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Calculus of Variations

Inverse mean curvature evolution of entire graphs

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Abstract

We study the evolution of strictly mean-convex entire graphs over $\mathbb{R}^n$ by *Inverse Mean Curvature flow*. First we establish the *global existence* of starshaped entire graphs with *superlinear growth* at infinity. The main result in this work concerns the critical case of *asymptotically conical* entire convex graphs. In this case we show that there exists a time $T < +\infty$, which depends on the growth at infinity of the initial data, such that the unique solution of the flow exists for all $t < T$. Moreover, as $t \to T$ the solution converges to a flat plane. Our techniques exploit the *ultra-fast diffusion* character of the fully-nonlinear flow, a property that implies that the asymptotic behavior at spatial infinity of our solution plays a crucial influence on the maximal time of existence, as such behavior propagates *infinitely fast* towards the interior.

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Contents

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1 Introduction

We consider a family of immersions $F_t : M^n \to \mathbb{R}^{n+1}$ of n-dimensional mean convex hypersurfaces in $\mathbb{R}^{n+1}$. We say that $M_t := F_t(M^n)$ moves by *inverse mean curvature flow* if

$$\frac{\partial}{\partial t} F(z,t) = H^{-1}(z,t)\,\nu(z,t), \quad z \in M^n \tag{1.1}$$

where $H(z,t) > 0$ and $\nu(z,t)$ denote the mean curvature and exterior unit normal of the surface M_t at the point $F(z,t)$.

The compact case is well understood. It was shown by Gerhardt [13] that for smooth compact star-shaped initial data of strictly positive mean curvature, the inverse mean curvature flow admits a smooth solution for all times which approaches a homothetically expanding spherical solution as $t \to +\infty$, see also Urbas [22]. For non-starshaped initial data it is well known that singularities may develop; in the case n = 2 Smoczyk [21] proved that such singularities can only occur if the speed becomes unbounded, or, equivalently, when the mean curvature tends to zero somewhere during the evolution. In [16,17], Huisken and Ilmanen developed a new level set approach to weak solutions of the flow, allowing "jumps" of the surfaces and solutions of weakly positive mean curvature. Weak solutions of the flow can be used to derive energy estimates in General Relativity, see [17] and the references therein.

In [18], Huisken and Ilmanen studied further regularity properties of inverse mean curvature flow with compact starshaped initial data of nonnegative mean curvature by a more classical approach than their works in [16,17]. They showed that starshapedness combined with the ultra fast-diffusion character of the equation, imply that at time $t > 0$ the mean curvature of the surface becomes strictly positive yielding to C^∞ regularity. No extra regularity assumptions on the initial data are necessary. This work is reminiscent of well known estimates for the fast-diffusion equation

$$\varphi_t = \nabla_i(\varphi^{m-1}\nabla_i\varphi), \quad \text{on} \quad \Omega \times (0,T) \tag{1.2}$$

on a domain $\Omega \subset \mathbb{R}^n$ and with exponents $m < 1$. We will actually see in the next section that under inverse mean curvature flow, the mean curvature H satisfies an ultra-fast diffusion equation modeled on (1.2) with $m = -1$. The work in [18] heavily uses that the initial surface is compact which corresponds to the domain Ω in (1.2) being bounded. However, the case of non-compact initial data has never been studied before.

Motivated by the theory for the Cauchy problem for the ultra-fast diffusion equation (1.2) on $\mathbb{R}^n \times (0,T)$, we will study in this work equation (1.1) in the case that the initial surface M_0 is an entire graph over $\mathbb{R}^n$, i.e. there exists a vector $\omega \in \mathbb{R}^{n+1}$, $|\omega| = 1$ such that

$$\langle \omega, \nu \rangle < 0, \quad \text{on} \quad M_0.$$

We will take from now on ω to be the direction of the x_{n+1} axis, namely $\omega = e_{n+1} \in \mathbb{R}^{n+1}$. A solution M_t of (1.1) can then be expressed (at each instant t) as the graph $\bar{F}(x,t) = (x, \bar{u}(x,t))$ of a function $\bar{u} : \mathbb{R}^n \times [0,T) \to \mathbb{R}$. In this parametrization, the inverse mean curvature flow (1.1) is, up to diffeomorphisms, equivalent to

$$\left(\frac{\partial}{\partial t}\bar{F}(x,t)\right)^{\perp} = H^{-1}(x,t)\,\nu(x,t), \quad x \in \mathbb{R}^n \tag{1.3}$$

where $\perp$ denotes the normal component of the vector. Equation (1.3) can then be expressed in terms of the height function $x_{n+1} = \bar{u}(x,t)$ as the fully nonlinear equation

$$\bar{u}_t = -\sqrt{1+|D\bar{u}|^2}\left(\operatorname{div}\left(\frac{D\bar{u}}{\sqrt{1+|D\bar{u}|^2}}\right)\right)^{-1}. \tag{1.4}$$

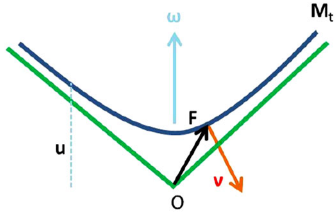
Entire graph solutions of the mean curvature flow have been studied by Ecker and Huisken in [8,9]. It follows from these works, which are based on local a'priori estimates, that the mean curvature flow behaves in some sense better than the heat equation on $\mathbb{R}^n$: for an initial data M_0 which is an entire graph over $\mathbb{R}^n$, no growth conditions are necessary to guarantee the long time existence of the flow for all times $t \in (0, +\infty)$. See also in [11] for evolution of entire convex graphs by powers of mean curvature. In the case of other flows, the evolution of entire convex graphs was studied in the work by Alessandroni and Sinestrari [1] and Holland [15]. Recently, entire graph solutions of fully-nonlinear flows by powers of Gaussian curvature were studied by Choi, Daskalopoulos, Kim and Lee in [4]. This is an example of slow diffusion which becomes degenerate at spatial infinity. Finally, entire convex graph solutions of other fully-nonlinear flows which are homogeneous of degree one were recently studied by Choi and Daskalopoulos in [3].

This work concerns with the long time existence of inverse mean curvature flow for an initial data M_0 which is an entire graph. In a first step we will establish in Theorem 4.1 the existence for all times $t \in (0, +\infty)$ of solutions with strictly meanconvex initial data $M_0 = \{x_{n+1} = \bar{u}_0(x)\}$ having superlinear growth at infinity, namely $\lim_{|x|\to+\infty} |D\bar{u}_0(x)| = +\infty$, and satisfying a uniform "δ-star-shaped" condition $\langle F - \bar{x}_0, \nu\rangle\, H \geq \delta > 0$ for some $\bar{x}_0 \in \mathbb{R}^{n+1}$. These conditions for example hold for initial data $\bar{u}_0(x) = |x|^q$, for $q > 1$. The proof of this result uses in a crucial way the evolution of $\langle F - \bar{x}_0, \nu\rangle\, H$ and the maximum principle which guarantees that this quantity remains bounded from below at all times.

