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Calculus of Variations

Stabilizing effect of magnetic field on the 2D ideal magnetohydrodynamic flow with mixed partial damping

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Abstract

This paper examines the stability of a 2D inviscid MHD system with anisotropic damping near a background magnetic field. It is well known that solutions of the incompressible Euler equations can grow rapidly in time and are thus unstable while solutions of the Euler equations with full damping are stable. Then naturally arises the question of whether solutions of the Euler equations with partial damping are stable. The main purpose of this paper is to give an affirmative answer to this question in the case when the fluid is coupled with the magnetic field through the MHD system with one-component damping. The result presented in this paper especially confirms the stabilizing effects of the magnetic field on the electrically conducting fluids, a phenomenon that has been observed in physical experiments and numerical simulations.

Mathematics Subject Classification 35A01 · 35B35 · 35B65 · 76D03 · 76E25

1 Introduction

The MHD system is composed of the Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. It describes the motion of electrically conducting fluids such as plasmas, liquid metals and electrolytes in an electromagnetic field and has

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a wide range of applications in astrophysics, geophysics, cosmology and engineering (see, e.g., [5, 13, 38]). The MHD equations not only share some mathematically important features with the Euler/Navier–Stokes equations, but also exhibit many more fascinating properties than the fluid equations without the magnetic field. Inspired by the phenomenon observed in physical experiments and numerical simulations that the magnetic field can stabilize electrically conducting fluids (see, e.g., [2, 3, 22, 23]), we aim to explore the smoothing and stabilizing effects of magnetic field on the fluid motion. For this purpose, we consider the following 2D MHD equations with only partial damping in the velocity and the magnetic field,

$$\begin{cases} \partial_t U + U\cdot\nabla U + \nabla P + \nu(U_1, 0)^\top = B\cdot\nabla B, \quad x\in\mathbb{R}^2,\ t>0, \\ \partial_t B + U\cdot\nabla B + \eta(0, B_2)^\top = B\cdot\nabla U, \\ \nabla\cdot U = \nabla\cdot B = 0, \end{cases} \tag{1.1}$$

where $U = (U_1, U_2)^\top$, $B = (B_1, B_2)^\top$ and P are the velocity field, the magnetic field, and the pressure, respectively. The positive constants $\nu > 0$ and $\eta > 0$ are the damping coefficients.

There have been substantial developments on two fundamental problems concerning the MHD equations, the global (in time) regularity and stability. In particular, the stability problem near a background magnetic field have recently attracted considerable interests. For the ideal MHD equations, Bardos et al. [4] took advantage of the Elsässer variables to establish the global regularity (in Hölder setting) of perturbations near a strong background magnetic field. Cai and Lei [7] and He et al. [25], via different approaches, successfully solved the stability problem on both the ideal MHD system and its fully dissipative counterpart (with identical viscosity and resistivity) near a background magnetic field. Wei and Zhang [47] allowed the viscosity and resistivity coefficients to be slightly different. The paper of Lin et al. [33] pioneered the study of the stability problem on the incompressible non-resistive MHD equation near a background magnetic field. The 3D problem together with the large-time behavior was solved by Abidi and Zhang [1] and Deng and Zhang [14] in the whole spaces case. Pan et al. [37] dealt with this problem when the spatial domain is a 3D periodic box $\mathbb{T}^3$. Tan and Wang [42] examined the case with the horizontally infinite flat layer $\mathbb{R}^2 \times (0, 1)$. The approach of Lin et al. [33] on the 2D non-resistive MHD problem is Lagrangian. Ren et al. [39] revisited the stability problem by resorting to the Eulerian energy estimates in anisotropic Sobolev space and obtained explicit time decay rates. Ren et al. [40] proved the global stability in a strip domain, and Chen and Ren [12] considered two types of periodic domains $\mathbb{T} \times \mathbb{R}$ and $\mathbb{T} \times (0, 1)$. Zhang [56] proved the global existence of strong solutions to the Cauchy problem with large initial perturbations, provided that the background magnetic field is sufficiently large. Recently, Jiang and Jiang [28] extended the results [56] to the 2D periodic domains $\mathbb{T}^2$ by using the Lagrangian approach and the odevity conditions proposed in [37], and obtained the asymptotic behaviors of global strong solutions with large initial perturbations. For the 2D inviscid and resistive MHD equations, Zhou and Zhu [57] investigated the stability of perturbations near a background magnetic field on the periodic domain. For the ideal MHD system with velocity damping, Wu et al. [52] studied the stability via the approach of wave equations, and Du et al. [18] proved the exponential stability of a stratified flow in the strip-type doamin $\mathbb{R} \times [0, 1]$. We also refer to [51] for the stability and large-time behavior of the 2D compressible MHD system without magnetic diffusion.

Due to its physical relevance and remarkable enhanced smoothing properties, the stability problem for the incompressible MHD equations with partial dissipation has recently generated a rich array of results. Lin et al. [34] obtained the stability of the 2D MHD equations with vertical velocity dissipation and horizontal magnetic diffusion (see also [31]). A new stability result on 3D MHD equations with horizontal dissipation and vertical magnetic diffusion was achieved by Wu and Zhu [53]. Boardman et al. [6] studied the stability of 2D inviscid and resistive MHD equations with only vertical velocity damping. The stability and large-time behavior of the 2D MHD equations with only vertical velocity dissipation and a damping magnetic field was investigated in [21]. The paper [30] dealt with the anisotropic equations with only (partially) vertical damping magnetic field. In comparison with [21] and [30], the MHD system considered in this current paper contains the least dissipation and damping. It appears that the anisotropic damping required in this paper can not be further reduced.

Many more results on the well-posedness and related issues concerning the incompressible MHD equations are available in the literature. For example, various partial dissipation cases are dealt with in [8, 9, 16, 17, 36], the non-resistive case in [11, 20, 32, 45, 55], the only magnetic diffusion case in [10, 29] and the fractional dissipation case in [15, 44, 48–50, 54].

This paper aims to understand the stability of the 2D ideal MHD system (1.1) near the equilibrium state $(U^{(0)}, B^{(0)})$,

$$U^{(0)} \equiv 0, \quad B^{(0)} \equiv e_1 := (1, 0).$$

Let (u, b) be the perturbation of (U, B) near the steady state $(U^{(0)}, B^{(0)})$,

$$u := U - U^{(0)}, \qquad b := B - B^{(0)}.$$

The system governing the perturbation is taken to be the following system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P + \nu (u_1, 0)^\top = b \cdot \nabla b + \partial_1 b, \quad x \in \mathbb{R}^2,\ t > 0, \\ \partial_t b + u \cdot \nabla b + \eta (0, b_2)^\top = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0. \end{cases} \tag{1.2}$$

We shall focus on an initial value problem of (1.2) with the Cauchy data:

$$u(x, 0) = u_0(x), \ \ b(x, 0) = b_0(x). \tag{1.3}$$

The motivation for studying the stability problem of (1.2)–(1.3) is twofold. The first is to reveal the phenomenon that the coupling and interaction between the velocity and the magnetic field actually stabilize the fluid motion. Indeed, when $B = 0$, (1.1) becomes the 2D incompressible Euler equation with only horizontally damping velocity,

$$\begin{cases} \partial_t U_1 + U \cdot \nabla U_1 + \partial_1 P + \nu U_1 = 0, \\ \partial_t U_2 + U \cdot \nabla U_2 + \partial_2 P = 0, \\ \nabla \cdot U = 0. \end{cases} \tag{1.4}$$

The stability problem of (1.4) remains unsolved. To understand the difficulty, we reformulate (1.4) in terms of the following vorticity equation

$$\begin{cases} \partial_t \omega + U \cdot \nabla \omega = \nu \mathcal{R}_2^2 \omega, \\ U = \nabla^\perp \Delta^{-1} \omega, \end{cases} \tag{1.5}$$

where $\mathcal{R}_k = \partial_k (-\Delta)^{-\frac{1}{2}}$ with $k = 1, 2$ denotes the standard Riesz transform (see, e.g., [24, 41]) and the fractional Laplacian operator is defined via the Fourier transform,

$$\widehat{(-\Delta)^\beta f}(\xi) = |\xi|^{2\beta} \widehat{f}(\xi).$$

and $\nabla^{\perp} = (-\partial_2, \partial_1)$. Unfortunately, the classical Yudovich's approach used to study the 2D incompressible Euler-like equations do not appear to work for (1.5), since the Riesz transform $\mathcal{R}_2$ is not known to be bounded in L^{∞}. In fact, as pointed out by Elgindi [19], the L^q-norms of ω are bounded for any $1 < q < \infty$, but these L^q-norms may grow exponentially in q. Therefore, the question of whether the solutions of (1.5) will develop singularity in finite time is an interesting and challenging problem. The first and main purpose of this paper is to show that the magnetic field is able to stabilize the velocity field through the MHD system (1.1). For the recent works on the magnetic inhibition phenomenon (or stability result), we refer to [26, 27, 46] and the references cited therein.

The second motivation is to explore the hidden wave structure and to understand the stability mechanism. To explain this clearly, we apply the Leray projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the equation (1.2) and separate it into the linear part and the nonlinear part. Due to $\nabla \cdot u = \nabla \cdot b = 0$,

$$\mathbb{P}(u_1, 0)^{\top} = (u_1, 0)^{\top} - \nabla\Delta^{-1}\nabla \cdot (u_1, 0)^{\top} = \partial_2^2\Delta^{-1}u = -\mathcal{R}_2^2 u,$$

and

$$\mathbb{P}(0, b_2)^{\top} = (0, b_2)^{\top} - \nabla\Delta^{-1}\nabla \cdot (0, b_2)^{\top} = \partial_1^2\Delta^{-1}b = -\mathcal{R}_1^2 b.$$

Thus the system (1.2) can be written as

$$\begin{cases} \partial_t u = \nu\mathcal{R}_2^2 u + \partial_1 b + \mathbb{P}(b\cdot\nabla b - u\cdot\nabla u), \\ \partial_t b = \eta\mathcal{R}_1^2 b + \partial_1 u + \mathbb{P}(b\cdot\nabla u - u\cdot\nabla b), \\ \nabla\cdot u = \nabla\cdot b = 0. \end{cases} \tag{1.6}$$

Differentiating (1.6) in t and making several substitutions, we find

$$\begin{cases} \partial_{tt} u - (\nu\mathcal{R}_2^2 + \eta\mathcal{R}_1^2)\partial_t u - \partial_1^2 u + \nu\eta\mathcal{R}_1^2\mathcal{R}_2^2 u = N_1, \\ \partial_{tt} b - (\nu\mathcal{R}_2^2 + \eta\mathcal{R}_1^2)\partial_t b - \partial_1^2 b + \nu\eta\mathcal{R}_1^2\mathcal{R}_2^2 b = N_2, \\ \nabla\cdot u = \nabla\cdot b = 0, \end{cases} \tag{1.7}$$

where N_1 and N_2 are the nonlinear terms,

$$\begin{aligned} N_1 &= (\partial_t - \eta\mathcal{R}_1^2)\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u) + \partial_1\mathbb{P}(b\cdot\nabla u - u\cdot\nabla b), \\ N_2 &= (\partial_t - \nu\mathcal{R}_2^2)\mathbb{P}(b\cdot\nabla u - u\cdot\nabla b) + \partial_1\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u). \end{aligned}$$

It is surprising that $u,\ b$ satisfy the same degenerate damped wave equation. The wave structure of (1.7) for (u, b) provides much more stabilization and regularization properties than the original system (1.1). In fact, the wave equation (1.7) indicates that there is a horizontal regularization via the coupling and interaction, and hence, the stability result of the solutions becomes possible.

The main result of this paper is the following stability theorem of global solutions to the Cauchy problem (1.2)–(1.3).

Theorem 1.1 *Assume the initial data $(u_0, b_0) \in H^3$ with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Then there exists a positive constant $\varepsilon > 0$, depending only on ν and η, such that if*

$$\|(u_0, b_0)\|_{H^3} \le \varepsilon,$$

then the problem (1.2)–(1.3) has a unique global solution (u, b) on $\mathbb{R}^2 \times [0, \infty)$, satisfying

$$\|(u, b)(t)\|_{H^3}^2 + \int_0^t \left(\|(u_1, b_2)(\tau)\|_{H^3}^2 + \|\partial_1 u(\tau)\|_{H^2}^2\right) d\tau \le C\varepsilon^2, \quad \forall\, t \ge 0,$$

where $C>0$ is a generic positive constant independent of ε and t.

Since the local-in-time existence result can be shown by the standard method (see, e.g., [35]), our main task is to derive the global-in-time a *prior* estimates of the solutions. The framework is the bootstrapping argument [43]. Due to the lack of full damping, some serious difficulties arise. To overcome these difficulties, we have to construct a suitable energy functional. It consists of two parts. The first part is the natural H^3-energy functional $\mathcal{E}_1(t)$,

$$\mathcal{E}_1(t) := \sup_{0\le\tau\le t} \|(u,b)(\tau)\|_{H^3}^2 + 2\int_0^t \Big(\nu\|u_1(\tau)\|_{H^3}^2 + \eta\|b_2(\tau)\|_{H^3}^2\Big)d\tau, \tag{1.8}$$

The second part $\mathcal{E}_2(t)$ includes the horizontal dissipation piece generated from $\partial_1 u$ and indicated by the wave structure of (1.7),

$$\mathcal{E}_2(t) := \int_0^t \|\partial_1 u(\tau)\|_{H^2}^2 d\tau. \tag{1.9}$$

When applying the standard L^2-method to estimate $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$, we encounter four of the most difficult terms:

$$\begin{aligned}
&\mathrm{Diff}_1 := \int \partial_1 u_1 |\partial_2^3 b_1|^2\,dx, \quad \mathrm{Diff}_2 := \int b_1 \partial_2^3 b_1 \partial_2^3 \partial_1 u_1\,dx,\\
&\mathrm{Diff}_3 := \int b_1 \partial_1 u_1 |\partial_2^3 b_1|^2\,dx, \quad \mathrm{Diff}_4 := \int b_1^2 \partial_2^3 b_1 \partial_2^3 \partial_1 u_1\,dx,
\end{aligned}$$

which cannot be well controlled by $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ directly. The strategy here is to use $(1.2)_2$ and $(1.2)_1$ to replace $\partial_1 u_1$ and $\partial_1 b_1$ by

$$\partial_1 u_1 = \partial_t b_1 + u\cdot\nabla b_1 - b\cdot\nabla u_1, \tag{1.10}$$

$$\partial_1 b_1 = \partial_t u_1 + u\cdot\nabla u_1 + \partial_1 P + \nu u_1 - b\cdot\nabla b_1. \tag{1.11}$$

For example, with the help of (1.10) and (1.11), we find

$$\begin{aligned}
\mathrm{Diff}_1 &= \int (\partial_t b_1 + u\cdot\nabla b_1 - b\cdot\nabla u_1)\,|\partial_2^3 b_1|^2\,dx\\
&= \frac{d}{dt}\int b_1|\partial_2^3 b_1|^2\,dx - 2\int b_1 \partial_2^3 b_1 \partial_2^3 \partial_t b_1\,dx\\
&\quad + \int u\cdot\nabla b_1 |\partial_2^3 b_1|^2\,dx - \int b\cdot\nabla u_1 |\partial_2^3 b_1|^2\,dx,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Diff}_2 &= -\int \partial_1 b_1 \partial_2^3 b_1 \partial_2^3 u_1\,dx - \int b_1 \partial_2^3 u_1 \partial_2^3 \partial_1 b_1\,dx\\
&= -\int \partial_1 b_1 \partial_2^3 b_1 \partial_2^3 u_1\,dx\\
&\quad - \int b_1 \partial_2^3 u_1 \partial_2^3\left(\partial_t u_1 + u\cdot\nabla u_1 + \partial_1 P + \nu u_1 - b\cdot\nabla b_1\right)\,dx
\end{aligned}$$

$$= -\int \partial_1 b_1 \partial_2^3 b_1 \partial_2^3 u_1 \, dx - \frac{1}{2}\frac{d}{dt}\int b_1 |\partial_2^3 u_1|^2 \, dx + \frac{1}{2}\int |\partial_2^3 u_1|^2 \partial_t b_1 \, dx$$
$$- \int b_1 \partial_2^3 u_1 \partial_2^3 \left(u \cdot \nabla u_1 + \partial_1 P + \nu u_1 - b \cdot \nabla b_1\right) \, dx.$$

The items associated with $\partial_t b_1$ will be handled by using (1.10) again. This process generates many terms. Based upon integration by parts and the anisotropic Sobolev inequalities, it is incredible that all the terms can be bounded by $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$, although the process is complicated and lengthy. For the details, we refer to the treatments of D_i with $i = 1, \ldots, 4$ in Sect. 2. Collecting these estimates, we are able to establish the energy inequalities stated in Proposition 2.1.

