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Calculus of Variations

Stability of smooth multi-solitons for the Camassa–Holm equation

Zhong Wang[1] · Yue Liu[2]

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Abstract

Consideration in this paper is the stability of exact smooth multi-solitons for the Camassa–Holm equation. By constructing a suitable Lyapunov functional, it is found that the smooth multi-solitons are non-isolated constrained minimizers satisfying a suitable variational non-local elliptic equation and the dynamical stability issue is reduced to study of the spectrum of explicit linearized systems. Our approach in the spectral analysis consists in an invariant for the multi-solitons and new operator identities motivated by the bi-Hamilton structure of the Camassa–Holm equation. The key ingredient in the spectral analysis is to use integrable property of the recursion operator of the Camassa–Holm equation. It is demonstrated here that orbital stability of shape of smooth single soliton implies that the shapes of all smooth multi-solitons are dynamically stable under small disturbances in a suitable Sobolev space.

Mathematics Subject Classification 35Q35 · 35Q51 · 37K05 · 37K10

1 Introduction

We consider the Camassa–Holm (CH) equation [5,27]

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad t > 0,\ x \in \mathbb{R} \tag{1.1}$$

with $\omega \geq 0$ and the function $u(t, x)$ in dimensionless space-time variables (x, t), which is a model to describe the unidirectional propagation of shallow water waves over a flat bottom [5,31] (see also [19] for a rigorous justification in shallow water approximation). The CH equation (1.1) is a completely integrable equation in the sense that it has an infinite number of conserved quantities and a Lax pair [1,5,9,11,20], describing permanent and breaking waves

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✉ Yue Liu
yliu@uta.edu

Zhong Wang
wangzh79@fosu.edu.cn

1 School of Mathematics and Big Data, Foshan University, Foshan 528000, China

2 Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019, USA

[10,12,44]. Its solitary waves are orbitally stable smooth solitons ($\omega > 0$) [23] or peakons ($\omega = 0$) [22,47] in the energy space. Equation (1.1) arises also as an equation of the geodesic flow for the H^1 right-invariant metric on the Bott-Virasoro group (if $\omega > 0$) [45] and on the diffeomorphism group (if $\omega = 0$) [17,18]. The CH equation (1.1) has the bi-Hamiltonian structure of the form (1.1) [5,27]:

$$\frac{\partial m}{\partial t} = \mathcal{J}_2 \frac{\delta H_2[m]}{\delta m} = \mathcal{J}_1 \frac{\delta H_1[m]}{\delta m}, \tag{1.2}$$

$$\mathcal{J}_1 := -(2\omega\partial_x + m\partial_x + \partial_x m), \quad \mathcal{J}_2 := -(\partial_x - \partial_x^3), \tag{1.3}$$

with the momentum density $m := u - u_{xx}$ and the two Hamiltonians

$$H_1[m] = H_1(u) = \frac{1}{2}\int_{\mathbb{R}} mu\,\mathrm{d}x, \quad \textit{and} \tag{1.4}$$

$$H_2[m] = H_2(u) = \frac{1}{2}\int_{\mathbb{R}} (u^3 + uu_x^2 + 2\omega u^2)\,\mathrm{d}x. \tag{1.5}$$

The CH equation (1.1) can be rewritten as an infinite dimensional Hamiltonian PDE as follows,

$$u_t = \mathcal{J}\frac{\delta H_2(u)}{\delta u}, \quad \mathcal{J} := (1-\partial_x^2)^{-1}\mathcal{J}_2(1-\partial_x^2)^{-1} = -\partial_x(1-\partial_x^2)^{-1}, \tag{1.6}$$

the operator $\mathcal{J}$ is skew symmetric and bounded in $L^2(\mathbb{R})$.

In general, there exist infinite many conservation laws (multi-Hamiltonian structures) $H_n[m]$, $n = 0, \pm 1, \pm 2, \ldots$, including (1.4) and (1.5), such that [35]

$$\mathcal{J}_2 \frac{\delta H_n[m]}{\delta m} = \mathcal{J}_1 \frac{\delta H_{n-1}[m]}{\delta m}. \tag{1.7}$$

Schemes for the computation of the conservation laws can be found in [8,26,30,35].

From the Inverse Scattering Theory, the evolution of a rapidly decaying initial data can be described by purely algebraic methods. Solutions are shown to decompose into a very particular set of solutions.*soliton resolution conjecture* states that any global solutions of dispersive equations will decompose as $t \to +\infty$ as a finite sum of (re-scaled and translated) solitons plus a radiation (solution of the corresponding linear equation). For the CH equation, such types of solutions consist of multi-solitons, which will describe in detail below.

It is known that the CH equation (1.1) possesses smooth solitary-wave solutions called *solitons* if $\omega > 0$ [6] or peaked solitons if $\omega = 0$ [5]. These profiles are often regarded as minimizers of a constrained functional in the H^1-topology. In particular, when $\omega > 0$, the CH equation (1.1) possesses the smooth soliton for some $x_0 \in \mathbb{R}$,

$$u(t,x) = \varphi_c(x - ct + x_0), \; c > 2\omega, \; t \geq 0, \; x \in \mathbb{R} \tag{1.8}$$

in a parametric form as follows [32,37],

$$u(t,x) = \frac{c - 2\omega}{1 + (2\omega/c)\sinh^2\theta},$$

$$\theta = \frac{1}{2\sqrt{\omega}}\sqrt{1 - \frac{2\omega}{c}}\left(y - c\sqrt{\omega}t\right),$$

$$x = \frac{y}{\sqrt{\omega}} + \ln\frac{\cosh(\theta - \theta_0)}{\cosh(\theta + \theta_0)}, \quad \theta_0 := \tanh^{-1}\sqrt{1 - \frac{2\omega}{c}}.$$

By inserting (1.8) into (1.1), it is observed that $\varphi_c > 0$ satisfies the following equation

$$-c\varphi_c + c\varphi_{cxx} + \frac{3}{2}\varphi_c^2 + 2\omega\varphi_c = \varphi\varphi_{cxx} + \frac{1}{2}\varphi_{cx}^2, \quad x \in \mathbb{R} \tag{1.9}$$

For $\omega > 0$, solitary wave $\varphi_c > 0$ propagates to the right exist only with speed $c > 2\omega$, and, conversely, each such a speed c determines uniquely the profile φ_c of the soliton up to translations. It is shown in [23] that these smooth solitary-wave profiles φ_c have the following properties.

1) φ_c is smooth and positive with an even profile decreasing from its peak height $c - 2\omega$.
2) φ_c is concave for values in the interval $\left[c - \omega/2 - \sqrt{c\omega + \omega^2/4},\ c - 2\omega\right]$ and convex elsewhere.
3) Multiplying both sides of (1.9) by φ_c' and integration, one has

$$\varphi_{cx}^2(c - \varphi_c) = \varphi_c^2(c - 2\omega - \varphi_c).$$

   Then it is found that $|\varphi_c'| \leq \varphi_c$ on $\mathbb{R}$ and

$$\varphi_c(x) = O\Big(\exp\Big(-\sqrt{1 - \frac{2\omega}{c}}|x|\Big)\Big) \text{ for } |x| \to \infty.$$

   Moreover, by combining (1.9), it is observed that

$$\varphi_c - \varphi_c'' = \frac{\omega\varphi_c(2c - \varphi_c)}{(c - \varphi_c)^2} > 0. \tag{1.10}$$

4) Requiring that the profile φ_c reaches its maximum at $x = 0$, φ_c converges uniformly on every compact subset of $\mathbb{R}$ as ω tends to zero to the peakon profile $ce^{-|x|}$.

The CH equation (1.1) possesses even more complex solutions, such as *multi-solitons* which can be given also in a parametric form like one soliton [7,14,37,43]. Moreover, in the limit of $t \to \infty$, the CH N-solitons $U_{\mathbf{c}}^{(N)}$ behave like N decoupled one soliton φ_{c_j} with wave speeds $c_j > 0$, $j = 1, 2, \ldots, N$, in particular, $U_{\mathbf{c}}^{(N)}$ is represented by a superposition of N-solitons and has the following asymptotic behavior (see [43])

$$U_{\mathbf{c}}^{(N)}(t) \sim \sum_{n=1}^{N} \varphi_{c_j}(\cdot - c_j t - x_j^{\pm}), \quad t \to \pm\infty, \tag{1.11}$$

for some $x_j^{\pm} \in \mathbb{R}$ depending on c_j. As a consequence of the integrability property, the multi-solitons interact elastically during the dynamics, and no dispersive effects are present at infinity. The CH equation (1.1) also has the invariant property, $\omega > 0$ and $m(0, x) + \omega > 0$, then $m(t, x) + \omega > 0$ for all the time t [9,11,20].

The definition of the stability of solitons may be classified according to the following four categories: (i) linear (or spectral) stability, (ii) Lyapunov (dynamical) stability, (iii) orbital (nonlinear) stability, (iv) asymptotic stability. The dynamical stability implies that the second variation of certain Lyapunov functional becomes strictly positive when evaluated at the soliton solutions. It would also imply the linear stability since the second variation is preserved for the linearized equation. In order to extend the Lyapunov stability to the orbital stability which deals with small but finite amplitude perturbations, one must take into account the higher-order nonlinear terms neglected in evaluating the Lyapunov functional and this makes the analysis more difficult to deal with, we refer to [39] for a nice exposition on this issue for the general linear Hamiltonian PDEs. In accordance with the above classification of the stability, we shall briefly review some known results associated with the stability characteristics of the CH solitons and multi-solitons. The nonlinear stability of the CH 1-solitons φ_c is proved by Constantin and Strauss [23] by applying the general spectral method developed by Benjamin [2] and Grillakis et al. [28]. It is noted that a general method to prove the orbital stability of the train of N solitary waves for nonlinear Hamiltonian dispersive equations was also introduced by Martel et al. [42], while the stability of the trains of N-solitons for the generalized Korteweg-de Vries (gKdV) equation was proved. This was the first result related to the stability of N solitary waves in the energy space $H^1(\mathbb{R})$. Using this approach, El Dika and Molinet [24] investigated the orbital stability of the train of N solitary waves of the CH equation in energy space $H^1(\mathbb{R})$, the main step in the proof is an almost monotonicity property for the localized conservation laws related to H_1 and H_2. However, to the best knowledge of the authors, there is no result for the stability of exact N-solitons currently in the literature. The purpose of the present paper is to establish the dynamical and orbital stability of the smooth multi-solitons of the CH equation. In particular, for $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$, and $\mathbf{c} \in S_N$, where

$$S_N = \{\mathbf{c}; \mathbf{c} = (c_1, c_2, \dots, c_N) \in (2\omega, +\infty)^N,\ c_i \neq c_j \text{ for } 1 \leq i < j \leq N\},$$

without loss of generality, we assume that $0 < 2\omega < c_1 < c_2 < c_3 < \cdots < c_N$. Denote N-solitons by $U^{(N)}(t, x; \mathbf{c}, \mathbf{x})$. Define

$$\begin{aligned} G_{\mathbf{c}} &= \{u \in H^N(\mathbb{R});\ H_k(u) = H_k(U^{(N)}(t, x; \mathbf{c}, \mathbf{x})) \text{ for } 1 \leq k \leq N+1\}, \\ M_{\mathbf{c}} &= \{u \in H^N(\mathbb{R});\ u = U^{(N)}(t, x; \mathbf{c}, \mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^N\}. \end{aligned}$$

Our goal is to prove the following dynamical stability of the smooth N-solitons to the integrable CH equation.

Theorem 1.1 (dynamical stability of smooth N-solitons) *For every $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^N$ with $m_0(x) + \omega = (1 - \partial_x^2)u_0(x) + \omega > 0$, $\mathbf{x}_0 \in \mathbb{R}^N$ and $\mathbf{c} \in S_N$, such that $\|u_0(x) - U^N(0, x; \mathbf{c}, \mathbf{x}_0)\|_{H^N(\mathbb{R})} < \delta$, then the corresponding solution $u(t, x)$ of the CH equation (1.1) with the initial data $u(0) = u_0$ satisfies $u(t) \in C([0, +\infty), H^N(\mathbb{R}))$ and for all $t > 0$,*

$$\inf_{\psi \in G_{\mathbf{c}}} \|u(t) - \psi(t)\|_{H^N(\mathbb{R})} < \epsilon.$$

Remark 1.1 By the definition, the set $G_{\mathbf{c}}$ is independent of $\mathbf{x}$. It is easy to verify that $M_{\mathbf{c}} \subseteq G_{\mathbf{c}}$. In particular, if $N = 1$, then $M_{\mathbf{c}} = G_{\mathbf{c}}$, Theorem 1.1 recovers the classical orbital stability result in [23]. However, when $N \geq 2$, the question $M_{\mathbf{c}} = G_{\mathbf{c}}$ appears to be open. In view of the statements above, orbital stability of 1-solitons implies dynamical stability directly. Dynamical stability of multi-solitons of other integrable systems are proposed in [33] for the NLS systems, in [46] for the BO equation and [36] for the mKdV equation.

As a direct consequence, we have the following result of the orbital stability of smooth double solitons to the CH equation (1.1).

Theorem 1.2 (orbital stability of smooth double solitons) *The CH smooth double solitons $U^{(2)}_{c_1,c_2}(t, x; x_1, x_2)$ with $2\omega < c_1 < c_2$ are orbitally stable in $H^2(\mathbb{R})$ in the following sense: There exist parameters ϵ_0 and A_0, depending on c_1 and c_2. If there exists $\epsilon \in (0, \epsilon_0)$ such that for any $u_0 \in H^2(\mathbb{R})$ with $m_0 + \omega > 0$,*

$$\|u_0 - U^{(2)}_{c_1,c_2}(0; 0, 0)\|_{H^2(\mathbb{R})} < \epsilon, \tag{1.12}$$

then there exist $x_1(t), x_2(t) \in \mathbb{R}$, *such that the corresponding solution* $u(t,x)$ *of the CH equation (1.1) with the initial data* $u(0) = u_0$ *satisfies* $u(t) \in C([0,+\infty), H^2(\mathbb{R}))$ *and*

$$\sup_{t\in(0,+\infty)} \|u(t) - U^{(2)}_{c_1,c_2}(t; x_1(t), x_2(t))\|_{H^2(\mathbb{R})} < A_0\epsilon, \tag{1.13}$$

with

$$\sup_{t\in(0,+\infty)} (|x_1'(t)| + |x_2'(t)|) \le CA_0\epsilon. \tag{1.14}$$

Remark 1.2 There are some interesting results of the stability of trains of smooth N-solitons or peakons for the CH equations obtained in [24,25]. Such type of stability (which holds also for other non-integrable models, see [42] for subcritical gKdV equations) does not include the dynamical stability of smooth N-solitons in Theorem 1.1 or the orbital stability result in Theorem 1.2. By minimizing the conserved quantities, we get the stability of the whole orbit of smooth N-solitons for all the time.

The approach used in this paper originates from the stability analysis of the multi-solitons of the Korteweg-de Vries (KdV) equation by means of the constrained variational principle [41]. We first demonstrate that the Lyapunov functional of the CH N-solitons I_N is given by

$$I_N(u) = (-1)^N \left(H_{N+1}(u) + \sum_{n=1}^{N} \mu_n H_n(u) \right) \tag{1.15}$$

and μ_n, $n = 1, 2, \ldots, N$ are the Lagrange multipliers which will be expressed in terms of the elementary symmetric functions of $c_1, c_2, \ldots, c_N$. The factor $(-1)^N$ ensures $(-1)^N \mu_1 > 0$. See Sect. 2 for the detail. Then we show that $U^{(N)}$ is realized as a critical point of the functional I_N. Using (1.15), this condition can be written as the following Euler-Lagrange equation

$$\frac{\delta H_{N+1}}{\delta u} + \sum_{n=1}^{N} \mu_n \frac{\delta H_n}{\delta u} = 0, \text{ at } u = U^{(N)}. \tag{1.16}$$

The Lyapunov stability of $U^{(N)}$ may characterize $U^{(N)}$ as a minimal point of the functional H_{N+1} subjected to N constraints

$$H_n(u) = H_n(U^{(N)}), \quad n = 1, 2, \ldots, N, \tag{1.17}$$

and consequently the second variation of I_N is strictly positive at $U^{(N)}$ when one modules several directions.

