



# Prescribing discrete Gaussian curvature on polyhedral surfaces

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## Abstract

Vertex scaling of piecewise linear metrics on surfaces introduced by Luo (Commun Contemp Math 6: 765–780, 2004) is a straightforward discretization of smooth conformal structures on surfaces. Combinatorial  $\alpha$ -curvature for vertex scaling of piecewise linear metrics on surfaces is a discretization of Gaussian curvature on surfaces. In this paper, we investigate the prescribing combinatorial  $\alpha$ -curvature problem on polyhedral surfaces. Using Gu-Luo-Sun-Wu's discrete conformal theory (J. Differ. Geom. 109: 223–256, 2018) for piecewise linear metrics on surfaces and variational principles with constraints, we prove some Kazdan-Warner type theorems for prescribing combinatorial  $\alpha$ -curvature problem, which generalize the results obtained in Gu-Luo-Sun-Wu (J. Differ. Geom. 109: 223–256, 2018), Xu (Parameterized discrete uniformization theorems and curvature flows for polyhedral surfaces, I, [arXiv:1806.04516v2](https://arxiv.org/abs/1806.04516v2)) on prescribing combinatorial curvatures on surfaces. Gu-Luo-Sun-Wu (J. Differ. Geom. 109: 223–256, 2018) conjectured that one can prove Kazdan-Warner's theorems in Kazdan (Ann Math 99: 14–47, 1974), Kazdan (Ann Math 101: 317–331, 1975) via approximating smooth surfaces by polyhedral surfaces. This paper takes the first step in this direction.

**Mathematics Subject Classification** 52C26

## 1 Introduction

### 1.1 Motivation

A classical problem in differential geometry asks the following question on prescribing Gaussian curvature on closed surfaces.

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*Prescribing Gaussian curvature problem.* Let  $S$  be a smooth connected closed surface. Which functions are Gaussian curvatures of Riemannian metrics on  $S$ ?

This problem has been extensively studied by lots of mathematicians by deforming Riemannian metric in its conformal structure. See, for instance, [3, 6, 24, 25, 30] and others.

Polyhedral metric on surfaces is a discrete analogue of the Riemannian metric on surfaces, which is a constant curvature (1, 0 or  $-1$ ) metric with cone singularities. The classical discrete Gaussian curvature for a polyhedral metric on a surface is the corresponding discrete analogue of the smooth Gaussian curvature, which is defined to be the angle defect at the cone points. Prescribing discrete Gaussian curvature problem on polyhedral surface asks the following question paralleling to the classical prescribing Gaussian curvature problem on smooth surfaces.

*Prescribing discrete Gaussian curvature problem.* Suppose  $S$  is a connected closed surface and  $V$  is a finite nonempty subset of  $S$ . Which functions defined on  $V$  are discrete Gaussian curvatures of polyhedral metrics on  $(S, V)$ ?

In discrete conformal geometry, paralleling to the smooth case, the prescribing discrete Gaussian curvature problem on surfaces is usually studied by deforming a polyhedral metric in its discrete conformal structure, which is a discrete analogue of the smooth conformal structure on surfaces defining polyhedral metrics by functions defined on the cone points. The discrete conformal structures on surfaces that have been extensively studied. These include Thurston's circle packings [36], Luo's vertex scaling [26] and others. For Thurston's circle packings on surfaces, the solution of prescribing discrete Gaussian curvature problem gives rise to the famous Koebe-Andreev-Thurston Theorem, which plays a fundamental role in the study of geometry and topology of 3-dimensional manifolds [36]. For vertex scaling of piecewise linear and piecewise hyperbolic metrics on surfaces, the recent complete solution of prescribing discrete Gaussian curvature problem by Gu-Luo-Sun-Wu [19] and Gu-Guo-Luo-Sun-Wu [18] provides a new constructive proof of the classical uniformization theorem on closed surfaces with genus  $g \geq 1$ , which is computable and has lots of applications [10, 20, 21, 28, 35, 45]. In the framework of vertex scaling, Springborn [34] recently proved that the discrete uniformization theorem on the sphere is equivalent to Rivin's realization theorem for ideal hyperbolic polyhedra [33].

However, the classical discrete Gaussian curvature is not a proper discretization of the smooth Gaussian curvature on surfaces. For example, it is scaling invariant, which is different from the transformation of smooth Gaussian curvatures under scaling of Riemannian metrics, and does not approximate the smooth Gaussian curvature pointwisely as the triangulations become finer and finer [17]. In [40], the first author introduced combinatorial  $\alpha$ -curvature for vertex scaling of piecewise linear metrics (PL metrics for short in the following) on surfaces to avoid the disadvantages of classical discrete Gaussian curvature alluded to above, where the rigidity and Yamabe problem for combinatorial  $\alpha$ -curvature were also studied. In this paper, we study the prescribing combinatorial  $\alpha$ -curvature problem for PL metrics on surfaces and prove some Kazdan-Warner type theorems. It is conjectured by Gu-Luo-Sun-Wu [19] that one can prove Kazdan-Warner's theorems in [24, 25] by approximating smooth surfaces by polyhedral surfaces involving vertex scaling. This paper takes the first step in this direction.

## 1.2 Statements of main results

Suppose  $S$  is a connected closed surface and  $V$  is a finite non-empty subset of  $S$ . We call  $(S, V)$  a marked surface. A PL metric  $d$  on the marked surface  $(S, V)$  is a flat cone metric on  $S$  with conic singularities contained in  $V$ . A triangulation  $\mathcal{T}$  of the marked surface  $(S, V)$

is a triangulation of  $S$  with the vertex set equal to  $V$ . Denote a vertex, an edge and a face in the triangulation  $\mathcal{T}$  by  $i, \{ij\}, \{ijk\}$  respectively and denote the sets of edges and faces of  $\mathcal{T}$  by  $E_{\mathcal{T}}$  and  $F_{\mathcal{T}}$  respectively. A triangulation  $\mathcal{T}$  of  $(S, V, d)$  is geometric if every edge in  $E_{\mathcal{T}}$  is a geodesic in the PL metric  $d$ . A PL metric  $d$  on a triangulated surface  $(S, V, \mathcal{T})$  with  $\mathcal{T}$  geometric defines a map  $l : E_{\mathcal{T}} \rightarrow (0, +\infty)$  such that  $l_{ij}, l_{ik}, l_{jk}$  satisfy the triangle inequalities for any triangle  $\{ijk\} \in F_{\mathcal{T}}$ . The map  $l : E_{\mathcal{T}} \rightarrow (0, +\infty)$  is called as a discrete metric and the logarithm of the discrete metric  $l, \lambda_{ij} = 2 \log l_{ij}$ , is called as the logarithmic length. One can also obtain discrete PL metrics on a triangulated surface  $(S, V, \mathcal{T})$  by gluing triangles in 2-dimensional Euclidean space isometrically along the edges in pair, which gives rise to PL metrics on the marked surface  $(S, V)$ . Note that a PL metric  $d$  on a marked surface  $(S, V)$  is intrinsic in the sense that it is independent of the geometric triangulations of  $(S, V, d)$ . Every PL metric  $d$  on  $(S, V)$  has a Delaunay triangulation  $\mathcal{T}$  of  $(S, V, d)$  such that each triangle in  $\mathcal{T}$  is Euclidean and the sum of two angles facing each edge is at most  $\pi$ . See [2, 5, 19, 32] for further discussion on Delaunay triangulation of polyhedral surfaces.

Suppose  $(S, V, d)$  is a marked surface with a PL metric  $d$  and  $\mathcal{T}$  is a geometric triangulation of  $(S, V, d)$ . The classical discrete Gaussian curvature  $K : V \rightarrow (-\infty, 2\pi)$  for  $(S, V, d)$  is defined as

$$K_i = 2\pi - \sum_{\{ijk\} \in F_{\mathcal{T}}} \theta_i^{jk} \tag{1}$$

with summation taken over all the triangles at  $i \in V$  and  $\theta_i^{jk}$  being the inner angle of the triangle  $\{ijk\} \in F_{\mathcal{T}}$  at the vertex  $i$ . Sometimes we call  $K$  as combinatorial curvature for simplicity. Note that the classical discrete Gaussian curvature  $K$  is intrinsic in the sense that it is independent of the geometric triangulations of  $(S, V, d)$ . The classical combinatorial curvature  $K$  for PL metrics on  $(S, V)$  satisfies the following discrete Gauss-Bonnet formula ([7], Proposition 3.1)

$$\sum_{i \in V} K_i = 2\pi \chi(S). \tag{2}$$

The discrete Gauss-Bonnet formula (2) provides a necessary condition for a function to be the discrete Gaussian curvature of some PL metric on  $(S, V)$ . The classical prescribing discrete Gaussian curvature problem on surfaces can be taken as a converse problem to the discrete Gauss-Bonnet formula (2), which asks the following question.