We also make efforts to exploit the full regularization and stabilization effects from the wave structure to understand the large-time behavior of the linearized system. The linearized system of (1.6) reads

$$\begin{cases} \partial_t u - \nu \mathcal{R}_2^2 u - \partial_1 b = 0, \\ \partial_t b - \eta \mathcal{R}_1^2 b - \partial_1 u = 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \; b(x,0) = b_0(x), \end{cases} \tag{1.12}$$

which can be converted to the linearized system of wave equations (1.7):

$$\begin{cases} \partial_{tt} u - (\nu \mathcal{R}_2^2 + \eta \mathcal{R}_1^2)\partial_t u - \partial_1^2 u + \nu\eta \mathcal{R}_1^2 \mathcal{R}_2^2 u = 0, \\ \partial_{tt} b - (\nu \mathcal{R}_2^2 + \eta \mathcal{R}_1^2)\partial_t b - \partial_1^2 b + \nu\eta \mathcal{R}_1^2 \mathcal{R}_2^2 b = 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \; b(x,0) = b_0(x). \end{cases} \tag{1.13}$$

We first aim to establish the decay rate of solution for the linearized system (1.12) in negative Sobolev space by careful energy estimates. To state our result precisely, we first define the fractional partial derivative operator Λ_i^γ with $i = 1, \ 2$ and $\gamma \in \mathbb{R}$ by

$$\widehat{\Lambda_i^\gamma f}(\xi) = |\xi_i|^\gamma \widehat{f}(\xi).$$

Theorem 1.2 *For $\sigma > 0$, assume that (u_0, b_0) satisfies*

$$(\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma}) u_0 \in H^{1+\sigma}, \; (\Lambda_1^{-\sigma}, \Lambda_2^{-\sigma}) b_0 \in H^{1+\sigma}, \; \nabla \cdot u_0 = \nabla \cdot b_0 = 0.$$

Then the corresponding solution (u, b) of (1.12) satisfies

$$(u, b) \in L^\infty(0, \infty; H^1), \; (\mathcal{R}_2 u, \mathcal{R}_1 b) \in L^2(0, \infty; H^1).$$

Moreover,

$$\|(u, b)(t)\|_{H^1} \le C(1+t)^{-\frac{\sigma}{2}}, \quad \forall \, t > 0,$$

where C is a generic positive constant depending only on ν, η, σ and the initial norms.

When the initial data is not in any Sobolev space of negative indices, we can still manage to show the precise decay rates for several quantities.

Theorem 1.3 *Assume that*

$$(u_0, b_0) \in L^2, \quad (\partial_1 u_0, \partial_1 b_0) \in L^2, \quad \nabla \cdot u_0 = \nabla \cdot b_0 = 0,$$
$$(\mathcal{R}_1 \mathcal{R}_2 u_0, \mathcal{R}_1 \mathcal{R}_2 b_0) \in L^2, \quad (\mathcal{R}_2^2 u_0, \mathcal{R}_1^2 b_0) \in L^2.$$

Then for any $t \geq 0$, the solution (u, b) of (1.12) satisfies,

$$\|\partial_t u(t)\|_{L^2} + \|\partial_1 u(t)\|_{L^2} + \|\mathcal{R}_1\mathcal{R}_2 u(t)\|_{L^2} \leq C\,(1+t)^{-\frac{1}{2}},$$
$$\|\partial_t b(t)\|_{L^2} + \|\partial_1 b(t)\|_{L^2} + \|\mathcal{R}_1\mathcal{R}_2 b(t)\|_{L^2} \leq C\,(1+t)^{-\frac{1}{2}},$$

where C is a generic positive constant depending only on ν, η and the initial norms.

Finally we show that any frequency away from a given area D decays exponentially in time. To do this, we define D by

$$D := \left\{\xi \in \mathbb{R}^2 : \ |\xi_1| < \alpha \ \text{ and } \ |\xi|^2 > \beta|\xi_1||\xi_2| \ \right\}, \tag{1.14}$$

where $\alpha > 0$ and $\beta > 2$ are fixed positive constants. In addition, we set $\widehat{\psi}(\xi)$ to be the following cutoff function in the frequency space,

$$\widehat{\psi}(\xi) = \begin{cases} 0, & \text{if} \quad \xi \in D, \\ 1, & \text{if} \quad \xi \in D^c. \end{cases}$$

Obviously,

$$\widehat{\psi * f}(\xi) = \widehat{\psi}(\xi)\,\widehat{f}(\xi). \tag{1.15}$$

Theorem 1.4 *Assume the initial data (u_0, b_0) with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ satisfies*

$$(\psi * u_0, \ \psi * b_0, \ \psi * \partial_1 u_0, \ \psi * \partial_1 b_0) \in L^2,$$
$$(\psi * \mathcal{R}_1\mathcal{R}_2 u_0, \ \psi * \mathcal{R}_1\mathcal{R}_2 b_0, \ \psi * \mathcal{R}_2^2 u_0, \ \psi * \mathcal{R}_1^2 b_0) \in L^2.$$

Then the corresponding solution (u, b) of (1.12) obeys the following exponential decay estimates,

$$\begin{aligned} &\|(\psi * u, \psi * b)\|_{L^2} + \|(\psi * \partial_1 u, \psi * \partial_1 b)\|_{L^2} \\ &\quad + \|(\psi * \mathcal{R}_1\mathcal{R}_2 u, \psi * \mathcal{R}_1\mathcal{R}_2 b)\|_{L^2} + \|(\psi * \partial_t u, \psi * \partial_t b)\|_{L^2} \\ &\leq C\, e^{-c(\eta,\nu,\alpha,\beta)\, t}, \end{aligned}$$

where $c = c(\nu, \eta, \alpha, \beta) > 0$ depends on ν, η, α and β, and $C = C(u_0, b_0, \nu, \eta, \alpha, \beta) > 0$ depends additionally on the initial norms.

Remark 1.1 It is an interesting problem to study the decay rates of the solutions to the nonlinear system (1.2). Unfortunately, this seems not easy and is left for the future. In fact, the large-time behavior of the solution depends crucially on the eigenvalues of the wave equation (1.13). Indeed, the characteristic polynomial associated with (1.13) reads

$$\lambda^2 + \left(\frac{\nu\xi_2^2}{|\xi|^2} + \frac{\eta\xi_1^2}{|\xi|^2}\right)\lambda + \nu\eta\frac{\xi_1^2\xi_2^2}{|\xi|^4} + \xi_1^2 = 0,$$

and the roots $\lambda_\mp$ are given by

$$\lambda_\mp := \frac{-\frac{\nu\xi_2^2+\eta\xi_1^2}{|\xi|^2} \mp \sqrt{\Gamma}}{2} \quad \textit{with} \quad \Gamma := \left(\frac{\nu\xi_2^2 + \eta\xi_1^2}{|\xi|^2}\right)^2 - 4\left(\nu\eta\frac{\xi_1^2\xi_2^2}{|\xi|^4} + \xi_1^2\right).$$

By direct calculations, we find

$$\lambda_+ = -\frac{2\xi_1^2\left(\nu\eta\frac{\xi_2^2}{|\xi|^4} + 1\right)}{\sqrt{\Gamma} + \left(\frac{\nu\xi_2^2}{|\xi|^2} + \frac{\eta\xi_1^2}{|\xi|^2}\right)} \lesssim -\xi_1^2,$$

provided $\Gamma \geq 0$ and $|\xi_1|$ is sufficiently small. As a result, the heat kernel only admits "one-component" decay. This is the inherent difficulty in the decay analysis of the solutions. Actually, it is also the reason that why we can only obtain the exponential decay away from the domain D.

The rest of this paper is organized as follows. Theorem 1.1 is proven in Sect. 2. The proof of Theorem 1.2 will be carried out in Sect. 3. Section 4 is devoted to the proofs of Theorems 1.3 and 1.4, based on the wave structure (1.13).

2 Proof of Theorem 1.1

This section aims to prove Theorem 1.1. As aforementioned, to establish the stability result in Theorem 1.1, it suffices to prove Proposition 2.1 below.

Proposition 2.1 *Let $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ be the same ones as defined in (1.8) and (1.9), respectively. Then there exists a generic positive constant C, depending only on ν and η, such that*

$$\begin{aligned}\mathcal{E}_1(t) \leq\, & C\left(\mathcal{E}_1(0)+\mathcal{E}_1(0)^{\frac{3}{2}}+\mathcal{E}_1(0)^2\right)\\ & +C\left(\mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}}\right)+C\left(\mathcal{E}_1(t)^3+\mathcal{E}_2(t)^3\right)\end{aligned} \tag{2.1}$$

and

$$\mathcal{E}_2(t) \leq C\mathcal{E}_1(0)+C\mathcal{E}_1(t)+C\mathcal{E}_1(t)^{\frac{3}{2}}+C\mathcal{E}_2(t)^{\frac{3}{2}}. \tag{2.2}$$

With Proposition 2.1 at our disposal, Theorem 1.1 can be easily achieved by the bootstrapping argument. For simplicity, we denote by C and C_i $(i = 1, 2, 3)$ various generic positive constants, which may depend only on ν and η, and may change from line to line.

Proof of Theorem 1.1 It follows from (2.1) and (2.2) that

$$\begin{aligned}\mathcal{E}_1(t)+\mathcal{E}_2(t) \leq\, & C_1\left(\mathcal{E}_1(0)+\mathcal{E}_1(0)^{\frac{3}{2}}+\mathcal{E}_1(0)^2\right)\\ & +C_2\left(\mathcal{E}_1(t)^{\frac{3}{2}}+\mathcal{E}_2(t)^{\frac{3}{2}}\right)+C_3\left(\mathcal{E}_1(t)^3+\mathcal{E}_2(t)^3\right).\end{aligned} \tag{2.3}$$

The bootstrapping argument then allows us to establish the stability result of Theorem 1.1, provided the initial data $\mathcal{E}_1(0)$ is chosen to be sufficiently small such that

$$C_1\left(\mathcal{E}_1(0)+\mathcal{E}_1(0)^{\frac{3}{2}}+\mathcal{E}_1(0)^2\right) \leq \frac{1}{4}\min\left\{\frac{1}{16C_2^2}, \left(\frac{1}{4C_3}\right)^{\frac{1}{2}}\right\}. \tag{2.4}$$

In fact, if we make the ansatz that for $0 < T \leq \infty$,

$$\mathcal{E}_1(t)+\mathcal{E}_2(t) \leq \min\left\{\frac{1}{16C_2^2}, \left(\frac{1}{4C_3}\right)^{\frac{1}{2}}\right\},$$

then (2.3) implies

$$
\begin{aligned}
\mathcal{E}_1(t)+\mathcal{E}_2(t) \leq\, & C_1\left(\mathcal{E}_1(0)+\mathcal{E}_1(0)^{\frac{3}{2}}+\mathcal{E}_1(0)^2\right) \\
& +C_2\left(\mathcal{E}_1(t)+\mathcal{E}_2(t)\right)^{\frac{3}{2}}+C_3\left(\mathcal{E}_1(t)+\mathcal{E}_2(t)\right)^3 \\
\leq\, & C_1\left(\mathcal{E}_1(0)+\mathcal{E}_1(0)^{\frac{3}{2}}+\mathcal{E}_1(0)^2\right)+\frac{1}{2}\left(\mathcal{E}_1(t)+\mathcal{E}_2(t)\right),
\end{aligned}
$$

or

$$
\mathcal{E}_1(t)+\mathcal{E}_2(t) \leq 2C_1\left(\mathcal{E}_1(0)+\mathcal{E}_1(0)^{\frac{3}{2}}+\mathcal{E}_1(0)^2\right), \tag{2.5}
$$

which, combined with the smallness assumption (2.4) on the initial data, leads to

$$
\mathcal{E}_1(t)+\mathcal{E}_2(t) \leq \frac{1}{2}\min\left\{\frac{1}{16C_2^2},\left(\frac{1}{4C_3}\right)^{\frac{1}{2}}\right\}.
$$

Thus, the bootstrapping argument then asserts that (2.5) holds for all time, provided $\mathcal{E}_1(0)$ fulfills (2.4). The proof of Theorem 1.1 is therefore complete. □

It remains to prove Proposition 2.1. To deal with the nonlinear terms, we need to make use of the anisotropic inequalities (cf. Lemmas 2.1 and 2.2), whose proofs rely on the basic one-dimensional Sobolev inequality

$$
\|g\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\|g\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|g'\|_{L^2(\mathbb{R})}^{\frac{1}{2}},
$$

and the Minkowski inequality

$$
\big\|\|f\|_{L^q_y(\mathbb{R}^n)}\big\|_{L^p_x(\mathbb{R}^m)} \leq \big\|\|f\|_{L^p_x(\mathbb{R}^m)}\big\|_{L^q_y(\mathbb{R}^n)}, \quad \forall\, 1 \leq q \leq p \leq \infty,
$$

where $f=f(x,y)$ with $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^n$ is a measurable function on $\mathbb{R}^m\times\mathbb{R}^n$.