The main result of the paper proves long-time existence and uniform finite time singular convergence for convex entire graphs with *conical* behavior at infinity. We will assume that M_0 lies between two rotationally symmetric cones $x_{n+1} = \zeta_i(\cdot, 0)$, with $\zeta_1(\cdot, 0) := \alpha_0\,|x|$ and $\zeta_2(\cdot, 0) := \alpha_0\,|x| + \kappa$, $x \in \mathbb{R}^n$, namely $\bar{u}$ satisfies

$$\alpha_0\,|x| \leq \bar{u}(\cdot, 0) \leq \alpha_0\,|x| + \kappa, \quad \text{on} \quad \mathbb{R}^n \tag{1.5}$$

for some constants $\alpha_0 > 0$ and $\kappa > 0$.



(1.6)

We will see that M_t will remain convex and will lie between the cones $\zeta_1(x,t) = \alpha(t)\,|x|$ and $\zeta_2(x,t) = \alpha(t)\,|x| + \kappa$ which are explicit solutions of (1.1), namely

$$\alpha(t)\,|x| \leq \bar{u}(\cdot, t) \leq \alpha(t)\,|x| + \kappa, \quad \text{on} \quad \mathbb{R}^n. \tag{1.7}$$

The coefficient $\alpha(t)$ is determined in terms of α_0 by the ordinary differential equation (3.3). We will see in Sect. 3 that $\alpha(t) \equiv 0$ at a finite time $T = T(\alpha_0)$, which means that the cone solutions ζ_i become flat at time $T(\alpha_0)$. Our goal in this work is to establish that the solution M_t of (1.1) with initial data M_0 will also exist up to this *critical time* T, as stated next.

Theorem 1.1 *Let M_0 be an entire convex C^2 graph $x_{n+1} = \bar{u}_0(x)$ over $\mathbb{R}^n$ which lies between the two cones as in condition (1.5). Assume in addition that the mean curvature H of M_0 satisfies the global bound*

$$c_0 \leq H \langle F, \omega \rangle \leq C_0. \tag{1.8}$$

Let $T = T(\alpha_0)$ denote the lifetime of the cone with initial slope α_0. Then, there exists a unique C^∞ smooth solution M_t of the (1.1) for $t \in (0, T)$ which is an entire convex graph $x_{n+1} = \bar{u}(x, t)$ over $\mathbb{R}^n$ and satisfies estimate (1.7) and has $H > 0$ for all $t \in (0, T)$. As $t \to T$, the solution converges in $C^{1,\alpha}$ to some horizontal plane of height $h \in [0, \kappa]$.

Remark 1.1 (Vanishing mean curvature) Condition (1.8) implies that the initial data M_0 has strictly positive mean curvature $H > 0$. Actually for generic initial data which is a graph $x_{n+1} = \bar{u}_0(x)$ satisfying (1.5) one expects that $H(x, \bar{u}_0(x)) \sim |x|^{-1}$ as $|x| \to +\infty$. Hence, under the extra assumption $H > 0$ one has that (1.8) holds. It would be interesting to see if it is possible that the result of Theorem 1.1 is valid under the weaker assumption that (1.8) holds only near spatial infinity, thus allowing the mean curvature H to vanish on a compact set of M_0.

Remark 1.2 The solutions in Theorem 1.1 have linear growth at infinity and they are critical in the sense that all other solutions are expected to live longer. For conical at infinity initial data, one has $\langle F - \bar{x}_0, \nu \rangle\, H \sim |x|^{-1}$ as $x \to +\infty$. We will see that maximum principle arguments do not apply in this case to give us the required bound from below on H which will guarantee existence. One needs to use integral bounds involving H. Since our solutions are non-compact special account needs to be given to the behavior at infinity of our solution. This is one of the challenges in this work.

Remark 1.3 (Graphical parametrization) While we use the graphical parametrization in conditions (1.5) and (1.7) and to establish the short time existence of our solution, for all the a priori estimates, which will occupy the majority of this work, we will use the geometric parametrization in (1.1) where $\partial F(z, t)/\partial t$ is assumed to be in the direction of the normal ν. This is because the evolution of the various geometric quantities becomes more simplified in the case of equation (1.1). In particular, $\bar{u} := \bar{u}(x, t)$, $x \in \mathbb{R}^n$ will denote the height function in the graph parametrization, while $u := \langle F, \omega \rangle$ will denote the height function in the geometric parametrization.

2 The geometric equations and preliminaries

We recall the evolution equations for various geometric quantities under the inverse mean curvature flow. Let $g = \{g_{ij}\}_{1 \leq i,j \leq n}$ and $A = \{h_{ij}\}_{1 \leq i,j \leq n}$ be the first and second fundamental form of the evolving surfaces, let $H = g^{ij}\, h_{ij}$ be the mean curvature, $\langle F - \bar{x}_0, \nu \rangle$ be the support function with respect to a point $\bar{x}_0 \in \mathbb{R}^{n+1}$ and $d\mu$ the induced measure on M_t.

Lemma 2.1 (Huisken, Ilmanen [18]) *Smooth solutions of* (1.1) *with $H > 0$ satisfy*

(1) $\dfrac{\partial}{\partial t} g_{ij} = \dfrac{2}{H}\, h_{ij}$

(2) $\dfrac{\partial}{\partial t} d\mu = d\mu$

(3) $\dfrac{\partial}{\partial t} \nu = -\nabla H^{-1} = \dfrac{1}{H^2}\, \nabla H$

(4) $\frac{\partial}{\partial t} h_{ij} = \frac{1}{H^2} \Delta h_{ij} - \frac{2}{H^3} \nabla_i H \nabla_j H + \frac{|A|^2}{H^2} h_{ij}$

(5) $\frac{\partial}{\partial t} H = \nabla_i \big(\frac{1}{H^2} \nabla_i H \big) - \frac{|A|^2}{H} = \frac{1}{H^2} \Delta H - \frac{2}{H^3} |\nabla H|^2 - \frac{|A|^2}{H}$

(6) $\frac{\partial}{\partial t} H^{-1} = \frac{1}{H^2} \Delta H^{-1} + \frac{|A|^2}{H^2} H^{-1}$