**Question 1.1** Is the discrete Gauss-Bonnet formula (2) sufficient condition for a function  $K : V \rightarrow (-\infty, 2\pi)$  to be the discrete Gaussian curvature of some PL metric on  $(S, V)$ ?

The prescribing discrete Gaussian curvature problem is usually studied in the frame of discrete conformality of polyhedral metrics on surfaces. See, for instance, [18, 19, 26, 36] and others. Vertex scaling of PL metrics on surfaces introduced by Luo [26] is a straightforward discrete analogue of the conformal transformation in Riemannian geometry.

**Definition 1.2** ([26]) Suppose  $l, \tilde{l} : E_{\mathcal{T}} \rightarrow (0, +\infty)$  are two discrete PL metrics on a triangulated surface  $(S, V, \mathcal{T})$ .  $\tilde{l}$  is called as a vertex scaling of  $l$  if there exists a function  $u : V \rightarrow \mathbb{R}$  such that

$$\tilde{l}_{ij} = l_{ij} \exp\left(\frac{u_i + u_j}{2}\right)$$

for any edge  $\{ij\} \in E_{\mathcal{T}}$ .

Among other things, Luo [26] further established a variational principle for vertex scaling of PL metrics on surfaces and proved that there exists some combinatorial obstructions for the existence of PL metric with constant discrete Gaussian curvature  $K$  in a discrete conformal class on a triangulated surface  $(S, V, \mathcal{T})$  in the sense of Definition 1.2. This implies that there exist some combinatorial obstructions for the solvability of the prescribing discrete Gaussian curvature problem in a discrete conformal class on a triangulated surface  $(S, V, \mathcal{T})$  in the sense of Definition 1.2. To overcome this difficulty, Gu-Luo-Sun-Wu [19] introduced the following new definition of discrete conformality of PL metrics on marked surfaces, which allows the triangulation of the marked surface to be changed under the Delaunay condition.

**Definition 1.3** ([19], Definition 1.1) Two PL metrics  $d, d'$  on a marked surface  $(S, V)$  are discrete conformal if there exist a sequence of PL metrics  $d_1 = d, d_2, \dots, d_m = d'$  on  $(S, V)$  and triangulations  $\mathcal{T}_1, \dots, \mathcal{T}_m$  of  $(S, V)$  satisfying

- (a) (Delaunay condition) each  $\mathcal{T}_i$  is Delaunay in  $d_i$ ,
- (b) (Vertex scaling condition) if  $\mathcal{T}_i = \mathcal{T}_{i+1}$ , there exists a function  $u : V \rightarrow \mathbb{R}$ , called a conformal factor, so that if  $e$  is an edge in  $\mathcal{T}_i$  with end points  $v$  and  $v'$ , then the lengths  $l_{d_i}(e)$  and  $l_{d_{i+1}}(e)$  of  $e$  in metrics  $d_i$  and  $d_{i+1}$  are related by

$$l_{d_{i+1}}(e) = l_{d_i}(e) \exp\left(\frac{u(v) + u(v')}{2}\right),$$

- (c) if  $\mathcal{T}_i \neq \mathcal{T}_{i+1}$ , then  $(S, d_i)$  is isometric to  $(S, d_{i+1})$  by an isometry homotopic to the identity in  $(S, V)$ .

Definition 1.3 defines an equivalence relationship for PL metrics on a marked surface  $(S, V)$ . The equivalence class of a PL metric  $d$  on  $(S, V)$  is called as the discrete conformal class of  $d$  and denoted by  $\mathcal{D}(d)$ . Using the new discrete conformality in Definition 1.3, Gu-Luo-Sun-Wu [19] completely solved the prescribing discrete Gaussian curvature problem for PL metrics on closed surfaces in the following well-known theorem. This theorem shows that the discrete Gauss-Bonnet formula (2) is a necessary and sufficient condition for a function  $\bar{K} : V \rightarrow (-\infty, 2\pi)$  to be the classical discrete Gaussian curvature of some PL metric on  $(S, V)$ .

**Theorem 1.4** ([19] Theorem 1.2) *Suppose  $(S, V)$  is a closed connected marked surface and  $d$  is a PL metric on  $(S, V)$ . Then for any  $\bar{K} : V \rightarrow (-\infty, 2\pi)$  with  $\sum_{v \in V} \bar{K}(v) = 2\pi \chi(M)$ , there exists a PL metric  $\bar{d}$ , unique up to scaling and isometry homotopic to the identity on  $(S, V)$ , such that  $\bar{d}$  is discrete conformal to  $d$  and the discrete curvature of  $\bar{d}$  is  $\bar{K}$ .*

The classical discrete Gaussian curvature defined by (1) is not a proper discretization of the smooth Gaussian curvature on surfaces, which is supported by the discussions in [4, 17]. To overcome the disadvantages of the classical discrete Gaussian curvature, Ge and the first author [17] introduced the combinatorial  $\alpha$ -curvature for Thurston’s Euclidean circle packing metrics on surfaces. After that, there are lots of research activities on combinatorial  $\alpha$ -curvature on surfaces and 3-manifolds. See, for instance, [9, 13–16, 37, 38, 40, 43, 44] and others. Following [17], the first author [40] introduced the following combinatorial  $\alpha$ -curvature for vertex scaling of PL metrics on triangulated surfaces.

**Definition 1.5** ([40]) Suppose  $(S, V, \mathcal{T})$  is a triangulated surface with a discrete PL metric  $l, \alpha \in \mathbb{R}$  is a constant and  $u : V \rightarrow \mathbb{R}$  is a discrete conformal factor for  $l$ . The combinatorial  $\alpha$ -curvature at  $i \in V$  is defined to be

$$R_{\alpha,i} = \frac{K_i}{e^{\alpha u_i}},$$

where  $K_i$  is the classical combinatorial curvature at  $i \in V$  defined by (1).

**Remark 1** If  $\alpha = 0$ , the combinatorial  $\alpha$ -curvature  $R_\alpha$  in Definition 1.5 is the classical discrete Gaussian curvature  $K$ . Taking  $g_i = e^{u_i}$  as a discrete analogue of the smooth Riemannian metric, we have  $R_{\alpha,i}(\lambda g_1, \dots, \lambda g_V) = \lambda^{-\alpha} R_{\alpha,i}(g_1, \dots, g_V)$  for any constant  $\lambda > 0$ . In the special case of  $\alpha = 1$ , we have  $R_{1,i}(\lambda g_1, \dots, \lambda g_V) = \lambda^{-1} R_{1,i}(g_1, \dots, g_V)$ , which is parallelling to the transformation of smooth Gaussian curvature  $K_{\lambda g} = \lambda^{-1} K_g$  with  $g$  being the Riemannian metric.

By the discrete Gauss-Bonnet formula (2) for the classical discrete Gaussian curvature  $K$ , the combinatorial  $\alpha$ -curvature  $R_\alpha$  in Definition 1.5 satisfies the following discrete Gauss-Bonnet formula  $\sum_{i \in V} R_{\alpha,i} e^{\alpha u_i} = 2\pi \chi(S)$ . Therefore, if  $\bar{R} \in \mathbb{R}^V$  is the combinatorial  $\alpha$ -curvature of some discrete PL metric discrete conformal to  $l$  on  $(S, V, T)$  with conformal factor  $u$ , then

$$\sum_{i \in V} \bar{R}_i e^{\alpha u_i} = 2\pi \chi(S), \tag{3}$$

which is a discrete analogue of the constraint equation  $\int_S K e^{2u} dV = 2\pi \chi(S)$  in the smooth case [3, 24]. Following Kazdan-Warner’s arguments in [24], the constraint equation (3) imposes the following sign conditions on  $\bar{R}$  depending on  $\chi(S)$ :

- (a)  $\chi(M) > 0$ :  $\bar{R}$  is positive somewhere,
- (b)  $\chi(M) = 0$ :  $\bar{R}$  changes sign (unless  $\bar{R} \equiv 0$ ),
- (c)  $\chi(M) < 0$ :  $\bar{R}$  is negative somewhere.

It is natural to ask the following discrete version of Kazdan-Warner’s question for combinatorial  $\alpha$ -curvature.

**Question 1.6** (Discrete Kazdan-Warner Question) Suppose  $(S, V)$  is a marked surface with a PL metric  $d$ . Are the sign conditions, depending on  $\chi(S)$ , sufficient conditions for a function  $\bar{R}$  defined on  $V$  to be the combinatorial  $\alpha$ -curvature of some polyhedral metric  $d'$  discrete conformal to  $d$ ?

We prove the following discrete Kazdan-Warner type theorem for Discrete Kazdan-Warner Question 1.6.