Lemma 2.1 *Assume that f, $\partial_1 f$, g and $\partial_2 g$ are all in $L^2(\mathbb{R}^2)$. Then,*

$$
\|fg\|_{L^2(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\partial_2 g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}.
$$

Lemma 2.2 *The following estimates hold when the right-hand sides are all bounded,*

$$
\|f\|_{L^\infty(\mathbb{R}^2)} \leq C\|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\|\partial_{12} f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.
$$

In particular,

$$
\begin{aligned}
\|f\|_{L^\infty} &\leq C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_1 f\|_{H^1}^{\frac{1}{2}}, \\
\|f\|_{L^\infty} &\leq C\|f\|_{H^1}^{\frac{1}{2}}\|\partial_2 f\|_{H^1}^{\frac{1}{2}}.
\end{aligned}
$$

We are now ready to prove Proposition 2.1. The proofs are split into two steps, which are concerned with the derivations of (2.1) and (2.2), respectively.

2.1 Proof of (2.1)

Due to the equivalence of $\|(u,b)\|_{H^3}$ with $\|(u,b)\|_{L^2}+\|(u,b)\|_{\dot{H}^3}$, it suffices to bound the L^2-norm and the homogeneous $\dot{H}^3$-norm of (u,b). First, based on the divergence-free conditions $\nabla\cdot u=\nabla\cdot b=0$, it is easy to check that

$$
\|(u,b)\|_{L^2}^2+2\int_0^t\left(\nu\|u_1\|_{L^2}^2+\eta\|b_2\|_{L^2}^2\right)d\tau=\|(u_0,b_0)\|_{L^2}^2. \tag{2.6}
$$

Next, to estimate the $\dot{H}^3$-norm, applying $\partial_i^3 (i = 1, 2)$ to (1.2) and dotting them with $(\partial_i^3 u, \partial_i^3 b)$ in L^2, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2}\|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 + \nu\sum_{i=1}^{2}\|\partial_i^3 u_1\|_{L^2}^2 + \eta\sum_{i=1}^{2}\|\partial_i^3 b_2\|_{L^2}^2\\ &\quad := K_1 + K_2 + K_3 + K_4 + K_5,\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}K_1 &:= \sum_{i=1}^{2}\int \left(\partial_i^3\partial_1 b\cdot\partial_i^3 u + \partial_i^3\partial_1 u\cdot\partial_i^3 b\right)dx,\\ K_2 &:= -\sum_{i=1}^{2}\int \partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,dx,\\ K_3 &:= \sum_{i=1}^{2}\int \left(\partial_i^3(b\cdot\nabla b) - b\cdot\nabla\partial_i^3 b\right)\cdot\partial_i^3 u\,dx,\\ K_4 &:= -\sum_{i=1}^{2}\int \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\,dx,\\ K_5 &:= \sum_{i=1}^{2}\int \left(\partial_i^3(b\cdot\nabla u) - b\cdot\nabla\partial_i^3 u\right)\cdot\partial_i^3 b\,dx.\end{aligned}$$

We are now in a position of estimating $K_1, \ldots, K_5$ term by term. First, integration by parts directly gives

$$K_1 = 0. \tag{2.8}$$

To bound K_2, we divide it into two parts,

$$K_2 = -\int \partial_1^3(u\cdot\nabla u)\cdot\partial_1^3 u\,dx - \int \partial_2^3(u\cdot\nabla u)\cdot\partial_2^3 u\,dx := K_{21} + K_{22}.$$

Due to $\nabla\cdot u = 0$, by Hölder's and Sobolev's inequalities, we obtain

$$\begin{aligned}K_{21} &= -\int (\partial_1^3 u\cdot\nabla u + 3\partial_1^2 u\cdot\nabla\partial_1 u + 3\partial_1 u\cdot\nabla\partial_1^2 u)\cdot\partial_1^3 u\,dx\\ &\le C\|\partial_1^3 u\|_{L^2}\left(\|\nabla u\|_{L^\infty}\|\nabla\partial_1^2 u\|_{L^2} + \|\partial_1^2 u\|_{L^4}\|\nabla\partial_1 u\|_{L^4}\right)\\ &\le C\|u\|_{H^3}\|\partial_1 u\|_{H^2}^2,\end{aligned} \tag{2.9}$$

and similarly,

$$K_{22} \le C\|u\|_{H^3}\|\partial_2 u\|_{H^2}^2,$$

which, together with (2.9), yields

$$K_2 \le C\|u\|_{H^3}\left(\|\partial_1 u\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right). \tag{2.10}$$

To estimate K_3, we rewrite it into three items,

$$K_3=\sum_{i=1}^{2}3\int \partial_i b\cdot\nabla\partial_i^2 b\cdot\partial_i^3 u\,dx+\sum_{i=1}^{2}3\int \partial_i^2 b\cdot\nabla\partial_i b\cdot\partial_i^3 u\,dx$$
$$+\sum_{i=1}^{2}\int \partial_i^3 b\cdot\nabla b\cdot\partial_i^3 u\,dx:=K_{31}+K_{32}+K_{33},$$

where the first term K_{31} on the right-hand side can be bounded as follows,

$$\begin{aligned}K_{31}&=3\int\left(\partial_1 b\cdot\nabla\partial_1^2 b\cdot\partial_1^3 u+\partial_2 b_1\partial_1\partial_2^2 b\cdot\partial_2^3 u-\partial_1 b_1\partial_2^3 b\cdot\partial_2^3 u\right)dx\\&\le C\|\partial_1 b\|_{L^\infty}\|\nabla\partial_1^2 b\|_{L^2}\|\partial_1^3 u\|_{L^2}\\&\quad+C\left(\|\partial_2 b_1\|_{L^\infty}\|\partial_1\partial_2^2 b\|_{L^2}+\|\partial_1 b_1\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\right)\|\partial_2^3 u\|_{L^2}\\&\le C\|b\|_{H^3}\left(\|\partial_1 b\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).\end{aligned}$$

In a similar manner,

$$\begin{aligned}K_{32}&=3\int\left(\partial_1^2 b\cdot\nabla\partial_1 b\cdot\partial_1^3 u+\partial_2^2 b_1\partial_1\partial_2 b\cdot\partial_2^3 u-\partial_1\partial_2 b_1\partial_2^2 b\cdot\partial_2^3 u\right)dx\\&\le C\|\partial_1^2 b\|_{L^4}\|\nabla\partial_1 b\|_{L^4}\|\partial_1^3 u\|_{L^2}+C\|\partial_1\partial_2 b\|_{L^4}\|\partial_2^2 b\|_{L^4}\|\partial_2^3 u\|_{L^2}\\&\le C\|b\|_{H^3}\left(\|\partial_1 b\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right),\end{aligned}$$

and

$$\begin{aligned}K_{33}&=\int\left(\partial_1^3 b\cdot\nabla b\cdot\partial_1^3 u+\partial_2^3 b_1\partial_1 b\cdot\partial_2^3 u-\partial_1\partial_2^2 b_1\partial_2 b\cdot\partial_2^3 u\right)dx\\&\le C\|\nabla b\|_{L^\infty}\|\partial_1^3 b\|_{L^2}\|\partial_1^3 u\|_{L^2}\\&\quad+C\left(\|\partial_1 b\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}+\|\partial_2 b\|_{L^\infty}\|\partial_1\partial_2^2 b_1\|_{L^2}\right)\|\partial_2^3 u\|_{L^2}\\&\le C\|b\|_{H^3}\left(\|\partial_1 b\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).\end{aligned}$$

Therefore,

$$K_3\le C\|b\|_{H^3}\left(\|\partial_1 b\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).\tag{2.11}$$

In order to estimate K_4, we write it in the form:

$$K_4=-\int\partial_1^3(u\cdot\nabla b)\cdot\partial_1^3 b\,dx-\int\partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\,dx:=K_{41}+K_{42},$$

where the first term K_{41} can be easily bounded by

$$\begin{aligned}K_{41}&=-\int\partial_1^3 u\cdot\nabla b\cdot\partial_1^3 b\,dx-3\int\left(\partial_1^2 u\cdot\nabla\partial_1 b+\partial_1 u\cdot\nabla\partial_1^2 b\right)\cdot\partial_1^3 b\,dx\\&\le C\|\nabla b\|_{L^\infty}\|\partial_1^3 b\|_{L^2}\|\partial_1^3 u\|_{L^2}\\&\quad+C\left(\|\partial_1^2 u\|_{L^4}\|\nabla\partial_1 b\|_{L^4}+\|\partial_1 u\|_{L^\infty}\|\nabla\partial_1^2 b\|_{L^2}\right)\|\partial_1^3 b\|_{L^2}\\&\le C\|b\|_{H^3}\left(\|\partial_1 b\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\right).\end{aligned}\tag{2.12}$$

The second term K_{42} needs more work. First, by virtue of the divergence-free condition $\nabla\cdot u=0$, we split it into three parts:

$$K_{42} = -\int \partial_2^3 u \cdot \nabla b \cdot \partial_2^3 b \, dx - 3\int \partial_2^2 u \cdot \nabla \partial_2 b \cdot \partial_2^3 b \, dx$$
$$- 3\int \partial_2 u \cdot \nabla \partial_2^2 b \cdot \partial_2^3 b \, dx := K_{421} + K_{422} + K_{423}.$$

For K_{421}, we have

$$\begin{aligned} K_{421} &= -\int \partial_2^3 u_1 \partial_1 b \cdot \partial_2^3 b \, dx - \int \partial_2^3 u_2 \partial_2 b \cdot \partial_2^3 b \, dx \\ &= -\int \partial_2^3 u_1 \partial_1 b \cdot \partial_2^3 b \, dx + \int \partial_2^3 u_2 \partial_1 b_1 \partial_2^3 b_2 \, dx \\ &\quad + \int \partial_1 \partial_2^2 u_1 \partial_2 b_1 \partial_2^3 b_1 \, dx := K_{4211} + K_{4212} + K_{4213}. \end{aligned}$$

where the first two terms K_{4211} and K_{4212} are bounded by

$$\begin{aligned} K_{4211} + K_{4212} &\le C\|\partial_1 b\|_{L^\infty} \|\partial_2^3 u\|_{L^2} \|\partial_2^3 b\|_{L^2} \\ &\le C\|b\|_{H^3} \left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right). \end{aligned}$$

For K_{4213}, integration by parts twice gives

$$\begin{aligned} K_{4213} &= -\int \partial_2^2 u_1 \partial_1 \partial_2 b_1 \partial_2^3 b_1 \, dx + \int \partial_2^3 u_1 \partial_2 b_1 \partial_1 \partial_2^2 b_1 \, dx \\ &\quad + \int \partial_2^2 u_1 \partial_2^2 b_1 \partial_1 \partial_2^2 b_1 \, dx \\ &\le C\|\partial_2^2 u_1\|_{L^4} \|\partial_1 \partial_2 b_1\|_{L^4} \|\partial_2^3 b_1\|_{L^2} \\ &\quad + C\left(\|\partial_2^3 u_1\|_{L^2} \|\partial_2 b_1\|_{L^\infty} + \|\partial_2^2 u_1\|_{L^4} \|\partial_2^2 b_1\|_{L^4}\right) \|\partial_1 \partial_2^2 b\|_{L^2} \\ &\le C\|b\|_{H^3} \left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right), \end{aligned}$$

which, together with the estimates of K_{4211} and K_{4212}, shows that

$$K_{421} \le C\|b\|_{H^3} \left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right). \tag{2.13}$$

Analogously,

$$\begin{aligned} K_{422} &= -3\int \partial_2^2 u_1 \partial_1 \partial_2 b \cdot \partial_2^3 b \, dx + 3\int \partial_2^2 u_1 \partial_1 \partial_2^2 b \cdot \partial_2^2 b \, dx \\ &\le C\|b\|_{H^3} \left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right). \end{aligned} \tag{2.14}$$

For K_{423}, due to $\nabla \cdot u = \nabla \cdot b = 0$, we have

$$\begin{aligned} K_{423} &= -3\int \partial_2 u_1 \partial_1 \partial_2^2 b \cdot \partial_2^3 b \, dx - 3\int \partial_2 u_2 \partial_1 \partial_2^2 b_1 \partial_1 \partial_2^2 b_1 \, dx \\ &\quad + 3\int \partial_1 u_1 \partial_2^3 b_1 \partial_2^3 b_1 \, dx := K_{4231} + K_{4232} + D_1. \end{aligned}$$

Based upon the Hölder's and Sobolev's inequalities, it is easily deduced that

$$\begin{aligned} K_{4231} + K_{4232} &\le C\|\partial_2 u\|_{L^\infty} \|\partial_1 \partial_2^2 b\|_{L^2} \|\nabla^3 b\|_{L^2} \\ &\le C\|b\|_{H^3} \left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right). \end{aligned} \tag{2.15}$$

We now turn to deal with D_1, which is one of the most difficult terms. The strategy here is to replace $\partial_1 u_1$ by using the equation of magnetic field,

$$\partial_1 u_1 = \partial_t b_1 + u \cdot \nabla b_1 - b \cdot \nabla u_1. \tag{2.16}$$

In terms of (2.16), we can rewrite D_1 in the form:

$$
\begin{aligned}
D_1 &= 3\int (\partial_t b_1 + u\cdot\nabla b_1 - b\cdot\nabla u_1)\,|\partial_2^3 b_1|^2\,dx\\
&= 3\frac{d}{dt}\int b_1|\partial_2^3 b_1|^2\,dx - 6\int b_1\partial_2^3 b_1\partial_2^3\partial_t b_1\,dx\\
&\quad + 3\int (u\cdot\nabla b_1)\,|\partial_2^3 b_1|^2\,dx - 3\int (b\cdot\nabla u_1)\,|\partial_2^3 b_1|^2\,dx,
\end{aligned}\tag{2.17}
$$

where the second term associated with $\partial_t b_1$ on the right side can be written as

$$
\begin{aligned}
&-6\int b_1\partial_2^3 b_1\partial_2^3\partial_t b_1\,dx\\
&= -6\int b_1\partial_2^3 b_1\partial_2^3(\partial_1 u_1 - u\cdot\nabla b_1 + b\cdot\nabla u_1)\,dx\\
&= -6\int b_1\partial_2^3 b_1\partial_2^3\partial_1 u_1\,dx + 6\int b_1\partial_2^3 b_1\partial_2^3 u\cdot\nabla b_1\,dx\\
&\quad + 18\int b_1\partial_2^3 b_1\partial_2^2 u\cdot\nabla\partial_2 b_1\,dx + 18\int b_1\partial_2^3 b_1\partial_2 u\cdot\nabla\partial_2^2 b_1 dx\\
&\quad + 3\int b_1 u\cdot\nabla|\partial_2^3 b_1|^2\,dx - 6\int b_1\partial_2^3 b_1\partial_2^3(b\cdot\nabla u_1)\,dx.
\end{aligned}\tag{2.18}
$$