(7) $\frac{\partial}{\partial t} \langle F - \bar{x}_0, \nu \rangle = \frac{1}{H^2} \Delta \langle F - \bar{x}_0, \nu \rangle + \frac{|A|^2}{H^2} \langle F - \bar{x}_0, \nu \rangle.$

We will next assume that M_t is a graph in the direction of the vector ω and a smooth solution of (1.1) on $0 < t \le \tau$ with $H > 0$ and we will derive the evolution of other useful geometric quantities under the IMCF. We will use the following identities that hold in terms of a local orthonormal frame $\{\mathbf{e_i}\}_{1 \le i \le n}$ on M_t:

$$\nabla_{\mathbf{e_i}} \nu = h_{ij} \, \mathbf{e_j}, \quad \nabla_{\mathbf{e_i}} \mathbf{e_j} = -h_{ij} \, \nu, \quad \nabla_{\mathbf{e_i}} \mathbf{e_i} = -H \, \nu. \tag{2.1}$$

Lemma 2.2 *The norm of the position vector $|F|^2$ satisfies*

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2} \Delta \right) |F|^2 = -2n H^{-2} + 4 H^{-1} \langle F, \nu \rangle. \tag{2.2}$$

Proof We have

$$\nabla_i |F|^2 = 2 \langle F, \mathbf{e_i} \rangle, \quad \Delta |F|^2 = 2n - 2H \langle F, \nu \rangle$$

and

$$\frac{\partial}{\partial t} |F|^2 = 2 \, \langle F, F_t \rangle = 2 \, H^{-1} \, \langle F, \nu \rangle$$

which readily gives (2.2). □

Lemma 2.3 *For any $\bar{x}_0 \in \mathbb{R}^{n+1}$ the support function $H \, \langle F - \bar{x}_0, \nu \rangle$ satisfies the equation*

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2} \Delta \right) (H \, \langle F - \bar{x}_0, \nu \rangle) = - \frac{2}{H^3} \nabla H \cdot \nabla \big(H \, \langle F - \bar{x}_0, \nu \rangle \big). \tag{2.3}$$

Proof Readily follows by combining the evolution equations of H and $\langle F - \bar{x}_0, \nu \rangle$. □

Lemma 2.4 *For a graph solution M_t, the quantity $\langle \omega, \nu \rangle$ satisfies*

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2} \Delta \right) \langle \omega, \nu \rangle = \frac{|A|^2}{H^2} \langle \omega, \nu \rangle. \tag{2.4}$$

Proof We have

$$\frac{\partial}{\partial t} \langle \omega, \nu \rangle = \langle \omega, \frac{\partial}{\partial t} \nu \rangle = \frac{1}{H^2} \langle \omega, \nabla H \rangle.$$

On the other hand

$$\frac{1}{H^2} \Delta \langle \omega, \nu \rangle = \frac{1}{H^2} \nabla_i \big(h_{ik} \, \langle \omega, \mathbf{e_k} \rangle \big) = \frac{1}{H^2} \langle \omega, \nabla H \rangle - \frac{|A|^2}{H^2} \langle w, \nu \rangle.$$

Hence, (2.4) holds. □

Lemma 2.5 *For a graph solution M_t, the function $\varphi := -H \langle \omega, \nu \rangle > 0$ satisfies*

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2} \Delta \right) \varphi = - \frac{2}{H^3} \nabla H \cdot \nabla \varphi. \tag{2.5}$$

Proof Readily follows by combining the evolution of H and (2.4). □

Lemma 2.6 *For a graph solution M_t, the height function $u := \langle F, \omega\rangle$ satisfies the evolution equation*

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right) u = \frac{2}{H}\,\langle \omega, \nu\rangle. \tag{2.6}$$

Proof It follows from

$$\frac{\partial}{\partial t}\langle F, \omega\rangle = \frac{1}{H}\,\langle \omega, \nu\rangle \quad \text{and} \quad \Delta\langle F, \omega\rangle = \nabla_i\langle \mathbf{e_i}, \omega\rangle = -H\,\langle \omega, \nu\rangle.$$

□

We next consider the quantity

$$\langle \hat{F}, \nu\rangle := -\langle F, \omega\rangle\,\langle \omega, \nu\rangle$$

which will play a crucial role in this work. We will assume that our origin $0 \in \mathbb{R}^{n+1}$ is chosen so that $\langle F, \omega\rangle > 0$ (in particular this holds if M_t lies above the cone $x_{n+1} = \alpha(t)\,|x|$ as in the picture (1.6)). Since $\langle \omega, \nu\rangle < 0$, we have $\langle \hat{F}, \nu\rangle > 0$ on M_t for all $0 \le t < \tau$.

Lemma 2.7 *The quantity $\langle \hat{F}, \nu\rangle := -\langle F, \omega\rangle\,\langle \omega, \nu\rangle > 0$ satisfies the equation*

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right)\langle \hat{F}, \nu\rangle = \frac{|A|^2}{H^2}\,\langle \hat{F}, \nu\rangle - \frac{2}{H}\,\langle \omega, \nu\rangle^2 + \frac{h_{ij}}{H^2}\,\langle \mathbf{e_i}, \omega\rangle\,\langle \mathbf{e_j}, \omega\rangle. \tag{2.7}$$

Proof Using the evolution equations for $\langle F, \omega\rangle$ and $\langle \omega, \nu\rangle$ shown in Lemmas 2.4 and 2.6 respectively, we conclude that

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right)\langle \hat{F}, \nu\rangle = \frac{|A|^2}{H^2}\,\langle \hat{F}, \nu\rangle - \frac{2}{H}\,\langle \omega, \nu\rangle^2 + \frac{1}{H^2}\nabla_i\langle F, \omega\rangle\nabla_i\langle \omega, \nu\rangle.$$

Since

$$\nabla_i\langle F, \omega\rangle\nabla_i\langle \omega, \nu\rangle = h_{ij}\,\langle \mathbf{e_i}, \omega\rangle\,\langle \mathbf{e_j}, \omega\rangle$$

the above readily yields (2.7). □

Lemma 2.8 *The product $v := \langle \hat{F}, \nu\rangle\,H$ satisfies the evolution equation*

$$\frac{\partial}{\partial t}v - \nabla_i\left(\frac{1}{H^2}\nabla_i v\right) = -2\,\langle \omega, \nu\rangle^2 + \frac{h_{ij}}{H}\,\langle \mathbf{e_i}, \omega\rangle\,\langle \mathbf{e_j}, \omega\rangle. \tag{2.8}$$