**Theorem 1.7** Suppose  $(S, V, d)$  is a marked surface with a PL metric  $d$ ,  $\alpha \in \mathbb{R}$  is a constant and  $\bar{R}$  is a given function defined on  $V$ . Then there exists a PL metric with combinatorial  $\alpha$ -curvature  $\bar{R}$  in the discrete conformal class  $\mathcal{D}(d)$  if one of the following conditions is satisfied :

- (1)  $\chi(S) > 0, \alpha < 0, \bar{R} > 0$ ;
- (2)  $\chi(S) < 0, \alpha \neq 0, \bar{R} \leq 0, \bar{R} \not\equiv 0$ ;
- (3)  $\chi(S) = 0, \alpha \neq 0, \bar{R} \equiv 0$ ;
- (4)  $\alpha = 0, \bar{R} \in (-\infty, 2\pi), \sum_{i \in V} \bar{R}_i = 2\pi \chi(S)$ .

**Remark 2** If  $\bar{R}$  is a constant and  $\alpha \bar{R} \leq 0$ , the existence of PL metric with combinatorial  $\alpha$ -curvature  $\bar{R}$  in the discrete conformal class  $\mathcal{D}(d)$  has been proved in [40]. In the case of  $\alpha \bar{R} \leq 0$ , the uniqueness of PL metric with combinatorial  $\alpha$ -curvature  $\bar{R}$  in the discrete conformal class  $\mathcal{D}(d)$  has been proved in [19, 40]. For the case  $\alpha \bar{R} > 0$ , the uniqueness is unknown. By the relationship of combinatorial  $\alpha$ -curvature and the classical discrete Gaussian curvature, the cases (3) and (4) in Theorem 1.7 are covered by Gu-Luo-Sun-Wu [19]. Therefore, we just need to prove the cases (1) and (2) of Theorem 1.7.

The main tools for the proof of Theorem 1.7 are Gu-Luo-Sun-Wu’s discrete conformal theory for PL metrics on surfaces [19] and variational principles with constraints. The main ideas of the paper come from reading of Gu-Luo-Sun-Wu [19] and Kouřimská [23]. A hyperbolic version of Theorem 1.4 has been proved by Gu-Guo-Luo-Sun-Wu [18], which perfectly solves the classical prescribing discrete Gaussian curvature problem in the hyperbolic background geometry. For prescribing combinatorial  $\alpha$ -curvature problem in the hyperbolic background geometry, the authors [43] recently obtained a hyperbolic version of Theorem 1.7 using Luo’s combinatorial Yamabe flow and Gu-Guo-Luo-Sun-Wu’s discrete conformal theory for piecewise hyperbolic metrics on surfaces [18].

### 1.3 Organization of the paper

The paper is organized as follows. In Sect. 2, we recall the variational principle introduced by Luo [26] for vertex scaling and Bobenko-Pinkall-Spingborn’s development of Luo’s variational principle, and then recall the discrete conformal theory established by Gu-Luo-Sun-Wu [19]. In Sect. 3, we translate Theorem 1.7 into an optimization problem with constraints. In Sect. 4, we prove Theorem 1.7.

## 2 Discrete conformality of PL metrics on surfaces

In this section, we recall some facts about discrete conformality of PL metrics that we need to use in the proof of Theorem 1.7.

### 2.1 The energy functions

Suppose  $\{ijk\} \in F_{\mathcal{T}}$  is a triangle and  $l : E_{\mathcal{T}} \rightarrow \mathbb{R}_{>0}$  is a discrete PL metric on  $(S, V, \mathcal{T})$ . Denote the inner angle in the triangle  $\{ijk\}$  at the vertex  $i$  as  $\theta_i$ . Luo [26] proved the following result.

**Lemma 2.1** ([26]) *Suppose  $\{ijk\} \in F_{\mathcal{T}}$  is a triangle and  $l : E_{\mathcal{T}} \rightarrow \mathbb{R}_{>0}$  is a discrete PL metric on  $(S, V, \mathcal{T})$ .*

(1) *The admissible space of the discrete conformal factors for a triangle  $\{ijk\} \in F_{\mathcal{T}}$*

$$\Omega_{ijk} = \{(u_i, u_j, u_k) \in \mathbb{R}^3 | \tilde{l}_{ij}, \tilde{l}_{ik}, \tilde{l}_{jk} \text{ satisfy the triangle inequality}\}$$

*is simply connected.*

(2) *The Jacobian matrix  $\frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)}$  is symmetric and negative semi-definite with kernel  $\{t(1, 1, 1)^T | t \in \mathbb{R}\}$  for any discrete conformal factor  $(u_i, u_j, u_k) \in \Omega_{ijk}$ .*

Based on Lemma 2.1, Luo [26] constructed the following energy function for a triangle  $\{ijk\} \in F_{\mathcal{T}}$

$$F_{ijk}(u_i, u_j, u_k) = - \int_{(0,0,0)}^{(u_i, u_j, u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k, \tag{4}$$

which is a well-defined smooth locally convex function defined on  $\Omega_{ijk}$  with  $\nabla F_{ijk} = (-\theta_i, -\theta_j, -\theta_k)$  and

$$F_{ijk}((u_i, u_j, u_k) + t(1, 1, 1)) = F_{ijk}(u_i, u_j, u_k) - \pi t \tag{5}$$

for  $t \in \mathbb{R}$ . Motivated by [8], Luo [26] further introduced the following energy function

$$\mathbb{F}_{\mathcal{T}}(u) = \int_0^u \sum_{i \in V} K_i du_i = \sum_{\{ijk\} \in F_{\mathcal{T}}} F_{ijk}(u_i, u_j, u_k) + 2\pi \sum_{i \in V} u_i, \tag{6}$$

which is a locally convex smooth function of the discrete conformal factors  $u \in \cap_{\{ijk\} \in F_{\mathcal{T}}} \Omega_{ijk}$  with  $\nabla \mathbb{F}_{\mathcal{T}}(u) = K$ . By (5) and the Gauss-Bonnet formula, the energy function  $\mathbb{F}_{\mathcal{T}}(u)$  has the following property

$$\mathbb{F}_{\mathcal{T}}(u + c(1, \dots, 1)) = \mathbb{F}_{\mathcal{T}}(u) + 2c\pi \chi(S) \tag{7}$$

for  $c \in \mathbb{R}$  [26].

In the following of the paper, we need to use the explicit expression of  $\mathbb{F}_{\mathcal{T}}(u)$ , which was first constructed up to a constant by Bobenko-Pinkall-Springborn [4] as follows. Set

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid e^x + e^y > e^z, e^y + e^z > e^x, e^z + e^x > e^y\}.$$

For  $(x, y, z) \in \Omega$ ,  $e^x, e^y, e^z$  form the edge lengths of a Euclidean triangle. Denote the inner angles facing  $e^x, e^y, e^z$  as  $\alpha, \beta, \gamma$  respectively and set

$$\mathbb{L}(x) = - \int_0^x \log |2 \sin(t)| dt$$

to be Milnor’s Lobachevsky function [29]. Bobenko-Pinkall-Springborn [4] then defined the following function

$$f : \Omega \rightarrow \mathbb{R} \\ (x, y, z) \mapsto f(x, y, z) = \alpha x + \beta y + \gamma z + \mathbb{L}(\alpha) + \mathbb{L}(\beta) + \mathbb{L}(\gamma).$$

Using  $f$  as building blocks, Bobenko-Pinkall-Springborn [4] further constructed the following function

$$\mathbb{E}_{\mathcal{T}}(u) = \sum_{\{ijk\} \in F_{\mathcal{T}}} \left( 2f\left(\frac{\tilde{\lambda}_{ij}}{2}, \frac{\tilde{\lambda}_{jk}}{2}, \frac{\tilde{\lambda}_{ki}}{2}\right) - \frac{\pi}{2}(\tilde{\lambda}_{ij} + \tilde{\lambda}_{jk} + \tilde{\lambda}_{ki}) \right) + 2\pi \sum_{i \in V} u_i \tag{8}$$

for  $u \in \cap_{\{ijk\} \in F_{\mathcal{T}}} \Omega_{ijk}$ , where  $\tilde{\lambda}_{ij}$  is the logarithm length of  $\tilde{l}_{ij} = l_{ij} e^{\frac{u_i + u_j}{2}}$ . Bobenko-Pinkall-Springborn [4] proved that  $\nabla \mathbb{E}_{\mathcal{T}} = K$ , which implies that  $\mathbb{E}_{\mathcal{T}}(u)$  and  $\mathbb{F}_{\mathcal{T}}(u)$  differs by some constant.