Noting that

$$
\int b_1 u\cdot\nabla|\partial_2^3 b_1|^2\,dx + \int u\cdot\nabla b_1|\partial_2^3 b_1|^2\,dx = 0,
$$

we obtain after plugging (2.18) into (2.17) that

$$
\begin{aligned}
D_1 &= 3\frac{d}{dt}\int b_1|\partial_2^3 b_1|^2\,dx - 6\int b_1\partial_2^3 b_1\partial_2^3\partial_1 u_1\,dx\\
&\quad + 6\int b_1\partial_2^3 b_1\partial_2^3 u\cdot\nabla b_1\,dx + 18\int b_1\partial_2^3 b_1\partial_2^2 u\cdot\nabla\partial_2 b_1\,dx\\
&\quad + 18\int b_1\partial_2^3 b_1\partial_2 u_1\partial_1\partial_2^2 b_1\,dx + 27\int b_1\partial_2 u_2|\partial_2^3 b_1|^2\,dx\\
&\quad - 3\int b_2\partial_2 u_1|\partial_2^3 b_1|^2\,dx - 6\int b_1\partial_2^3 b_1\partial_2^3 b_2\partial_2 u_1\,dx\\
&\quad - 18\int b_1\partial_2^3 b_1\partial_2^2 b\cdot\nabla\partial_2 u_1\,dx - 18\int b_1\partial_2^3 b_1\partial_2 b\cdot\nabla\partial_2^2 u_1\,dx\\
&\quad - 6\int b_1\partial_2^3 b_1 b\cdot\nabla\partial_2^3 u_1\,dx.
\end{aligned}\tag{2.19}
$$

Two of the most difficult terms on the right-hand side of (2.19) are the second and sixth terms,

$$
D_2 := -6\int b_1\partial_2^3 b_1\partial_2^3\partial_1 u_1\,dx,\quad D_3 := 27\int b_1\partial_2 u_2|\partial_2^3 b_1|^2\,dx,
$$

which will be handled by using (2.16) and the equation of velocity,

$$
\partial_1 b_1 = \partial_t u_1 + u\cdot\nabla u_1 + \partial_1 P + \nu u_1 - b\cdot\nabla b_1.\tag{2.20}
$$

For D_2, using (2.16), (2.20) and integrating by parts, we have

$$
\begin{aligned}
D_2 =& 6\int \partial_1 b_1 \partial_2^3 b_1 \partial_2^3 u_1 \, dx + 6\int b_1 \partial_2^3 u_1 \partial_2^3 \partial_1 b_1 \, dx \\
:=& J_1 + 6\int b_1 \partial_2^3 u_1 \partial_2^3 \left(\partial_t u_1 + u\cdot\nabla u_1 + \partial_1 P + \nu u_1 - b\cdot\nabla b_1\right) \, dx \\
=& J_1 + 3\frac{d}{dt}\int b_1 |\partial_2^3 u_1|^2 \, dx - 3\int |\partial_2^3 u_1|^2 (\partial_1 u_1 + b\cdot\nabla u_1) \, dx \\
&+ 6\sum_{k=1}^{3} C_3^k \int b_1 \partial_2^3 u_1 \partial_2^k u\cdot\nabla\partial_2^{3-k} u_1 \, dx \\
&+ 6\int b_1 \partial_2^3 u_1 \partial_2^3 \partial_1 P \, dx + 6\nu \int b_1 |\partial_2^3 u_1|^2 \, dx \\
&- 6\sum_{k=1}^{3} C_3^k \int b_1 \partial_2^3 u_1 \partial_2^k b\cdot\nabla\partial_2^{3-k} b_1 \, dx - 6\int b_1 \partial_2^3 u_1 b\cdot\nabla\partial_2^3 b_1 \, dx,
\end{aligned}
\tag{2.21}
$$

where the symbol C_n^k denotes the standard combination number, and

$$
J_1 := 6\int \partial_1 b_1 \partial_2^3 b_1 \partial_2^3 u_1 \, dx.
$$

Here, we have also used the divergence-free condition $\nabla\cdot u = 0$ to get that

$$
\int b_1 u\cdot\nabla|\partial_2^3 u_1|^2 \, dx + \int u\cdot\nabla b_1 |\partial_2^3 u_1|^2 \, dx = 0.
$$

To deal with D_3, we first infer from (2.16) that

$$
\begin{aligned}
D_3 :=& 27\int b_1 \partial_2 u_2 |\partial_2^3 b_1|^2 \, dx = -27\int b_1 \partial_1 u_1 |\partial_2^3 b_1|^2 \, dx \\
=& -27\int b_1 |\partial_2^3 b_1|^2 (\partial_t b_1 + u\cdot\nabla b_1 - b\cdot\nabla u_1) \, dx \\
=& -\frac{27}{2}\frac{d}{dt}\int b_1^2 |\partial_2^3 b_1|^2 \, dx + 27\int b_1^2 \partial_2^3 b_1 \partial_2^3 \partial_t b_1 \, dx \\
&- \frac{27}{2}\int |\partial_2^3 b_1|^2 u\cdot\nabla b_1^2 \, dx + 27\int b_1 |\partial_2^3 b_1|^2 b\cdot\nabla u_1 \, dx,
\end{aligned}
\tag{2.22}
$$

where, similarly to the derivation of (2.21), the second term on the right-hand side can be written as

$$
\begin{aligned}
&27\int b_1^2 \partial_2^3 b_1 \partial_2^3 \partial_t b_1 \, dx = 27\int b_1^2 \partial_2^3 b_1 \partial_2^3 (\partial_1 u_1 - u\cdot\nabla b_1 + b\cdot\nabla u_1) \, dx \\
&= 27\int b_1^2 \partial_2^3 b_1 \partial_2^3 \partial_1 u_1 \, dx - \frac{27}{2}\int b_1^2 u\cdot\nabla|\partial_2^3 b_1|^2 \, dx \\
&\quad - 27\sum_{k=1}^{3} C_3^k \int b_1^2 \partial_2^3 b_1 \partial_2^k u\cdot\nabla\partial_2^{3-k} b_1 \, dx \\
&\quad + 27\sum_{k=1}^{3} C_3^k \int b_1^2 \partial_2^3 b_1 \partial_2^k b\cdot\nabla\partial_2^{3-k} u_1 \, dx + 27\int b_1^2 \partial_2^3 b_1 b\cdot\nabla\partial_2^3 u_1 \, dx.
\end{aligned}
\tag{2.23}
$$

Thus, inserting (2.23) into (2.22) and noting that

$$\int b_1^2 u\cdot\nabla|\partial_2^3 b_1|^2\,dx+\int u\cdot\nabla b_1^2|\partial_2^3 b_1|^2\,dx=0,$$

we find

$$\begin{aligned}
D_3=&-\frac{27}{2}\frac{d}{dt}\int b_1^2|\partial_2^3 b_1|^2\,dx+27\int b_1^2\partial_2^3 b_1\partial_2^3\partial_1 u_1\,dx\\
&+27\int b_1|\partial_2^3 b_1|^2 b\cdot\nabla u_1\,dx-27\sum_{k=1}^{3}C_3^k\int b_1^2\partial_2^3 b_1\partial_2^k u\cdot\nabla\partial_2^{3-k}b_1\,dx\\
&+27\sum_{k=1}^{3}C_3^k\int b_1^2\partial_2^3 b_1\partial_2^k b\cdot\nabla\partial_2^{3-k}u_1\,dx+27\int b_1^2\partial_2^3 b_1 b\cdot\nabla\partial_2^3 u_1\,dx. \qquad (2.24)
\end{aligned}$$

Clearly, we still need to deal with the second term on the right-hand side of (2.24). In fact, using (2.16) and (2.20) again, we have from integration by parts that

$$\begin{aligned}
D_4:=&\,27\int b_1^2\partial_2^3 b_1\partial_2^3\partial_1 u_1\,dx\\
=&-54\int b_1\partial_1 b_1\partial_2^3 b_1\partial_2^3 u_1\,dx-27\int b_1^2\partial_2^3 u_1\partial_2^3\partial_1 b_1\,dx\\
:=&\,J_2-27\int b_1^2\partial_2^3 u_1\partial_2^3\left(\partial_t u_1+u\cdot\nabla u_1+\partial_1 P+\nu u_1-b\cdot\nabla b_1\right)\,dx\\
=&\,J_2-\frac{27}{2}\frac{d}{dt}\int b_1^2|\partial_2^3 u_1|^2\,dx+27\int|\partial_2^3 u_1|^2 b_1(\partial_1 u_1-u\cdot\nabla b_1+b\cdot\nabla u_1)\,dx\\
&-27\int b_1^2\partial_2^3 u_1 u\cdot\nabla\partial_2^3 u_1\,dx-27\sum_{k=1}^{3}C_3^k\int b_1^2\partial_2^3 u_1\partial_2^k u\cdot\nabla\partial_2^{3-k}u_1\,dx\\
&-27\int b_1^2\partial_2^3 u_1\partial_2^3\partial_1 P\,dx-27\nu\int b_1^2|\partial_2^3 u_1|^2\,dx\\
&+27\int b_1^2\partial_2^3 u_1 b\cdot\nabla\partial_2^3 b_1\,dx+27\sum_{k=1}^{3}C_3^k\int b_1^2\partial_2^3 u_1\partial_2^k b\cdot\nabla\partial_2^{3-k}b_1\,dx, \qquad (2.25)
\end{aligned}$$

where J_2 is given by

$$J_2:=-54\int b_1\partial_1 b_1\partial_2^3 b_1\partial_2^3 u_1\,dx.$$

Now, plugging (2.21), (2.24) and (2.25) into (2.19), we obtain after careful rearrangement that

$$\begin{aligned}
D_1=&\,3\frac{d}{dt}\int b_1\left(|\partial_2^3 b_1|^2+|\partial_2^3 u_1|^2\right)dx-\frac{27}{2}\frac{d}{dt}\int b_1^2\left(|\partial_2^3 b_1|^2\,dx+|\partial_2^3 u_1|^2\right)dx\\
&+J_1+J_2+6\int b_1\partial_2^3 b_1\partial_2^3 u\cdot\nabla b_1\,dx+18\int b_1\partial_2^3 b_1\partial_2^2 u\cdot\nabla\partial_2 b_1\,dx\\
&-24\int b_1\partial_2^3 b_1\partial_2 u_1\partial_2^3 b_2\,dx-3\int b_2\partial_2 u_1|\partial_2^3 b_1|^2\,dx\\
&-18\int b_1\partial_2^3 b_1\partial_2^2 b\cdot\nabla\partial_2 u_1\,dx-18\int b_1\partial_2^3 b_1\partial_2 b\cdot\nabla\partial_2^2 u_1\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ 6\int b\cdot\nabla b_1(\partial_2^3u_1\partial_2^3b_1)\,dx - 3\int|\partial_2^3u_1|^2(\partial_1u_1+b\cdot\nabla u_1)\,dx\\
&+ 6\sum_{k=1}^{3}C_3^k\int b_1\partial_2^3u_1\partial_2^k u\cdot\nabla\partial_2^{3-k}u_1\,dx + 6\int b_1\partial_2^3u_1\partial_2^3\partial_1P\,dx\\
&+ 6\nu\int b_1|\partial_2^3u_1|^2\,dx - 6\sum_{k=1}^{3}C_3^k\int b_1\partial_2^3u_1\partial_2^k b\cdot\nabla\partial_2^{3-k}b_1\,dx\\
&+ 27\int|\partial_2^3u_1|^2b_1(\partial_1u_1-u\cdot\nabla b_1+b\cdot\nabla u_1)\,dx - 27\int b_1^2\partial_2^3u_1u\cdot\nabla\partial_2^3u_1\,dx\\
&- 27\sum_{k=1}^{3}C_3^k\int b_1^2\partial_2^3u_1\partial_2^k u\cdot\nabla\partial_2^{3-k}u_1\,dx - 27\int b_1^2\partial_2^3u_1\partial_2^3\partial_1P\,dx\\
&- 27\nu\int b_1^2|\partial_2^3u_1|^2\,dx + 27\sum_{k=1}^{3}C_3^k\int b_1^2\partial_2^3u_1\partial_2^k b\cdot\nabla\partial_2^{3-k}b_1\,dx\\
&+ 27\int b_1|\partial_2^3b_1|^2b\cdot\nabla u_1\,dx - 27\sum_{k=1}^{3}C_3^k\int b_1^2\partial_2^3b_1\partial_2^k u\cdot\nabla\partial_2^{3-k}b_1\,dx\\
&+ 27\sum_{k=1}^{3}C_3^k\int b_1^2\partial_2^3b_1\partial_2^k b\cdot\nabla\partial_2^{3-k}u_1\,dx - 54\int b\cdot\nabla b_1(b_1\partial_2^3u_1\partial_2^3b_1)\,dx\\
:=\;& I'(t) + J_1 + J_2 + \cdots + J_{24},
\end{aligned}
\tag{2.26}
$$

where we have also used $\nabla\cdot b = 0$ and the following simple facts that

$$
\begin{aligned}
&27\int b_1^2\partial_2^3b_1b\cdot\nabla\partial_2^3u_1\,dx + 27\int b_1^2\partial_2^3u_1b\cdot\nabla\partial_2^3b_1\,dx\\
&\quad = 27\int b_1^2b\cdot\nabla(\partial_2^3u_1\partial_2^3b_1)\,dx = -54\int b\cdot\nabla b_1(b_1\partial_2^3u_1\partial_2^3b_1)\,dx,
\end{aligned}
$$

and

$$
\begin{aligned}
&-6\int b_1\partial_2^3b_1b\cdot\nabla\partial_2^3u_1\,dx - 6\int b_1\partial_2^3u_1b\cdot\nabla\partial_2^3b_1\,dx\\
&\quad = -6\int b_1b\cdot\nabla(\partial_2^3u_1\partial_2^3b_1)\,dx = 6\int b\cdot\nabla b_1(\partial_2^3u_1\partial_2^3b_1)\,dx.
\end{aligned}
$$

Next, we need to bound J_1, J_2, ... and J_{24} one by one. First, it follows from the Sobolev's embedding inequality that

$$
\begin{aligned}
J_1 + J_2 &\le C\|\partial_1 b\|_{L^\infty}\|\partial_2^3u\|_{L^2}\|\partial_2^3b\|_{L^2}(1+\|b_1\|_{L^\infty})\\
&\le C(\|b\|_{H^3}+\|b\|_{H^3}^2)\left(\|\partial_1b\|_{H^2}^2+\|\partial_2u\|_{H^2}^2\right).
\end{aligned}
$$