The proof of Theorem 1.1 reduces to the spectrum analysis of the second variation of I_N (called $\mathcal{L}_N := I_N''(U^{(N)})$) around the smooth N-solitons $U^{(N)}$ and the computation of the eigenvalues of a Hessian matrix $D := \left\{ \frac{\partial^2 I_N(U^{(N)})}{\partial \mu_i \partial \mu_j} \right\}$, we need to show that the number of the negative eigenvalues of $\mathcal{L}_N$ equals to the number of the positive eigenvalues of D (see Propostion 3.4). The main ingredient in the proof is the spectrum analysis of linearized operator $\mathcal{L}_N$ around the N-solitons $U^{(N)}$. It is noted that in the KdV case or similar semi-linear models, the associated linearized operator is a $2N$-th order self-adjoint linear ordinary differential operator and the spectral information of which is obtained by the generalized Sturm-Liouville theory in [41]. For the special case $N = 2$, Neves and Lopes [46] have given an alternate method motivated by the Sylvester Law of Inertia to the spectral analysis part, their method also works to the nonlocal self-adjoint operators and leads to a similar result for the Benjamin-Ono equation in the double solitons. To consider the spectra of $\mathcal{L}_N$ for arbitrary N to the CH equation or similar quasilinear model equations, $\mathcal{L}_N$ is nonlocal and difficult to write the explicit form (even for $N = 2$). Moreover, the conjugate operator identities in the work of [46] (for $N = 2$) and [36] (for arbitrary N) seem difficult to derive for the CH equation (1.1). To overcome this difficulty, we build up some new operator identities (see Lemma 3.1). The spectral analysis of $\mathcal{L}_N$ is then reduced to that of the operators $\mathcal{J}\mathcal{L}_N$ and the CH recursion operators.

The main steps of our spectral analysis could be outlined in the following. Firstly, by making use of the CH recursion operator $\mathcal{R}$ (see (2.45)), we establish an iterative operator identity (see the definition in (2.73)) between the higher order linearized Hamiltonians $-H''_{n+1}(\varphi_c) + cH''_n(\varphi_c)$ and $-H''_n(\varphi_c) + cH''_{n-1}(\varphi_c)$. Secondly, motivated by the identity derived in (3.4) which reduces to the spectrum problem of the adjoint recursion operator $\mathcal{R}^*(\varphi_c)$ (see the definition in (2.51)), we realize that the spectral problem of the operator $\mathcal{J}\mathcal{L}_N$ is much easier to deal with than that of $\mathcal{L}_N$. Thirdly, we show that the eigenfunctions of $\mathcal{R}^*(\varphi_c)$ ($\mathcal{J}\mathcal{L}_N$) plus a generalized kernel of $\mathcal{J}\mathcal{L}_N$ form an orthogonal basis in $L^2(\mathbb{R})$, which can be viewed a completeness relation similar to (2.38) and (2.39). Lastly, we calculate the quadratic form $\langle \mathcal{L}_N z, z\rangle$ with function z to have a decomposition in the above basis. Hence the inertia of L_n can be derived directly. In all, the above four steps make it possible for us to show that for all $N \in \mathbb{N}$, the operators $\mathcal{L}_N$ possess $[\frac{N+1}{2}]$ simple negative, N-fold zero eigenvalue and the rest of the spectra is positive.

The proof of Theorem 1.2 relies heavily on the spectral analysis of the linearized operator $\mathcal{L}_2$ around the double solitons $U^{(2)}$, it follows that $\mathcal{L}_2$ possesses one simple negative eigenvalue and one zero eigenvalue which is double. Therefore, the CH smooth double-solitons have 3 directions of instability, two of which are associated to translation invariance and the third one to the scaling parameters c_1 and c_2. We then modulate in time in order to remove the spatial instabilities. This is a necessary condition in order to gain an orbital stability property. To handle the scaling instability, we do not modulate it but instead replace the associated negative mode by a more tractable direction $U^{(2)} - U^{(2)}_{xx}$, then control the dynamics by employing the conservation law of mass H_1.

The reminder of the paper is organized as follows. In Sect. 2, we summarize the basic properties of the the Hamiltonian formulation of the CH equation and present some results with the help of inverse scattering method, which provide the necessary machinery in carrying out the stability analysis. In Sect. 3, we give a detailed spectral analysis of the Hessian of I_N and hence establish the dynamical stability of the N-soliton solutions of the CH equation. The proof of Theorem 1.2 will be given in Sect. 4. For the sake of completeness, in Sect. 1 which is an appendix, an invariant of the multi-solitons and abstract framework are introduced to handle the spectral analysis part of the linearized operator $\mathcal{L}_N$ around the CH N-solitons.

2 Preliminaries

In this section we collect some preliminaries for the CH equation. Let us first introduce some notations, given $s \in \mathbb{R}$, by $H^s := H^s(\mathbb{R})$ we denote the usual Sobolev space. In particular $H^0(\mathbb{R}) \simeq L^2(\mathbb{R})$. The scalar product in H^s will be denoted by $(\cdot, \cdot)_{H^s}$. This Section is divided into five parts. At the first part, we present the equivalent eigenvalue problem of the CH equation and the basic facts of which through the inverse scattering transform, the conservation laws and multi-solitons of the CH equation are derived. Second, the bi-Hamiltonian formation of the CH equation is considered, the recursion operators are introduced to the computation of the conservation laws at the multi-solitons; In Sect. 2.3, we find the Euler-Lagrange equation of the CH multi-solitons by employing the squared eigenfunctions which are functionally independent and thus the variational characterization of the N-soliton profile. Section 2.4 devotes to the iteration formula of the linearized operators $(-H''_{n+1}(\varphi)+cH''_n(\varphi)$ for all $n\in\mathbb{N}$, it follows that the recursion operators play an important role. In the last subsection, some well-posedness results of the CH equations are presented which will be of use in the proof of the main results.

2.1 Eigenvalue problem, conservation laws and multi-solitons

The CH equation (1.1) can be expressed as a compatibility condition of the following two linear problems [5]

$$\Psi_{xx}=\left(\frac{1}{4}+\lambda(m+\omega)\right)\Psi \tag{2.1}$$

$$\Psi_t=\left(\frac{1}{2\lambda}-u\right)\Psi_x+\frac{u_x}{2}\Psi+\eta\Psi \tag{2.2}$$

with a spectral parameter λ and a constant η for a proper normalization of the eigenfunctions. (2.1) is the spectral problem associated to (1.1). Let $k^2=-\frac{1}{4}-\lambda\omega$, i.e.

$$\lambda(k)=-\frac{1}{\omega}\left(k^2+\frac{1}{4}\right). \tag{2.3}$$

The spectrum of the problem (2.1) is described in [9,11]. The continuous spectrum in terms of k corresponds to $k\in\mathbb{R}$. The discrete spectrum consists of finitely many points $k_n=i\kappa_n$, $n=1,\dots,N$ where κ_n is real and $0<\kappa_n<1/2$.

A basis in the space of solutions of (2.1) can be introduced by the analogs of the Jost solutions of the CH equation, $f^+(x,k)$ and $\bar{f}^+(x,\bar{k})$. For all real $k\neq 0$ it is fixed by its asymptotic behavior when $x\to\infty$ [11]:

$$\lim_{x\to\infty}e^{-ikx}f^+(x,k)=1. \tag{2.4}$$

Another basis can be introduced, $f^-(x,k)$ and $\bar{f}^-(x,\bar{k})$ fixed by its asymptotic when $x\to-\infty$ for all real $k\neq 0$:

$$\lim_{x\to-\infty}e^{ikx}f^-(x,k)=1. \tag{2.5}$$

Since $m(x)$ and ω are real one gets that if $f^+(x,k)$ and $f^-(x,k)$ are solutions of (1.1) then

$$\bar{f}^+(x,\bar{k})=f^+(x,-k),\qquad \text{and}\qquad \bar{f}^-(x,\bar{k})=f^-(x,-k), \tag{2.6}$$

are also solutions of (1.1). The relations (2.6) are known as involutions. In particular, for real $k\neq 0$ we get:

$$\bar{f}^{\pm}(x,k)=f^{\pm}(x,-k), \tag{2.7}$$

and the vectors of the two bases are related:

$$f^-(x,k)=a(k)f^+(x,-k)+b(k)f^+(x,k),\qquad \text{Im}\quad k=0. \tag{2.8}$$

From (2.8) with $x \to \pm\infty$ one has

$$\lim_{x\to-\infty} \left(f^{+}(x,k) - a(k)e^{ikx} + b(-k)e^{-ikx}\right) = 0, \tag{2.9}$$

$$\lim_{x\to+\infty} \left(f^{-}(x,k) - a(k)e^{-ikx} - b(k)e^{ikx}\right) = 0. \tag{2.10}$$

The Wronskian $W(f_1, f_2) \equiv f_1\partial_x f_2 - f_2\partial_x f_1$ of any pair of solutions of (2.1) does not depend on x. Therefore

$$W(f^{-}(x,k), f^{-}(x,-k)) = W(f^{+}(x,-k), f^{+}(x,k)) = 2ik \tag{2.11}$$

Computing the Wronskians $W(f^{-}, f^{+})$ and $W(\bar{f}^{+}, f^{-})$ and using (2.8), (2.11) we obtain:

$$a(k) = (2ik)^{-1}W(f^{-}(x,k), f^{+}(x,k)), \tag{2.12}$$

$$b(k) = -(2ik)^{-1}W(f^{-}(x,k), f^{+}(x,-k)). \tag{2.13}$$

From (2.8) and (2.11) it follows that for real k

$$a(k)a(-k) - b(k)b(-k) = 1. \tag{2.14}$$

It is well known [11] that $f^{+}(x,k)e^{-ikx}$ and $f^{-}(x,k)e^{ikx}$ have analytic extensions in the upper half of the complex k-plane. Likewise $\bar{f}^{+}(x,\bar{k})e^{i\bar{k}x}$ and $\bar{f}^{-}(x,\bar{k})e^{-i\bar{k}x}$ allow analytic extension in the lower half of the complex k-plane. An important consequence of these properties is that $a(k)$ also allows analytic extension in the upper half of the complex k-plane and

$$\bar{a}(\bar{k}) = a(-k), \qquad \bar{b}(\bar{k}) = b(-k), \tag{2.15}$$

As a result (2.14) reduces to the form:

$$|a(k)|^2 - |b(k)|^2 = 1. \tag{2.16}$$

At the points κ_n of the discrete spectrum, $a(k)$ has simple zeroes [11],

$$a(k) = (k - i\kappa_n)\dot{a}_n + \frac{1}{2}(k - i\kappa_n)^2\ddot{a}_n + \cdots, \tag{2.17}$$

and the Wronskian $W(f^{-}, f^{+})$, (2.12) vanishes. Thus f^{-} and f^{+} are linearly dependent:

$$f^{-}(x, i\kappa_n) = b_n f^{+}(x, i\kappa_n). \tag{2.18}$$

In other words, the discrete spectrum is simple, there is only one (real) linearly independent eigenfunction, corresponding to each eigenvalue $i\kappa_n$,

$$f_n^{-}(x) := f^{-}(x, i\kappa_n). \tag{2.19}$$

From (2.19) and (2.4), (2.5) it follows that $f_n^{-}(x)$ falls off exponentially for $x \to \pm\infty$, which allows one to show that $f_n(x)$ is square integrable. Moreover, for compactly supported potentials $m(x)$ (cf. (2.18) and (2.8))

$$b_n = b(i\kappa_n), \qquad b(-i\kappa_n) = -\frac{1}{b_n}. \tag{2.20}$$

The above results can be extended to Schwarz-class potentials by an appropriate limiting procedure. The asymptotic of f_n^{-}, according to (2.7), (2.4), (2.18) is

$$f_n^{-}(x) = e^{\kappa_n x} + o(e^{\kappa_n x}), \qquad x \to -\infty; \tag{2.21}$$

$$f_n^{-}(x) = b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \qquad x \to \infty. \tag{2.22}$$

The sign of b_n obviously depends on the number of the zeroes of f_n^-. Suppose that $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_N < 1/2$. Then from the oscillation theorem for the Sturm-Liouville problem, f_n^- has exactly $n-1$ zeroes. Therefore

$$b_n = (-1)^{n-1}|b_n|. \tag{2.23}$$

The sets

$$\mathcal{S}^{\pm} := \left\{ \frac{b(\pm k)}{a(k)} \quad (k>0), \quad \kappa_n, \quad C_n^{\pm} \equiv \frac{b_n^{\pm 1}}{i\dot{a}_n}, \quad n = 1, \ldots N \right\} \tag{2.24}$$

are called scattering data. Here the dot stands for a derivative with respect to k and $\dot{a}_n \equiv \dot{a}(i\kappa_n)$, $\ddot{a}_n \equiv \ddot{a}(i\kappa_n)$, etc. The time evolution of the scattering data are obtained in [15] as follows.

$$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)e^{\frac{ik}{\lambda}t}; \tag{2.25}$$

$$\frac{1}{a(k,t)} = \frac{1}{a(k,0)}, \qquad \frac{b(\pm k,t)}{a(k,t)} = \frac{b(\pm k,0)}{a(k,0)} e^{\pm\frac{ik}{\lambda}t}; \tag{2.26}$$

$$C_n^{\pm}(t) = C_n^{\pm}(0) \exp\left(\pm \frac{4\omega\kappa_n}{1-4\kappa_n^2} t \right). \tag{2.27}$$

In other words, $a(k)$ is independent on t and will serve as a generating function of the conservation laws. In particular, the integral

$$\alpha = \int_{-\infty}^{\infty} \left(\sqrt{\frac{m(x)+\omega}{\omega}} - 1 \right) \mathrm{d}x, \tag{2.28}$$

as well as all the coefficients $\mathcal{I}_k$ in the asymptotic expansion

$$\ln a(k) = -i\alpha k + \sum_{s=1}^{\infty} \frac{\mathcal{I}_s}{k^{2s+1}} \tag{2.29}$$

must be integrals of motion. The integral α is the unique Casimir function for the CH equation. The densities p_s of $\mathcal{I}_s = \int_{-\infty}^{\infty} p_s \mathrm{d}x$ can be expressed in terms of $m(x)$ using a set of recurrent relations obtained in [30].

Using the analyticity properties of $a(k)$ one can prove that it satisfies the following dispersion relation ($k \in \mathcal{C}_+$) [15],

$$\ln a(k) = -i\alpha k + \sum_{n=1}^{N} \ln \frac{k - i\kappa_n}{k + i\kappa_n} - \frac{k}{\pi i} \int_0^{\infty} \frac{\ln(1 - |\frac{b(\pm k')}{a(k')}|^2)}{k'^2 - k^2} \mathrm{d}k', \tag{2.30}$$

where $i\kappa_n$ are the zeroes of $a(k)$. The dispersion relation (2.30) allows one to express the integrals of motion also in terms of the scattering data [16]:

$$\mathcal{I}_s = \frac{1}{\pi i} \int_0^{\infty} \ln(1 - |\frac{b(\pm k)}{a(k)}|^2)\, k^{2s} \mathrm{d}k - \sum_{n=1}^{N} \frac{2i(-1)^s \kappa_n^{2s+1}}{2s+1}, \tag{2.31}$$

which are known as the *trace identities*. In addition the integral α is expressed through the scattering data as follows. Note that for $k = i/2$, $\lambda(i/2) = 0$ from (2.3). In this case therefore the spectral problem (2.1) does not depend on m, and the eigenfunctions are equal to their asymptotics: $f^{\pm}(x, i/2) = e^{\mp\frac{x}{2}}$. It is inferred from (2.12) that $a(i/2) = 1$ and it is deduced from (2.30) for $k = i/2$ that

$$\alpha = \sum_{n=1}^{N} \ln\left(\frac{1+2\kappa_n}{1-2\kappa_n}\right)^2 + \frac{4}{\pi}\int_0^{\infty} \frac{\ln(1-|\frac{b(\widetilde{k})}{a(\widetilde{k})}|^2)}{4\widetilde{k}^2+1}\mathrm{d}\widetilde{k}. \tag{2.32}$$

For what follows, let us define the following *squared eigenfunctions*

$$F^{\pm}(x,k) := (f^{\pm}(x,k))^2, \qquad F_n^{\pm}(x) := F^{\pm}(x, i\kappa_n). \tag{2.33}$$

Another type of Wronskian relations is proposed in [16] which relates the variations of the potential $m(x)$ with the variation of the scattering data:

$$(f(x,k)\delta f_x - f_x \delta f(x,k))|_{x=-\infty}^{\infty} = \int_{-\infty}^{\infty} \lambda \delta m(x) f^2(x,k)\mathrm{d}x, \tag{2.34}$$

where $\delta f(x,k)$ is the variation of the Jost solution $f(x,k)$ corresponding to the variation $\delta m(x)$ of the potential. Using (2.34) one can also derive the following relations for the variations of the scattering data, for details see [16]:

$$\frac{\delta a(k)}{\delta m(x)} = -\frac{\lambda}{2ik} f^+(x,k) f^-(x,k) \tag{2.35}$$

$$\frac{\delta b(k)}{\delta m(x)} = \frac{\lambda}{2ik} f^+(x,-k) f^-(x,k) \tag{2.36}$$

$$\frac{\delta \ln \lambda_n}{\delta m(x)} = \frac{iF_n^-(x)}{\omega b_n \dot{a}_n} \tag{2.37}$$