Bobenko-Pinkall-Springborn’s construction of  $\mathbb{E}_{\mathcal{T}}(u)$  in [4] is very elegant, but a little mysterious. It seems that Bobenko-Pinkall-Springborn’s construction comes from their observation on the relationships of vertex scaling and 3-dimensional hyperbolic geometry and can be taken as a consequence of the Schläfli formula [32]. Once one has the explicit form of the function  $\mathbb{E}_{\mathcal{T}}(u)$ , it is easy to check that  $\nabla \mathbb{E}_{\mathcal{T}} = K$ . However, the construction of  $\mathbb{E}_{\mathcal{T}}(u)$  is not so easy. As Luo’s construction of the energy function  $\mathbb{F}_{\mathcal{T}}(u)$  in [26] is relatively direct and easy, a natural question of independent interest is whether one can derive an explicit form of  $\mathbb{F}_{\mathcal{T}}(u)$  directly, which differs from the explicit form of  $\mathbb{E}_{\mathcal{T}}(u)$  by a constant. In the following, we give such an argument, the idea of which comes from Yuhao Xue from summer school 2017 at Tsinghua University. In fact, we just need to derive an explicit form of the energy function  $\mathbb{F}_{ijk}(u_i, u_j, u_k)$  for a triangle  $\{ijk\} \in F_{\mathcal{T}}$ .

**Proposition 2.2** *Suppose  $\tilde{l}$  is vertex scaling of  $l$  on the triangle  $\{ijk\} \in F_{\mathcal{T}}$  with a conformal factor  $(u_i, u_j, u_k) \in \Omega_{ijk}$ . Denote the inner angle of the triangle with edge lengths  $l_{ij}, l_{ik}, l_{jk}$  as  $\overline{\Theta}_i, \overline{\Theta}_j, \overline{\Theta}_k$  respectively and denote the inner angle of the triangle with edge lengths*

$\tilde{l}_{ij}, \tilde{l}_{ik}, \tilde{l}_{jk}$  as  $\Theta_i, \Theta_j, \Theta_k$  respectively. Then the energy function  $F_{ijk}(u_i, u_j, u_k)$  defined by (4) has the following explicit form

$$\begin{aligned}
 F_{ijk}(u_i, u_j, u_k) &= -(\Theta_i u_i + \Theta_j u_j + \Theta_k u_k) \\
 &\quad + 2\mathbb{L}(\Theta_i) + 2\mathbb{L}(\Theta_j) + 2\mathbb{L}(\Theta_k) + 2\Theta_i \ln l_{jk} + 2\Theta_j \ln l_{ik} + 2\Theta_k \ln l_{ij} \\
 &\quad - 2\mathbb{L}(\bar{\Theta}_i) - 2\mathbb{L}(\bar{\Theta}_j) - 2\mathbb{L}(\bar{\Theta}_k) - 2\bar{\Theta}_k \ln l_{ij} - 2\bar{\Theta}_j \ln l_{ik} - 2\bar{\Theta}_i \ln l_{jk}.
 \end{aligned}$$

**Proof** As  $\Omega_{ijk}$  is simply connected by Lemma 2.1 and  $(0, 0, 0), (u_i, u_j, u_k) \in \Omega_{ijk}$ , we can assume that  $\gamma : [0, 1] \rightarrow \Omega_{ijk}$  is a smooth path from  $(0, 0, 0)$  to  $(u_i, u_j, u_k)$ . Denote the corresponding quantities along the path  $\gamma(t)$  as  $u_i(t), \theta_i(t), l_{ij}(t)$  et al. Then  $u_i(1) = u_i, u_i(0) = 0, \theta_i(1) = \Theta_i, \theta_i(0) = \bar{\Theta}_i, l_{ij}(1) = \tilde{l}_{ij}, l_{ij}(0) = l_{ij}$ . By Definition 1.2, one can solve  $u_i(t), u_j(t), u_k(t)$  as follows

$$u_i(t) = \ln \frac{l_{ij}(t)l_{ik}(t)l_{jk}(0)}{l_{ij}(0)l_{ik}(0)l_{jk}(t)}, u_j(t) = \ln \frac{l_{ij}(t)l_{ik}(0)l_{jk}(t)}{l_{ij}(0)l_{ik}(t)l_{jk}(0)}, u_k(t) = \ln \frac{l_{ij}(0)l_{ik}(t)l_{jk}(t)}{l_{ij}(t)l_{ik}(0)l_{jk}(0)} \tag{9}$$

Suppose the circumcircle radius of the triangle with edge lengths  $l_{ij}(t), l_{ik}(t), l_{jk}(t)$  is  $R(t)$ , then

$$l_{ij}(t) = 2R(t) \sin \theta_k(t), l_{ik}(t) = 2R(t) \sin \theta_j(t), l_{jk}(t) = 2R(t) \sin \theta_i(t). \tag{10}$$

Submitting (10) into (9) gives

$$\begin{aligned}
 u_i(t) &= 3 \ln R(t) + \ln \sin \theta_j(t) + \ln \sin \theta_k(t) - \ln \sin \theta_i(t) - \ln l_{ij}(0) \\
 &\quad - \ln l_{ik}(0) + \ln l_{jk}(0), \\
 u_j(t) &= 3 \ln R(t) + \ln \sin \theta_i(t) + \ln \sin \theta_k(t) - \ln \sin \theta_j(t) - \ln l_{ij}(0) \\
 &\quad - \ln l_{jk}(0) + \ln l_{ik}(0), \\
 u_k(t) &= 3 \ln R(t) + \ln \sin \theta_i(t) + \ln \sin \theta_j(t) - \ln \sin \theta_k(t) - \ln l_{ik}(0) \\
 &\quad - \ln l_{jk}(0) + \ln l_{ij}(0).
 \end{aligned} \tag{11}$$

By the definition of  $F_{ijk}(u_i, u_j, u_k)$  in (4), we have

$$\begin{aligned}
 F_{ijk}(u_i, u_j, u_k) &= - \int_0^1 \theta_i(t) du_i(t) + \theta_j(t) du_j(t) + \theta_k(t) du_k(t) \\
 &= -[\theta_i(t)u_i(t) + \theta_j(t)u_j(t) + \theta_k(t)u_k(t)] \Big|_0^1 \\
 &\quad + \int_0^1 u_i(t) d\theta_i(t) + u_j(t) d\theta_j(t) + u_k(t) d\theta_k(t)
 \end{aligned} \tag{12}$$



by integration by parts. Submitting (11) into (12) gives

$$\begin{aligned}
 &F_{ijk}(u_i, u_j, u_k) \\
 &= -[\theta_i(t)u_i(t) + \theta_j(t)u_j(t) + \theta_k(t)u_k(t)]|_0^1 \\
 &\quad - 2 \int_0^1 [\theta'_i(t) \ln \sin \theta_i(t) + \theta'_j(t) \ln \sin \theta_j(t) + \theta'_k(t) \ln \sin \theta_k(t)]dt \\
 &\quad + 2 \int_0^1 [\ln l_{ij}(0)\theta'_k(t) + \ln l_{ik}(0)\theta'_j(t) + \ln l_{jk}(0)\theta'_i(t)]dt \\
 &= -(\Theta_i u_i + \Theta_j u_j + \Theta_k u_k) \\
 &\quad + 2\mathbb{L}(\Theta_i) + 2\mathbb{L}(\Theta_j) + 2\mathbb{L}(\Theta_k) - 2\mathbb{L}(\overline{\Theta}_i) - 2\mathbb{L}(\overline{\Theta}_j) - 2\mathbb{L}(\overline{\Theta}_k) \\
 &\quad + 2 \ln l_{ij}(0)(\Theta_k - \overline{\Theta}_k) + 2 \ln l_{ik}(0)(\Theta_j - \overline{\Theta}_j) + 2 \ln l_{jk}(0)(\Theta_i - \overline{\Theta}_i),
 \end{aligned}$$

where the identity  $\theta'_i(t) + \theta'_j(t) + \theta'_k(t) \equiv 0$  is used in the first equality. □

**Remark 3** Set

$$h(u_i, u_j, u_j) = 2f\left(\frac{\tilde{\lambda}_{ij}}{2}, \frac{\tilde{\lambda}_{jk}}{2}, \frac{\tilde{\lambda}_{ki}}{2}\right) - \frac{\pi}{2}(\tilde{\lambda}_{ij} + \tilde{\lambda}_{jk} + \tilde{\lambda}_{ki}),$$

which is a building block of  $\mathbb{E}_{\mathcal{T}}(u)$  in (8). By direct calculations, we have

$$\begin{aligned}
 h(u_i, u_j, u_j) &= -(\Theta_i u_i + \Theta_j u_j + \Theta_k u_k) \\
 &\quad + 2\mathbb{L}(\Theta_i) + 2\mathbb{L}(\Theta_j) + 2\mathbb{L}(\Theta_k) + 2\Theta_i \ln l_{jk} + 2\Theta_j \ln l_{ik} + 2\Theta_k \ln l_{ij} \\
 &\quad - \pi(\ln l_{ij} + \ln l_{ik} + \ln l_{jk}),
 \end{aligned}$$

which implies that  $F_{ijk}(u_i, u_j, u_k)$  differ from  $h(u_i, u_j, u_j)$  by a constant depending on the background discrete PL metric  $l$ . As a direct consequence, Luo’s energy function  $\mathbb{F}_{\mathcal{T}}(u)$  in (6) differs from Bobenko-Pinkall-Springborn’s energy function  $\mathbb{E}_{\mathcal{T}}(u)$  in (8) by a constant depending on the background discrete PL metric  $l$ .