Proof Combining the evolution equation of H given in Lemma 2.1 with (2.7), gives

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right) v &= -\frac{2}{H^2}\nabla\langle \hat{F}, \nu\rangle\nabla H - \frac{2}{H^3}\,\langle \hat{F}, \nu\rangle\,|\nabla H|^2 - 2\,\langle \omega, \nu\rangle^2 + \frac{h_{ij}}{H}\,\langle \mathbf{e_i}, \omega\rangle\,\langle \mathbf{e_j}, \omega\rangle\\ &= -\frac{2}{H^3}\nabla v\nabla H - 2\,\langle \omega, \nu\rangle^2 + \frac{h_{ij}}{H}\,\langle \mathbf{e_i}, \omega\rangle\,\langle \mathbf{e_j}, \omega\rangle\end{aligned}$$

from which (2.8) readily follows. □

Lemma 2.9 *Under the additional assumption that M_t is convex, the function $v^{-1} := (\langle \hat{F}, \nu\rangle\,H)^{-1}$ satisfies*

$$\frac{\partial}{\partial t}v^{-1} - \nabla_i\left(\frac{1}{H^2}\nabla_i v^{-1}\right) \le -\frac{2}{H^2 v^{-1}}|\nabla v^{-1}|^2 + 2\,\langle \omega, \nu\rangle^2 v^{-2}. \tag{2.9}$$

Proof Let $v := \langle \hat{F}, \nu \rangle H$ as in Lemma 2.8. We have

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right) v^{-1} = -\frac{1}{v^2}\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right) v - \frac{2}{H^2 v^3}|\nabla v|^2$$
$$= -\frac{2}{H^3}\nabla v^{-1}\nabla H - \frac{2}{H^2 v^3}|\nabla v|^2$$
$$+ 2\langle \omega, \nu\rangle^2 v^{-2} - \frac{h_{ij}}{H}\langle \mathbf{e_i}, \omega\rangle \langle \mathbf{e_j}, \omega\rangle v^{-2}$$

which implies (2.9) since $h_{ij}\langle \mathbf{e_i}, \omega\rangle \langle \mathbf{e_j}, \omega\rangle \geq 0$ by convexity. □

Throughout this paper we will make use of the comparison principle in our non-compact setting. Although rather standard under our assumptions, for the convenience of the reader we will show next a proposition which justifies this. The assumptions are made so that it is applicable in our setting.

Proposition 2.10 (Comparison principle) *Assume that* $f \in C^2(\mathbb{R}^n \times (0,\tau)) \cap C^0(\mathbb{R}^n \times (0,\tau))$ *satisfies the linear parabolic inequality*

$$f_t \leq a_{ij}\, D_{ij} f + b_i\, D_i f + c\, f, \quad \text{on} \quad \mathbb{R}^n \times (0,\tau)$$

for some $\tau > 0$ *with coefficients which are measurable functions and satisfy the bounds*

$$\lambda|\xi|^2 \leq a_{ij}(x,t)\,\xi_i\xi_j \leq \Lambda|\xi|^2\,(|x|^2+1), \quad (x,t) \in \mathbb{R}^n \times [0,\tau], \quad \xi \in \mathbb{R}^n \tag{2.10}$$

and

$$|b_i(x,t)| \leq \Lambda\,(|x|^2+1)^{1/2}, \quad |c(x,t)| \leq \Lambda, \quad (x,t) \in \mathbb{R}^n \times [0,\tau] \tag{2.11}$$

for some constants $0 < \lambda < \Lambda < +\infty$*. Assume in addition that the solution* f *satisfies the polynomial growth upper bound*

$$f(x,t) \leq C\,(|x|^2+1)^p, \quad \text{on} \quad \mathbb{R}^n \times [0,\tau]$$

for some $p > 0$*. If* $f(\cdot,0) \leq 0$ *on* $\mathbb{R}^n$*, then* $f \leq 0$ *on* $\mathbb{R}^n \times [0,\tau]$*.*

Proof To justify the application of the maximum principle it is sufficient to construct an appropriate supersolution φ of our equation. We look for such a supersolution in the form

$$\varphi(x,t) = e^{\theta t}\,(|x|^2+1)^q$$

for some exponent $q > p$ and $\theta = \theta(\Lambda, q) > 0$ to be determined in the sequel. Defining the operator

$$L\varphi := a_{ij}\, D_{ij}\varphi + b_i\, D_i\varphi + c\,\varphi$$

a direct calculation shows that under the assumptions on our coefficients we have

$$\varphi_t - L\varphi \geq \big(\theta - C(\Lambda, q)\big)\,\varphi$$

for some constant $C(\Lambda, q)$ depending only on Λ, q and the dimension n. Hence, by choosing $\theta := 2C(\Lambda, q)$ we conclude that φ satisfies the inequality

$$\varphi_t - L\varphi > 0.$$

Now, setting $\varphi_\epsilon := \epsilon\,\varphi$, we have

$$f(x,t) \leq \varphi_\epsilon(x,t), \quad \text{for} \quad |x| \geq R_\epsilon, \quad 0 \leq t \leq \tau$$

for $R_\epsilon >> 1$, since we have taken $q > p$. Since, $f_\epsilon \leq 0 \leq \varphi_\epsilon$ by assumption, the maximum principle guarantees that $f \leq \varphi_\epsilon$ on $\mathbb{R}^n \times (0, \tau)$ and by letting $\epsilon \to 0$ we conclude that $f \leq 0$ on on $\mathbb{R}^n \times [0, \tau]$ as stated in our proposition. □

We will establish next, using the maximum principle, local and global L^∞ bounds from above on the mean curvature of our solution M_t. We begin with the local bound. For the fixed point $\bar{x}_0 \in \mathbb{R}^{n+1}$ and number $r > 1$, we consider the cut off function

$$\eta := (r^2 - |F - \bar{x}_0|^2)_+^2.$$

Proposition 2.11 (Local bound from above on H) *For a solution M_t of* (1.1) *on $t \in [0, \tau]$, $\tau > 0$, if* $\sup_{M_0} \eta(F(\cdot, 0))\, H(\cdot, 0) \leq C_0$, *then*

$$\sup_{M_t} \eta(F(\cdot, t))\, H(\cdot, t) \leq \max(C_0, 2n\, r^3). \tag{2.12}$$