Moreover, from the Proposition 5 of [15], there holds the following useful completeness relation for the squared eigenfunctions.

$$\begin{aligned}\frac{\omega}{(\sqrt{m(x)+\omega})(\sqrt{m(y)+\omega})}\theta(x-y) &= -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F^-(x,k)F^+(y,k)}{ka^2(k)}\mathrm{d}k \\ &+ \sum_{n=1}^{N} \frac{1}{i\kappa_n \dot{a}_n^2}\left[F_n^-(x)\dot{F}_n^+(y) + F_n^-(x)\dot{F}_n^+(y) - \left(\frac{1}{i\kappa_n} + \frac{\ddot{a}_n}{\dot{a}_n}\right)F_n^-(x)F_n^+(y)\right],\end{aligned} \tag{2.38}$$

where $\theta(x)$ is the step function. Now for any function $f(x)$ which vanishes for $x \to \pm\infty$, one can expand which over the squared eigenfunctions $F^+(x,k)$ and $F^-(x,k)$. Indeed, we multiply (2.38) with $\frac{1}{2}m_y f(y) + (m(y)+\omega)f_y$ and integrate over $\mathrm{d}y$ to have

$$\pm\omega f(x) = -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F^{\mp}(x,k)\xi_f^{\pm}(k)}{ka^2(k)}\mathrm{d}k + \sum_{n=1}^{N}\frac{1}{i\kappa_n \dot{a}_n^2}\left[\dot{F}_n^{\mp}(x)\xi_{f,n}^{\pm} + F_n^{\mp}(x)\dot{\xi}_{f,n}^{\pm}\right], \tag{2.39}$$

$$\xi_f^{\pm}(k) = \int_{\mathbb{R}} F^{\pm}(y,k)\left[\frac{1}{2}m_y f(y) + (m(y)+\omega)f_y\right]\mathrm{d}y; \tag{2.40}$$

$$\xi_{f,n}^{\pm} = \int_{\mathbb{R}} F_n^{\pm}(y)\left[\frac{1}{2}m_y f(y) + (m(y)+\omega)f_y\right]\mathrm{d}y; \tag{2.41}$$

$$\dot{\xi}_{f,n}^{\pm} = \int_{\mathbb{R}} \dot{F}_n^{\pm}(y)\left[\frac{1}{2}m_y f(y) + (m(y)+\omega)f_y\right]\mathrm{d}y - \left(\frac{1}{i\kappa_n} + \frac{\ddot{a}_n}{\dot{a}_n}\right)\xi_{f,n}^{\pm}. \tag{2.42}$$

The inverse scattering is simplified into the important case of the so-called reflectionless potentials, when the scattering data is confined to the case the reflection coefficient $\frac{b(k)}{a(k)} = 0$ for all real k. This class of potentials corresponds to the N-solitons of the CH equation. In this case $b(k) = 0$ and $|a(k)| = 1$ (see (2.16)), the N-solitons can be calculated in a parametric form [14] which depends on the discreet spectrum $k_n = i\kappa_n$ of (2.1) for $n = 1, 2, \ldots, N$ and $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_N < 1/2$, which are the zeros $a(k)$ in the imaginary axis. In particular, $b(k) = 0$, $|a(k)| = 1$ and $i\dot{a}_n$ is real, by (2.32), one has

$$i\dot{a}_n = \frac{1}{2\kappa_n} e^{\alpha\kappa_n} \prod_{j\neq n} \frac{\kappa_n - \kappa_j}{\kappa_n + \kappa_j}, \quad \text{where} \quad \alpha = \sum_{n=1}^{N} \ln\left(\frac{1+2\kappa_n}{1-2\kappa_n}\right)^2.$$

The CH N-solitons can be expressed in a parametric form as follows [14]

$$u(t,x) = \frac{\omega}{2}\int_0^\infty e^{-|x-g(t,\xi)|}\xi^{-2} g_\xi^{-1}(t,\xi)\mathrm{d}\xi - \omega, \tag{2.43}$$

where $g(t,\xi)$ can be expressed through the scattering data as

$$g(t,\xi) := \ln \int_0^\xi \left(1 - \sum_{n,p} \frac{C_n^+(t) y^{-2\kappa_n}}{\kappa_n + 1/2} A_{np}(t,y)\right)^{-2} \mathrm{d}y,$$

with

$$A_{np}(t,y) := \delta_{pn} + \frac{C_n^+(t) y^{-2\kappa_n}}{\kappa_n + \kappa_p}.$$

The CH N-solitons are showed also in [43] with a parametric form by elementary theory of determinants and the author shows that the CH N-solitons possess the asymptotic behavior (1.11) at the infinity time with the formula for the phase shift.

2.2 Hamiltonian formation

From the bi-Hamiltonian structure (1.2), Lenard relation (1.7) and the relation $(1 - \partial^2)\frac{\delta H_n[m]}{\delta m} = \frac{\delta H_n(u)}{\delta u}$, one has the following relations for H_n,

$$\frac{\delta H_{n+1}[m]}{\delta m} = (1-\partial_x^2)^{-1}(2\omega + m + \partial_x^{-1}(m\partial))\frac{\delta H_n[m]}{\delta m} := \mathcal{K}[m]\frac{\delta H_n[m]}{\delta m}, \tag{2.44}$$

$$\frac{\delta H_{n+1}(u)}{\delta u} = (2\omega + m + \partial_x^{-1}(m\partial))(1-\partial_x^2)^{-1}\frac{\delta H_n(u)}{\delta u} := \mathcal{R}(u)\frac{\delta H_n(u)}{\delta u}, \tag{2.45}$$

where ∂^{-1} is the inverse derivative operator defined by $\partial^{-1} f = \int_{-\infty}^x f \mathrm{d}x$ with $f \in \mathcal{S}(\mathbb{R})$. If we take $\mathcal{S}^*(\mathbb{R}) := \{\partial^{-1} f |\ f \in \mathcal{S}(\mathbb{R})\}$, with a bilinear form given by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\mathrm{d}x,$$

then $\partial_x^{-1} : \mathcal{S}^*(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is skew-symmetric and $\partial_x\partial_x^{-1} = id$. The occurrence of ∂_x^{-1} in the expression of various operators is defined only modulo functions of the constants $C \in \mathbb{R}$. For simplicity, whenever ∂_x^{-1} appears, we will choose the integration constant C to be zero (see for example an exposition in [35]). We emphasize that this convention is only a technical issue and does not cause any controversy and ambiguity in our main results.

It is not difficult (from the bi-Hamiltonian structure (1.2)) to see

$$\mathcal{K}[m] = \mathcal{J}_2^{-1}\mathcal{J}_1. \tag{2.46}$$

Moreover, the operator $\mathcal{K}(m)$ and $\mathcal{R}(u)$ are linear but nonlocal, satisfy

$$\mathcal{R}(u) = (1-\partial_x^2)\mathcal{K}[m](1-\partial_x^2)^{-1}, \tag{2.47}$$

namely, they are similar to each other. Moreover, one can check that there holds

$$\mathcal{K}[m]\left(1-\sqrt{\frac{\omega}{m+w}}\right) = u, \tag{2.48}$$

$$\mathcal{R}(u)(1-\partial_x^2)\left(1-\sqrt{\frac{\omega}{m+w}}\right) = m. \tag{2.49}$$

(2.48) and (2.49) can be viewed as in a special case $n = 0$ in (2.44) and (2.45) respectively, the associated conservation law

$$H_0[m] := \int_{\mathbb{R}}\left(\sqrt{m(x)+\omega}-\sqrt{\omega}\right)^2 \mathrm{d}x, \quad \frac{\delta H_0[m]}{\delta m} = 1-\sqrt{\frac{\omega}{m+w}}.$$

Therefore, the CH hierarchy with the choice of the dispersion law $\Omega(z)$ can be expressed as follows:

$$m_t + \sqrt{m+\omega}\left(\sqrt{m+\omega}\,\Omega(2\mathcal{K}[m])\left(1-\sqrt{\frac{\omega}{m+w}}\right)\right)_x = 0. \tag{2.50}$$

In particular, if the dispersion law $\Omega(z) = z$, then (2.50) becomes the CH equation.

The adjoint operator of $\mathcal{K}[m]$ which denoted by $\mathcal{K}^*[m] := \mathcal{J}_1\mathcal{J}_2^{-1}$, then the adjoint of $\mathcal{R}(u)$ is

$$\mathcal{R}^*(u) = (1-\partial_x^2)^{-1}\mathcal{K}^*[m](1-\partial_x^2) = (1-\partial_x^2)^{-1}\mathcal{J}_1\mathcal{J}_2^{-1}(1-\partial_x^2), \tag{2.51}$$

and it is not difficult to see that the operators $\mathcal{R}(u)$ and $\mathcal{R}^*(u)$ satisfy

$$\mathcal{R}^*(u)\mathcal{J} = \mathcal{J}\mathcal{R}(u). \tag{2.52}$$

It will be shown in Sect. 3 that understanding the spectral information of the recursion operators $\mathcal{R}(u)$ and $\mathcal{R}^*(u)$ is essential in the (spectral) stability of multi-solitons of the CH equation (1.1).

It is shown in [15] that the conservation laws H_j can be expressed as follows:

$$H_j(u) = -\int_0^\infty (-2)^{1-j}\frac{k\rho(k)}{\lambda^j}dk + \frac{(-2)^{2-j}}{\omega}\sum_{n=1}^N\int\frac{\kappa_n^2}{\lambda_n^{j+2}}d\kappa_n, \tag{2.53}$$

where $\rho(k) := \frac{2k}{\pi\omega\lambda^2}\ln|a(k)|$. The first term on the right-hand side of (2.53) is the contribution from radiations and the second term comes from solitons.

Let us consider the quantities $H_j(\varphi_c)$ which are related to 1-soliton profile φ_c. Since (1.9) and (2.45) imply that 1-soliton φ_c with speed c satisfies the following variational principle

$$\frac{\delta}{\delta u}\left(-H_{j+1}(u)+cH_j(u)\right) = 0, \quad j \in \mathbb{Z}_+ \tag{2.54}$$

that is,

$$\left(-H'_{j+1}(\varphi_c)+cH'_j(\varphi_c)\right) = 0. \tag{2.55}$$

Now multiply (2.55) with $\frac{\mathrm{d}\varphi_c}{\mathrm{d}c}$, for each j one has

$$\frac{\mathrm{d}H_{j+1}(\varphi_c)}{\mathrm{d}c} = c\frac{\mathrm{d}H_j(\varphi_c)}{\mathrm{d}c} = \cdots = c^j\frac{\mathrm{d}H_1(\varphi_c)}{\mathrm{d}c},$$

and therefore

$$H_{j+1}(\varphi_c) = \int_0^c y^j \frac{\mathrm{d}H_1(\varphi_y)}{\mathrm{d}y}\mathrm{d}y, \tag{2.56}$$

The calculation of $H_1(\varphi_c)$ is not easy since φ_c is not so explicit. However, taking account of the fact that the reflection coefficient $\rho(k)$ becomes zero for $u = U^{(N)}$, with the associated discreet eigenvalue $i\kappa_n$ for $n = 1, 2, \ldots, N$ and $0 < \kappa_n < 1/2$. In particular, for one soliton φ_c with discreet eigenvalue $i\kappa$, we can derive from (2.53) the formula

$$H_j(\varphi_c) = \frac{(-2)^{2-j}}{\omega}\int \frac{\kappa^2}{\lambda^{j+2}}d\kappa, \quad \lambda = -\frac{1}{\omega}\left(\frac{1}{4} - \kappa^2\right) < 0. \tag{2.57}$$

More precisely, one has the following for the conservation laws H_1 and H_2

$$H_1(\varphi_c) = \omega^2\left(\ln\frac{1-2\kappa}{1-2\kappa} + \frac{4\kappa(1+4\kappa^2)}{(1-4\kappa^2)^2}\right); \tag{2.58}$$

$$H_2(\varphi_c) = \omega^3\left(\ln\frac{1-2\kappa}{1-2\kappa} + \frac{4\kappa(3+32\kappa^2-48\kappa^4)}{3(1-4\kappa^2)^3}\right). \tag{2.59}$$

From (2.55) (multiply it with $\frac{\mathrm{d}\varphi_c}{\mathrm{d}\kappa}$) and (2.57), one has

$$\frac{(-2)^{1-j}\kappa^2}{\omega\lambda^{j+3}} = \frac{\mathrm{d}H_{j+1}(\varphi_c)}{\mathrm{d}\kappa} = c\frac{\mathrm{d}H_j(\varphi_c)}{\mathrm{d}\kappa} = c\frac{(-2)^{2-j}\kappa^2}{\omega\lambda^{j+2}}.$$

On the other hand, one can represent the wave velocity c with respect to κ as follows,

$$c = -\frac{1}{2\lambda} = \frac{2\omega}{1-4\kappa^2} > 2\omega. \tag{2.60}$$

Therefore, the quantities $H_j(\varphi_c)$ can be computed explicitly with respect to the wave velocity c. In particular, by (2.57) and (2.60) , the derivative of $H_1(\varphi_c)$ with respect to the wave speed c can be computed in the following form,

$$\frac{\mathrm{d}H_1(\varphi_c)}{\mathrm{d}c} = \frac{\mathrm{d}H_1(\varphi_c)}{\mathrm{d}\kappa}\frac{\mathrm{d}\kappa}{\mathrm{d}c} = -\frac{\kappa(1-4\kappa^2)^2}{8\omega^2\lambda^3} = 4\kappa c > 0. \tag{2.61}$$

2.3 Variational characterization of the *N*-solitons

We first show that the CH N-solitons $U^{(N)}$ satisfies (1.16) if one prescribes the Lagrange multipliers μ_n appropriately. This provides a variational characterization of $U^{(N)}$. The idea is by employing the variational derivatives of the scattering date with respect to the potential (see (2.37)) and trace formula (2.53). The main result in this subsection is as follows:

Proposition 2.1 *The profiles of the CH N-solitons $U^{(N)}$ satisfy* (1.16) *if μ_n are symmetric functions of wave velocities $c_1, c_2, \ldots, c_N$ which satisfy the following:*

$$\prod_{n=1}^{N}(x - c_n) = x^N + \sum_{n=1}^{N}\mu_n x^{N-n}, \quad x \in \mathbb{R}.$$

Proof Since from the trace formula (2.53), one has the following variation derivative with respect to the potential at the CH N-soliton profile $U^{(N)}$:

$$\left(\frac{\delta H_j(u)}{\delta u}\right)|_{u=U^{(N)}} = \frac{(-2)^{2-j}}{\omega}\sum_{n=1}^{N}\frac{\kappa_n^2}{\lambda_n^{j+2}}\left(\frac{\delta\kappa_n}{\delta u}\right)|_{u=U^{(N)}}. \tag{2.62}$$

By (2.37), one has

$$\left(\frac{\delta\kappa_n}{\delta u}\right)|_{u=U^{(N)}} = (1-\partial_x^2)\left(\frac{\delta\kappa_n}{\delta m}\right)|_{u=U^{(N)}} = \frac{i\lambda_n(1-\partial^2)F_n^-(x)}{2\kappa_n b_n\dot{a}_n}.$$

Therefore, (1.16) and (2.62) give a linear relation among $\left(\frac{\delta\kappa_n}{\delta u}\right)|_{u=U^{(N)}}$,

$$\left((-2)^{1-N}\sum_{j=1}^{N}\frac{\kappa_j^2}{\lambda_j^{N+3}} + \sum_{n=1}^{N}(-2)^{2-n}\mu_n\sum_{j=1}^{N}\frac{\kappa_j^2}{\lambda_j^{n+2}}\right)\left(\frac{\delta\kappa_n}{\delta u}\right)|_{u=U^{(N)}} = 0. \tag{2.63}$$

In view of the fact that $\left(\frac{\delta\kappa_n}{\delta u}\right)|_{u=U^{(N)}}$ are functionally independent squared eigenfunctions $F_n^-(x)$, μ_n must satisfy the following system of linear algebraic equations:

$$\sum_{n=1}^{N}(-2)^{-n}\mu_n\frac{1}{\lambda_j^{n+2}} + (-2)^{-N-1}\frac{1}{\lambda_j^{N+3}} = 0.$$

Recall from (2.60) that the wave velocities $c_j = -\frac{1}{2\lambda_j}$, then we obtain from above

$$\sum_{n=1}^{N}\mu_n c_j^{n+2} + c_j^{N+3} = 0, \quad j = 1, 2, \ldots, N. \tag{2.64}$$

It thus follows that $\mu_n = (-1)^{N-n+1}\sigma_{N-n+1}$, where $\sigma_s(1 \le s \le N)$ are elementary symmetric functions of $c_1, c_2, \ldots, c_N$

$$\sigma_1 = \sum_{j=1}^{N}c_j,\ \sigma_2 = \sum_{j<k}c_jc_k, \ldots,\ \sigma_N = \prod_{j=1}^{N}c_j.$$

This completes the proof of Proposition 2.1. □

We consider now the following CH N-solitons $U^{(N)}(t, x; \mathbf{c}, \mathbf{x})$ variational principle

$$I_N'(U^{(N)}) := (-1)^N\left(H_{N+1}'(U^{(N)}) + \sum_{n=1}^{N}\mu_n H_n'(U^{(N)})\right) = 0, \tag{2.65}$$

where $\mu_j = (-1)^{N-j+1}\sigma_{N-j+1}$ is the associated Lagrange multipliers proposed in Proposition 2.1. The equation (2.65) is the gradient of the following functional

$$I_N(u) := (-1)^N\left(H_{N+1}(u) + \sum_{n=1}^{N}\mu_n H_n(u)\right),$$

evaluated at $u = U^{(N)}$. In general, $U^{(N)}$ is not a minimum of I_N, rather, it is at best a constrained and non-isolated minimum of the following variational problem

$$\min H_{N+1}(u) \qquad \text{subject to} \quad H_j(u) = H_j(U^{(N)}), \quad j = 1, 2, \ldots, N. \tag{2.66}$$

Now define the self-adjoint second variation operator

$$\mathcal{L}_N := I_N''(U^{(N)}), \tag{2.67}$$

and $n(\mathcal{L}_N)$ the number of negative eigenvalues of $\mathcal{L}_N$. We define also the $N \times N$ Hessian matrix as follows

$$D := \left\{ \frac{\partial^2 I_N(U^{(N)})}{\partial \mu_i \partial \mu_j} \right\},$$

and $p(D)$ the number of positive eigenvalues of D. Then by employing the approach in [41] which deals with the stability of the KdV N-solitons case, the proof of Theorem 1.1 reduces to calculate the exact value of $n(\mathcal{L}_N)$ and $p(D)$. Concerning $p(D)$, one has the following.