**Remark 4** There are two approaches to extend the energy function  $\mathbb{F}_{\mathcal{T}}(u)$  or  $\mathbb{E}_{\mathcal{T}}(u)$  to be a convex function defined on  $\mathbb{R}^V$  introduced by Bobenko-Pinkall-Springborn [4] and Gu-Luo-Sun-Wu [19] independently. In [4], the function  $\mathbb{E}_{\mathcal{T}}(u)$  is extended to be a  $C^1$  smooth convex function defined on  $\mathbb{R}^V$  by exploiting the fact that the function  $f$  defined on  $\Omega$  could be extended to a  $C^1$  smooth convex function on  $\mathbb{R}^3$  by extending the inner angles of a triangle by constants. Using Luo’ development [27] of Bobenko-Pinkall-Springborn’s extension [4], Ge-Jiang [11] similarly extended the potential function  $\mathbb{F}_{\mathcal{T}}(u)$  defined by (6) to be a  $C^1$ -smooth convex function defined on  $\mathbb{R}^V$ . Luo’s development [27] of Bobenko-Pinkall-Springborn’s extension [4] has recently been further developed to handle other cases. See, for instance, [38, 39, 41, 42] and others. In [19], the function  $\mathbb{F}_{\mathcal{T}}(u)$  is extended to be a  $C^2$ -smooth convex function defined on  $\mathbb{R}^V$  by changing the triangulation of the marked surface under the Delaunay condition, based on which Gu-Luo-Sun-Wu proved the Discrete Uniformization Theorem 1.4 for PL metrics on closed surfaces. The first approach could not ensure the triangles being non-degenerate, while the second approach could.

## 2.2 Gu-Luo-Sun-Wu’s discrete conformal theory

Based on Penner’s decorated Teichmüller space theory [31], Gu-Luo-Sun-Wu [19] established the discrete conformal theory for PL metrics on compact surfaces and proved Theorem

1.4. In the following, we recall some results in [19] that we need. We will not formally involve too much notions and results on decorated Teichmüller space. For more details of these results, please refer to Gu-Luo-Sun-Wu’s original work [19].

**Theorem 2.3** ([19] Corollary 4.7) *Suppose  $d$  is a PL metric on the marked surface  $(S, V)$ . Then there exists a  $C^1$  diffeomorphism  $\phi : \mathcal{D}(d) \rightarrow \mathbb{R}^V$ .*

By Theorem 2.3, the discrete conformal class  $\mathcal{D}(d)$  is parameterized by  $\mathbb{R}^V$ . For simplicity, for any  $d' \in \mathcal{D}(d)$ , we denote it by  $d(u)$  for some  $u \in \mathbb{R}^V$ . Suppose  $\mathcal{T}$  is a triangulation of the marked surface  $(S, V)$ . Set

$$\mathcal{A}_{\mathcal{T}} = \{u \in \mathbb{R}^V \mid \mathcal{T} \text{ is isotopic to a Delaunay triangulation of } (S, V, d(u))\}.$$

Based on Akiyoshi’s finiteness theorem in [1] (see also Appendix in [19] for a new proof), Gu-Luo-Sun-Wu [19] proved the following result.

**Theorem 2.4** ([19] Lemma 5.1) *Let*

$$J = \{\mathcal{T} \mid \mathcal{A}_{\mathcal{T}} \text{ has nonempty interior in } \mathbb{R}^V\}.$$

*Then  $J$  is a finite set,  $\mathbb{R}^V = \cup_{\mathcal{T} \in J} \mathcal{A}_{\mathcal{T}}$  and  $\mathcal{A}_{\mathcal{T}}$  is real analytically diffeomorphic to a closed convex polytope in  $\mathbb{R}^V$ .*

By Theorem 2.3, Gu-Luo-Sun-Wu [19] introduced the following globally defined  $C^1$  smooth combinatorial curvature function

$$\begin{aligned} \mathbf{F} : \mathbb{R}^V &\rightarrow (-\infty, 2\pi)^V \\ u &\mapsto K(d(u)). \end{aligned}$$

Combining with Lemma 2.1, Gu-Luo-Sun-Wu [19] further constructed a globally defined energy function with  $\mathbf{F}$  as gradient, which plays an important role in the proof of Theorem 1.4.

**Theorem 2.5** ([19] Proposition 5.2) *There exists a  $C^2$ -smooth convex function*

$$\begin{aligned} \mathbb{E} : \mathbb{R}^V &\rightarrow \mathbb{R} \\ u &\mapsto \int_0^u \sum_{i \in V} \mathbf{F}_i du_i \end{aligned} \tag{13}$$

*so that its gradient  $\nabla \mathbb{E} = \mathbf{F}$  and the restriction of  $\mathbb{E}$  to the hyperplane  $\{u \in \mathbb{R}^V \mid \sum_{i \in V} u_i = 0\}$  is strictly convex.*

**Remark 5** By the construction in Theorem 2.5, for  $\mathcal{T} \in J$ , the restriction  $\mathbb{E}|_{\mathcal{A}_{\mathcal{T}}}$  differs from  $\mathbb{F}_{\mathcal{T}}$  in (6) by a constant. As a consequence of (7),  $\mathbb{E}$  has the following property

$$\mathbb{E}(u + c(1, \dots, 1)) = \mathbb{E}(u) + 2c\pi\chi(S) \tag{14}$$

for any  $c \in \mathbb{R}$ .

### 3 Variational principles with constraints

In this section, we translate Theorem 1.7 into an optimization problem with inequality constraints by variational principles, which involve the function  $\mathbb{E}$  defined in (13).

Suppose  $(S, V, d)$  is a marked surface with a PL metric  $d$ ,  $\alpha \in \mathbb{R}$  is a non-zero constant and  $\bar{R}$  is a given function defined on  $V$ . Set

$$\mathcal{A} = \left\{ u \in \mathbb{R}^V \mid 0 > \sum_{i \in V} \bar{R}_i e^{\alpha u_i} \geq 2\pi \chi(S), \bar{R} \leq 0, \bar{R} \neq 0 \right\}, \tag{15}$$

$$\mathcal{B} = \left\{ u \in \mathbb{R}^V \mid 0 < \sum_{i \in V} \bar{R}_i e^{\alpha u_i} \leq 2\pi \chi(S), \bar{R} > 0 \right\}, \tag{16}$$

$$\mathcal{C} = \left\{ u \in \mathbb{R}^V \mid \sum_{i \in V} \bar{R}_i e^{\alpha u_i} \leq 2\pi \chi(S) < 0, \bar{R} \leq 0, \bar{R} \neq 0 \right\}. \tag{17}$$

**Proposition 3.1** *The sets  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are unbounded closed subsets of  $\mathbb{R}^V$ .*

**Proof** It is obvious that the sets  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are closed subsets of  $\mathbb{R}^V$ . For the set  $\mathcal{A}$ , by direct calculations,

$$\sum_{i \in V} \bar{R}_i e^{\alpha(u_i+c)} = e^{\alpha c} \sum_{i \in V} \bar{R}_i e^{\alpha u_i} \geq 2\pi \chi(S)$$

is equivalent to

$$c \geq \frac{1}{\alpha} \log \frac{2\pi \chi(S)}{\sum_{i \in V} \bar{R}_i e^{\alpha u_i}}$$

under the condition  $\alpha < 0$ ;

$$\sum_{i \in V} \bar{R}_i e^{\alpha(u_i+c)} = e^{\alpha c} \sum_{i \in V} \bar{R}_i e^{\alpha u_i} \geq 2\pi \chi(S)$$

is equivalent to

$$c \leq \frac{1}{\alpha} \log \frac{2\pi \chi(S)}{\sum_{i \in V} \bar{R}_i e^{\alpha u_i}}$$

under the condition  $\alpha > 0$ . Therefore, the set  $\mathcal{A}$  is unbounded. Similarly, the sets  $\mathcal{B}$  and  $\mathcal{C}$  are unbounded. □

According to Proposition 3.1, we have following result.

