L^\infty(\Omega)}\int_0^T\int_\Omega |n_\varepsilon|^{\frac{10}{3}} \\ \le C_3(T+1) \ \text{ for all } \ T>0 \quad \text{with some positive constant } \ C_3 \tag{7.15}$$

by (2.8) and (5.8). According to a well-established argument (see e.g. [16, 49, 58]), one can infer from (7.5) and the Lebesgue dominated convergence theorem that

$$Y_\varepsilon u_\varepsilon\otimes u_\varepsilon \to u\otimes u \ \text{ in } L_{loc}^1(\bar\Omega\times[0,\infty)) \quad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{7.16}$$

Now (7.2), (7.11), (7.12), (7.13) and (7.14) firstly warrant that the integrability requirements in (2.2) are satisfied. Secondly, the regularity properties (2.1) therein are obvious from (7.3)–(7.7). Finally, relying on the above convergence properties, one can pass to the limit in each term of weak formulation for (2.5) to construct a global weak solution of (1.1) and thereby completes the proof. □

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