Lemma 2.1

$$p(D) = \left[\frac{N+1}{2} \right], \tag{2.68}$$

where $[x]$ *is the largest integer part of* x.

Proof The proof is inspired in the KdV N-solitons case in [41]. The matrix D a real symmetric matrix whose elements are calculated explicitly for the N-solitons. Indeed, by taking $\rho = 0$ in (2.53), the j-th conservation law corresponding to $u = U^{(N)}$ reduces to

$$H_j(U^{(N)}) = \frac{(-2)^{2-j}}{\omega} \sum_{n=1}^{N} \int \frac{\kappa_n^2}{\lambda_n^{j+2}} d\kappa_n. \tag{2.69}$$

The Hessian matrix D of the solution surface is given by

$$D_{ij} = \frac{\partial^2 I_N}{\partial \mu_i \partial \mu_j} = \sum_k \frac{\partial}{\partial c_k} \left(\frac{\partial I_N}{\partial \mu_i} \right) \frac{\partial c_k}{\partial \mu_j} = (-1)^N \sum_k \frac{\partial H_{N+1-i}}{\partial c_k} \frac{\partial c_k}{\partial \mu_j} = (-1)^N (AB^{-1})_{ij}.$$

where the matrices A and B are $N \times N$ with elements

$$A_{ik} = \frac{\partial H_{N+1-i}}{\partial c_k} = \frac{\partial H_{N+1-i}}{\partial \kappa_k} \frac{\partial \kappa_k}{\partial c_k} = 4\kappa_k c_k^{N+1-i},$$
$$B_{jk} = \frac{\partial \mu_j}{\partial c_k} = (-1)^{N-j+1} \frac{\partial \sigma_{N-j+1}}{\partial c_k}.$$

We find that $B^T M B = B^T A$, by Sylvester's law of inertia, to find the number of positive eigenvalues of D it suffices to consider the number of positive eigenvalues of the matrix $(-1)^N B^T A$, we next evaluate this matrix product explicitly and observe that the answer is a diagonal matrix with entries of alternating sign, which facts follow immediately from the binomial expansion. The diagonal (j, j) entry is

$$(-1)^{N+1} 4\kappa_j c_j \prod_{k \neq j} (c_k - c_j)$$

with all off-diagonal entries zero. With the assumed ordering of the speeds c_j these diagonal entries are of alternating sign, with $\left[\frac{N+1}{2}\right]$ positive entries. □

2.4 Recursion operators around the smooth solitons

Let us recall that the soliton $\varphi_c(x-ct)$ is a solution of the CH equation. For simplicity, we denote φ_c by φ. Then by (2.45), we have

$$H'_{n+1}(\varphi)=\mathcal{R}(\varphi)H'_n(\varphi), \tag{2.70}$$

where $\mathcal{R}(\varphi)$ is the associated operator

$$\mathcal{R}(\varphi)=(2\omega+m_\varphi+\partial^{-1}(m_\varphi\partial_x))(1-\partial_x^2)^{-1}, \tag{2.71}$$

with $m_\varphi:=\varphi-\varphi_{xx}$.

To analyze the second variation of the actions, we linearize the equation (2.45) to let $u=\varphi+\varepsilon z$, and obtain a relation between linearized Hamiltonian $H''_{n+1}(\varphi)$ and $H''_n(\varphi)$ for all $n\geq 1$. One has

Proposition 2.2 *Suppose that φ is a soliton of the CH equation (1.1) with speed $c>2\omega$, if $z\in H^n$, then there hold*

$$H''_{n+1}(\varphi)z=\mathcal{R}(\varphi)H''_n(\varphi)z+c^{n-1}\left(z\varphi-z_{xx}\varphi+\partial_x^{-1}(z\varphi-z_{xx}\varphi)\right), \tag{2.72}$$

and the following iteration operator identity

$$\left(-H''_{n+1}(\varphi)+cH''_n(\varphi)\right)z=\mathcal{R}(\varphi)\left(-H''_n(\varphi)+cH''_{n-1}(\varphi)\right)z. \tag{2.73}$$

Proof Let $u=\varphi+\varepsilon z$, by (2.45) and the definition of Gateaux derivative, one has

$$H''_{n+1}(\varphi)z=\mathcal{R}(\varphi)H''_n(\varphi)z+\left(\mathcal{R}'(\varphi)z\right)(H'_n(\varphi)), \tag{2.74}$$

where

$$\mathcal{R}'(\varphi)z=\lim_{\varepsilon\to 0}\frac{\mathcal{R}(\varphi+\varepsilon z)-\mathcal{R}(\varphi)}{\varepsilon}=\left(z-z_{xx}+\partial_x^{-1}(z-z_{xx})\right)(1-\partial_x^2)^{-1}.$$

Notice that by the variational principle of 1-soliton φ, one has
$H'_n(\varphi)=c^{n-1}H'_1(\varphi)=c^{n-1}m_\varphi$, then

$$\left(\mathcal{R}'(\varphi)z\right)(m_\varphi)=\left(z\varphi-z_{xx}\varphi+\partial^{-1}(z\varphi-z_{xx}\varphi)\right). \tag{2.75}$$

Combining (2.74) and (2.75), (2.72) is verified. (2.73) follows directly from (2.72). □

Remark 2.1 One can also linearize (2.44) around m_φ to obtain the second variation of the action with respect to m_φ, we have the associated iteration operator identity as follows

$$\left(-H''_{n+1}[m_\varphi]+cH''_n[m_\varphi]\right)z=\mathcal{K}[m_\varphi]\left(-H''_n[m_\varphi]+cH''_{n-1}[m_\varphi]\right)z. \tag{2.76}$$

Clearly, the operators $\mathcal{K}[m_\varphi]=(1-\partial_x^2)^{-1}\mathcal{R}(\varphi)(1-\partial_x^2)$ is similar to $\mathcal{R}(\varphi)$.

2.5 Well-posedness results

In this subsection, we recall some well-posedness results of the CH equation(1.1) which will be of use in the proof of the main results. The Cauchy problem associated to the CH equation (1.1) has been extensively investigated. Without trying to be exhaustive we quote only a few of results and references therein for more literature about well-posedness to the Camassa–Holm equation.

For initial profiles $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, it is known [38] that the CH equation (1.1) has a unique local solution in $C([0,T);H^s(\mathbb{R})) \cap C^1\big([0,T);H^{s-1}(\mathbb{R})\big)$ for some $T > 0$ with H_1 and H_2 conserved. Moreover, if the initial momentum potential $m_0 + \omega > 0$ with $m_0 = u_0 - \partial_x^2 u_0$, then u is global in time [13]. However, if m_0 changes sign ($\omega = 0$), singularities may appear in the solution in finite time in the form of wave breaking (the wave profile remains bounded but its slope becomes unbounded) [10,12,38].

More recently, the unique local weak solution in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ was established in the following result.

Proposition 2.3 [40] *Let $u_0 \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. Then there exists $T > 0$ and a unique solution to the CH equation (1.1) such that*

$$u \in C\big([0,T);H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})\big) \cap C^1\big((0,T);L^2(\mathbb{R})\big).$$

The following existence and uniqueness result is derived in [21] (see also [48] for global weak solution in H^1).

Proposition 2.4 [21] *Let $u_0 \in H^1(\mathbb{R})$ with $m_0 = (1-\partial_x^2)u_0 \in \mathcal{M}(\mathbb{R})$, where $\mathcal{M}(\mathbb{R})$ denotes the set of Radon measures with bounded total variation. Then there exists $T = T(\|m_0\|_{\mathcal{M}})$ and a unique solution to the CH equation (1.1) such that*

$$\begin{aligned} &u \in C\big([-T,T];H^1(\mathbb{R})\big) \cap L^\infty\big((-T,T);W^{1,1}(\mathbb{R})\big);\\ &u_x \in L^\infty\big((-T,T);BV(\mathbb{R})\big), \end{aligned}$$

with initial data u_0. The functionals H_1 and H_2 are constant along the trajectory. Moreover, if m_0 is positive, then the weak solution u is uniquely global in time.

The existence and uniqueness of a H^1 global solution of the CH equation (1.1) have been established in [3,4] (see also [29]).

For initial profiles that are more regular, $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, one has [12,13,38],

Proposition 2.5 *Suppose that $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then there exist $T = T(\|u_0\|_{H^s})$ and a unique solution to CH equation with $u \in C\big([0,T],H^s(\mathbb{R})\big)$. When $s \geq 3$, u becomes a classical solution. Moreover, the solution u depends continuously on the initial data u_0 in the sense that the mapping of the initial data to the solution is continuous from H^s to the space $C\big([0,T],H^s(\mathbb{R})\big)$. The functionals H_1 and H_2 are constant along the trajectory and if m_0 has a definite sign, then u is global in time.*

3 Spectral analysis

Let $U = U^{(N)}_{\mathbf{c},\mathbf{x}}$ be any N-soliton solution with shift parameter $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and wave speed $\mathbf{c} = (c_1, c_2, \ldots, c_N)$. Our attention in this section is focused on a total spectral analysis of the linearized operator around N-solitons $\mathcal{L}_N$ (defined in (2.67)) by employing the recursion operator $\mathcal{R}(\varphi)$ in Sect. 2.

To obtain the spectral of $\mathcal{L}_N$, we follow the approach in [36,46] to study the iso-inertial family of operators, which was used to prove stability of double solitons of the BO equations and multi-solitons of the mKdV equation. This approach consists in using a new invariant for multi-soliton solution (see Proposition 4.1), and certain new identities motivated by the Sylvester Law of Inertia.

Definition 1 The inertia $in(L)$ of a self-adjoint operator L is the pair (n, z) of nonnegative integers, where n is the dimension of the negative subspace of L (counted with geometric multiplicities) and z is the dimension of the null space of L.

Since the multi-soliton solution fits in the framework of Proposition 4.1 in Appendix (see also Theorem 3 in [46]), we conclude that the inertia $in(\mathcal{L}_N(t))$ of $\mathcal{L}_N(t)$ is independent of t. Therefore, we can choose a convenient t to calculate the inertia and the best way is to find out the inertia $in(\mathcal{L}_N(t))$ as t goes to infinity. More precisely, the N-soliton solution U splits into N one-soliton φ_{c_j} far apart. By Theorem 4.1 we infer that, as t goes to infinity, the spectrum $\sigma(\mathcal{L}_N(t))$ of $\mathcal{L}_N(t)$ converges to the union of the spectra $\sigma(L_{N,j})$ of $L_{N,j} := I_N''(\varphi_{c_j})$. In this section, we show that the inertia of the linearized operator $\mathcal{L}_N$ related to the N-soliton solution U has exactly $[\frac{N+1}{2}]$ negative eigenvalues and the dimension of the null space equals to N, namely, $in(\mathcal{L}_N(t)) = ([\frac{N+1}{2}], N)$.

In view of the form of $L_{N,j}$, it is nothing but the summation of the operators

$$-H_{n+1}''(\varphi_{c_j}) + c_j H_n''(\varphi_{c_j}) \quad 1 \le n \le N.$$

Or, what is the same,

$$L_{N,j} = \sum_{n=1}^{N} (-1)^{n-1} \sigma_{j,N-n} \big(- H_{n+1}''(\varphi_{c_j}) + c_j H_n''(\varphi_{c_j}) \big), \tag{3.1}$$

where $\sigma_{j,k}$ are the elementally symmetric functions of $c_1, c_2, \ldots, c_{j-1}, c_{j+1}, \ldots, c_N$ defined in the following,

$$\sigma_{j,0} = 1, \ \sigma_{j,1} = \sum_{l=1, l \ne j}^{N} c_l, \ \sigma_{j,2} = \sum_{l<k, k, l \ne j} c_l c_k, \ldots, \ \sigma_{j,N} = \prod_{l=1, l \ne j}^{N} c_l.$$

We now first deal with the linearized operator around one soliton φ_{c_j} associated linearized operator

$$L_1 = L_{1,1} = -H_2''(\varphi) + cH_1''(\varphi) = (\varphi - c)\partial_x^2 + \varphi' \partial_x - 3\varphi + \varphi'' + c - 2\omega,$$

here we denote φ_{c_j} by φ in the rest of this manuscript for simplicity. By the Liouville substitution, the linearized operator L_1 with respect to the soliton profile φ, defined on $H^2(\mathbb{R})$, is transformed into a regular self-adjoint Sturm-Liouville operator, which is a relatively compact perturbation of a second order differential operator with constant coefficients, then the spectral information of L_1, namely, $in(L_1) = (1, 1)$, follows directly from the Sturm-Liouville theory. In particular, define $y = \partial_x^{-1}\left(\frac{1}{\sqrt{2c-2\varphi}}\right)$. Then one has the following factorization

$$(2c - 2\varphi)^{\frac{1}{4}} \big(2L_1\big)(2c - 2\varphi)^{-\frac{1}{4}} = \mathcal{L}_0 := -\partial_y^2 + 2c - 4\omega + \frac{3}{2}\varphi_{xx} - 6\varphi - \frac{\varphi_x^2}{8(c - \varphi)}.$$

This indicates that the operator $2L_1$ is similar to $\mathcal{L}_0$ and both share the same inertia.

To obtain the spectrum of the operator $L_{N,j}$ (3.1), let us consider the spectral analysis of the linearized Hamiltonian

$$L_n := -H_{n+1}''(\varphi) + cH_n''(\varphi), \tag{3.2}$$

for all integers $n \ge 1$, one of crucial ingredients to deal with this spectrum problem is the following operator identities related to the recursion operator $\mathcal{R}(\varphi)$ and the adjoint recursion operator $\mathcal{R}^*(\varphi)$ (see (2.51)).

Lemma 3.1 *The recursion operator $\mathcal{R}(\varphi)$, the adjoint recursion operator $\mathcal{R}^*(\varphi)$ and the linearized Hamiltonian L_n for all integers $n \geq 1$ satisfy the following operator identities.*

$$L_n \mathcal{J} \mathcal{R}(\varphi) = \mathcal{R}(\varphi) L_n \mathcal{J}, \tag{3.3}$$

$$\mathcal{J} L_n \mathcal{R}^*(\varphi) = \mathcal{R}^*(\varphi) \mathcal{J} L_n, \tag{3.4}$$

where $\mathcal{J}$ is the skew symmetric operator defined in (1.6).