**Lemma 3.2** *Suppose  $(S, V, d)$  is a marked surface with a PL metric  $d$ ,  $\alpha \in \mathbb{R}$  is a constant and  $\bar{R}$  is a given function defined on  $V$ . If one of the following three conditions is satisfied*

- (1)  $\alpha > 0$  and the energy function  $\mathbb{E}$  attains a minimum in the set  $\mathcal{A}$ ,
- (2)  $\alpha < 0$  and the energy function  $\mathbb{E}$  attains a minimum in the set  $\mathcal{B}$ ,
- (3)  $\alpha < 0$  and the energy function  $\mathbb{E}$  attains a minimum in the set  $\mathcal{C}$ ,

*then the minimum value point of  $\mathbb{E}$  lies in the set  $\{u \in \mathbb{R}^V \mid \sum_{i \in V} \bar{R}_i e^{\alpha u_i} = 2\pi \chi(S)\}$ .*

**Proof** Let  $\alpha > 0$  and suppose the function  $\mathbb{E}$  attains a minimum at  $u \in \mathcal{A}$ . The definition of  $\mathcal{A}$  in (15) implies  $\chi(S) < 0$ . Taking  $c_0 = \frac{1}{\alpha} \log \frac{2\pi \chi(S)}{\sum_{i \in V} \bar{R}_i e^{\alpha u_i}}$ , then  $c_0 \geq 0$ . By the proof of Proposition 3.1,  $u + c_0 \mathbb{1} \in \mathcal{A}$ . Therefore, by the additive property of the function  $\mathbb{E}$  in (14), we have

$$\mathbb{E}(u) \leq \mathbb{E}(u + c_0 \mathbb{1}) = \mathbb{E}(u) + 2\pi c_0 \chi(S),$$

which implies  $c_0 \leq 0$  by  $\chi(S) < 0$ . Hence  $c_0 = 0$  and  $\sum_{i \in V} \bar{R}_i e^{\alpha u_i} = 2\pi \chi(S)$ . This proves the case of (1). The proofs for the cases (2) and (3) are similar, we omit the details here.  $\square$

By Lemma 3.2, we translate Theorem 1.7 into the following theorem, which is a non-convex optimization problem with inequality constraints.

**Theorem 3.3** *Suppose  $(S, V, d)$  is a marked surface with a PL metric  $d$  and  $\chi(S) \neq 0$ ,  $\alpha \in \mathbb{R}$  is a non-zero constant and  $\bar{R}$  is a given function defined on  $V$ .*

- (1) *If  $\alpha > 0$  and the energy function  $\mathbb{E}$  attains a minimum in  $\mathcal{A}$ , then there exists a PL metric in the conformal class  $\mathcal{D}(d)$  with combinatorial  $\alpha$ -curvature  $\bar{R} \leq 0$  and  $\bar{R} \not\equiv 0$ ;*
- (2) *If  $\alpha < 0$  and the energy function  $\mathbb{E}$  attains a minimum in  $\mathcal{B}$ , then there exists a PL metric in the conformal class  $\mathcal{D}(d)$  with combinatorial  $\alpha$ -curvature  $\bar{R} > 0$ ;*
- (3) *If  $\alpha < 0$  and the energy function  $\mathbb{E}$  attains a minimum in  $\mathcal{C}$ , then there exists a PL metric in the conformal class  $\mathcal{D}(d)$  with combinatorial  $\alpha$ -curvature  $\bar{R} \leq 0$  and  $\bar{R} \not\equiv 0$ .*

**Proof** Lemma 3.2 shows that if  $u \in \mathbb{R}^V$  is a minimum of the energy function  $\mathbb{E}$  defined on one of these sets, then  $\sum_{i \in V} \bar{R}_i e^{\alpha u_i} = 2\pi \chi(S)$ . The conclusion follows from the following claim.

**Claim :** Up to scaling, the PL-metrics with prescribed combinatorial  $\alpha$ -curvature in the discrete conformal class  $\mathcal{D}(d)$  are in one-to-one correspondence with the critical points of the function  $\mathbb{E}$  under the constraint  $\sum_{i \in V} \bar{R}_i e^{\alpha u_i} = 2\pi \chi(S)$ .

We use the method of Lagrange multipliers to prove this claim. Set

$$H(u, \lambda) = \mathbb{E}(u) + \lambda \left( \sum_{i \in V} \bar{R}_i e^{\alpha u_i} - 2\pi \chi(S) \right),$$

where  $\lambda \in \mathbb{R}$  is a Lagrange multiplier. If  $u$  is a critical point of the function  $\mathbb{E}$  under the constraint  $\sum_{i \in V} \bar{R}_i e^{\alpha u_i} = 2\pi \chi(S)$ , then

$$0 = \frac{\partial H(u, \lambda)}{\partial u_i} = K_i + \lambda \alpha \bar{R}_i e^{\alpha u_i},$$

which implies

$$R_{\alpha,i} = \frac{K_i}{e^{\alpha u_i}} = -\lambda \alpha \bar{R}_i.$$

By the discrete Gauss-Bonnet formula (2), the Lagrange multiplier  $\lambda$  satisfies

$$\lambda = -\frac{2\pi \chi(S)}{\alpha \sum_{i \in V} \bar{R}_i e^{\alpha u_i}} = -\frac{1}{\alpha}$$

under the constraint  $\sum_{i \in V} \bar{R}_i e^{\alpha u_i} = 2\pi \chi(S)$ , which implies the combinatorial  $\alpha$ -curvature

$$R_{\alpha,i} = -\lambda \alpha \bar{R}_i = \frac{2\pi \chi(S)}{\sum_{i \in V} \bar{R}_i e^{\alpha u_i}} \bar{R}_i = \bar{R}_i$$

under the constraint  $\sum_{i \in V} \bar{R}_i e^{\alpha u_i} = 2\pi \chi(S)$ .  $\square$

### 4 Proof of Theorem 1.7

In this section, we give a proof for Theorem 1.7. By Theorem 3.3, we just need to prove that the energy function  $\mathbb{E}$  attains the minimum in the sets  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ . Recall the following classical result from calculus.

**Theorem 4.1** *Let  $A \subseteq \mathbb{R}^m$  be a closed set and  $f : A \rightarrow \mathbb{R}$  be a continuous function. If every unbounded sequence  $\{u_n\}_{n \in \mathbb{N}}$  in  $A$  has a subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$  such that  $\lim_{k \rightarrow +\infty} f(x_{n_k}) = +\infty$ , then  $f$  attains a minimum in  $A$ .*

One can refer to [22] (Section 4.1) for a proof of Theorem 4.1. The majority of the conditions in Theorem 4.1 are satisfied, including the sets  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are closed subsets of  $\mathbb{R}^V$  by Proposition 3.1 and the energy function  $\mathbb{E}$  is continuous by Theorem 2.5. To prove Theorem 1.7, we just need to prove the following theorem by Theorem 3.3 and Theorem 4.1.

**Theorem 4.2** *Suppose  $(S, V, d)$  is a marked surface with a PL metric  $d$ ,  $\alpha \in \mathbb{R}$  is a constant and  $\bar{R}$  is a given function defined on  $V$ . If one of the following three conditions is satisfied,*

- (1)  $\alpha > 0$  and  $\{u_n\}_{n \in \mathbb{N}}$  is an unbounded sequence in  $\mathcal{A}$ ,
- (2)  $\alpha < 0$  and  $\{u_n\}_{n \in \mathbb{N}}$  is an unbounded sequence in  $\mathcal{B}$ ,
- (3)  $\alpha < 0$  and  $\{u_n\}_{n \in \mathbb{N}}$  is an unbounded sequence in  $\mathcal{C}$ ,

*then there exist a subsequence  $\{u_{n_k}\}_{k \in \mathbb{N}}$  of  $\{u_n\}_{n \in \mathbb{N}}$  such that  $\lim_{k \rightarrow +\infty} \mathbb{E}(u_{n_k}) = +\infty$ .*

Let  $\{u_n\}_{n \in \mathbb{N}}$  be an unbounded sequence in  $\mathbb{R}^V$ , denote its coordinate sequence at  $j \in V$  by  $\{u_{j,n}\}_{n \in \mathbb{N}}$ . Motivated by [23], we call the sequence  $\{u_n\}_{n \in \mathbb{N}}$  with the following properties as a “good” sequence.

- (1) It lies in one cell  $\mathcal{A}_{\mathcal{T}}$  of  $\mathbb{R}^V$  for some  $\mathcal{T} \in J$  given by Theorem 2.4;
- (2) There exists a vertex  $i^* \in V$  such that  $u_{i^*,n} \leq u_{j,n}$  for all  $j \in V$  and  $n \in \mathbb{N}$ ;
- (3) Each coordinate sequence  $\{u_{j,n}\}_{n \in \mathbb{N}}$  either converge, diverge properly to  $+\infty$ , or diverges properly to  $-\infty$ ;
- (4) For all  $j \in V$ , the sequence  $\{u_{j,n} - u_{i^*,n}\}_{n \in \mathbb{N}}$  either converge or diverge properly to  $+\infty$ .

By Theorem 2.4, it is obvious that every sequence in  $\mathbb{R}^V$  possesses a “good” subsequence, hence the “good” sequence could be chosen without loss of generality. To prove Theorem 4.2, we further need the following two results obtained by Kouřimská [23].