For J_3, J_8 and J_9, by Lemma 2.2, we have

$$
\begin{aligned}
J_3 + J_8 + J_9 &\le C\|b\|_{L^\infty}\|\nabla b\|_{L^\infty}\|\partial_2^3b_1\|_{L^2}\|\nabla\partial_2^2u\|_{L^2}\\
&\le C\|b\|_{H^1}^{\frac12}\|\partial_1 b\|_{H^1}^{\frac12}\|\nabla b\|_{H^1}^{\frac12}\|\partial_1\nabla b\|_{H^1}^{\frac12}\|b\|_{H^3}\|\partial_2u\|_{H^2}\\
&\le C\|b\|_{H^3}^2\left(\|\partial_1b\|_{H^2}^2+\|\partial_2u\|_{H^2}^2\right).
\end{aligned}
$$

For J_4 and J_7, we use Lemmas 2.1 and 2.2 to deduce

$$\begin{aligned}
J_4 + J_7 &\le C\|b\|_{L^\infty}\|\partial_2^3 b\|_{L^2}(\|\partial_2^2 u\cdot\nabla\partial_2 b_1\|_{L^2} + \|\partial_2^2 b\cdot\nabla\partial_2 u_1\|_{L^2})\\
&\le C\|b\|_{H^1}^{\frac12}\|\partial_1 b\|_{H^1}^{\frac12}\|b\|_{H^3}\|\nabla\partial_2 u\|_{L^2}^{\frac12}\|\nabla\partial_2^2 u\|_{L^2}^{\frac12}\|\nabla\partial_2 b\|_{L^2}^{\frac12}\|\nabla\partial_1\partial_2 b\|_{L^2}^{\frac12}\\
&\le C\|b\|_{H^3}^2\left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right).
\end{aligned}$$

Using $\nabla\cdot b = 0$ and the Sobolev's embedding inequality, we obtain

$$\begin{aligned}
J_5 + J_6 &\le C\|b_1\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty}\|\partial_2^3 b_2\|_{L^2}\\
&\quad + C\|b_2\|_{L^\infty}\|\partial_2 u_1\|_{L^\infty}\|\partial_2^3 b_1\|_{L^2}^2\\
&\le C\|b\|_{H^3}^2\left(\|b_2\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right).
\end{aligned}$$

For J_{10}, J_{13}, J_{15} and J_{19}, the Sobolev's embedding inequality yields

$$\begin{aligned}
&J_{10} + J_{13} + J_{15} + J_{19}\\
&\quad\le C\|\partial_2^3 u_1\|_{L^2}^2\left(\|\partial_1 u_1\|_{L^\infty} + \|b\|_{L^\infty}\|\nabla u_1\|_{L^\infty} + \|b_1\|_{L^\infty} + \|b_1\|_{L^\infty}^2\right)\\
&\qquad + C\|\partial_2^3 u_1\|_{L^2}^2\|b_1\|_{L^\infty}(\|u\|_{L^\infty}\|\nabla b_1\|_{L^\infty} + \|b\|_{L^\infty}\|\nabla u_1\|_{L^\infty})\\
&\quad\le C\left(\|(u,b)\|_{H^3} + \|(u,b)\|_{H^3}^2 + \|b\|_{H^3}^4\right)\|\partial_2 u\|_{H^2}^2,
\end{aligned}$$

and similarly,

$$\begin{aligned}
J_{11} &\le C\|\partial_2^3 u_1\|_{L^2}\|b_1\|_{L^\infty}(\|\nabla u\|_{L^\infty}\|\nabla\partial_2^2 u\|_{L^2} + \|\partial_2^2 u\|_{L^4}\|\nabla\partial_2 u\|_{L^4})\\
&\le C\|(u,b)\|_{H^3}^2\|\partial_2 u\|_{H^2}^2.
\end{aligned}$$

To estimate J_{12} and J_{18}, we first need to deal with $\|\partial_1\partial_2^3 P\|_{L^2}$. In fact, operating $\nabla\cdot$ to $(1.2)_1$ yields

$$\Delta P = \nabla\cdot(b\cdot\nabla b) - \nabla\cdot(u\cdot\nabla u) - \nu\partial_1 u_1,$$

from which it follows that

$$\partial_1\partial_2^3 P = \partial_1\partial_2^3\Delta^{-1}\nabla\cdot(b\cdot\nabla b) - \partial_1\partial_2^3\Delta^{-1}\nabla\cdot(u\cdot\nabla u) - \nu\partial_1\partial_2^3\Delta^{-1}\partial_1 u_1. \tag{2.27}$$

Due to $\nabla\cdot b = 0$, one has

$$\nabla\cdot(b\cdot\nabla b) = \partial_j(b_i\partial_i b_j) = \partial_j b_i\partial_i b_j.$$

So, using the well known fact that the Riesz operator $\partial_i(-\Delta)^{-\frac12}$ with $i = 1, 2$ is bounded in L^r for any $1 < r < \infty$, we deduce

$$\|\partial_1\partial_2^3\Delta^{-1}\nabla\cdot(b\cdot\nabla b)\|_{L^2} = \|\partial_1\partial_2^3\Delta^{-1}(\partial_j b_i\partial_i b_j)\|_{L^2} \le \|\partial_1\partial_2(\partial_j b_i\partial_i b_j)\|_{L^2}.$$

Noting that

$$\partial_1\partial_2(\partial_j b_i\partial_i b_j) = \partial_1\partial_2\partial_j b_i\partial_i b_j + \partial_2\partial_j b_i\partial_1\partial_i b_j + \partial_1\partial_j b_i\partial_2\partial_i b_j + \partial_j b_i\partial_1\partial_2\partial_i b_j,$$

and hence,

$$\begin{aligned}
\|\partial_1\partial_2^3\Delta^{-1}\nabla\cdot(b\cdot\nabla b)\|_{L^2} &\le \|\partial_1\partial_2(\partial_j b_i\partial_i b_j)\|_{L^2}\\
&\le C(\|\nabla b\|_{L^\infty}\|\partial_1\partial_2\nabla b\|_{L^2} + \|\partial_2\nabla b\|_{L^4}\|\partial_1\nabla b\|_{L^4})\\
&\le C\|\nabla b\|_{H^2}\|\partial_1 b\|_{H^2}.
\end{aligned} \tag{2.28}$$

The analogous estimate also holds for $\|\partial_1\partial_2^3\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\|_{L^2}$, that is,

$$\begin{aligned}\|\partial_1\partial_2^3\Delta^{-1}\nabla\cdot(u\cdot\nabla u)\|_{L^2} &\le C(\|\nabla u\|_{L^\infty}\|\partial_1\partial_2\nabla u\|_{L^2}+\|\partial_2\nabla u\|_{L^4}\|\partial_1\nabla u\|_{L^4})\\ &\le C\|\nabla u\|_{H^2}\|\partial_2 u\|_{H^2}.\end{aligned}\tag{2.29}$$

Thus, inserting (2.28) and (2.29) into (2.27), we arrive at

$$\begin{aligned}\|\partial_1\partial_2^3 P\|_{L^2} &\le \|\partial_1\partial_2(\partial_j b_i\partial_i b_j)\|_{L^2}+\|\partial_1\partial_2(\partial_j u_i\partial_i u_j)\|_{L^2}+\nu\|\partial_1^2\partial_2 u_1\|_{L^2}\\ &\le C(\|\nabla b\|_{H^2}\|\partial_1 b\|_{H^2}+\|\nabla u\|_{H^2}\|\partial_2 u\|_{H^2}+\|\partial_2 u\|_{H^2}).\end{aligned}\tag{2.30}$$

With (2.30) at our disposal, we can now bound J_{12} and J_{18} by

$$\begin{aligned}J_{12}+J_{18} &\le C\|\partial_2^3 u_1\|_{L^2}\|\partial_2^3\partial_1 P\|_{L^2}\left(\|b_1\|_{L^\infty}+\|b_1\|_{L^\infty}^2\right)\\ &\le C\left(\|b\|_{H^3}+\|(u,b)\|_{H^3}^2+\|b\|_{H^3}^3+\|b\|_{H^3}^4\right)\left(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).\end{aligned}$$

For J_{14}, using Lemma 2.1 and Lemma 2.2, we find

$$\begin{aligned}J_{14} &\le C\|\partial_2^3 u_1\|_{L^2}\|b_1\|_{L^\infty}\left(\|\nabla b\|_{L^\infty}\|\nabla\partial_2^2 b\|_{L^2}+\|\partial_2^2 b\cdot\nabla\partial_2 b_1\|_{L^2}\right)\\ &\le C\|\partial_2^3 u_1\|_{L^2}\|b_1\|_{H^1}^{\frac12}\|\partial_1 b_1\|_{H^1}^{\frac12}\|\nabla b\|_{H^1}^{\frac12}\|\partial_1\nabla b\|_{H^1}^{\frac12}\|\nabla\partial_2^2 b_1\|_{L^2}\\ &\quad+C\|\partial_2^3 u_1\|_{L^2}\|b_1\|_{H^1}^{\frac12}\|\partial_1 b_1\|_{H^1}^{\frac12}\|\partial_2^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2^2 b\|_{L^2}^{\frac12}\|\nabla\partial_2 b_1\|_{L^2}^{\frac12}\|\nabla\partial_2^2 b_1\|_{L^2}^{\frac12}\\ &\le C\|b\|_{H^3}^2\left(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).\end{aligned}$$

For J_{16}, it is easily seen that

$$\begin{aligned}J_{16} &= -\frac{27}{2}\int b_1^2 u\cdot\nabla|\partial_2^3 u_1|^2\,dx = 27\int|\partial_2^3 u_1|^2 b_1 u\cdot\nabla b_1\,dx\\ &\le C\|\partial_2^3 u_1\|_{L^2}^2\|b_1\|_{L^\infty}\|u\|_{L^\infty}\|\nabla b_1\|_{L^\infty}\\ &\le C\left(\|u\|_{H^3}^2+\|b\|_{H^3}^4\right)\|\partial_2 u\|_{H^2}^2.\end{aligned}$$

As the treatment of J_{14}, we have

$$\begin{aligned}J_{17} &\le C\|b_1\|_{L^\infty}^2\|\partial_2^3 u_1\|_{L^2}(\|\nabla u\|_{L^\infty}\|\nabla\partial_2^2 u\|_{L^2}+\|\partial_2^2 u\|_{L^4}\|\nabla\partial_2 u_1\|_{L^4})\\ &\le C\left(\|u\|_{H^3}^2+\|b\|_{H^3}^4\right)\|\partial_2 u\|_{H^2}^2,\end{aligned}$$

and

$$
\begin{aligned}
J_{20} &\le C\|b_1\|_{L^\infty}^2\|\partial_2^3 u_1\|_{L^2}(\|\nabla b\|_{L^\infty}\|\nabla\partial_2^2 b\|_{L^2}+\|\partial_2^2 b\|_{L^4}\|\nabla\partial_2 b_1\|_{L^4})\\
&\le C\|b\|_{H^3}^2\|b\|_{H^1}\|\partial_1 b\|_{H^1}\|\partial_2^3 u_1\|_{L^2}\\
&\le C\|b\|_{H^3}^3\left(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}
$$

Due to $\nabla\cdot u=0$, it holds that $\|\nabla u_1\|_{L^\infty}=\|\partial_2 u\|_{L^\infty}$. Thus,

$$
\begin{aligned}
J_{21}+J_{24} &\le C\|b\|_{L^\infty}^2\|\partial_2^3 b_1\|_{L^2}^2\|\nabla u_1\|_{L^\infty}\\
&\quad + C\|b\|_{L^\infty}^2\|\partial_2^3 u_1\|_{L^2}\|\partial_2^3 b_1\|_{L^2}\|\nabla b_1\|_{L^\infty}\\
&\le C\|b\|_{H^1}\|\partial_1 b\|_{H^1}\|b\|_{H^3}^2\|\partial_2 u\|_{H^2}\\
&\le C\|b\|_{H^3}^3\left(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
J_{22}+J_{23} &\le C\|b_1\|_{L^\infty}^2\|\partial_2^3 b_1\|_{L^2}\|\nabla u_1\|_{L^\infty}\|\nabla\partial_2^2 b\|_{L^2}\\
&\quad + C\|b_1\|_{L^\infty}^2\|\partial_2^3 b_1\|_{L^2}\|\nabla\partial_2 u\|_{L^4}\|\nabla\partial_2 b\|_{L^4}\\
&\quad + C\|b_1\|_{L^\infty}^2\|\partial_2^3 b_1\|_{L^2}\|\nabla\partial_2^2 u\|_{L^2}\|\nabla b\|_{L^\infty}\\
&\le C\|b\|_{H^3}^2\|b\|_{H^1}\|\partial_1 b\|_{H^1}\|\partial_2 u\|_{H^2}\\
&\le C\|b\|_{H^3}^3\left(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}
$$

Thus, noting that $\|\partial_1 b\|_{H^2}=\|\nabla b_2\|_{H^2}$, we conclude after inserting the above estimates of $J_1,\dots,J_{24}$ in (2.26) and using the Cauchy–Schwarz's inequality that

$$
\begin{aligned}
D_1 &\le 3\frac{d}{dt}\int b_1\left(|\partial_2^3 b_1|^2+|\partial_2^3 u_1|^2\right)dx-\frac{27}{2}\frac{d}{dt}\int b_1^2\left(|\partial_2^3 b_1|^2+|\partial_2^3 u_1|^2\right)dx\\
&\quad + C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^4\right)\left(\|b_2\|_{H^3}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}
\tag{2.31}
$$

In view of (2.12), (2.13), (2.14), (2.15) and (2.31), we obtain

$$
\begin{aligned}
K_4 &\le 3\frac{d}{dt}\int b_1\left(|\partial_2^3 b_1|^2+|\partial_2^3 u_1|^2\right)dx-\frac{27}{2}\frac{d}{dt}\int b_1^2\left(|\partial_2^3 b_1|^2+|\partial_2^3 u_1|^2\right)dx\\
&\quad + C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^4\right)\left(\|b_2\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}
\tag{2.32}
$$

It remains to estimate K_5. To do this, noting that

$$
\begin{aligned}
K_5 &= \int\left(\partial_1^3(b\cdot\nabla u)-b\cdot\nabla\partial_1^3 u\right)\cdot\partial_1^3 b\,dx\\
&\quad + \int\left(\partial_2^3(b\cdot\nabla u)-b\cdot\nabla\partial_2^3 u\right)\cdot\partial_2^3 b\,dx := K_{51}+K_{52},
\end{aligned}
$$

where the first term on the right-hand side can be easily bounded by

$$
\begin{aligned}
K_{51} &= \int(3\partial_1 b\cdot\nabla\partial_1^2 u+3\partial_1^2 b\cdot\nabla\partial_1 u+\partial_1^3 b\cdot\nabla u)\cdot\partial_1^3 b\,dx\\
&\le C\left(\|\partial_1 b\|_{L^\infty}\|\nabla\partial_1^2 u\|_{L^2}+\|\partial_1^2 b\|_{L^4}\|\nabla\partial_1 u\|_{L^4}+\|\nabla u\|_{L^\infty}\|\partial_1^3 b\|_{L^2}\right)\|\partial_1^3 b\|_{L^2}\\
&\le C\|u\|_{H^3}\|\partial_1 b\|_{H^2}^2.
\end{aligned}
\tag{2.33}
$$