Proof We need only to prove (3.4), since one takes the adjoint operation on (3.4) to have (3.3). Notice that from Proposition 2.2, one has that the operator $\mathcal{R}(\varphi)L_n = L_{n+1}$ is self-adjoint. This in turn implies that

$$(\mathcal{R}(\varphi)L_n)^* = \mathcal{R}(\varphi)L_n = L_n \mathcal{R}^*(\varphi),$$

On the other hand, in view of (1.6) and (2.52), one has

$$\mathcal{J} L_n \mathcal{R}^*(\varphi) = \mathcal{J} \mathcal{R}(\varphi) L_n = \mathcal{R}^*(\varphi) \mathcal{J} L_n,$$

as the advertised result in the lemma. □

Remark 3.1 (3.3) and (3.4) hold for any solutions of the CH equation. In particular, let $U^{(N)}$ be the CH N-soliton profile and $\mathcal{L}_N$ be the second variation of the action in (1.15) with the associated Lagrange multipliers μ_n given by Proposition 2.1. Then it is easy to see that similar to Lemma 3.1, the following operator identities hold

$$\mathcal{L}_N \mathcal{J} \mathcal{R}(U^{(N)}) = \mathcal{R}(U^{(N)}) \mathcal{L}_N \mathcal{J}; \tag{3.5}$$

$$\mathcal{J} \mathcal{L}_N \mathcal{R}^*(U^{(N)}) = \mathcal{R}^*(U^{(N)}) \mathcal{J} \mathcal{L}_N. \tag{3.6}$$

An immediate consequence of the factorization results (3.3) and (3.4) is that the (adjoint) recursion operators $\mathcal{R}(\varphi)(\mathcal{R}^*(\varphi))$ and $L_n\mathcal{J}$ $(\mathcal{J}L_n)$ are commutable. This in turn implies that the operators $\mathcal{J}L_n$ and $\mathcal{R}^*(\varphi)$ share the same eigenfunctions, and $L_n\mathcal{J}$ shares the same eigenfunctions with the recursion operator $\mathcal{R}(\varphi)$.

Our approach for the spectral analysis of the linearized Hamiltonian L_n is as follows. Firstly, we derive the spectra of the operator $\mathcal{J}L_n$ which is easier than to have the spectra of L_n. The idea is motivated by (3.4) to reduce to the spectra of the adjoint recursion operator $\mathcal{R}^*(\varphi)$. We then show that the eigenfunctions of $\mathcal{R}^*(\varphi)$ $(\mathcal{J}L_n)$ plus a generalized kernel of $\mathcal{J}L_n$ form an orthogonal basis in $L^2(\mathbb{R})$, which can be viewed a completeness relation similar to (2.38) and (2.39). Finally we calculate the quadratic form $\langle L_n z, z\rangle$ with function z has a decomposition in the above basis, and the inertia of L_n can be derived directly.

3.1 The spectra of the recursion operator around the CH one soliton

The spectra of the recursion operator

$$\mathcal{R}(\varphi) = (m_\varphi + 2\omega + \partial_x^{-1}(m_\varphi \partial_x))(1 - \partial_x^2)^{-1},$$

and its adjoint operator $\mathcal{R}^*(\varphi)$ are essential to analyze the linearized Hamiltonian L_n defined in (3.2). Note that the recursion operators are nonlocal which are not easy to study directly. However, by employing the operator identity (2.47) and the properties of the squared eigenfunctions $F^{\pm}(x, k)$, one could have the following result.

Lemma 3.2 *The recursion operator $\mathcal{R}(\varphi)$ has only one eigenvalue c associated with the eigenfunction m_φ, the essential spectrum is the interval $(0, 2\omega]$, and the corresponding* *eigenfunctions do not have spatial decay and not in $L^2(\mathbb{R})$. Moreover, the kernel of $\mathcal{R}(\varphi)$ is empty, and the inverse of $\mathcal{R}(\varphi)$ reads*

$$\mathcal{R}^{-1}(\varphi) = \frac{1}{2}(1-\partial_x^2)\frac{1}{\sqrt{m_\varphi+\omega}}\partial_x^{-1}\frac{1}{\sqrt{m_\varphi+\omega}}\partial_x.$$

Proof Consider the Jost solutions $f^{\pm}(x,k)$ of the spectral problem (2.1) with the potential $m(x) = m_\varphi = \varphi - \varphi_{xx}$ and the asymptotic expressions in (2.4) and (2.5). In this case there is an eigenvalue $k = i\kappa_1$ $(0 < \kappa_1 < \frac{1}{2})$ which generates the soliton. It is then found that the squared eigenfunctions $F^{\pm}(x,k) = (f^{\pm}(x,k))^2$ satisfy

$$\Big(-\partial_x^2 + 1 + 2\lambda\big(m_\varphi(x) + 2\omega + \partial_x^{-1}(m_\varphi(x)\partial_x)\big)\Big)F^{\pm}(x,k) = 0. \tag{3.7}$$

This in turn implies that

$$\begin{aligned}&(1-\partial_x^2)^{-1}\big(m_\varphi(x) + 2\omega + \partial_x^{-1}(m_\varphi(x)\partial_x)\big)F^{\pm}(x,k)\\ &= \mathcal{K}[m_\varphi]F^{\pm}(x,k) = -\frac{1}{2\lambda}F^{\pm}(x,k).\end{aligned} \tag{3.8}$$

By (3.8) and (2.47), it is adduced that

$$\mathcal{R}(\varphi)(1-\partial_x^2)F^{\pm}(x,k) = -\frac{1}{2\lambda}(1-\partial_x^2)F^{\pm}(x,k), \quad \text{for } k \in \mathbb{R}, \tag{3.9}$$

$$\mathcal{R}(\varphi)(1-\partial_x^2)F_1^{\pm}(x) = -\frac{1}{2\lambda_1}(1-\partial_x^2)F_1^{\pm}(x) = c(1-\partial_x^2)F_1^{\pm}(x). \tag{3.10}$$

Moreover, in view of the element $\dot{F}_1^{\pm}(x)$ in the completeness relation (2.38), it follows that there holds

$$\mathcal{R}(\varphi)(1-\partial_x^2)\dot{F}_1^{\pm}(x) = -\frac{1}{2\lambda_1}(1-\partial_x^2)\dot{F}_1^{\pm}(x) - \frac{i\kappa_1}{\omega\lambda_1^2}F_1^{\pm}(x). \tag{3.11}$$

Since $\mathcal{R}(\varphi)m_\varphi = cm_\varphi$, from (3.10), it is inferred from (3.10) that $F_1^{\pm}(x) \sim \varphi(x)$. On account of (3.9), the essential spectra of $\mathcal{R}(\varphi)$ given by the set $-\frac{1}{2\lambda} = \frac{\omega}{2k^2+\frac{1}{2}}$ for $k \in \mathbb{R}$, which is equal to the interval $(0, 2\omega]$. The associated generalized eigenfunctions $(1-\partial_x^2)F^{\pm}(x,k)$ possess no spatial decay and not in $L^2(\mathbb{R})$ which can be seen from the asymptotic formulas of $f^{\pm}(x,k)$ in (2.4) and (2.5).

On the other hand, a simple direct computation shows that the kernel of $\mathcal{R}(\varphi)$ is empty except for $\omega = 0$. In particular, for the function $v(x) := (1-\partial_x^2)(1/\sqrt{m_\varphi+\omega})$, one deduces that $\mathcal{R}(\varphi)v = \omega > 0$. The inverse of $\mathcal{R}(\varphi)$ can also be verified directly. The proof of the lemma is complete. □

Remark 3.2 In view of (2.47) or (3.8), one can conclude that the operator $\mathcal{K}[m_\varphi]$ shares the same spectra with the operator $\mathcal{R}(\varphi)$, for instance, we have $\mathcal{K}(m_\varphi)\varphi = c\varphi$, the kernel of which is empty and the inverse is

$$\mathcal{K}^{-1}[m_\varphi] = \frac{1}{2\sqrt{m_\varphi+\omega}}\partial_x^{-1}\frac{1}{\sqrt{m_\varphi+\omega}}(1-\partial_x^2)^{-1}\partial_x.$$

Similar to the proof of Lemma 3.2, we have the following result concerning the spectra of the composite operators $\mathcal{R}^n(\varphi)$ for $n \geq 1$.

Corollary 3.1 *The composite operator $\mathcal{R}^n(\varphi)$ has only one eigenvalue c^n associated with the eigenfunction m_φ, the essential spectrum is the interval $(0, 2^n\omega^n]$, and the corresponding generalized eigenfunctions do not have spatial decay and not in $L^2(\mathbb{R})$.*

We now consider the adjoint recursion operator $\mathcal{R}^*(\varphi)$. In view of the factorization (3.4), it shares the same eigenfunctions of $\mathcal{J}L_n$ and thus is more relevant to the spectral stability problems of solitons. Recall from (2.51) that

$$\mathcal{R}^*(u) = (1-\partial_x^2)^{-1}\mathcal{J}_1\mathcal{J}_2^{-1}(1-\partial_x^2).$$

The observation to (3.8) reveals that

$$\mathcal{J}_1\mathcal{J}_2^{-1}\mathcal{J}_1 F^{\pm}(x,k) = \mathcal{J}_1\mathcal{K}[m]F^{\pm}(x,k) = -\frac{1}{2\lambda}\mathcal{J}_1 F^{\pm}(x,k).$$

From above one can verify that the eigenfunctions of $\mathcal{R}^*(u)$ are

$$(1-\partial_x^2)^{-1}\mathcal{J}_1 F^{\pm}(x,k) = -\partial_x\mathcal{K}[m]F^{\pm}(x,k) = \frac{1}{2\lambda}(F^{\pm}(x,k))_x. \tag{3.12}$$

Lemma 3.3 *The adjoint recursion operator $\mathcal{R}^*(\varphi)$ has only one eigenvalue c associated with the eigenfunction φ_x, the essential spectrum is the interval $(0, 2\omega]$, and the corresponding eigenfunctions do not have spatial decay and not in $L^2(\mathbb{R})$. Moreover, the kernel of $\mathcal{R}^*(\varphi)$ is empty, and the inverse of $\mathcal{R}^*(\varphi)$ is*

$$\big(\mathcal{R}^*(\varphi)\big)^{-1} = \frac{1}{2}\partial_x\frac{1}{\sqrt{m_\varphi+\omega}}\partial_x^{-1}\frac{1}{\sqrt{m_\varphi+\omega}}(1-\partial_x^2)^{-1}.$$

Proof Consider the Jost solutions $f^{\pm}(x,k)$ of the spectral problem (2.1) with the potential m_φ and the asymptotic formulas in (2.4) and (2.5). The soliton profile φ is generated by the eigenvalue $k = i\kappa_1$ $(0<\kappa_1<\frac{1}{2})$. By (3.12), the following relations hold

$$\mathcal{R}^*(\varphi)\big(F^{\pm}(x,k)\big)_x = -\frac{1}{2\lambda}\big(F^{\pm}(x,k)\big)_x, \quad \text{for } k\in\mathbb{R}, \tag{3.13}$$

$$\mathcal{R}^*(\varphi)\big(F_1^{\pm}(x)\big)_x = -\frac{1}{2\lambda_1}\big(F_1^{\pm}(x)\big)_x = c\big(F_1^{\pm}(x)\big)_x, \tag{3.14}$$

$$\mathcal{R}^*(\varphi)\big(\dot F_1^{\pm}(x)\big)_x = -\frac{1}{2\lambda_1}\big(\dot F_1^{\pm}(x)\big)_x - \frac{i\kappa_1}{\omega\lambda_1^2}\big(F_1^{\pm}(x)\big)_x. \tag{3.15}$$

Since $\mathcal{R}^*(\varphi)\varphi_x = c\varphi_x$, by (3.14), c is the only one eigenvalue. In view of (3.13), the essential spectra of $\mathcal{R}^*(\varphi)$ is $-\frac{1}{2\lambda} = \frac{\omega}{2k^2+\frac{1}{2}}$ for $k\in\mathbb{R}$, which is the interval $(0, 2\omega]$. The associated generalized eigenfunctions $\big(F^{\pm}(x,k)\big)_x$ possess no spatial decay and not in $L^2(\mathbb{R})$ which can be seen from the asymptotic formulas of $f^{\pm}(x,k)$ in (2.4) and (2.5).

Similarly, a direct computation shows that the kernel of $\mathcal{R}^*(\varphi)$ is empty except for $\omega = 0$. The inverse of $\mathcal{R}^*(\varphi)$ can also be verified directly. This completes the proof of Lemma 3.3. □

3.2 The spectra of $\mathcal{J}L_n$, $L_n\mathcal{J}$ and L_n

In this subsection our attention is focused on the spectral analysis of the operators $\mathcal{J}L_n$, $L_n\mathcal{J}$ and L_n (3.2), $\mathcal{J}$ is defined in (1.6). The main ingredients are (3.4) and the observation that the eigenfunctions of the adjoint recursion operator $\mathcal{R}^*(\varphi)$ (see (3.13), (3.14) and (3.15)) form an orthogonal basis in $L^2(\mathbb{R})$. It follows that the spectra of $\mathcal{J}L_n$ lies on the imaginary axis which implies directly the spectral stability of solitons.

We now consider the operator $\mathcal{J}L_n$. Since $L_n = \mathcal{R}^{n-1}(\varphi)L_1$ and the principle part of which is

$$\big(2\omega(1-\partial_x^2)^{-1}\big)^{n-1}(-c\partial_x^2 + c - 2\omega),$$

it thus transpires that the symbol of the principle part of the operator $\mathcal{J}L_n$ is

$$\varrho_{n,c}(\zeta) := -2^{n-1}\omega^{n-1}\frac{i\zeta(c\zeta^2 + c - 2\omega)}{(1+\zeta^2)^n} \tag{3.16}$$

We have the following statement related to the spectra of the operator $\mathcal{J}L_n$.

Proposition 3.1 *The operators $\mathcal{J}L_n$ for $n \geq 1$ and the adjoint recursion operator $\mathcal{R}^*(\varphi)$ share the same eigenfunctions. Moreover, the essential spectra of $\mathcal{J}L_n$ are contained in $i\mathbb{R}$, the kernel is spanned by the function φ_x and the generalized kernel is spanned by $\frac{\partial\varphi}{\partial c}$.*

Proof The operators $\mathcal{J}L_n$ for $n \geq 1$ and the adjoint recursion operator $\mathcal{R}^*(\varphi)$ share the same eigenfunctions are inferred by the operator identity (3.4). By Lemma 3.3, one can compute the spectra of the operator $\mathcal{J}L_n$ directly by employing the squared eigenfunctions as follows

$$\mathcal{J}L_n\big(F^{\pm}(x,k)\big)_x = \varrho_{n,c}(\pm 2k)\big(F^{\pm}(x,k)\big)_x, \quad \text{for } k \in \mathbb{R}; \tag{3.17}$$

$$\mathcal{J}L_n\big(F_1^{\pm}(x)\big)_x \sim \mathcal{J}L_n\varphi_x = 0, \tag{3.18}$$

$$\mathcal{J}L_n\big(\dot{F}_1^{\pm}(x)\big)_x \sim \mathcal{J}L_n\frac{\partial\varphi}{\partial c} = c^{n-1}\varphi_x. \tag{3.19}$$

In view of (3.16), the essential spectra of $\mathcal{J}L_n$ are $\varrho_{n,c}(\pm 2k)$ for $k \in \mathbb{R}$ which are contained in the whole imaginary axis, which gives the desired result in Proposition 3.1. □

For the adjoint operator of $\mathcal{J}L_n$, namely, the operator $-L_n\mathcal{J}$ which is commutative with the recursion operator $\mathcal{R}(\varphi)$(3.3), we have the following result.