**Lemma 4.3** ([23] Corollary 5.6) *In every triangle  $\{ijk\} \in F_{\mathcal{T}}$ , at least two of the three sequences  $(u_{i,n} - u_{i^*,n})_{n \in \mathbb{N}}$ ,  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  and  $(u_{k,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converge.*

**Lemma 4.4** ([23] Lemma 5.11) *There exists a convergent sequence  $\{C_n\}_{n \in \mathbb{N}}$  such that the energy function  $\mathbb{E}$  satisfies*

$$\mathbb{E}(u_n) = C_n + 2\pi \left( u_{i^*,n} \chi(S) + \sum_{j \in V} (u_{j,n} - u_{i^*,n}) \right).$$

The proof of Lemma 4.4 is based on an interesting analysis of the explicit form of the energy function  $\mathbb{F}_{\mathcal{T}}$  or  $\mathbb{E}_{\mathcal{T}}$ . Readers can refer to [22, 23] for the proof.

**Proof of Theorem 4.2** Let  $\{u_n\}_{n \in \mathbb{N}}$  be an unbounded “good” sequence. We just need to prove that  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$ .

- (1) Let  $\alpha > 0$  and  $\{u_n\}_{n \in \mathbb{N}}$  be an unbounded sequence in  $\mathcal{A}$ . The definition of  $\mathcal{A}$  in (15) implies  $\chi(S) < 0$ ,  $\bar{R} \leq 0$  and  $\bar{R} \neq 0$ . Since the sequence  $\{u_n\}_{n \in \mathbb{N}}$  lies in  $\mathcal{A}$ , we have

$$0 > \sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})} = e^{-\alpha u_{i^*,n}} \cdot \sum_{j \in V} \bar{R}_j e^{\alpha u_{j,n}} \geq 2\pi \chi(S) e^{-\alpha u_{i^*,n}}. \tag{18}$$

Note that  $(\sum_{j \in V} (u_{j,n} - u_{i^*,n}))_{n \in \mathbb{N}}$  converges to a finite number or diverges properly to  $+\infty$  by the definition of ‘‘good’’ sequence.

If  $(\sum_{j \in V} (u_{j,n} - u_{i^*,n}))_{n \in \mathbb{N}}$  converges to a finite number, then the sequence  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converges for all  $j \in V$ , which implies  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$  converges to a finite negative number by  $\bar{R} \leq 0$  and  $\bar{R} \neq 0$ . Therefore,  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  is bounded from above by (18),  $\alpha > 0$  and  $\chi(S) < 0$ , which implies  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  converges to a finite number or diverges to  $-\infty$ . If  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  converges to a finite number, then  $\{u_{j,n}\}_{n \in \mathbb{N}}$  are bounded for all  $j \in V$  by  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converges for all  $j \in V$ , which implies  $\{u_n\}_{n \in \mathbb{N}}$  is bounded. This contradicts the assumption that  $\{u_n\}_{n \in \mathbb{N}}$  is unbounded. Therefore, the sequence  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  diverges properly to  $-\infty$ . Combining this with  $\chi(S) < 0$  and Lemma 4.4, we have  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$ .

If  $(\sum_{j \in V} (u_{j,n} - u_{i^*,n}))_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ , then there exists at least one vertex  $j \in V$  such that the sequence  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ . By Lemma 4.3, there exists at least one vertex  $k \in V$  such that the sequence  $(u_{k,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converges (for example  $(u_{i^*,n} - u_{i^*,n})_{n \in \mathbb{N}}$ ). Therefore,  $e^{\alpha(u_{j,n} - u_{i^*,n})}$  converges to a finite positive number or diverges properly to  $+\infty$  and for at least one vertex  $j \in V$  the term  $e^{\alpha(u_{j,n} - u_{i^*,n})}$  converges to a finite positive number. Combining  $\bar{R} \leq 0$  and  $\bar{R} \neq 0$ ,  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$  tends to either a finite negative number or  $-\infty$ . If  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$  tends to a finite negative number, then  $u_{i^*,n}$  is bounded from above by (18), which implies that  $u_{i^*,n} \chi(S)$  is bounded from below. Combining with the assumption that  $(\sum_{j \in V} (u_{j,n} - u_{i^*,n}))_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ , we have  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$  by Lemma 4.4. If  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$  tends to  $-\infty$ , we have  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  diverges properly to  $-\infty$  by (18), which implies  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$  by  $\chi(S) < 0$  and Lemma 4.4.

- (2) Let  $\alpha < 0$  and  $\{u_n\}_{n \in \mathbb{N}}$  be an unbounded sequence in  $\mathcal{B}$ . The definition of  $\mathcal{B}$  in (16) implies  $\chi(S) > 0$  and  $\bar{R} > 0$ . Since the sequence  $\{u_n\}_{n \in \mathbb{N}}$  lies in  $\mathcal{B}$ , we have

$$0 < \sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})} = e^{-\alpha u_{i^*,n}} \cdot \sum_{j \in V} \bar{R}_j e^{\alpha u_{j,n}} \leq 2\pi \chi(S) e^{-\alpha u_{i^*,n}}. \tag{19}$$

If  $(\sum_{j \in V} (u_{j,n} - u_{i^*,n}))_{n \in \mathbb{N}}$  converges, then the sequence  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converges for all  $j \in V$ . Note that  $\alpha < 0$ ,  $\chi(S) > 0$  and  $\bar{R} > 0$ , we have  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  is bounded from below by (19), which implies  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  converges to a finite number or diverges properly to  $+\infty$ . Combining this with  $\{u_n\}_{n \in \mathbb{N}}$  is unbounded and  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converges for all  $j \in V$ , we have the sequence  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ . As a result, we have  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$  by Lemma 4.4 and  $\chi(S) > 0$ .

If the sequence  $(\sum_{j \in V} (u_{j,n} - u_{i^*,n}))_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ , then there exists at least one vertex  $j \in V$  such that the sequence  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ . By Lemma 4.3, there exists at least one vertex  $k \in V$  such that the sequence  $(u_{k,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converges (for example  $(u_{i^*,n} - u_{i^*,n})_{n \in \mathbb{N}}$ ). Therefore,  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$

has a positive lower bound by  $\bar{R} > 0$  and  $\alpha < 0$ , which implies  $2\pi\chi(S)e^{-\alpha u_{i^*,n}}$  has a positive lower bound by (19). Therefore,  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  is bounded from below by  $\alpha < 0$  and  $\chi(S) > 0$  and hence  $u_{i^*,n}\chi(S)$  is bounded from below. Combining this with  $\left(\sum_{j \in V}(u_{j,n} - u_{i^*,n})\right)_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ , we have  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$  by Lemma 4.4.

- (3) Let  $\alpha < 0$  and  $\{u_n\}_{n \in \mathbb{N}}$  be an unbounded sequence in  $\mathcal{C}$ . The definition of  $\mathcal{C}$  in (17) implies  $\chi(S) < 0$ ,  $\bar{R} \leq 0$  and  $\bar{R} \neq 0$ . Since the sequence  $\{u_n\}_{n \in \mathbb{N}}$  lies in  $\mathcal{C}$ , we have

$$\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})} = e^{-\alpha u_{i^*,n}} \cdot \sum_{j \in V} \bar{R}_j e^{\alpha u_{j,n}} \leq 2\pi\chi(S)e^{-\alpha u_{i^*,n}} < 0. \tag{20}$$

If  $\left(\sum_{j \in V}(u_{j,n} - u_{i^*,n})\right)_{n \in \mathbb{N}}$  converges, then the sequence  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converges for all  $j \in V$ , which implies that  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$  converges to a finite negative number by  $\bar{R} \leq 0$  and  $\bar{R} \neq 0$ . Combining this with  $\alpha < 0$  and  $\chi(S) < 0$ , we have  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  is bounded from above by (20). As  $\{u_n\}_{n \in \mathbb{N}}$  is unbounded, then similar to the arguments above, the sequence  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  diverges properly to  $-\infty$ . Combining this with  $\chi(S) < 0$  and Lemma 4.4, we have  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$ .