Proof We work on a local orthonormal frame $\{\mathbf{e_i}\}_{1 \leq i \leq n}$ on M_t where identities (2.1) hold. We have

$$\nabla_i \eta = -4\, (r^2 - |F - \bar{x}_0|^2)_+ \,\langle F - \bar{x}_0, \mathbf{e_i} \rangle = -4\eta^{1/2} \,\langle F - \bar{x}_0, \mathbf{e_i} \rangle$$

and

$$\Delta \eta = 8|(F - \bar{x}_0)^T|^2 - 4n\, \eta^{1/2} + 4\eta^{1/2} \langle F - \bar{x}_0, \nu \rangle\, H$$

and

$$\frac{\partial \eta}{\partial t} = -4\eta^{1/2} \,\langle F - \bar{x}_0, F_t \rangle = -4\eta^{1/2} \,\langle F - \bar{x}_0, \frac{1}{H}\, \nu \rangle = -4\eta^{1/2} \,\frac{1}{H} \,\langle F - \bar{x}_0, \nu \rangle.$$

We recall the H evolves by the equation

$$\frac{\partial}{\partial t} H = \frac{1}{H^2} \Delta H - \frac{2}{H^3} |\nabla H|^2 - \frac{|A|^2}{H}.$$

Using also the bound $|A|^2/H \geq H/n$, it follows that

$$\frac{\partial (\eta H)}{\partial t} \leq \frac{1}{H^2} \eta\, \Delta H - \frac{2}{H^3}\, |\nabla H|^2\, \eta - \frac{H}{n} \eta - 4\eta^{1/2} \langle F - \bar{x}_0, \nu \rangle.$$

Since

$$\frac{1}{H^2}\, \Delta(\eta H) = \frac{1}{H^2} \eta\, \Delta H + \frac{2}{H^2}\, \nabla_i H\, \nabla_i \eta + \frac{1}{H}\, \Delta \eta$$

the above yields

$$\begin{aligned} \frac{\partial (\eta H)}{\partial t} \leq \frac{1}{H^2}\, \Delta(\eta H) - \frac{2}{H^2}\, \nabla_i H\, \nabla_i \eta - \frac{8}{H}\, |(F - \bar{x}_0)^T|^2 \\ + \frac{4n}{H} \eta^{1/2} - \frac{2}{H^3}\, |\nabla H|^2 \eta - \frac{H}{n} \eta. \end{aligned}$$

Using

$$\frac{2}{H^3} \nabla_i H \nabla_i (\eta H) = \frac{2}{H^2}\, \nabla_i H\, \nabla_i \eta + \frac{2}{H^3}\, |\nabla H|^2\, \eta$$

we conclude that $\varphi := \eta H$ satisfies

$$\frac{\partial \varphi}{\partial t} \leq \frac{1}{H^2}\,\Delta\varphi - \frac{2}{H^3}\nabla_i H \nabla_i \varphi + \frac{4n}{\varphi}\,\eta^{3/2} - \frac{\varphi}{n}.$$

For the fixed $r > 0$ and $\bar{x}_0 \in \mathbb{R}^{n+1}$, let

$$m(t) := \max_{M_t} H\eta = \max_{M_t} \varphi.$$

Since $\eta \leq r^4$, it follows from the above differential inequality that $m(t)$ will decrease if

$$r^6\,\frac{4n}{m(t)} - \frac{m(t)}{n} \leq 0 \iff m^2(t) \geq 4n^2\,r^6 \iff m(t) \geq 2n\,r^3.$$

Hence

$$m(t) \leq \max\,(\,m(0), 2n\,r^3).$$

□

Remark 2.1 We note that Proposition 2.11 does not require the convexity of M_t.

Proposition 2.12 (Global bound from above on H) *For a convex graph solution M_t of* (1.1) *on $t \in [0, \tau]$, if $\sup_{M_0} \langle F, \omega\rangle H(\cdot, 0) < \infty$, then*

$$\sup_{t\in[0,\tau]}\ \sup_{M_t} \langle F, \omega\rangle H \leq \sup_{M_0} \langle F, \omega\rangle H. \tag{2.13}$$

Proof We will compute the evolution of $\langle F, \omega\rangle H \geq 0$ from the evolution of H given in Lemma 2.1 and the evolution of the height function $\langle F, \omega\rangle$ given by (2.6). Indeed, combining these two equations leads

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right)(\langle F, \omega\rangle H) = -\frac{2}{H^3}|\nabla H|^2\,\langle F, \omega\rangle - \frac{2}{H^2}\nabla H \cdot \nabla\langle F, \omega\rangle$$
$$-\frac{|A|^2}{H}\,\langle F, \omega\rangle + 2\,\langle \omega, \nu\rangle.$$

Writing

$$\frac{2}{H^3}|\nabla H|^2\,\langle F, \omega\rangle + \frac{2}{H^2}\nabla H \cdot \nabla\langle F, \omega\rangle = \frac{2}{H^3}\nabla H \cdot \nabla(\langle F, \omega\rangle H)$$

and using $\langle F, \omega\rangle \geq 0$ and $\langle \omega, \nu\rangle \leq 0$, we conclude that $\langle F, \omega\rangle H$ satisfies

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\right)\big(\langle F, \omega\rangle H\big) \leq -\frac{2}{H^3}\nabla H \cdot \nabla\big(\langle F, \omega\rangle H\big)$$

and the bound (2.13) readily follows by the comparison principle. □

3 Self-similar solutions

We will study in this section self-similar entire graph solutions $x_{n+1} = \bar{u}(x, t)$ of the IMCF equation (1.4) which have polynomial growth at infinity, namely $\bar{u}(x, t) \sim |x|^q$, with $q \geq 1$. These solutions are all rotationally symmetric.

First, we consider rotationally symmetric infinite cones in the direction of the vector $\omega = e_{n+1}$. If the vertex $P \in \mathbb{R}^{n+1}$ of the cone is the origin $0 \in \mathbb{R}^{n+1}$ for its position vector

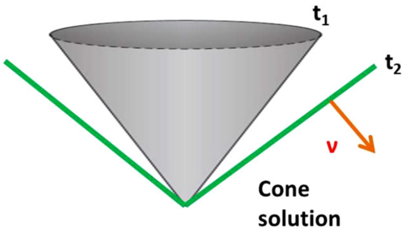
F, then $\langle F, \nu\rangle = 0$, otherwise $\langle F, \nu\rangle = \langle P, \nu\rangle$. Since the cone is a rotationally symmetric graph, in its graph parametrization $\bar{F}(x,t) := (x, \zeta(|x|,t))$, $x \in \mathbb{R}^n$ it is given by a height function

$$\zeta(r,t) = \alpha(t)\, r + \kappa, \quad r := |x|, \quad x \in \mathbb{R}^n \tag{3.1}$$

for a constant $\kappa \in \mathbb{R}$. The function ζ is a solution of the equation

$$\zeta_t = -\frac{(1+\zeta_r^2)^2}{\zeta_{rr} + (n-1)\,(1+\zeta_r^2)\,\zeta_r/r} \tag{3.2}$$

which is satisfied by any rotationally symmetric graph $\bar{F}(x,t) := (x, \bar{u}(|x|,t))$, $x \in \mathbb{R}^n$ which evolves by equation (1.3).