Proposition 3.2 *The operators $L_n\mathcal{J}$ for $n \geq 1$ and the recursion operator $\mathcal{R}(\varphi)$ share the same eigenfunctions. Moreover, the essential spectra of $L_n\mathcal{J}$ are contained in $i\mathbb{R}$, the kernel is spanned by the function m_φ and the generalized kernel is spanned by $\partial_x^{-1}\big(\frac{\partial m_\varphi}{\partial c}\big)$.*

Proof The operators $L_n\mathcal{J}$ for $n \geq 1$ and the adjoint recursion operator $\mathcal{R}(\varphi)$ share the same eigenfunctions are inferred by the operator identity (3.3). By Lemma 3.2, one can compute the spectra of the operator $L_n\mathcal{J}$ directly by employing the squared eigenfunctions as follows

$$L_n\mathcal{J}(1-\partial_x^2)F^{\pm}(x,k) = -L_n\big(F^{\pm}(x,k)\big)_x = \varrho_{n,c}(\pm 2k)(1-\partial_x^2)F^{\pm}(x,k), \tag{3.20}$$

$$L_n\mathcal{J}(1-\partial_x^2)F_1^{\pm}(x) \sim L_n\mathcal{J}m_\varphi = -L_n\varphi_x = 0, \tag{3.21}$$

$$L_n\mathcal{J}(1-\partial_x^2)\dot{F}_1^{\pm}(x) = -L_n\big(\dot{F}_1^{\pm}(x)\big)_x \sim L_n\frac{\partial\varphi}{\partial c} = -c^{n-1}m_\varphi. \tag{3.22}$$

In view of (3.16), the essential spectra of $L_n\mathcal{J}$ is $\varrho_{n,c}(\pm 2k)$ for $k \in \mathbb{R}$ which is contained in the whole imaginary axis. On the other hand, it is inferred from (3.22) that $\big(\dot{F}_1^{\pm}(x)\big)_x \sim \frac{\partial\varphi}{\partial c}$. Hence the generalized kernel is

$$(1-\partial_x^2)\dot{F}_1^{\pm}(x) \sim \partial_x^{-1}\Big(\frac{\partial m_\varphi}{\partial c}\Big),$$

which implies the advertised result in the proposition. □

On account of Propositions 3.1 and 3.2, we now have the two function sets as follows. The first set

$$\left\{ \left(F^{\pm}(x,k)\right)_x, \quad \text{for} \quad k \in \mathbb{R}; \quad \varphi_x; \quad \frac{\partial \varphi}{\partial c} \right\} \tag{3.23}$$

consists of linearly independent eigenfunctions and generalized kernel of the operator $\mathcal{J}L_n$. Moreover, they are essentially orthogonal under the L^2-inner product. The second set

$$\left\{ (1-\partial_x^2)F^{\pm}(x,k), \quad \text{for} \quad k \in \mathbb{R}; \quad m_\varphi; \quad \partial_x^{-1}\left(\frac{\partial m_\varphi}{\partial c}\right) \right\} \tag{3.24}$$

consists of linearly independent eigenfunctions and generalized kernel of the operator $L_n\mathcal{J}$. Notice that from the expression of soliton profile (1.8) or (1.9), as the functions φ, m_φ are even and localized, one sees that $L_n \frac{\partial \varphi(-x)}{\partial c} = -m_\varphi$ also holds, then the function $\frac{\partial \varphi}{\partial c}$ is also localized and even. The nonzero inner product of the elements of the sets (3.23) and (3.24) are the following:

$$\int_{\mathbb{R}} \left(F^{\pm}(x,k)\right)_x (1-\partial_x^2)\overline{F^{\pm}(x,l)}\mathrm{d}x = \mp 2\pi i k |a(k)|^2 \delta(k-l), \quad \text{for } k, l \in \mathbb{R}; \tag{3.25}$$

$$\int_{\mathbb{R}} \varphi_x \partial_x^{-1}\left(\frac{\partial m_\varphi}{\partial c}\right)\mathrm{d}x = -\int_{\mathbb{R}} \varphi \frac{\partial m_\varphi}{\partial c}\mathrm{d}x = -\frac{\mathrm{d}H_1(\varphi)}{\mathrm{d}c} = -4\kappa c, \tag{3.26}$$

$$\int_{\mathbb{R}} \frac{\partial \varphi}{\partial c} m_\varphi \mathrm{d}x = \frac{\mathrm{d}H_1(\varphi)}{\mathrm{d}c} = 4\kappa c. \tag{3.27}$$

The corresponding closure relation is

$$\begin{aligned} &\mp \int_{\mathbb{R}} \frac{1}{2\pi i k |a(k)|^2} \left(F^{\pm}(x,k)\right)_x (1-\partial_y^2)\overline{F^{\pm}(y,k)}\mathrm{d}k \\ &+ \frac{1}{4\kappa c}\left(\varphi_x \partial_y^{-1}\left(\frac{\partial m_\varphi}{\partial c}\right) + \frac{\partial \varphi}{\partial c} m_\varphi(y)\right) = \delta(x-y), \end{aligned} \tag{3.28}$$

which indicates that any function $z(x)$ which vanishes for $x \to \pm\infty$ can be expanded over the above two bases (3.23) and (3.24). By comparing (3.28) to (2.39), one can take the derivative of (2.39) with respect to x and insert $z(x) = f'(x)$ to have the following decomposition:

$$z(x) = \int_{\mathbb{R}} \left(F^{\pm}(x,k)\right)_x P^{\pm}(k)\mathrm{d}k + \beta\varphi_x + \gamma \frac{\partial \varphi}{\partial c}, \tag{3.29}$$

with the coefficients $P^{\pm}$, β and γ which are related to the coefficients in (2.39). Similarly, one can also decompose the function $z(x)$ on the second set (3.24) by taking the operator $1-\partial_x^2$ upon (2.39).

With the decomposition of function $z(x)$, we can compute the quadratic form related to the operator L_n and illustrate the spectral information. The following statement describes the full spectrum of linearized Hamiltonian $L_n = -H''_{n+1}(\varphi) + cH''_n(\varphi)$ (3.2) for $n \geq 1$.

Lemma 3.4 *For $n \geq 1$ and any $z \in H^n_{odd}$, we have $\langle L_n z, z\rangle \geq 0$ and $\langle L_n z, z\rangle = 0$ if and only if z is a multiple of φ_x, and in H^n_{ev} the operator L_n has exactly one negative eigenvalue and zero is not an eigenvalue any more.*

Proof For any $z(x) \in H^n(\mathbb{R})$, in view of the decomposition (3.29), we can evaluate the quadratic form $\langle L_n z, z\rangle$ as follows,

$$\begin{aligned}\langle L_n z, z\rangle &= \left\langle \int_{\mathbb{R}} L_n\big(F^{\pm}(x,k)\big)_x P^{\pm}(k)\mathrm{d}k, \int_{\mathbb{R}} \big(F^{\pm}(x,k)\big)_x P^{\pm}(k)\mathrm{d}k \right\rangle \\ &\quad +2\gamma \left\langle \int_{\mathbb{R}} L_n\big(F^{\pm}(x,k)\big)_x P^{\pm}(k)\mathrm{d}k, \frac{\partial\varphi}{\partial c}\right\rangle + \gamma^2 \langle L_n \frac{\partial \varphi}{\partial c}, \frac{\partial\varphi}{\partial c}\rangle \\ &= I + II + III. \end{aligned} \tag{3.30}$$

First it is noticed from (3.20) and the zero inner product property of the two sets (3.23) and (3.24) that

$$\begin{aligned} II &= -2\gamma \left\langle \int_{\mathbb{R}} (1-\partial_x^2) F^{\pm}(x,k) P^{\pm}(k) \varrho_{n,c}(\pm 2k)\mathrm{d}k, \frac{\partial\varphi}{\partial c}\right\rangle \\ &= -2\gamma \int_{\mathbb{R}} \langle (1-\partial_x^2)F^{\pm}(x,k), \frac{\partial\varphi}{\partial c}\rangle P^{\pm}(k)\varrho_{n,c}(\pm 2k)\mathrm{d}k = 0. \end{aligned} \tag{3.31}$$

For the third term of (3.30), a direct computation by (2.61) shows that,

$$III = \gamma^2 \langle -c^{n-1} m_{\varphi}, \frac{\partial\varphi}{\partial c}\rangle = -\gamma^2 c^{n-1}\frac{\mathrm{d}H_1(\varphi)}{\mathrm{d}c} = -4\gamma^2\kappa c^n < 0, \quad \text{for} \quad \gamma \neq 0. \tag{3.32}$$

To deal with the first term in (3.30), using (3.20) and (3.25) yields that

$$\begin{aligned} I &= -\left\langle \int_{\mathbb{R}} (1-\partial_x^2)F^{\pm}(x,k)P^{\pm}(k)\varrho_{n,c}(\pm 2k)\mathrm{d}k, \int_{\mathbb{R}} \big(F^{\pm}(x,k)\big)_x P^{\pm}(k)\mathrm{d}k\right\rangle \\ &= -\int_{\mathbb{R}^2} \varrho_{n,c}(\pm 2k) P^{\pm}(k)\overline{P^{\pm}(k_1)} \langle (1-\partial_x^2)F^{\pm}(x,k), \big(F^{\pm}(x,k_1)\big)_x\rangle \mathrm{d}k\mathrm{d}k_1 \\ &= \int_{\mathbb{R}} \pm 2\pi i k |a(k)|^2 \varrho_{n,c}(\pm 2k)|P^{\pm}(k)|^2\mathrm{d}k \\ &= \int_{\mathbb{R}} \frac{2^n \pi\omega^{n-1}k^2|a(k)|^2(4ck^2+c-2\omega)}{(1+4k^2)^n}|P^{\pm}(k)|^2\mathrm{d}k \geq 0, \end{aligned} \tag{3.33}$$

where $I=0$ holds if and only if $P^{\pm}(k)=0$. Combining (3.33), (3.31) and (3.32), one has

$$(3.30) = \int_{\mathbb{R}} \frac{2^n \pi\omega^{n-1}k^2|a(k)|^2(4ck^2+c-2\omega)}{(1+4k^2)^n}|P^{\pm}(k)|^2\mathrm{d}k - 4\gamma^2\kappa c^n. \tag{3.34}$$

For $z \in H^n_{odd}$, we have $\gamma = 0$, then (3.34) and (3.33) reveal that $\langle L_n z, z\rangle \geq 0$. Moreover, $\langle L_n z, z\rangle = 0$ infers that $P^{\pm}(k) = 0$, therefore, $z = \beta\varphi_x$ for $\beta \neq 0$. If $z \in H^n_{even}$, we then have $\beta = 0$, In the hyperplane $\gamma = 0$, $\langle L_n z, z\rangle \geq 0$ and $\langle L_n z, z\rangle = 0$ if and only if $P^{\pm}(k) = 0$, then one has $z = 0$. Therefore, $\langle L_n z, z\rangle > 0$ in the hyperplane $\gamma = 0$ and which implies that L_n can have at most one negative eigenvalue. Since $L_n\frac{\partial\varphi}{\partial c} = -c^{n-1}m_{\varphi} < 0$ and $\left\langle L_n\frac{\partial\varphi}{\partial c}, \frac{\partial\varphi}{\partial c}\right\rangle = -c^{n-1}\frac{\mathrm{d}H_1(\varphi)}{\mathrm{d}c} = -4\kappa c^n < 0$. Therefore, L_n has exactly one negative eigenvalue. This completes the proof of Lemma 3.4. □

As a direct consequence, one has the following spectrum information of higher order linearized Hamiltonian $\mathcal{T}_{n,j} := H''_{n+2}(\varphi_{c_j}) - (c_1+c_2)H''_{n+1}(\varphi_{c_j}) + c_1c_2H''_n(\varphi_{c_j})$ with $n \geq 1$, $j = 1, 2$ and $c_1 \leq c_2$, which are related closely to stability problem of the double solitons $U^{(2)}$. Following the same line of the proof of Lemma 3.4, we have

Corollary 3.2 *For $n \geq 1$ and $c_1 = c_2 = c$, we have $\mathcal{T}_{n,1} = \mathcal{T}_{n,2} \geq 0$, and the eigenvalue zero is double with eigenfunctions φ_c' and $\frac{\partial \varphi_c}{\partial c}$. For $n \geq 1$ and $c_1 < c_2$, the operator $\mathcal{T}_{n,1}$ has one negative eigenvalue and $\mathcal{T}_{n,2} \geq 0$ is positive. $\mathcal{T}_{n,j}$ has zero as a simple eigenvalue with associated eigenfunctions φ_{c_j}'.*

Proof By (2.73), one has

$$\mathcal{T}_{n,j} = \mathcal{R}^{n-1}(\varphi_{c_j})\mathcal{T}_{1,j} = \mathcal{R}^{n-1}(\varphi_{c_j})\big(\mathcal{H}_3''(\varphi_{c_j}) - (c_1 + c_2)H_2''(\varphi_{c_j}) + c_1c_2H_1''(\varphi_{c_j})\big).$$

Similar to the proof of Lemma 3.4, the study of the operator $\mathcal{T}_{n,j}$ is reduced to consider the operator $\mathcal{T}_{1,j} = \big(-\mathcal{R}(\varphi_{c_j}) + c_k\big)\big(-H_2''(\varphi_{c_j}) + c_jH_1''(\varphi_{c_j})\big)$ where $k \neq j$. One can verify that

$$\mathcal{T}_{1,j}\frac{\partial \varphi_{c_j}}{\partial c_j} = (c_j - c_k)\big(\varphi_{c_j} - \varphi_{c_j}''\big). \tag{3.35}$$

In particular, if $c_1 = c_2 = c$, the function $\frac{\partial \varphi_c}{\partial c}$ degenerates to belong to the kernel of $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$. Notice that φ_c' belongs always to the kernel of which, therefore, zero eigenvalue is double with eigenfunctions φ_c' and $\frac{\partial \varphi_c}{\partial c}$. The non-negativeness of $\mathcal{T}_{n,1}$ and $\mathcal{T}_{n,2}$ follow from the same argument of Lemma 3.4.

If $c_1 < c_2$, then by (3.35), the operator $\mathcal{T}_{1,1}$ has a negative eigenvalue and $\mathcal{T}_{1,2} \geq 0$, their zero eigenvalue are simple with associated eigenfunction φ_{c_j}'. □

3.3 The spectra of linearized operator around the CH *N*-solitons

In order to prove Theorem 1.1, we need to know the spectral information of the operator $\mathcal{L}_N$ (2.67). More precisely, the inertia of $\mathcal{L}_N$ called $in(\mathcal{L}_N)$ has to be determined. The aim of this subsection is to show the following result.

Lemma 3.5 *The linearized operator around the CH N-solitons $\mathcal{L}_N$ verifies*

$$in(\mathcal{L}_N) = \big(n(\mathcal{L}_N), z(\mathcal{L}_N)\big) = \left(\left[\frac{N+1}{2}\right], N\right). \tag{3.36}$$

To this aim, for $j = 1, 2\ldots, N$ and recall that the operator $L_{N,j} = I_N''(\varphi_{c_j})$ defined in (3.1). By Theorem 4.1 (see Theorem 4 in [46] for the case $N = 2$), the spectrum of $\mathcal{L}_N$ tends to the unions of $L_{N,j}$, that is $\sigma(\mathcal{L}_N) \to \bigcup_{j=1}^N \sigma(L_{N,j})$ as $t \to +\infty$. The result (3.36) follows directly from the following claim.