If  $\left(\sum_{j \in V}(u_{j,n} - u_{i^*,n})\right)_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ , then there exists at least one vertex  $j \in V$  such that the sequence  $(u_{j,n} - u_{i^*,n})_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ . By Lemma 4.3, there exists at least one vertex  $k \in V$  such that the sequence  $(u_{k,n} - u_{i^*,n})_{n \in \mathbb{N}}$  converges (for example  $(u_{i^*,n} - u_{i^*,n})_{n \in \mathbb{N}}$ ). Therefore,  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$  either tends to zero or to a finite negative number by  $\alpha < 0$ ,  $\bar{R} \leq 0$  and  $\bar{R} \neq 0$ . If  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$  tends to zero, then  $2\pi\chi(S)e^{-\alpha u_{i^*,n}}$  tends to zero by (20), which implies  $\{u_{i^*,n}\}_{n \in \mathbb{N}}$  diverges properly to  $-\infty$  by  $\alpha < 0$  and  $\chi(S) < 0$ . Combining this with  $\chi(S) < 0$  and Lemma 4.4, we have  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$ . If  $\sum_{j \in V} \bar{R}_j e^{\alpha(u_{j,n} - u_{i^*,n})}$  tends to a finite negative number, we have  $2\pi\chi(S)e^{-\alpha u_{i^*,n}}$  has a negative lower bound by (20), which implies  $u_{i^*,n}$  is bounded from above by  $\alpha < 0$  and  $\chi(S) < 0$ . Combining this with  $\chi(S) < 0$  and  $\left(\sum_{j \in V}(u_{j,n} - u_{i^*,n})\right)_{n \in \mathbb{N}}$  diverges properly to  $+\infty$ , we have  $\lim_{n \rightarrow +\infty} \mathbb{E}(u_n) = +\infty$  by Lemma 4.4.

□

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## References

1. Akiyoshi, H.: Finiteness of polyhedral decompositions of cusped hyperbolic manifolds obtained by the Epstein-Penner’s methods. Proc. Amer. Math. Soc. **129**(8), 2431–2439 (2001)
2. Aurenhammer, F., Kelin, R.: Voronoi diagrams, handbook of computational geometry, pp. 201–290. North-Holland, Amsterdam (2000)
3. Berger, M.S.: Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds. J. Differ. Geom. **5**, 325–332 (1971)
4. Bobenko, A., Pinkall, U., Springborn, B.: Discrete conformal maps and ideal hyperbolic polyhedra. Geom. Topol. **19**(4), 2155–2215 (2015)

5. Bobenko, A., Springborn, B.: A discrete Laplace-Beltrami operator for simplicial surfaces. *Discrete Comput. Geom.* **38**(4), 740–756 (2007)
6. Chang, S.Y.A., Yang, P.C.: Prescribing Gaussian curvature on  $S^2$ . *Acta Math.* **159**, 215–259 (1987)
7. Chow, B., Luo, F.: Combinatorial Ricci flows on surfaces. *J. Differ. Geom.* **63**(1), 97–129 (2003)
8. de Verdière, Y.C.: Un principe variationnel pour les empilements de cercles. *Invent. Math.* **104**(3), 655–666 (1991)
9. Dai, S., Ge, H.: Discrete Yamabe flows with R-curvature revisited. *J. Math. Anal. Appl.* **484**, 123681 (2020)
10. Dai, J., Gu, X., Luo, F.: *Variational principles for discrete surfaces*, Advanced Lectures in Mathematics (ALM), vol. 4, International Press/Higher Education Press, Somerville, MA/Beijing, 2008, iv+146 pp
11. Ge, H., Jiang, W.: On the deformation of discrete conformal factors on surfaces. *Cala Var Partial Differ. Equ.* **55**, 14 (2016)
12. Ge, H., Jiang, W.: On the deformation of inversive distance circle packings. III. *J. Funct. Anal.* **272**(9), 3596–3609 (2017)
13. Ge, H., Xu, X.: Discrete quasi-Einstein metrics and combinatorial curvature flows in 3- dimension. *Adv. Math.* **267**, 470–497 (2014)
14. Ge, H., Xu, X.:  $\alpha$ - curvatures and  $\alpha$ -flows on low dimensional triangulated manifolds. *Calc. Var. Partial Differ. Equ.* **55**, 16 (2016)
15. Ge, H., Xu, X.: A discrete Ricci flow on surfaces with hyperbolic background geometry. *Int. Math. Res. Not. IMRN* **11**, 3510–3527 (2017)
16. Ge, H., Xu, X.: On a combinatorial curvature for surfaces with inversive distance circle packing metrics. *J. Funct. Anal.* **275**(3), 523–558 (2018)
17. Ge, H., Xu, X.: A combinatorial Yamabe problem on two and three dimensional manifolds. *Calc Var Partial Differ. Equ.* **60**, 20 (2021)
18. Gu, X.D., Guo, R., Luo, F., Sun, J., Wu, T.: A discrete uniformization theorem for polyhedral surfaces II. *J. Differ. Geom.* **109**(3), 431–466 (2018)
19. Gu, X.D., Luo, F., Sun, J., Wu, T.: A discrete uniformization theorem for polyhedral surfaces. *J. Differ. Geom.* **109**(2), 223–256 (2018)
20. Gu, X.D., Luo, F., Yau, S.T.: Computational conformal geometry behind modern technologies. *Notices Amer. Math. Soc.* **67**(10), 1509–1525 (2020)
21. Gu, X. D., Yau, S.T.: *Computational conformal geometry*. Advanced Lectures in Mathematics, 3. International Press, Somerville, MA; Higher Education Press, Beijing, 2008. vi+295 pp
22. Kouřimská, H.: Polyhedral surfaces of constant curvature and discrete uniformization. PhD thesis, Technische Universität Berlin, Germany (2020)
23. Kouřimská, H.: *Discrete Yamabe problem for polyhedral surfaces*, [arXiv:2103.15693](https://arxiv.org/abs/2103.15693) [math.MG]
24. Kazdan, J.L., Warner, F.W.: Curvature functions for compact 2-manifolds. *Ann. Math.* **99**, 14–47 (1974)
25. Kazdan, J.L., Warner, F.W.: Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures. *Ann. Math.* **101**, 317–331 (1975)
26. Luo, F.: Combinatorial Yamabe flows on surfaces. *Commun. Contemp. Math.* **6**(5), 765–780 (2004)
27. Luo, F.: Rigidity of polyhedral surfaces, III. *Geom. Topol.* **15**, 2299–2319 (2011)
28. Luo, F.: The Riemann mapping theorem and its discrete counterparts. From Riemann to differential geometry and relativity, pp. 367–388. Springer, Cham (2017)
29. Milnor, J.: Hyperbolic geometry: the first 150 years. *Bull. Amer. Math. Soc.* **6**, 9–24 (1982)
30. Moser, J.: *On a nonlinear problem in differential geometry*, in Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), (Academic Press, New York, 1973), pp. 273–280
31. Penner, R.C.: The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.* **113**, 299–339 (1987)
32. Rivin, I.: Euclidean structures of simplicial surfaces and hyperbolic volume. *Ann. Math.* **139**, 553–580 (1994)
33. Rivin, I.: *Intrinsic geometry of convex ideal polyhedra in hyperbolic 3-space*. In: Gyllenberg, M., Persson, L.-E. (eds.) *Analysis, Algebra, and Computers in Mathematical Research*. Lecture Notes in Pure and Appl. Math., vol. 156, pp. 275–291. Dekker, New York (1994)
34. Springborn, B.: Ideal hyperbolic polyhedra and discrete uniformization. *Discrete Comput. Geom.* **64**(1), 63–108 (2020)
35. Sun, J., Wu, T., Gu, X., Luo, F.: Discrete conformal deformation: algorithm and experiments. *SIAM J. Imaging Sci.* **8**(3), 1421–1456 (2015)
36. Thurston, W.: *Geometry and topology of 3- manifolds*. Princeton lecture notes (1976). <http://www.msri.org/publications/books/gt3m>
37. Xu, X.: Rigidity of inversive distance circle packings revisited. *Adv. Math.* **332**, 476–509 (2018)



38. Xu, X.: On the global rigidity of sphere packings on 3-dimensional manifolds. *J. Differ. Geom.* **115**(1), 175–193 (2020)
39. Xu, X.: A new proof of Bowers-Stephenson conjecture. *Math. Res. Lett.* **28**(4), 1283–1306 (2021)
40. Xu, X.: *Parameterized discrete uniformization theorems and curvature flows for polyhedral surfaces, I*. [arXiv:1806.04516v2](https://arxiv.org/abs/1806.04516v2) [math.GT]
41. Xu, X.: *Rigidity and deformation of discrete conformal structures on polyhedral surfaces*, [arXiv:2103.05272](https://arxiv.org/abs/2103.05272) [math.DG]
42. Xu, X., Zheng, C.: A new proof for global rigidity of vertex scaling on polyhedral surfaces, accepted by. *Asian J. Math.* (2021)
43. Xu, X., Zheng, C.: Parameterized discrete uniformization theorems and curvature flows for polyhedral surfaces, II. [math.GT]. accepted by *Trans. Amer. Math. Soc.* (2021) [arXiv:2103.16077](https://arxiv.org/abs/2103.16077)
44. Xu, X., Zheng, C.: *Parameterized combinatorial curvatures and parameterized combinatorial curvature flows for discrete conformal structures on polyhedral surfaces*, [arXiv:2105.14714](https://arxiv.org/abs/2105.14714) [math.GT]
45. Zeng, W., Gu, X.: *Ricci flow for shape analysis and surface registration*. Springer Briefs in Mathematics. Springer, New York (2013)

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