It follows from (3.2) that $\alpha(t)$ satisfies the ODE

$$\alpha'(t) = -\frac{1}{n-1}\left(\alpha(t) + \frac{1}{\alpha(t)}\right). \tag{3.3}$$

On the conical solution we have

$$\langle \omega, \nu\rangle = -\frac{1}{\sqrt{1+\alpha(t)^2}} \tag{3.4}$$

and

$$H(r,t) = \frac{(n-1)\,\alpha(t)}{r\sqrt{1+\alpha(t)^2}}. \tag{3.5}$$

We conclude that on the conical solution

$$v := -\langle F, \omega\rangle\,\langle \omega, \nu\rangle\, H = \gamma(t) := \frac{(n-1)\,\alpha(t)^2}{1+\alpha(t)^2}.$$

Setting

$$\beta(t) := \langle \omega, \nu\rangle^2 = \frac{1}{1+\alpha(t)^2}$$

we have

$$\gamma(t) = (n-1)\,(1-\beta(t)).$$

To compute the evolution of $\gamma(t)$ and $\beta(t)$ it is simpler to use the equations (2.4) and (2.7) which directly give

$$\beta'(t) = \frac{2}{(n-1)}\beta(t) \quad \text{and} \quad \gamma'(t) = -2\,\beta(t).$$

This is correct since $\gamma(t)$ and $\beta(t)$ are independent of the parametrization. We conclude that

$$\beta'(t) = \frac{2}{(n-1)}\beta(t) \quad \text{and} \quad \gamma'(t) = 2\left(\frac{\gamma(t)}{n-1} - 1\right). \tag{3.6}$$

Solving the last ODE's with initial conditions $\beta_0 \in (0,1)$ and $\gamma_0 := (n-1)(1-\beta_0)$ yields

$$\beta(t) = \beta_0\, e^{2t/(n-1)} \quad \text{and} \quad \gamma(t) = (n-1)\left(1 - \left(1 - \frac{\gamma_0}{(n-1)}\right) e^{2t/(n-1)}\right). \tag{3.7}$$

Finally, recalling that $1 + \alpha^2(t) = \beta(t)^{-1}$, we conclude that the *slope* $\alpha(t)$ of the conical solution is

$$\alpha(t) = \left((1+\alpha_0^2)\, e^{-2t/(n-1)} - 1\right)^{1/2}.$$

It is clear from the above equations that the conical solution will become flat at time $T(\alpha_0)$ given by

$$T(\alpha_0) = \frac{n-1}{2}\log(1+\alpha_0^2) = \frac{n-1}{2}\log\left(\frac{n-1}{(n-1)-\gamma_0}\right). \tag{3.8}$$

Next, let us briefly discuss other self-similar solutions of equation (1.4) which exists for all time $t > 0$ and they are also rotationally symmetric. It is simple to observe that the time t cannot be scaled in the fully-nonlinear equation (1.4). Nevertheless, equation (1.4) admits (non-standard) self-similar solutions of the form

$$\bar{u}_\lambda(x,t) = e^{\lambda t}\, \bar{u}_\lambda(e^{-\lambda t}\, x), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{3.9}$$

for a suitable range of exponents $\lambda > 0$, where $x_{n+1} = \bar{u}_\lambda(x)$ are entire convex graphs over $\mathbb{R}^n$. The function $x_{n+1} = \bar{u}_\lambda(x)$, $x \in \mathbb{R}^n$ satisfies the fully-nonlinear elliptic equation

$$\operatorname{div}\left(\frac{D\bar{u}}{\sqrt{1+|D\bar{u}|^2}}\right) = \frac{1}{\lambda}\,\frac{\sqrt{1+|Du|^2}}{x\cdot D\bar{u} - \bar{u}}. \tag{3.10}$$

Although equation (3.10) may possess non-radial solutions, restricting ourselves to rotationally symmetric solutions $x_{n+1} = \bar{u}_\lambda(r)$, $r = |x|$, it follows that $\bar{u} := \bar{u}_\lambda(r)$ satisfies the ODE

$$\bar{u}_{rr} + (n-1)\,(1+u_r^2)\,\frac{\bar{u}_r}{r} - \frac{1}{\lambda}\,\frac{((1+\bar{u}_r^2)^2}{r\,\bar{u}_r - \bar{u}} = 0. \tag{3.11}$$

One needs to impose condition $\bar{u}(0) = \kappa < 0$ to guarantee the existence of an entire convex solution. The following was shown by the authors and J. King in [6].

Theorem 3.1 (The existence of self-similar solutions) *For every $\lambda > 1/(n-1)$ and $\kappa < 0$, there exists a unique rotationally symmetric entire convex solution $x_{n+1} = \bar{u}_\lambda(r)$ of (3.10) on $\mathbb{R}^n$ with $\bar{u}_\lambda(0) = \kappa$. In addition, $\bar{u} := \bar{u}_\lambda$ satisfies the following flux condition at $r = +\infty$*

$$\lim_{r\to\infty} \frac{r\,u_r(r)}{u(r)} = q, \quad q := \frac{\lambda\,(n-1)}{(n-1)\,\lambda - 1}. \tag{3.12}$$

The condition (3.12) shows that $u_\lambda(x) \sim |x|^{\frac{\lambda\,(n-1)}{(n-1)\,\lambda-1}}$ as $|x| \to \infty$. Notice that since λ is any number $\lambda > 1/(n-1)$, the exponent $q := \frac{\lambda\,(n-1)}{(n-1)\,\lambda-1}$ covers the whole range $q \in (1,+\infty)$, hence each solution u_λ has a polynomial growth at infinity larger than that of the conical solution $x_{n+1} = \alpha(t)\,|x| + \kappa$. It would be interesting to see whether the limit $\lim_{\lambda\to+\infty} u_\lambda$ gives the conical solution or possibly another solution with super linear behavior as $|x| \to \infty$.