To deal with K_{52}, we rewrite it as

$$
\begin{aligned}
K_{52} &= \int \left(3\partial_2 b \cdot \nabla \partial_2^2 u \cdot \partial_2^3 b + 3\partial_2^2 b \cdot \nabla \partial_2 u \cdot \partial_2^3 b + \partial_2^3 b \cdot \nabla u \cdot \partial_2^3 b\right) dx \\
&= 3\int \partial_2 b \cdot \nabla \partial_2^2 u \cdot \partial_2^3 b \, dx + 3\int \partial_2^2 b \cdot \nabla \partial_2 u \cdot \partial_2^3 b \, dx \\
&\quad - \int \partial_1 \partial_2^2 b_1 \partial_2 u \cdot \partial_2^3 b \, dx - \int \partial_2^3 b_1 \partial_1 u_2 \partial_1 \partial_2^2 b_1 \, dx \\
&\quad + \int \partial_1 u_1 |\partial_2^3 b_1|^2 \, dx := K_{521} + K_{522} + K_{523} + K_{524} + \frac{1}{3} D_1.
\end{aligned}
$$

Based upon integration by parts and the divergence-free condition $\nabla \cdot b = 0$, we deduce from the Sobolev's inequalities that

$$
\begin{aligned}
K_{521} &= 3\int \partial_2 b_1 \partial_1 \partial_2^2 u \cdot \partial_2^3 b \, dx + 3\int \partial_2 b_2 \partial_2^3 u \cdot \partial_2^3 b \, dx \\
&= -3\int \partial_1 \partial_2 b_1 \partial_2^2 u \cdot \partial_2^3 b \, dx + 3\int \partial_2^2 b_1 \partial_2^2 u \cdot \partial_1 \partial_2^2 b \, dx \\
&\quad + 3\int \partial_2 b_1 \partial_2^3 u \cdot \partial_1 \partial_2^2 b \, dx - 3\int \partial_1 b_1 \partial_2^3 u \cdot \partial_2^3 b \, dx \\
&\le C\|\partial_1 \partial_2 b_1\|_{L^4} \|\partial_2^2 u\|_{L^4} \|\partial_2^3 b\|_{L^2} + C\|\partial_2^2 b_1\|_{L^4} \|\partial_2^2 u\|_{L^4} \|\partial_1 \partial_2^2 b\|_{L^2} \\
&\quad + C\|\partial_2 b_1\|_{L^\infty} \|\partial_2^3 u\|_{L^2} \|\partial_1 \partial_2^2 b\|_{L^2} + C\|\partial_1 b_1\|_{L^\infty} \|\partial_2^3 u\|_{L^2} \|\partial_2^3 b\|_{L^2} \\
&\le C\|b\|_{H^3} \left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right),
\end{aligned} \tag{2.34}
$$

and similarly,

$$
\begin{aligned}
K_{522} &= 3\int \partial_2^2 b_1 \partial_1 \partial_2 u \cdot \partial_2^3 b \, dx + 3\int \partial_2^2 b_2 \partial_2^2 u \cdot \partial_2^3 b \, dx \\
&= -3\int \partial_1 \partial_2^2 b_1 \partial_2 u \cdot \partial_2^3 b \, dx + 3\int \partial_2^3 b_1 \partial_2 u \cdot \partial_1 \partial_2^2 b \, dx \\
&\quad + 3\int \partial_2^2 b_1 \partial_2^2 u \cdot \partial_1 \partial_2^2 b \, dx - 3\int \partial_1 \partial_2 b_1 \partial_2^2 u \cdot \partial_2^3 b \, dx \\
&\le C(\|\partial_1 \partial_2^2 b\|_{L^2} \|\partial_2 u\|_{L^\infty} + \|\partial_1 \partial_2 b_1\|_{L^4} \|\partial_2^2 u\|_{L^4}) \|\partial_2^3 b\|_{L^2} \\
&\quad + C\|\partial_2^2 b_1\|_{L^4} \|\partial_2^2 u\|_{L^4} \|\partial_1 \partial_2^2 b\|_{L^2} \\
&\le C\|b\|_{H^3} \left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right).
\end{aligned} \tag{2.35}
$$

For K_{523} and K_{524}, we have

$$
\begin{aligned}
K_{523} + K_{524} &\le C\|\partial_1 \partial_2^2 b_1\|_{L^2} \|\partial_2^3 b\|_{L^2} \left(\|\partial_2 u\|_{L^\infty} + \|\partial_1 u_2\|_{L^\infty}\right) \\
&\le C\|b\|_{H^3} \left(\|\partial_1 b\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\right).
\end{aligned} \tag{2.36}
$$

Thus, combining (2.33), (2.34), (2.35), (2.36) with (2.31) gives

$$
\begin{aligned}
K_5 &\le \frac{d}{dt}\int b_1 \left(|\partial_2^3 b_1|^2 + |\partial_2^3 u_1|^2\right) dx - \frac{9}{2}\frac{d}{dt}\int b_1^2 \left(|\partial_2^3 b_1|^2 + |\partial_2^3 u_1|^2\right) dx \\
&\quad + C\left(\|(u,b)\|_{H^3} + \|(u,b)\|_{H^3}^4\right)\left(\|b_2\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2\right).
\end{aligned} \tag{2.37}
$$

Now, substituting (2.8), (2.10), (2.11), (2.32) and (2.37) into (2.7), we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla^3 u, \nabla^3 b)\|_{L^2}^2 + \nu\|\nabla^3 u_1\|_{L^2}^2 + \eta\|\nabla^3 b_2\|_{L^2}^2 \\
&\quad \le 4\frac{d}{dt}\int b_1\left(|\partial_2^3 b_1|^2 + |\partial_2^3 u_1|^2\right)dx - 18\frac{d}{dt}\int b_1^2\left(|\partial_2^3 b_1|^2 + |\partial_2^3 u_1|^2\right)dx \\
&\qquad + C\left(\|(u,b)\|_{H^3} + \|(u,b)\|_{H^3}^4\right)\left(\|b_2\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\right),
\end{aligned}
$$

which, integrated over $[0, t]$ and combined with the Sobolev's inequalities, yields

$$
\begin{aligned}
&\|\nabla^3(u,b)(t)\|_{L^2}^2 + 2\int_0^t\left(\nu\|\nabla^3 u_1\|_{L^2}^2 + \eta\|\nabla^3 b_2\|_{L^2}^2\right)d\tau \\
&\quad \le C\left(\|(u_0,b_0)\|_{H^3}^2 + \|(u_0,b_0)\|_{H^3}^3 + \|(u_0,b_0)\|_{H^3}^4\right) \\
&\qquad + 8\int b_1\left(|\partial_2^3 b_1|^2 + |\partial_2^3 u_1|^2\right)dx - 36\int b_1^2\left(|\partial_2^3 b_1|^2 + |\partial_2^3 u_1|^2\right)dx \\
&\qquad + C\int_0^t\left(\|(u,b)\|_{H^3} + \|(u,b)\|_{H^3}^4\right)\left(\|b_2\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^2}^2\right)d\tau \\
&\quad \le C\left(\|(u_0,b_0)\|_{H^3}^2 + \|(u_0,b_0)\|_{H^3}^3 + \|(u_0,b_0)\|_{H^3}^4\right) \\
&\qquad + C\left(\|b_1(t)\|_{L^\infty} + \|b_1(t)\|_{L^\infty}^2\right)\|(u,b)(t)\|_{H^3}^2 \\
&\qquad + C\sup_{0\le\tau\le t}\left(\|(u,b)\|_{H^3} + \|(u,b)\|_{H^3}^4\right)\int_0^t\left(\|(u_1,b_2)\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2\right)d\tau,
\end{aligned}
\tag{2.38}
$$

due to the fact that $\|\partial_2 u\|_{H^2} = \|\nabla u_1\|_{H^2}$. Thus, it readily follows from (2.6) and (2.38) that

$$
\begin{aligned}
\mathcal{E}_1(t) &\le C\mathcal{E}_1(0) + C\mathcal{E}_1(0)^{\frac{3}{2}} + C\mathcal{E}_1(0)^2 \\
&\quad + C\mathcal{E}_1(t)^{\frac{3}{2}} + C\mathcal{E}_2(t)^{\frac{3}{2}} + C\mathcal{E}_1(t)^3 + C\mathcal{E}_2(t)^3.
\end{aligned}
$$

The proof of the first assertion (2.1) in Proposition 2.1 is therefore complete.

2.2 Proof of (2.2)

Since $\|\partial_1 u\|_{H^2} \sim \|\partial_1 u\|_{L^2} + \|\nabla^2\partial_1 u\|_{L^2}$, it suffices to establish the estimates of the following two items:

$$
\int_0^t \|\partial_1 u(\tau)\|_{L^2}^2 d\tau \quad \text{and} \quad \int_0^{\mathrm{t}} \|\nabla^2 @_1 \mathrm{u}(\o)\|_{\mathrm{L}^2}^2 \mathrm{d}\o,
$$

whose proofs are based on the special struture of equation $(1.2)_2$,

$$
\partial_1 u = \partial_t b + u\cdot\nabla b + \eta(0, b_2)^\top - b\cdot\nabla u. \tag{2.39}
$$

First, to bound $\|\partial_1 u(\tau)\|_{L^2}$, we multiply (2.39) by $\partial_1 u$ in L^2 and integrate by parts over $\mathbb{R}^2$ to get

$$
\begin{aligned}
\|\partial_1 u\|_{L^2}^2 &= \int \partial_1 u\cdot\partial_t b\,dx + \int u\cdot\nabla b\cdot\partial_1 u\,dx \\
&\quad + \eta\int b_2\partial_1 u_2\,dx - \int b\cdot\nabla u\cdot\partial_1 u\,dx \\
&:= L_1 + L_2 + L_3 + L_4.
\end{aligned}
\tag{2.40}
$$

Using the velocity equation in (1.2)$_1$ and the fact that $\nabla \cdot b = 0$, we have

$$
\begin{aligned}
L_1 &= \frac{d}{dt}\int \partial_1 u \cdot b\, dx - \int b \cdot \partial_1 \Big(\partial_1 b - \nu (u_1, 0)^\top + b \cdot \nabla b - u \cdot \nabla u\Big)\, dx \\
&:= L_{11} + L_{12} + L_{13} + L_{14} + L_{15}.
\end{aligned}
$$

It is easily seen that

$$
\begin{aligned}
L_{12} + L_{13} &= \int \partial_1 b \cdot \partial_1 b\, dx - \nu \int \partial_1 b_1 u_1\, dx \\
&\le C\|\partial_1 b\|_{H^1}^2 + C\|\partial_1 b\|_{L^2}\|u_1\|_{L^2}.
\end{aligned}
$$

Integrating by parts and using Sobolev's embedding inequality, we find

$$
\begin{aligned}
L_{14} &= -\int b \cdot \partial_1 (b \cdot \nabla b)\, dx = \int \partial_1 b \cdot (b \cdot \nabla b)\, dx \\
&= \int b_1 \partial_1 b \cdot \partial_1 b\, dx + \int b_2 \partial_2 b \cdot \partial_1 b\, dx \\
&\le C\|b_1\|_{L^\infty}\|\partial_1 b\|_{L^2}^2 + C\|b_2\|_{L^\infty}\|\partial_2 b\|_{L^2}\|\partial_1 b\|_{L^2} \\
&\le C\|b\|_{H^2}\|b_2\|_{H^2}^2,
\end{aligned}
$$

where we have used the fact that $\|\partial_1 b\|_{L^2} = \|\nabla b_2\|_{L^2}$ due to $\nabla \cdot b = 0$. By virtue of Lemma 2.1, we have

$$
\begin{aligned}
L_{15} &= \int b \cdot \partial_1 (u \cdot \nabla u)\, dx = -\int \partial_1 b \cdot (u \cdot \nabla u)\, dx \\
&\le C\|\partial_1 b\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_1 \nabla u\|_{L^2}^{\frac12} \\
&\le C\|u\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2\right).
\end{aligned}
$$

Thus, collecting the estimates of $L_{12}, \ldots, L_{15}$ together, we obtain

$$
\begin{aligned}
L_1 \le{}& \frac{d}{dt}\int \partial_1 u \cdot b\, dx + C\left(\|b_2\|_{H^2}^2 + \|u_1\|_{L^2}^2\right) \\
&+ C\|(u, b)\|_{H^2}\left(\|b_2\|_{H^2}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2\right),
\end{aligned}
$$

since $\|\partial_1 b\|_{H^1} \le \|b_2\|_{H^2}$. In a similar manner,

$$
\begin{aligned}
L_2 &\le C\|\partial_1 u\|_{L^2}\|u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_1 \nabla b\|_{L^2}^{\frac12} \\
&\le C\|(u, b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2\right), \\
L_3 &\le C\|b_2\|_{L^2}\|\partial_1 u_2\|_{L^2} \le \frac12\|\partial_1 u\|_{L^2}^2 + C\|b_2\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
L_4 &\le C\|\partial_1 u\|_{L^2}\|b\|_{L^2}^{\frac12}\|\partial_1 b\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2 \nabla u\|_{L^2}^{\frac12} \\
&\le C\|(u, b)\|_{H^2}\left(\|\partial_1 b\|_{H^1}^2 + \|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^1}^2\right),
\end{aligned}
$$

which, combined with the estimate of L_1 and (2.40), shows that

$$\|\partial_1 u\|_{L^2}^2 \le 2\frac{d}{dt}\int \partial_1 u\cdot b\,dx + C\left(\|b_2\|_{H^2}^2+\|u_1\|_{L^2}^2\right)$$
$$+ C\|(u,b)\|_{H^2}\left(\|b_2\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{H^1}^2\right). \tag{2.41}$$

This leads to the desired estimate of $\|\partial_1 u\|_{L^2}$.