Proposition 3.3 *(1). $L_{N,2k-1}$ has zero as a simple eigenvalue and exactly one negative eigenvalue for $1 \leq k \leq [\frac{N+1}{2}]$, i.e, $in(L_{N,2k-1}) = (1, 1)$; (2). $L_{N,2k}$ has zero as a simple eigenvalue and no negative eigenvalues for $1 \leq k \leq [\frac{N}{2}]$, i.e, $in(L_{N,2k}) = (0, 1)$.*

Proof The proof follows the same line of the proof of Lemma 3.4. We consider the operator $L_{N,j} = I_N''(\varphi_{c_j})$ for $1 \leq j \leq N$ and compute the quadratic form $\langle L_{N,j}z, z\rangle$ under a special decomposition of z (3.29). Recall from (3.1) that the form of $L_{N,j}$ which is a combination of the operators $-H_{n+1}''(\varphi_{c_j}) + c_jH_n''(\varphi_{c_j})$. Moreover, one has

$$L_{N,j}\frac{\partial \varphi_{c_j}}{\partial c_j} = -\prod_{k\neq j}^{N}(c_k - c_j)(\varphi_{c_j} - \varphi_{c_j}'') := \Gamma_j(\varphi_{c_j} - \varphi_{c_j}''). \tag{3.37}$$

The quadratic form $\langle L_{N,j}z, z\rangle$ can be evaluated similar to (3.30) as follows

$$\begin{aligned}\langle L_{N,j}z, z\rangle &= \left\langle \int_{\mathbb{R}} L_{N,j}\big(F^{\pm}(x,k)\big)_x P^{\pm}(k)\mathrm{d}k, \int_{\mathbb{R}} \big(F^{\pm}(x,k)\big)_x P^{\pm}(k)\mathrm{d}k \right\rangle \\ &+2\gamma\langle \int_{\mathbb{R}} L_{N,j}\big(F^{\pm}(x,k)\big)_x P^{\pm}(k)\mathrm{d}k, \frac{\partial\varphi_{c_j}}{\partial c_j}\rangle + \gamma^2\langle L_{N,j}\frac{\partial\varphi_{c_j}}{\partial c_j}, \frac{\partial\varphi_{c_j}}{\partial c_j}\rangle \\ &= \sum_{n=1}^{N}(-1)^{n-1}\sigma_{j,N-n}\int_{\mathbb{R}} \frac{2^n\pi\omega^{n-1}k^2|a(k)|^2(4k^2+c_j-2\omega)}{(1+4k^2)^n}|P^{\pm}(k)|^2\mathrm{d}k + 4\Gamma_j\gamma^2\kappa c_j.\end{aligned}$$

It reveals that the symbol of the principle part of $L_{N,j}$ evaluated at $2k$ is

$$\begin{aligned}\widehat{I''_N(0)}(2k) &= \sum_{n=1}^{N}(-1)^{n-1}\sigma_{j,N-n}\frac{\big(2\omega\big)^{n-1}(4c_jk^2+c_j-2\omega)}{(1+4k^2)^{n-1}} \\ &= (4c_jk^2+c_j-2\omega)\sum_{n=1}^{N}(-1)^{n-1}\sigma_{j,N-n}\Big(\frac{2\omega}{1+4k^2}\Big)^{n-1}. \end{aligned}\tag{3.38}$$

Recall that those $\sigma_{j,k}$ are the elementally symmetric functions of $c_1, c_2, \dots, c_{j-1}, c_{j+1}, \dots, c_N$ as follows

$$\sigma_{j,0} = 1,\ \sigma_{j,1} = \sum_{l=1,l\neq j}^{N} c_l,\ \sigma_{j,2} = \sum_{l<k,l\neq j} c_lc_k, \dots,\ \sigma_{j,N} = \prod_{l=1,l\neq j}^{N} c_l.$$

Then for $N \geq 1$, $n = 1, 2, 3, \dots, N$ and $c_n \geq c_1 > 2\omega > \frac{2\omega}{1+4k^2}$, we can easily have

$$\sigma_{j,N-n} - \sigma_{j,N-n-1}\frac{2\omega}{1+4k^2} \geq \sigma_{j,N-n} - 2\omega\sigma_{j,N-n-1} > 0.$$

Therefore

$$(3.38) \geq (4c_jk^2+c_j-2\omega)\sum_{n=1}^{N}(-1)^{n-1}\sigma_{j,N-n}\big(2\omega\big)^{n-1} > 0.$$

Then the first term of the quadratic form $\langle L_{N,j}z, z\rangle$ is nonnegative and equals to zero if and only if $P^{\pm}(k) = 0$.

If j is even, then in view of the definition of Γ_j (3.37), one has $\Gamma_j > 0$ and $\langle L_{N,j}z, z\rangle \geq 0$ and $\langle L_{N,j}z, z\rangle = 0$ if and only if $P^{\pm}(k) = 0$ and $\gamma = 0$, which indicates that $z = \beta\varphi'_{c_j}$. Hence $L_{N,j} \geq 0$ and zero is simple with associated eigenfunction φ'_{c_j}.

If j is odd, then one has $\Gamma_j < 0$, we investigate z in H^N_{ev} and H^N_{odd} respectively. If $z \in H^N_{odd}$, then $\gamma = 0$. Then one has $\langle L_{N,j}z, z\rangle \geq 0$ and $\langle L_{N,j}z, z\rangle = 0$ if and only if $P^{\pm}(k) = 0$. Then $z = \beta\varphi'_{c_j}$ with $\beta \neq 0$, which indicates that zero is simple with associated eigenfunction φ'_{c_j}.

If $z \in H^N_{ev}$, then $\beta = 0$. In the hyperplane $\gamma = 0$, $\langle L_{N,j}z, z\rangle \geq 0$ and $\langle L_{N,j}z, z\rangle = 0$ if and only if $P^{\pm}(k)$. Therefore, $\langle L_{N,j}z, z\rangle > 0$ in the hyperplane $\gamma = 0$ and which implies that $L_{N,j}$ can have at most one negative eigenvalue. Since $L_{N,j}\frac{\partial\varphi_{c_j}}{\partial c_j} = \Gamma_j(\varphi_{c_j} - \varphi''_{c_j}) < 0$ and $\langle L_{N,j}\frac{\partial\varphi_{c_j}}{\partial c_j}, \frac{\partial\varphi_{c_j}}{\partial c_j}\rangle = \Gamma_j\frac{\mathrm{d}H_1(\varphi_{c_j})}{\mathrm{d}c_j} < 0$. Therefore, $L_{N,j}$ has exactly one negative eigenvalue. This implies the desired result as advertised in the statement of Proposition 3.3. □

Proof of Lemma 3.5 From the invariance of inertia stated in Corollary 4.2 and the results of Proposition 3.3, we know that

$$in(\mathcal{L}_N) = \big(n(\mathcal{L}_N), z(\mathcal{L}_N)\big) = \sum_{j=1}^{N} in(\mathcal{L}_{N,j}) = \left(\left[\frac{N+1}{2}\right], N\right).$$

The proof is concluded. □

Remark 3.3 The spectral information of $L_{N,j}$ (3.1) indicates that the CH one-soliton φ_{c_j} is nonlinearly stable in the Sobolev space H^N. Indeed, we can choose I_N to be a Lyapunov functional, at the H^N level, which can describe the dynamics of small perturbations. Then we modulate the φ'_{c_j} direction to have the variation of space transition parameters $x_j(t)$, compare to [23], which gives an alternative proof of nonlinearly stability of φ_{c_j} in H^N.

To prove Theorem 1.1, it suffices to verify the following proposition which can be viewed as GSS framework [28] adapted to the multi-solitons case for nonlinear dispersive equations, see Lemma 2.3 in [41].

Proposition 3.4 [41] *Suppose that*

$$n(\mathcal{L}_N) = p(D). \tag{3.39}$$

Then there exists a constant $C > 0$ such that $U^{(N)}$ is a non-degenerate unconstrained minimum of the augmented Lagrangian (Lyapunov function)

$$I_N(u) + \frac{C}{2}\sum_{j=1}^{N}\big(H_j(u) - H_j(U^{(N)})\big)^2, \tag{3.40}$$

with I_N defined in (1.15). The N-dimensional family of all N-soliton profiles $U^{(N)}$ are dynamically stable in the sense of Theorem 1.1.

Proof of Theorem 1.1 By Lemma 2.1 and Proposition 3.3, one has that $n(\mathcal{L}_N) = p(D) = [\frac{N+1}{2}]$. The proof of Theorem 1.1 is obtained directly in view of Proposition 3.4, since $U^{(N)}$ is now an (non-isolated) unconstrained minimizers of the augmented Lagrangian (3.40) which therefore serves as a Lyapunov function. □

4 Orbital stability of the smooth double solitons

Our attention in this section is now turned to the proof of Theorem 1.2. We need to prove a coercivity property on the Hessian of action related to the double-solitons profile U

$$I_2(U) = H_3(U) - (c_1 + c_2)H_2(U) + c_1c_2H_1(U),$$

which is crucial to control the difference between double solitons and a function in a neighborhood of its orbit.

4.1 Coercivity of the linearized operator around the smooth CH double solitons

As showed in Corollary 3.2, the linearized operator $\mathcal{L}_2$ ((2.67) with $N = 2$) around the CH double soliton possesses only one negative eigenvalue and the inertia of $\mathcal{L}_2$ equals to $(1, 2)$. It is natural to verify that the kernel of $\mathcal{L}_2$ is spanned by functions $U_{(j)} := \partial_{x_j} U$, where $j = 1, 2$ and x_j the spatial transitions. Let U_{-1} be an eigenfunction associated to the unique negative eigenvalue of the operator $\mathcal{L}_2$, as stated in Corollary 3.2. We assume that $\|U_{-1}\|_{L^2} = 1$. Then U_{-1} is unique and one has $\mathcal{L}_2 U_{-1} = -\lambda_0^2 U_{-1}$ with $-\lambda_0^2$ the associated negative eigenvalue. It is now easy to see the following result holds.

Lemma 4.1 *Let U be the CH double-solitons with wave velocities $0 < 2\omega < c_1 < c_2$, and $U_{(1)}, U_{(2)}$ be in the corresponding kernel of the operator $\mathcal{L}_2$. There exists $\nu_1 > 0$ depending on c_1 and c_2 only, such that for any $z \in H^2(\mathbb{R})$ satisfying the following orthogonality conditions*

$$(z, U_{-1})_{L^2} = (z, U_{(1)})_{L^2} = (z, U_{(2)})_{L^2} = 0, \tag{4.1}$$

then we have $\langle \mathcal{L}_2 z, z\rangle \geq \nu_1 \|z\|_{H^2}^2$.

It is observed that U_{-1} is hard to handle in our case, so we need a more applicable version of Lemma 4.1. To see this, we consider the natural modes associated to the scaling parameters, which are the best candidates to generate negative directions for the related quadratic form defined from $\mathcal{L}_2$. More precisely, for t fixed, $i, j = 1, 2$ and $i \neq j$

$$\begin{aligned} \mathcal{L}_2 \partial_{c_j} U &= \frac{\partial}{\partial c_j}\big(H_3'(U) - (c_1 + c_2) H_2'(U) + c_1 c_2 H_1'(U)\big) + H_2'(U) - c_i H_1'(U) \\ &= H_2'(U) - c_i H_1'(U). \end{aligned} \tag{4.2}$$

We now define a function $\Psi := \frac{\partial_{c_1} U - \partial_{c_2} U}{c_1 - c_2}$. It is then found that Ψ is Schwartz and satisfies

$$\mathcal{L}_2 \Psi = -H_1'(U) = -(U - U_{xx}) := -m_U, \quad \text{and} \tag{4.3}$$

$$\begin{aligned} \langle \mathcal{L}_2 \Psi, \Psi\rangle &= \frac{-1}{(c_2 - c_1)}\Big(\frac{\partial H_1(U)}{\partial c_2} - \frac{\partial H_1(U)}{\partial c_1}\Big) \\ &= \frac{4}{c_2 - c_1}(\kappa_1 c_1 - \kappa_2 c_2) < 0, \end{aligned} \tag{4.4}$$

in view of (2.61) and $0 < \kappa_1 < \kappa_2$ when $0 < 2\omega < c_1 < c_2$.

We have the following result which gives a coercivity property of $\langle \mathcal{L}_2 z, z\rangle$.

Lemma 4.2 *Let U be the CH double solitons with wave velocities $0 < 2\omega < c_1 < c_2$, and $U_{(1)}$ and $U_{(2)}$ be in the corresponding kernel of the associated linearized operator $\mathcal{L}_2$. There exists $\nu_2 > 0$ depending only on c_1, c_2, such that for any $z \in H^2(R)$ satisfying the following orthogonality conditions*

$$(z, U_{(1)})_{L^2} = (z, U_{(2)})_{L^2} = 0, \tag{4.5}$$

then we have

$$\langle \mathcal{L}_2 z, z\rangle \geq \nu_2 \|z\|_{H^2}^2 - \frac{1}{\mu_1}(z, m_U)_{L^2}^2.$$

Proof It suffices to show that under the conditions (4.5) and the orthogonality condition $(z, m_U)_{L^2} = 0$, there holds

$$\langle \mathcal{L}_2 z, z\rangle \geq \nu_2 \|z\|_{H^2}^2.$$

First notice that by (4.3) and (4.4), we have

$$(\Psi, m_U)_{L^2} = -\langle \mathcal{L}_2 \Psi, \Psi\rangle > 0. \tag{4.6}$$

The next step is to decompose z and Ψ in $span(U_{-1}, U_{(1)}, U_{(2)})$ and the associated orthogonal subspace. We decompose

$$z = \tilde{z} + pU_{-1},\ \Psi = \Psi_0 + nU_{-1} + aU_{(1)} + bU_{(2)}, \quad p, n, a, b \in \mathbb{R},$$

where the following orthogonal conditions hold

$$(\tilde{z}, U_{-1})_{L^2} = (\tilde{z}, U_{(1)})_{L^2} = (\tilde{z}, U_{(2)})_{L^2} = (\Psi_0, U_{-1})_{L^2} = (\Psi_0, U_{(1)})_{L^2} = (\Psi_0, U_{(2)})_{L^2} = 0.$$

And in addition,

$$(U_{-1}, U_{(1)})_{L^2} = (U_{-1}, U_{(2)})_{L^2} = 0.$$

From the above identities, we obtain

$$\langle \mathcal{L}_2 z, z\rangle = \langle \mathcal{L}_2\tilde{z} - p\lambda_0^2 U_{-1}, \tilde{z} + pU_{-1}\rangle = \langle \mathcal{L}_2\tilde{z}, \tilde{z}\rangle - p^2\lambda_0^2. \tag{4.7}$$

In view of (4.3) and the self-adjointness of $\mathcal{L}_2$, we deduce that

$$0 = (z, m_U)_{L^2} = -\langle \mathcal{L}_2\Psi, z\rangle = -\langle \Psi, \mathcal{L}_2 z\rangle = -\langle \Psi_0, \mathcal{L}_2\tilde{z}\rangle + pn\lambda_0^2. \tag{4.8}$$

On the other hand,

$$(\Psi, m_U)_{L^2} = -\langle \mathcal{L}_2\Psi, \Psi\rangle = -\langle \mathcal{L}_2\Psi_0, \Psi_0\rangle + \lambda_0^2 n^2. \tag{4.9}$$

Combining (4.7),(4.8) and (4.9), it follows from the Cauchy-Schwartz inequality that

$$\begin{aligned}\langle \mathcal{L}_2 z, z\rangle &= \langle \mathcal{L}_2\tilde{z}, \tilde{z}\rangle - \frac{\langle \mathcal{L}_2\tilde{z}, \Psi_0\rangle^2}{\langle \mathcal{L}_2\Psi_0, \Psi_0\rangle + (\Psi, m_U)_{L^2}} \\ &\geq \frac{\langle \mathcal{L}_2\tilde{z}, \tilde{z}\rangle(\Psi, m_U)_{L^2}}{\langle \mathcal{L}_2\Psi_0, \Psi_0\rangle + (\Psi, m_U)_{L^2}} \geq \gamma_1\langle \mathcal{L}_2\tilde{z}, \tilde{z}\rangle, \end{aligned}\tag{4.10}$$

where $0 < \gamma_1 < 1$. From (4.7), it is inferred that $\langle \mathcal{L}_2\tilde{z}, \tilde{z}\rangle \geq p^2\lambda_0^2 \geq 0$. This in turn implies that there exists a constant $C > 0$ such that

$$\begin{aligned}\langle \mathcal{L}_2 z, z\rangle &\geq \gamma_1\langle \mathcal{L}_2\tilde{z}, \tilde{z}\rangle \geq \frac{\gamma_1}{2}\langle \mathcal{L}_2\tilde{z}, \tilde{z}\rangle + \gamma_1 p^2\lambda_0^2 \\ &\geq C(2\|\tilde{z}\|_{H^2}^2 + 2p^2\|U_{-1}\|_{H^2}^2) \geq C\|z\|_{H^2}^2, \end{aligned}\tag{4.11}$$

thereby concluding the proof of Lemma 4.2. □

4.2 Proof of Theorem 1.2

In this subsection we give the proof of Theorem 1.2. To this end, we employ a natural Lyapunov functional I_2 for the CH equation, which is well-defined at the natural H^2 level. Indeed, for any $u_0 \in H^2(\mathbb{R})$, we have global in time $H^2(\mathbb{R})$ solution $u(t)$ [13,38]. Recall the Lyapunov functional I_2 defined by

$$I_2(u(t)) = H_3(u(t)) - (c_1 + c_2)H_2(u(t)) + c_1c_2H_1(u(t)) = I_2(u_0). \tag{4.12}$$

It is clear that $I_2(u)$ represents a real-valued conserved quantity, well-defined for H^2-solutions of the CH equation. Moreover, one has the following Taylor-like expansion.

Lemma 4.3 *Let $z \in H^2(\mathbb{R})$ be any function with sufficiently small H^2-norm, and U be the 2-solitons. Then, for all $t \in \mathbb{R}$, one has*

$$I_2(U+z) = I_2(U) + \frac{1}{2}\langle \mathcal{L}_2 z, z\rangle + \mathcal{N}(z), \quad |\mathcal{N}(z)| = O(\|z\|^3_{H^2}), \tag{4.13}$$

where the operator $\mathcal{L}_2 = I_2''(U)$ and $\mathcal{N}(z)$ is the remaining higher order nonlinear term.

Proof The proof is a direct consequence of the fact that U is a critical point of the functional I_2, namely, $I_2'(U) = 0$. □

The Lyapunov functional I_2 allows us to describe the dynamics of small perturbations and a direct control of the corresponding instability modes. The degenerate directions are controlled by H^1 conservation law H_1.