4 The super-linear case and short time existence

In this section we assume that M_0 is an entire graph $\{x_{n+1} = \bar{u}(\cdot, 0)\}$ over $\mathbb{R}^n$ in the direction of the vector $\omega := e_{n+1}$. We first prove long-time existence for solutions to (1.1) with superlinear growth that are δ-*starshaped* (see below) and then we establish the short-time existence for the critical case of convex solutions that lie between the rotationally symmetric cones $x_{n+1} = \zeta_i(\cdot, 0)$, with $\zeta_1(\cdot, 0) := \alpha_0 |x|$ and $\zeta_2(\cdot, 0) := \alpha_0 |x| + \kappa$, $x \in \mathbb{R}^n$, as in (1.5). The solutions M_t are then given as the graph of $\bar{u}(\cdot, t)$ satisfying (1.4).

In our first result, Theorem 4.1 below, we will assume that the initial data M_0 is an entire graph $x_{n+1} = \bar{u}_0(x)$ over $\mathbb{R}^n$ with superlinear growth at infinity, i.e.

$$|D\bar{u}_0(x)| \to \infty, \quad \bar{u}_0(x) \to \infty, \quad \text{for} \quad |x| \to \infty \tag{4.1}$$

and is strictly starshaped with a uniformity condition:

We say that M_0 is δ-*starshaped* if there is a point $\bar{x}_0 \in \mathbb{R}^{n+1} \cap \{(x, x_{n+1}) | u_0(x) < x_{n+1}\}$ and a constant $\delta > 0$ such that the mean curvature H satisfies

$$H\langle F - \bar{x}_0, \nu\rangle \geq \delta > 0 \tag{4.2}$$

everywhere on M_0. By Lemma 2.3, this condition which provides a scaling invariant quantitative measure for the starshapedness of a hypersurface, is preserved under inverse mean curvature flow whenever the maximum principle can be applied.

Theorem 4.1 (Existence for superlinear initial data) *Assume that the initial surface M_0 is an entire graph $\{(x, x_{n+1}) | \, x \in \mathbb{R}^n, x_{n+1} = \bar{u}_0(x)\}$ with $\bar{u}_0 \in C^2(\mathbb{R}^n)$ and satisfying the assumptions (4.1) and (4.2), for some $\bar{x}_0 \in \mathbb{R}^{n+1}$. Then, there is a smooth solution $F : M^n \times [0, \infty) \to \mathbb{R}^{n+1}$ of the inverse mean curvature flow (1.1) for all times $t > 0$ that can be written as a graph $M_t = F(\cdot, t)(M^n) = \{x_{n+1} = \bar{u}(x, t)\}$ with initial data M_0. If $\bar{u}_0$ is convex, then the solution M_t is also convex for all time.*

Remark 4.1 (i) It is easy to see that the assumptions (4.1) and (4.2) are satisfied if $\bar{u}_0(x) = |x|^q$, provided $q > 1$.

(ii) The condition "δ-starshaped" is reminiscent but different from the "δ-non-collapsed" condition that has been used in mean curvature flow.

Remark 4.2 In the case of convex initial data, the condition $\bar{u}_0 \in C^2(\mathbb{R}^n)$ in Theorem 4.1 may be replaced by the weaker condition $\bar{u}_0 \in C^2_{\text{loc}}(\mathbb{R}^n)$, since the mean curvature is uniformly controlled on compact sets.

Proof By translating the surface we may assume that $\bar{x}_0 = 0$ is the origin of $\mathbb{R}^{n+1}$; then $\bar{u}_0(x) \geq C_0$ is bounded below everywhere by some negative constant. For the proof we will assume that $\bar{u}_0 \in C^{2,\alpha}(\mathbb{R}^n)$. For initial data just in C^2 as our theorem states, the result follows by approximation in view of the estimates we establish.

By the assumption (4.1) we may choose $R_0 > 1$ such that $|D\bar{u}_0(x)| \geq 100$ provided $|x| \geq R_0$. We want to approximate M_0 with compact surfaces by replacing the region $\{\bar{u}_0 \geq R\}$ of the surface with the reflection of the region $\{\bar{u}_0 \leq R\}$ on the plane at height R. Set, for each $R \geq R_0$

$$\hat{u}_{0,R}(x) := 2R - \bar{u}_0(x) \tag{4.3}$$

and set

$$E_{0,R} := \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \, \bar{u}_0(x) < x_{n+1} < \hat{u}_{0,R}(x)\}, \quad \Sigma_{0,R} := \partial E_{0,R}. \tag{4.4}$$

The outer unit normals ν and $\hat{\nu}$ to the lower and and upper part of $\Sigma_{0,R}$ are given by

$$\frac{(D\bar{u}_0, -1)}{\sqrt{1+|D\bar{u}_0|^2}} \quad \text{and} \quad \frac{(-D\hat{u}_{0,R}, 1)}{\sqrt{1+|D\hat{u}_{0,R}|^2}} \tag{4.5}$$

respectively, where by definition $D\hat{u}_{0,R}(x) = -D\bar{u}_0(x)$ and the mean curvatures H and $\hat{H}$ satisfy $\hat{H}(x, \hat{u}_{0,R}(x)) = H(x, \bar{u}_0(x))$. In particular we get that for $(x, x_{n+1}) \in \Sigma_{0,R}$ with $x_{n+1} = R$

$$\langle \nu, \hat{\nu} \rangle (x, x_{n+1}) = \frac{|D\bar{u}_0(x)|^2 - 1}{|D\bar{u}_0(x)|^2 + 1} \geq 1 - 10^{-3}. \tag{4.6}$$

In addition we compute that $\hat{F}_R(x, 0) := (x, \hat{u}_{0,R}(x)$ satisfies

$$\begin{aligned}
&\hat{H}(x, \hat{u}_{0,R}(x)) \langle \hat{F}_R(x,0), \hat{\nu} \rangle (x, \hat{u}_{0,R}(x)) \\
&= \frac{H(x, \bar{u}_{0,R}(x))}{\sqrt{1+|D\bar{u}_{0,R}(x)|^2}} \Big(\langle (x, \bar{u}_{0,R}(x), (-D\bar{u}_{0,R}(x), 1) \rangle \Big) \\
&= \frac{H(x, \bar{u}_0(x))}{\sqrt{1+|D\bar{u}_0(x)|^2}} \Big(\langle (x, 2R - \bar{u}_0(x), (D\bar{u}_0(x), 1) \rangle \Big) \\
&= H(x, \bar{u}_0(x)) \langle F(x), \nu \rangle (x, \bar{u}_0(x)) + \frac{2R}{\sqrt{1+|D\bar{u}_0(x)|^2}} \geq \delta > 0
\end{aligned}$$

such that the surface $\Sigma_R = \partial E_R$ is again uniformly δ-starshaped. If the initial function u is convex, all regions E_R are convex as well.