Next, we proceed to estimate $\|\nabla^2\partial_1 u\|_{L^2}$. To do this, applying ∇^2 to (2.39), and dotting it with $\nabla^2\partial_1 u$ in L^2, we deduce

$$\|\nabla^2\partial_1 u\|_{L^2}^2 = \int \nabla^2\partial_1 u\cdot\partial_t\nabla^2 b\,dx + \int \nabla^2(u\cdot\nabla b)\cdot\nabla^2\partial_1 u\,dx$$
$$+\eta\int\nabla^2\partial_1 u_2\cdot\nabla^2 b_2\,dx - \int\nabla^2(b\cdot\nabla u)\cdot\nabla^2\partial_1 u\,dx$$
$$:= M_1+M_2+M_3+M_4. \tag{2.42}$$

Owing to $(1.2)_1$ and $\nabla\cdot b=0$, we see that

$$M_1 = \frac{d}{dt}\int\nabla^2\partial_1 u\cdot\nabla^2 b\,dx$$
$$-\int\nabla^2 b\cdot\nabla^2\partial_1\left(\partial_1 b-\nu(u_1,0)^\top + b\cdot\nabla b - u\cdot\nabla u\right)dx$$
$$:= M_{11}+M_{12}+M_{13}+M_{14}+M_{15}.$$

Integrating by parts gives

$$M_{12}+M_{13} = \int\nabla^2\partial_1 b\cdot\nabla^2\partial_1 b\,dx - \nu\int\partial_1\nabla^2 b_1\cdot\nabla^2 u_1\,dx$$
$$\le C\|\partial_1 b\|_{H^2}^2 + C\|\partial_1 b\|_{H^2}\|u_1\|_{H^2}.$$

Due to $\|\nabla b_2\|_{H^k} = \|\partial_1 b\|_{H^k}$ for $k=1,2$, we have

$$M_{14} = -\int\nabla^2 b\cdot\nabla^2\partial_1(b\cdot\nabla b)\,dx = \int\partial_1\nabla^2 b\cdot\nabla^2(b\cdot\nabla b)\,dx$$
$$=\int\partial_1\nabla^2 b\cdot\left(\nabla^2 b_1\partial_1 b+\nabla^2 b_2\partial_2 b\right)dx$$
$$+2\int\partial_1\nabla^2 b\cdot(\nabla b_1\partial_1\nabla b+\nabla b_2\partial_2\nabla b)\,dx$$
$$+\int\left(b_1|\partial_1\nabla^2 b|^2 + b_2\partial_2\nabla^2 b\cdot\partial_1\nabla^2 b\right)dx$$
$$\le C\|\partial_1\nabla^2 b\|_{L^2}\left(\|\partial_1 b\|_{L^4}\|\nabla^2 b_1\|_{L^4}+\|\partial_2 b\|_{L^4}\|\nabla^2 b_2\|_{L^4}\right)$$
$$+C\|\partial_1\nabla^2 b\|_{L^2}\left(\|\nabla b_1\|_{L^4}\|\partial_1\nabla b\|_{L^4}+\|\nabla b_2\|_{L^4}\|\partial_2\nabla b\|_{L^4}\right)$$
$$+C\left(\|b_1\|_{L^\infty}\|\partial_1\nabla^2 b\|_{L^2}^2+\|b_2\|_{L^\infty}\|\partial_2\nabla^2 b\|_{L^2}\|\partial_1\nabla^2 b\|_{L^2}\right)$$
$$\le C\|b\|_{H^3}\|b_2\|_{H^3}^2.$$

Analogously, noting that $\|\nabla u_2\|_{H^k} = \|\partial_1 u\|_{H^k}$ and $\|\nabla u_1\|_{H^k} = \|\partial_2 u\|_{H^k}$ for $k=1,2$, we obtain

$$M_{15} = \int\nabla^2 b\cdot\nabla^2\partial_1(u\cdot\nabla u)\,dx = -\int\partial_1\nabla^2 b\cdot\nabla^2(u\cdot\nabla u)\,dx$$
$$\le C\|\partial_1\nabla^2 b\|_{L^2}\left(\|\partial_1 u\|_{L^4}\|\nabla^2 u_1\|_{L^4}+\|\partial_2 u\|_{L^4}\|\nabla^2 u_2\|_{L^4}\right)$$
$$+C\|\partial_1\nabla^2 b\|_{L^2}\left(\|\nabla u_1\|_{L^4}\|\partial_1\nabla u\|_{L^4}+\|\nabla u_2\|_{L^4}\|\partial_2\nabla u\|_{L^4}\right)$$

$$
\begin{aligned}
&+C\|\partial_1\nabla^2 b\|_{L^2}\left(\|u_1\|_{L^\infty}\|\partial_1\nabla^2 u\|_{L^2}+\|u_2\|_{L^\infty}\|\partial_2\nabla^2 u\|_{L^2}\right)\\
&\le C\|u\|_{H^3}\left(\|b_2\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}
$$

Hence, in terms of the estimates of M_{1i} with $i=2,\ldots,5$, we can bound M_1 by

$$
\begin{aligned}
M_1 \le& \frac{d}{dt}\int \nabla^2\partial_1 u\cdot\nabla^2 b\,dx + C\left(\|b_2\|_{H^3}^2+\|u_1\|_{H^3}^2\right)\\
&+C\|(u,b)\|_{H^3}\left(\|b_2\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}
$$

For M_2, by Lemma 2.1 we infer from integration by parts that

$$
\begin{aligned}
M_2 =& \int \nabla^2(u\cdot\nabla b)\cdot\nabla^2\partial_1 u\,dx\\
=& \int \nabla^2 u\cdot\nabla b\cdot\nabla^2\partial_1 u\,dx + 2\int \nabla u_i\cdot\partial_i\nabla b\cdot\nabla^2\partial_1 u\,dx\\
&+\int u_1\partial_1\nabla^2 b\cdot\partial_1\nabla^2 u\,dx - \int \partial_1 u_2\partial_2\nabla^2 b\cdot\nabla^2 u\,dx\\
&+\int \partial_2 u_2\partial_1\nabla^2 b\cdot\nabla^2 u\,dx + \int u_2\partial_1\nabla^2 b\cdot\partial_2\nabla^2 u\,dx\\
\le& C\|\partial_1\nabla^2 u\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac12}\|\partial_2\nabla^2 u\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_1\nabla b\|_{L^2}^{\frac12}\\
&+C\|\partial_1\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}\|\partial_1\nabla^2 b\|_{L^2}^{\frac12}\\
&+C\|u_1\|_{L^\infty}\|\partial_1\nabla^2 u\|_{L^2}\|\partial_1\nabla^2 b\|_{L^2}+C\|\partial_1 u_2\|_{L^4}\|\partial_2\nabla^2 b\|_{L^2}\|\nabla^2 u\|_{L^4}\\
&+C\|\partial_2 u_2\|_{L^4}\|\partial_1\nabla^2 b\|_{L^2}\|\nabla^2 u\|_{L^4}+C\|u_2\|_{L^\infty}\|\partial_1\nabla^2 b\|_{L^2}\|\partial_2\nabla^2 u\|_{L^2}\\
\le& C\|(u,b)\|_{H^3}\left(\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 b\|_{H^2}^2\right).
\end{aligned}
$$

Obviously, $M_3,\ M_4$ can be bounded as follows.

$$
M_3 \le C\|\nabla^2 b_2\|_{L^2}\|\nabla^2\partial_1 u_2\|_{L^2}\le \frac12\|\partial_1\nabla^2 u\|_{L^2}^2 + C\|\nabla^2 b_2\|_{L^2}^2,
$$

and

$$
\begin{aligned}
M_4 =& -\int \nabla^2\partial_1 u\cdot\left(\nabla^2 b\cdot\nabla u + 2\nabla b_i\cdot\partial_i\nabla u + b_i\partial_i\nabla^2 u\right)\,dx\\
\le& C\|\partial_1\nabla^2 u\|_{L^2}\|\nabla^2 b\|_{L^2}^{\frac12}\|\partial_1\nabla^2 b\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\\
&+C\|\partial_1\nabla^2 u\|_{L^2}\|\nabla b\|_{L^2}^{\frac12}\|\partial_1\nabla b\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12}\|\partial_2\nabla^2 u\|_{L^2}^{\frac12}\\
&+C\|b\|_{L^\infty}\|\nabla^2\partial_1 u\|_{L^2}\|\nabla^3 u\|_{L^2}\\
\le& C\|(u,b)\|_{H^3}\left(\|\partial_1 b\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{H^2}^2\right).
\end{aligned}
$$

Thus, it follows from (2.42) and the estimates of M_i $(i=1,\ldots,4)$ that

$$
\begin{aligned}
\|\partial_1\nabla^2 u\|_{L^2}^2 \le& 2\frac{d}{dt}\int \nabla^2\partial_1 u\cdot\nabla^2 b\,dx + C\left(\|b_2\|_{H^3}^2+\|u_1\|_{H^3}^2\right)\\
&+C\|(u,b)\|_{H^3}\left(\|b_2\|_{H^3}^2+\|\partial_1 u\|_{H^2}^2+\|\partial_2 u\|_{H^2}^2\right).
\end{aligned}
\tag{2.43}
$$

Now, adding up (2.41) and (2.43), we deduce

$$
\begin{aligned}
\|\partial_1 u\|_{H^2}^2 \leq 2\frac{d}{dt}\int \left(\partial_1 u\cdot b + \nabla^2\partial_1 u\cdot\nabla^2 b\right)dx + C\left(\|u_1\|_{H^3}^2 + \|b_2\|_{H^3}^2\right)\\
+ C\|(u,b)\|_{H^3}\left(\|b_2\|_{H^3}^2 + \|u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2\right),
\end{aligned}
$$

where we have also used $\|\nabla b_2\|_{H^k} = \|\partial_1 b\|_{H^k}$ and $\|\partial_2 u\|_{H^k} = \|\nabla u_1\|_{H^k}$ for $k = 1, 2$. As an immediate result,

$$
\begin{aligned}
\int_0^t \|\partial_1 u\|_{H^2}^2 d\tau \leq C\|(u_0,b_0)\|_{H^3}^2 + C\|(u,b)\|_{H^3}^2 + C\int_0^t \left(\|u_1\|_{H^3}^2 + \|b_2\|_{H^3}^2\right)d\tau\\
+ C\sup_{0\leq\tau\leq t}\|(u,b)\|_{H^3}\int_0^t\left(\|b_2\|_{H^3}^2 + \|u_1\|_{H^3}^2 + \|\partial_1 u\|_{H^2}^2\right)d\tau,
\end{aligned}
$$

from which it readily follows that

$$
\mathcal{E}_2(t) \leq C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1(t)^{\frac{3}{2}} + C\mathcal{E}_2(t)^{\frac{3}{2}}.
$$

The proof of (2.2) is therefore complete.

3 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2 by making full use of the symmetric structure of linearized system (1.12).

Proof of Theorem 1.2 Taking the inner product of (1.12) with (u, b) in H^1, we have

$$
\frac{d}{dt}A(t) + B(t) = 0, \tag{3.1}
$$

where

$$
\begin{aligned}
&A(t) = \|(u,b)(t)\|_{L^2}^2 + \|(\nabla u,\nabla b)(t)\|_{L^2}^2,\\
&B(t) = 2\nu\|\mathcal{R}_2 u(t)\|_{L^2}^2 + 2\eta\|\mathcal{R}_1 b(t)\|_{L^2}^2 + 2\nu\|\nabla\mathcal{R}_2 u(t)\|_{L^2}^2 + 2\eta\|\nabla\mathcal{R}_1 b(t)\|_{L^2}^2.
\end{aligned}
$$

Next, we compute the norm of (u, b) in anisotropic Sobolev space with negative indices. Applying $\Lambda_1^{-\sigma}$ and $\Lambda_2^{-\sigma}$ to (1.12) and dotting them with $(\Lambda_1^{-\sigma}u, \Lambda_1^{-\sigma}b)$ and $(\Lambda_2^{-\sigma}u, \Lambda_2^{-\sigma}b)$ in $H^{1+\sigma}$, respectively, we find

$$
\begin{aligned}
&\frac{d}{dt}H(t) + 2\nu\|\mathcal{R}_2(\Lambda_1^{-\sigma},\Lambda_2^{-\sigma})u(t)\|_{L^2}^2 + 2\eta\|\mathcal{R}_1(\Lambda_1^{-\sigma},\Lambda_2^{-\sigma})b(t)\|_{L^2}^2\\
&\quad + 2\nu\|\mathcal{R}_2\Lambda^{1+\sigma}(\Lambda_1^{-\sigma},\Lambda_2^{-\sigma})u(t)\|_{L^2}^2 + 2\eta\|\mathcal{R}_1\Lambda^{1+\sigma}(\Lambda_1^{-\sigma},\Lambda_2^{-\sigma})b(t)\|_{L^2}^2 = 0,
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
H(t) = \|(\Lambda_1^{-\sigma},\Lambda_2^{-\sigma})u(t)\|_{L^2}^2 + \|(\Lambda_1^{-\sigma},\Lambda_2^{-\sigma})b(t)\|_{L^2}^2\\
+ \|\Lambda^{1+\sigma}(\Lambda_1^{-\sigma},\Lambda_2^{-\sigma})u(t)\|_{L^2}^2 + \|\Lambda^{1+\sigma}(\Lambda_1^{-\sigma},\Lambda_2^{-\sigma})b(t)\|_{L^2}^2.
\end{aligned}
$$

We claim that there exists a generic positive constant $C > 0$, depending only on ν and η, such that

$$
A(t) \leq CB(t)^{\frac{\sigma}{1+\sigma}}H(t)^{\frac{1}{1+\sigma}}. \tag{3.3}
$$

In fact, using Plancherel theorem and Hölder's inequality, we have from direct calculations that

$$\|u(t)\|_{L^2}^2 \le C\|\nabla\mathcal{R}_2 u(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda_2^{-\sigma} u(t)\|_{L^2}^{\frac{2}{1+\sigma}} \le CB(t)^{\frac{\sigma}{1+\sigma}} H(t)^{\frac{1}{1+\sigma}},$$
$$\|\nabla u(t)\|_{L^2}^2 \le C\|\nabla\mathcal{R}_2 u(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda^{1+\sigma}\Lambda_2^{-\sigma} u(t)\|_{L^2}^{\frac{2}{1+\sigma}} \le CB(t)^{\frac{\sigma}{1+\sigma}} H(t)^{\frac{1}{1+\sigma}},$$
$$\|b(t)\|_{L^2}^2 \le C\|\nabla\mathcal{R}_1 b(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda_1^{-\sigma} b(t)\|_{L^2}^{\frac{2}{1+\sigma}} \le CB(t)^{\frac{\sigma}{1+\sigma}} H(t)^{\frac{1}{1+\sigma}},$$
$$\|\nabla b(t)\|_{L^2}^2 \le C\|\nabla\mathcal{R}_1 b(t)\|_{L^2}^{\frac{2\sigma}{1+\sigma}} \|\Lambda^{1+\sigma}\Lambda_1^{-\sigma} b(t)\|_{L^2}^{\frac{2}{1+\sigma}} \le CB(t)^{\frac{\sigma}{1+\sigma}} H(t)^{\frac{1}{1+\sigma}},$$

from which the assertion (3.3) follows.