Proof of Theorem 1.2 Let $u_0 \in H^2$ be a function such that u_0 satisfying (1.12). Assume $u(t)$ is the solution of the Cauchy problem associated to the CH equation with initial data u_0. In view of the continuity of the CH flow for H^2 data [13], there exists a time $T_0 > 0$ and continuous parameters $x_1(t), x_2(t) \in \mathbb{R}$, defined for all $t \in [0, T_0]$, and such that the solution $u(t)$ of the Cauchy problem associated with the CH equation with initial data u_0, satisfies

$$\sup_{t\in[0,T_0]} \|u(t) - U(t; x_1(t), x_2(t))\|_{H^2(\mathbb{R})} < 2\epsilon. \tag{4.14}$$

We want to show that $T_0 = +\infty$. To this aim, let $K > 2$ be a constant, to be fixed later. Let us suppose, by contradiction, that the maximal time of stability $T^\star$, that is

$$\begin{aligned} T^\star = \sup\{&T > 0, \text{ for all } t \in [0, T], \text{ there exists } \tilde{x}_1(t), \tilde{x}_2(t) \in \mathbb{R} \text{ such that} \\ &\sup_{t\in[0,T]} \|u(t) - U(t; \tilde{x}_1(t), \tilde{x}_2(t))\|_{H^2(\mathbb{R})} < K\epsilon\}, \end{aligned} \tag{4.15}$$

is finite. By (4.14), we see easily that $T^\star$ is well-defined. Our idea is to find a suitable contradiction to the assumption $T^\star < +\infty$.

By taking ϵ_0 smaller, if necessary, we can apply modulation theory for the solution $u(t)$. We now give the following claim.

Claim: *There exists $\epsilon_0 > 0$, depending only on U, such that for all $\epsilon \in (0, \epsilon_0)$, the following property is verified. There exist $x_1(t), x_2(t) \in \mathbb{R}$* defined on $[0, T^*]$, *such that if we denote*

$$\Upsilon(t) = u(t) - U(t), \;\; U(t) = U(t; x_1(t), x_2(t)), \tag{4.16}$$

then for all $t \in [0, T^*]$, Υ satisfies the orthogonality conditions

$$(\Upsilon, U_{(1)})_{L^2} = (\Upsilon, U_{(2)})_{L^2} = 0. \tag{4.17}$$

Moreover, for all $t \in [0, T^*]$, we have

$$\|\Upsilon(t)\|_{H^2} + |x_1'(t)| + |x_2'(t)| < CK\epsilon, \;\; \|\Upsilon(0)\|_{H^2} \le C\epsilon, \tag{4.18}$$

for some constant $C > 0$, independent of K.

The proof of this claim relies on the Implicit Function Theorem. Indeed, let

$$J_j(u(t), x_1, x_2) = \langle u - U(t; x_1, x_2), U_{(j)}(t; x_1, x_2)\rangle.$$

We clearly have

$$J_j\left(U(t; x_1, x_2), x_1, x_2\right) = 0.$$

On the other hand, for $j, k = 1, 2$, we have

$$\partial_{x_k} J_j(u(t,x), x_1, x_2)|_{(U,0,0)} = -\langle U_{(k)}(t; 0, 0), U_{(j)}(t; 0, 0)\rangle.$$

Let J be the 2×2 matrix with component $J_{j,k} := (\partial_{x_k} J_j)_{j,k=1,2}$. We now compute

$$det J = \|U_{(1)}U_{(2)}\|_{L^2}^2 - \|U_{(1)}\|_{L^2}^2\|U_{(2)}\|_{L^2}^2 \neq 0,$$

since the fact that $U_{(1)}$ and $U_{(2)}$ are not parallel for all time. Therefore, we have the desired invertibility, by the Implicit Function Theorem, we can write the decomposition (4.16) with property (4.17) in a small H^2 neighborhood of U, for $t \in [0, T^\star]$. Now we verify (4.18). The first bounds are consequence of the decomposition itself and the equations satisfied by the derivatives of the parameters x_1 and x_2, after taking time derivative in (4.17) and using the invertibility property of ∇J. More precisely, we first write the equation verified by Υ. Recall that u satisfies the CH equation, then one replaces u by $U(t) + \Upsilon(t)$ in the CH equation to obtain

$$\begin{aligned}(\Upsilon - \Upsilon_{xx})_t + 2\omega\Upsilon_x + 3(U\Upsilon)_x - 2(U_x\Upsilon_x)_x - (U\Upsilon_{xxx} + U_{xxx}\Upsilon)\\ +\big((U_{(1)} - U_{(1),xx})x_1'(t) + (U_{(2)} - U_{(2),xx})x_2'(t)\big) + \mathcal{N}(\Upsilon) = 0,\end{aligned} \tag{4.19}$$

where $\mathcal{N}(\Upsilon)$ is the remaining nonlinear part.

Take now the scalar product of (4.19) with $U_{(j)}$. By the definition of U and the orthogonality conditions (4.17), we have

$$A \cdot (x_1'(t), x_2'(t))^T = B(\Upsilon) + O(\|\Upsilon\|_{H^2}^2),$$
$$A = \begin{pmatrix}\|U_{(1)}\|_{H^1}^2 & 0\\ 0 & \|U_{(2)}\|_{H^1}^2\end{pmatrix} + small,$$

and $|B(\Upsilon)| \leq C\|\Upsilon\|_{H^2}$. As long as the modulation parameter do not vary too much and $\|\Upsilon\|_{H^2}$ remains small, A is invertible and we can deduce that

$$|x_1'(t)| + |x_2'(t)| \leq C\|\Upsilon\|_{H^2} + O(\|\Upsilon\|_{H^2}^2). \tag{4.20}$$

This completes the proof of the claim.

Next, since $\Upsilon(t)$ defined by (4.16) is small, by Lemma 4.3 and the claim above, it is deduced that

$$I_2(u(t)) = I_2(U(t)) + \frac{1}{2}\langle \mathcal{L}_2\Upsilon(t), \Upsilon(t)\rangle + O(\|\Upsilon(t)\|_{H^2}^3).$$

By Lemma 4.3, it follows from (4.12) that

$$\begin{aligned}\langle \mathcal{L}_2\Upsilon(t), \Upsilon(t)\rangle &\leq \langle \mathcal{L}_2\Upsilon(0), \Upsilon(0)\rangle + O(\|\Upsilon(t)\|_{H^2}^3) + O(\|\Upsilon(0)\|_{H^2}^3)\\ &\leq C\|\Upsilon(0)\|_{H^2}^2 + C\|\Upsilon(t)\|_{H^2}^3.\end{aligned}$$

Since $\Upsilon(t)$ satisfies (4.17), it is thus inferred from Lemma 4.2 that

$$\begin{aligned}\|\Upsilon(t)\|_{H^2}^2 &\leq C\|\Upsilon(0)\|_{H^2}^2 + C\|\Upsilon(t)\|_{H^2}^3 + C(m_U(t), \Upsilon(t))_{L^2}^2\\ &\leq C\epsilon^2 + CK^3\epsilon^3 + C(m_U(t), \Upsilon(t))_{L^2}^2.\end{aligned} \tag{4.21}$$

Using the conservation of mass, it is found that

$$\begin{aligned}\|u(t)\|_{H^1}^2 &= \int mu = \|U(t)\|_{H^1}^2 + \|\Upsilon(t)\|_{H^1}^2 + 2(m_U(t), \Upsilon(t))_{L^2}\\ &= \|U(0)\|_{H^1}^2 + \|\Upsilon(0)\|_{H^1}^2 + 2(m_U(0), \Upsilon(0))_{L^2} = \|u(0)\|_{H^1}^2.\end{aligned}$$

For $t \in [0, T^\star]$, it thus transpires that

$$|(m_U(t), \Upsilon(t))_{L^2}| \leq C|(m_U(0), \Upsilon(0))_{L^2}| + C\|\Upsilon(0)\|_{H^2}^2 + C\|\Upsilon(t)\|_{H^2}^2 \leq C(\epsilon + K^2\epsilon^2).$$

Replacing this last identity in (4.21), by choosing K large, one has

$$\|\Upsilon(t)\|_{H^2}^2 \leq C\epsilon^2(1 + K^3\epsilon + K^4\epsilon^2) \leq \frac{1}{2}K^2\epsilon^2,$$

that is, $\|\Upsilon(t)\|_{H^2} \leq \frac{\sqrt{2}}{2}K\epsilon$. However, this estimate contradicts the definition of $T^\star$ in (4.15) and therefore the stability of U in H^2 is established, which completes the proof of Theorem 1.2. □

Remark 4.1 As we consider the linearized operators around one soliton if the wave velocities $c_1 = c_2$ in Corollary 3.2, one may consider the solutions of the following differential equation

$$H_3'(u) - 2cH_2'(u) + c^2H_1'(u) = 0, \tag{4.22}$$

this equation has the usual one soliton φ_c as a solution. We can obtain another solution of this differential equation in the limit $c_1, c_2 \to c$ from the solution of the differential equation

$$H_3'(u) - (c_1 + c_2)H_2'(u) + c_1c_2H_1'(u) = 0.$$

One solution of this equation is the two-soliton $U^{(2)}_{c_1,c_2}(t, x; x_1, x_2)$ with asymptotic speeds c_1 and c_2. However, as we point out in Sect. 2.1 (see (2.17)), the zeros of $a(k)$ are all simple and the solution of (4.22) will be trivially zero if the space transitions $x_1, x_2 \in \mathbb{R}$ (see (2.43)). But this changes if we move the x_1, x_2 into the complex plane, such types of real solutions are singular called *resonant double solitons*, the time evolution of this solution is far from decomposing asymptotically into travelling waves. In the study of the KdV equation, solution like this one is sometimes neglected because it has a pole (of second order).

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Appendix

The tools presented in this section have been developed in [34,36,46]. It is noted that the work of Neves and Lopes [46] was devoted to the case of the double solitons and [36] extends their results to the case of N-solitons with N an arbitrary integer. For the sake of completeness, we give the most relevant elements of the statement only and refer to [34,36,46] for the details of the proof and further discussion.

Iso-inertial family of operators

We will be working with linearized operators around a multi-soliton, which fit in the following more generic framework.

Consider the abstract evolution equation

$$\frac{du}{dt} = f(u), \tag{4.23}$$

for $u : \mathbb{R} \to X$, and recall that the following framework was set in [34,46]. Let $X_2 \subset X_1 \subset X$ be Hilbert spaces and $V : X_1 \to \mathbb{R}$ be such that the following assumptions are verified.

(H1) $X_2 \subset X_1 \subset X$ are continuously embedded. The embedding from X_2 to X_1 is denoted by i.

(H2) The functional $V : X_1 \to \mathbb{R}$ is $\mathcal{C}^3$.

(H3) The function $f : X_2 \to X_1$ is $\mathcal{C}^2$.

(H4) For any $u \in X_2$, we have $V'(i(u))f(u) = 0$. Moreover, given $u \in \mathcal{C}^1(\mathbb{R}, X_1) \cap \mathcal{C}(\mathbb{R}, X_2)$ a strong solution of (4.23), we assume that there exists a self-adjoint operator $L(t) : D(L) \subset X \to X$ with domain $D(L)$ independent of t such that for $h, k \in Z$, where $Z \subset D(L) \cap X_2$ is a dense subspace of X, we have $\langle L(t)h, k\rangle = V''(u(t))(h, k)$. We also consider the operators $B(t) : D(B) \subset X \to X$ such that for any $h \in Z$ we have $B(t)h = -f'(u(t))h$. Then we assume moreover that

(H5) The closed operators $B(t)$ and $B^*(t)$ have a common domain $D(B)$ which is independent of t. The Cauchy problems

$$\frac{du}{dt} = B(t)u, \quad \frac{dv}{dt} = B^*(t)v,$$

are well-posed in X for positive and negative times.

We then have the following result (see [34,46]).

Proposition 4.1 *Let $u \in \mathcal{C}^1(\mathbb{R}, X_1) \cap \mathcal{C}(\mathbb{R}, X_2)$ be a strong solution of* (4.23) *and assume that (H1)-(H5) are satisfied. Then the following assertions hold.*

• *Invariance of the set of critical points. If there exists $t_0 \in \mathbb{R}$ such that $V'(u(t_0)) = 0$, then $V'(u(t)) = 0$ for any $t \in \mathbb{R}$.*

• *Invariance of the inertia. Assume that u is such that $V'(u(t)) = 0$ for all $t \in \mathbb{R}$. Then the inertia $in(L(t))$ of the operator $L(t)$ representing $V''(u(t))$ is independent of t.*

Calculation of the inertial

Given an t-dependent family of operators whose inertia we are interested in, Proposition 4.1 allows to choose for a specific t to perform the calculation of the inertia. This is however in most situations not sufficient, as we would like to let t go to infinity and relate the inertia of our family with the inertia of the asymptotic objects that we obtain. This is what is allowed in the following framework.

Let X be a real Hilbert space. Let $N \in \mathbb{N}$ and (τ_n^j) be sequences of isometries of X for $j = 1, \dots, N$. For brevity in notation, we denote the composition of an isometry τ_n^k and the inverse of τ_n^j by

$$\tau_n^{k/j} := \tau_n^k(\tau_n^j)^{-1}.$$

Let A, $(B^j)_{j=1,\dots,N}$ be linear operators and (R_n) be a sequence of linear operators. Define the sequences of operators based on (B^j) and (τ_n^j) by

$$B_n^j = (\tau_n^j)^{-1}B_j(\tau_n^j).$$

Define the operator $L_n : D(A) \subset X \to X$ by

$$L_n = A + \sum_{j=1}^{N} B_n^j + R_n.$$

We make the following assumptions.

(A1) For all $j = 1, \ldots, N$ and $n \in \mathbb{N}$, the operators A, $A + B^j$, $A + B_n^j$ and L_n are self-adjoint with the same domain $D(A)$.

(A2) The operator A is invertible. For all $j = 1, \ldots, N$ and $n \in \mathbb{N}$, the operator A commutes with τ_n^j (i.e. $A = (\tau_n^j)^{-1} A(\tau_n^j)$).

(A3) There exists $\delta > 0$ such that for all $j = 1, \ldots, N$ and $n \in \mathbb{N}$, the essential spectrum of A, $A + B_j$, $A + B_n^j$ and L_n are contained in $(\delta, +\infty)$.

(A4) For every $\lambda \in \cap_{j=1}^N \rho(A + B^j)$ and for all $j = 1, \ldots, N$ the operators $A(A + B^j - \lambda I)^{-1}$ are bounded.

(A5) In the operator norm, $\|R_n A^{-1}\| \to 0$ as $n \to +\infty$.

(A6) For all $u \in D(A)$ and $j, k = 1, \ldots, N$ and $j \neq k$ one has

$$\lim_{n \to +\infty} \|\tau_n^{j/k} B^k \tau_n^{k/j}\|_X = 0.$$

(A7) For all $u \in X$ and $j, k = 1, \ldots, N$ and $j \neq k$ we have $\tau_n^{j/k} u \rightharpoonup 0$ weakly in X as $n \to \infty$.

(A8) For all $j = 1, \ldots, N$, the operators $B^j A^{-1}$ is compact.

Theorem 4.1 *Assume that assumptions (A1)-(A8) hold and let $\lambda < \delta$. The following assertions hold.*

• *If $\lambda \in \cap_{j=1}^N \rho(A + B^j)$, then there exists $n_\lambda \in \mathbb{N}$ such that for all $n > n_\lambda$ we have $\lambda \in \rho(L_n)$.*

• *If $\lambda \in \cup_{j=1}^N \sigma(A + B^j)$, then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n > n_\varepsilon$ we have*

$$dim(Range(P_{\lambda,\varepsilon}(L_n))) = \sum_{j=1}^N dim(Range(P_{\lambda,\varepsilon}(A + B^j))),$$

where $P_{\lambda,\varepsilon}(L)$ is the spectral projection of L corresponding to the circle of center λ and radius ε.

Corollary 4.2 *Under the assumptions of Theorem 4.1, if there exists n_L such that for all $n > n_L$ we have*

$$dim(ker(L_n)) \geq \sum_{j=1}^N dim(ker(A + B^j)),$$

then for all $n > n_L$ we have

$$in(L_n) = \sum_{j=1}^N in(A + B^j).$$

Moreover, a non-zero eigenvalue of L_n cannot approach 0 as $n \to \infty$.

Theorem 4.1 and Corollary 4.2 were proved in [46] in the case $N = 2$. For the proof of general $N \in \mathbb{N}$ cases, we refer to [36] for details.

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