It is easily seen from (3.2) that $H(t)$ is non-increasing, and $H(t) \le H(0)$. Hence, by (3.3) we have

$$A(t) \le CB(t)^{\frac{\sigma}{1+\sigma}} H(0)^{\frac{1}{1+\sigma}} \quad \text{or} \quad B(t) \ge CH(0)^{-\frac{1}{\sigma}} A(t)^{1+\frac{1}{\sigma}},$$

which, inserted in(3.1), yields

$$\frac{d}{dt}A(t) + CH(0)^{-\frac{1}{\sigma}} A(t)^{1+\frac{1}{\sigma}} \le 0,$$

so that

$$A(t) \le \left(A(0)^{-\frac{1}{\sigma}} + \frac{C}{\sigma} H(0)^{-\frac{1}{\sigma}} t\right)^{-\sigma}.$$

This finishes the proof of Theorem 1.2. □

4 Proofs of Theorems 1.3 and 1.4

This section aims to prove Theorems 1.3 and 1.4, based on the special wave structure of the linearized system (1.13). To begin, we first recall the following elementary lemma, which provides a precise decay rate for a nonnegative integrable function when it decreases in a generalized sense.

Lemma 4.1 *For given positive constants $C_0 > 0$ and $C_1 > 0$, assume that $f = f(t)$ is a nonnegative function defined on $[0, \infty)$ and satisfies,*

$$\int_0^\infty f(\tau)\, d\tau \le C_0 < \infty, \quad \textit{and} \quad f(t) \le C_1 f(s), \quad \forall\, 0 \le s < t.$$

Then there exists a positive constant $C_2 := \max\{2C_1 f(0), 4C_0C_1\}$ such that

$$f(t) \le C_2(1+t)^{-1}, \quad \forall\, t \ge 0.$$

Proof On the one hand, when $0 \le t \le 1$, it holds that

$$f(t) \le C_1 f(0).$$

On the other hand, when $t \ge 1$, one has

$$C_0 \ge \int_{\frac{t}{2}}^t f(\tau)\, d\tau \ge C_1^{-1} \int_{\frac{t}{2}}^t f(t)\, d\tau = \frac{t}{2C_1} f(t),$$

which implies that

$$f(t) \le 2C_0C_1 t^{-1}, \quad \forall\, t \ge 1.$$

Combining the above two cases leads to the desired decay estimate. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 Dotting $(1.13)_1$ with $\partial_t u$ in L^2, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\partial_t u\|_{L^2}^2+\nu\eta\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2+\|\partial_1 u\|_{L^2}^2\right)$$
$$+\nu\|\partial_t\mathcal{R}_2 u\|_{L^2}^2+\eta\|\partial_t\mathcal{R}_1 u\|_{L^2}^2=0, \tag{4.1}$$

and hence,

$$\frac{d}{dt}\left(\|\partial_t u\|_{L^2}^2+\nu\eta\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2+\|\partial_1 u\|_{L^2}^2\right)\le 0. \tag{4.2}$$

Multiplying $(1.13)_1$ by u in L^2 and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\left(\eta\|\mathcal{R}_1 u\|_{L^2}^2+\nu\|\mathcal{R}_2 u\|_{L^2}^2+2\langle\partial_t u,u\rangle\right)$$
$$+\|\partial_1 u\|_{L^2}^2+\nu\eta\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2-\|\partial_t u\|_{L^2}^2=0. \tag{4.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard L^2-inner product.

Let $\delta:=\min\{\nu,\eta\}$. For a constant $\mu>0$ to be specified later, we obtain after adding (4.1) and $\mu\times$ (4.3) together that

$$\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t u\|_{L^2}^2+\mu\eta\|\mathcal{R}_1 u\|_{L^2}^2+\mu\nu\|\mathcal{R}_2 u\|_{L^2}^2+\nu\eta\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2+\|\partial_1 u\|_{L^2}^2+2\mu\langle\partial_t u,u\rangle\Big)$$
$$+(\delta-\mu)\|\partial_t u\|_{L^2}^2+\mu\nu\eta\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2+\mu\|\partial_1 u\|_{L^2}^2\le 0, \tag{4.4}$$

since $\|\partial_t\mathcal{R}_2 u\|_{L^2}^2+\|\partial_t\mathcal{R}_1 u\|_{L^2}^2=\|\partial_t u\|_{L^2}^2$. By choosing $\mu=\frac{\delta}{4}$, we see that

$$\begin{aligned}\frac{1}{2}\|\partial_t u\|_{L^2}^2+\frac{1}{8}\delta^2\|u\|_{L^2}^2&\le\|\partial_t u\|_{L^2}^2+\mu\delta\|u\|_{L^2}^2+2\mu\langle\partial_t u,u\rangle\\&\le\|\partial_t u\|_{L^2}^2+\mu\eta\|\mathcal{R}_1 u\|_{L^2}^2+\mu\nu\|\mathcal{R}_2 u\|_{L^2}^2+2\mu\langle\partial_t u,u\rangle,\end{aligned} \tag{4.5}$$

Thus, by virtue of (4.5), we deduce after integrating (4.4) over $(0,t)$ that

$$\begin{aligned}&\frac{1}{2}\|\partial_t u\|_{L^2}^2+\frac{1}{8}\delta^2\|u\|_{L^2}^2+\nu\eta\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2+\|\partial_1 u\|_{L^2}^2\\&\quad+2\int_0^t\left(\frac{3\delta}{4}\|\partial_t u\|_{L^2}^2+\mu\nu\eta\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2+\mu\|\partial_1 u\|_{L^2}^2\right)d\tau\\&\le C\left(\|\partial_t u_0\|_{L^2},\|u_0\|_{L^2},\|\mathcal{R}_1\mathcal{R}_2 u_0\|_{L^2},\|\partial_1 u_0\|_{L^2}\right),\end{aligned}$$

and consequently,

$$\int_0^\infty\left(\|\partial_t u\|_{L^2}^2+\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2+\|\partial_1 u\|_{L^2}^2\right)dt<\infty. \tag{4.6}$$

In view of (4.2) and (4.6), it readily follows from Lemma 4.1 that

$$\|\partial_t u\|_{L^2}^2+\|\mathcal{R}_1\mathcal{R}_2 u\|_{L^2}^2+\|\partial_1 u\|_{L^2}^2\le C(1+t)^{-1}.$$

Based upon $(1.13)_2$, one can obtain the same result for b. The proof of Theorm 1.3 is thus complete. □

We proceed to prove Theorem 1.4.

Proof of Theorem 1.4 Let ψ be the Fourier cutoff operator defined in (1.15). Taking the convolution of ψ with $(1.13)_1$ leads to

$$\partial_{tt}(\psi * u) - \left(\nu \mathcal{R}_2^2 + \eta \mathcal{R}_1^2\right) \partial_t(\psi * u) - \partial_1^2(\psi * u) + \nu\eta \mathcal{R}_1^2 \mathcal{R}_2^2(\psi * u) = 0. \tag{4.7}$$

Dotting (4.7) by $\partial_t(\psi * u)$ in L^2 and integrating it by parts, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|\partial_t(\psi * u)\|_{L^2}^2 + \|\partial_1(\psi * u)\|_{L^2}^2 + \nu\eta\|\mathcal{R}_1\mathcal{R}_2(\psi * u)\|_{L^2}^2\right) \\ &\quad + 2\nu\|\partial_t\mathcal{R}_2(\psi * u)\|_{L^2}^2 + 2\eta\|\partial_t\mathcal{R}_1(\psi * u)\|_{L^2}^2 = 0. \end{aligned} \tag{4.8}$$

Similarly, multiplying (4.7) by $\psi * u$ in L^2, we have

$$\begin{aligned} &\frac{d}{dt}\left(\nu\|\mathcal{R}_2(\psi * u)\|_{L^2}^2 + \eta\|\mathcal{R}_1(\psi * u)\|_{L^2}^2 + 2\left\langle \partial_t(\psi * u), \psi * u\right\rangle\right) \\ &\quad + 2\|\partial_1(\psi * u)\|_{L^2}^2 + 2\nu\eta\|\mathcal{R}_1\mathcal{R}_2(\psi * u)\|_{L^2}^2 - 2\|\partial_t(\psi * u)\|_{L^2}^2 = 0. \end{aligned} \tag{4.9}$$

Let $\delta := \min\{\nu, \eta\}$, and $\lambda > 0$ be a positive constant to be determined later. Then, operating (4.8)+$\lambda\times$ (4.9) yields

$$\begin{aligned} &\frac{d}{dt}F(t) + 2(\delta - \lambda)\|\partial_t(\psi * u)\|_{L^2}^2 \\ &\quad + 2\lambda\|\partial_1(\psi * u)\|_{L^2}^2 + 2\lambda\nu\eta\|\mathcal{R}_1\mathcal{R}_2(\psi * u)\|_{L^2}^2 \le 0, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} F(t) := &\|\partial_t(\psi * u)\|_{L^2}^2 + \|\partial_1(\psi * u)\|_{L^2}^2 + \nu\eta\|\mathcal{R}_1\mathcal{R}_2(\psi * u)\|_{L^2}^2 \\ &+ \lambda\nu\|\mathcal{R}_2(\psi * u)\|_{L^2}^2 + \lambda\eta\|\mathcal{R}_1(\psi * u)\|_{L^2}^2 + 2\lambda\left\langle \partial_t(\psi * u), \psi * u\right\rangle. \end{aligned}$$

Let D be the frequency domain defined in (1.14) and D^c be its complement. Moreover, we divide D^c into two regions:

$$A_1 = \left\{\xi \in \mathbb{R}^2 : \ |\xi_1| \ge \alpha\right\}, \quad A_2 = \left\{\xi \in \mathbb{R}^2 : \ |\xi_1| < \alpha \ \text{ and } \ |\xi|^2 \le \beta|\xi_1||\xi_2|\right\}.$$

We can now bound $\|\psi * u\|_{L^2}^2$ by $\|\partial_1(\psi * u)\|_{L^2}^2$ and $\|\mathcal{R}_1\mathcal{R}_2(\psi * u)\|_{L^2}^2$. Indeed,

$$\begin{aligned} \|\psi * u\|_{L^2}^2 = \left\|\widehat{\psi}\widehat{u}\right\|_{L^2}^2 &= \int_{A_1} |\widehat{\psi}\widehat{u}|^2 d\xi + \int_{A_2} |\widehat{\psi}\widehat{u}|^2 d\xi \\ &\le \alpha^{-2}\int_{A_1} \xi_1^2|\widehat{\psi}\widehat{u}|^2 d\xi + \beta^2\int_{A_2} \frac{\xi_1^2\xi_2^2}{|\xi|^4}|\widehat{\psi}\widehat{u}|^2 d\xi \\ &\le \alpha^{-2}\|\partial_1(\psi * u)\|_{L^2}^2 + \beta^2\|\mathcal{R}_1\mathcal{R}_2(\psi * u)\|_{L^2}^2. \end{aligned} \tag{4.11}$$

Then, multiplying (4.11) by λ^2 and then adding with (4.10), we obtain

$$\begin{aligned} &\frac{d}{dt}F(t) + 2(\delta - \lambda)\|\partial_t(\psi * u)\|_{L^2}^2 + (2\lambda - \lambda^2\alpha^{-2})\|\partial_1(\psi * u)\|_{L^2}^2 \\ &\quad + (2\lambda\nu\eta - \lambda^2\beta^2)\|\mathcal{R}_1\mathcal{R}_2(\psi * u)\|_{L^2}^2 + \lambda^2\|\psi * u\|_{L^2}^2 \le 0. \end{aligned} \tag{4.12}$$

Thus, if $\lambda > 0$ is chosen to be such that

$$\lambda \le \min\left\{\frac{1}{2}\delta, \alpha^2, \frac{\nu\eta}{\beta^2}\right\},$$

then we infer from (4.12) that

$$\frac{d}{dt}F(t)+\delta\|\partial_t(\psi*u)\|_{L^2}^2+\lambda\|\partial_1(\psi*u)\|_{L^2}^2 \\ +\lambda\nu\eta\|\mathcal{R}_1\mathcal{R}_2(\psi*u)\|_{L^2}^2+\lambda^2\|\psi*u\|_{L^2}^2\le 0. \tag{4.13}$$

Recalling the definition of F and noting that

$$2\lambda\,(\partial_t(\psi*u),\psi*u)\le\lambda\|\partial_t(\psi*u)\|_{L^2}^2+\lambda\|\psi*u\|_{L^2}^2, \tag{4.14}$$

we obtain after operating (4.13)+$\lambda^2\times$ (4.14) that ($\vartheta:=\max\{\nu,\eta\}$)

$$\frac{d}{dt}F+\lambda^2F+(\delta-\lambda^2-\lambda^3)\|\partial_t(\psi*u)\|_{L^2}^2+\left(\lambda-\lambda^2\right)\|\partial_1(\psi*u)\|_{L^2}^2 \\ +\left(\lambda\nu\eta-\lambda^2\nu\eta\right)\|\mathcal{R}_1\mathcal{R}_2(\psi*u)\|_{L^2}^2+\lambda^2(1-\vartheta\lambda-\lambda)\|\psi*u\|_{L^2}^2\le 0, \tag{4.15}$$

If $\lambda>0$ is taken to be sufficiently small such that

$$\lambda=\min\left\{\frac{1}{4}\delta,\alpha^2,1,\frac{\nu\eta}{\beta^2},\frac{1}{\vartheta+1}\right\}, \tag{4.16}$$

then it follows from (4.15) that

$$\frac{d}{dt}F+\lambda^2F\le 0\quad\text{or}\quad F(t)\le F(0)e^{-\lambda^2t}. \tag{4.17}$$

In view of the simple inequality,

$$2\lambda\,(\partial_t(\psi*u),\psi*u)\le\frac{1}{2}\|\partial_t(\psi*u)\|_{L^2}^2+2\lambda^2\|\psi*u\|_{L^2}^2,$$

one easily has

$$\frac{1}{2}\|\partial_t(\psi*u)\|_{L^2}^2+\frac{1}{2}\lambda\delta\|\psi*u\|_{L^2}^2+\nu\eta\|\mathcal{R}_1\mathcal{R}_2(\psi*u)\|_{L^2}^2+\|\partial_1(\psi*u)\|_{L^2}^2\le F(t).$$

As an immediate consequence of (4.17), we conclude that for λ satisfying (4.16),

$$\|\psi*u\|_{L^2}^2+\|\partial_t(\psi*u)\|_{L^2}^2+\|\partial_1(\psi*u)\|_{L^2}^2+\|\mathcal{R}_1\mathcal{R}_2(\psi*u)\|_{L^2}^2\le Ce^{-c(\eta,\nu,\alpha,\beta)t}$$

where

$$c(\eta,\nu,\alpha,\beta):=\left(\min\left\{\frac{1}{4}\delta,\alpha^2,1,\frac{\nu\eta}{\beta^2},\frac{1}{\vartheta+1}\right\}\right)^2.$$

The same result also holds for b. The proof of Theorem 1.4 is therefore finished. □

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