Next we smoothen out the region $x_{n+1} = \bar{u}_0(x) = R$ using mean curvature flow:

Lemma 4.2 *Given $\bar{u}_0 \in C^{2,\alpha}(\mathbb{R}^n)$, for each $\Sigma_R = \partial E_R$ as above there is a one-parameter family of hypersurfaces $\Phi : S^n \times [0, s_R] \to \mathbb{R}^{n+1}$, $\Phi(\cdot, s)(S^n) = \Sigma_R(s)$, $s_R > 0$, with initial data $\Sigma_R(0) = \Sigma_R$ satisfying mean curvature flow*

$$\frac{d}{ds}\Phi(p, s) = \overrightarrow{H}(\Phi(p, s)), \quad p \in S^n, \; s \in [0, s_R]. \tag{4.7}$$

The surfaces $\Sigma_R(s)$, $s \in [0, s_R]$, are smooth and approach Σ_R in $C^{0,1/2}$ as $s \to 0$. For small $s_R > 0$ they are $\tilde{\delta}$-starshaped with $\tilde{\delta} \geq \delta - o(s^{\alpha/2})$. We may choose $\sigma_R \in (0, s_R]$ such that $\sigma_R \to 0$ as $R \to \infty$ and all $\Sigma_R(\sigma_R)$ are uniformly bounded in $C^{2,\alpha}$. If u_0 is convex, then $\Sigma_R(\sigma_R)$ is strictly convex with some lower bound $\lambda_R > 0$ for all its principal curvatures.

Proof of Lemma 4.2 Σ_R is a uniformly Lipschitz hypersurface over its tangent spaces in view of (4.6), so we may solve mean curvature flow for a short time with Σ_R as initial data, compare ([9], Theorems 3.4 and 4.2), to obtain a smooth solution $\Sigma_R(s)$ for (4.7) on some time interval $(0, s_R]$, $s_R > 0$ which approaches the initial data in $C^{0,1/2}$ as $s \to 0$. For small $s > 0$ this solution has strictly positive mean curvature; this follows from the fact that Σ_R provides a barrier for Mean curvature flow and can be approximated by a smooth surface of strictly positive mean curvature from the inside, e.g. by gluing in arbitrary small sectors of an approximate cylinder along the edge $x_{n+1} = R$. Since $\bar{u}_0 \in C^{2,\alpha}(\mathbb{R}^n)$, the interior regularity estimates in [9] combined with Schauder theory yield

$$\Big| H(\Phi(p, s)) - H(\Phi(p, 0)) \Big| \leq c(R)\, r(p)^{-1-\alpha}\, s^{\alpha/2} \tag{4.8}$$

where $r(p) = |\Phi^{n+1}(p, 0) - R|$ is the distance to the singular set $\{x_{n+1} = R\}$ and $c(R) < \infty$ depends on the $C^{2,\alpha}$- norm of $\bar{u}_0$ on $B_R(0)$. If $y = \Phi(p_0, 0)$ with $y^{n+1} = R$ is a point on the edge formed between graph$(\bar{u}_0)$ and graph$(\hat{u}_{0,R})$, then in view of the uniform Lipschitz estimates for small s a rescaling of $\Sigma_R(s)$ around y for $s \to 0$ converges to the solution of mean curvature flow $\Gamma(s) \times \mathbb{R}^{n-1}, 0 < s < \infty$, where $\Gamma(s)$ is the unique selfsimilar expanding solution of curve-shortening flow in the 2-plane containing e_{n+1} and $D\bar{u}_0(p_0)$ that is associated with the angle between $\nu(p_0)$ and $\hat{\nu}(p_0)$. The unit normal to this solution interpolates between $\nu(p_0)$ and $\hat{\nu}(p_0)$ while its geodesic curvature decays exponentially, in fact it has been shown in ([12], Lemma 6.4) that its geodesic curvature $\kappa(r, s)$ at time s and distance r from y is given by

$$\kappa(r, s) = \frac{1}{\sqrt{2s}}\, \kappa_{\max}\left(\frac{1}{2}\right) \exp\left(\kappa^2_{\max}\left(\frac{1}{2}\right) s - \frac{r^2(p, s)}{4s}\right). \tag{4.9}$$

Here $\kappa_{\max}(1/2)$ is determined by the opening angle between $\nu, \hat{\nu}$ in such a way that $\kappa_{\max}(1/2) \to 0$ as this opening angle tends to 0, or, equivalently, $|D\bar{u}_0| \to \infty$ on the edge $\{x_{n+1} = R\}$ as $R \to \infty$. Let κ_R be the largest such $\kappa_{max}(1/2)$ arising from an opening angle on the edge $\{x_{n+1} = R\}$. If we then choose $\sigma_R \in (0, S_R]$ smaller than κ_R^2 we see that the surfaces $\Sigma_R(\sigma_R)$ are uniformly bounded in $C^{2,\alpha}$ in view of (4.9) and Schauder theory while approximating M_0 uniformly in $C^{2,\alpha}$ as $R \to \infty$ since $\sigma_R \le \kappa_R^2 \to 0$. Combining then the δ-starshapedness in (4.2), (4.7) with the estimates (4.8) and (4.9) we see that $\Sigma_R(\sigma_R)$ must be $\tilde{\delta}$-starshaped with $\tilde{\delta} \ge \delta - o(\sigma_R^{\alpha/2})$. If the function $\bar{u}_0$ is convex then $\Sigma_R(s), s > 0$ will be uniformly convex by the strong parabolic maximum principle, i.e. there will be $\lambda_R > 0$ such that the eigenvalues $\lambda_i, 1 \le i \le n$ of the second fundamental form all satisfy $\lambda_i \ge \lambda_R$ everywhere on $\Sigma_R(\sigma_R)$. □

Proof of Theorem 4.1 continued Given a sequence of radii $R_i \to \infty$ we may choose parameters $\sigma_i := \sigma_{R_i} \to 0$ as in the preceding lemma with corresponding smooth approximating surfaces $\Sigma^i := \Sigma_{R_i}(\sigma_i)$ with $\Sigma^i = \partial E^i$ such that $E^i \subset E^j$ for $i < j$ and Σ^i is δ_i-starshaped with $\delta_i \to \delta$ as $i \to \infty$. From the work of Gerhardt [13] (see also Urbas [22]), for each approximating surface Σ^i there is a smooth solution $\Sigma^i(t), t \in [0, \infty)$ of inverse mean curvature flow starting from Σ^i that approaches a homothetically expanding sphere as $t \to \infty$. We now combine the δ_i- starshapedness for each $R > 0$ with the local bound on the mean curvature obtained in proposition 2.11 such that

$$0 < \delta_i \le H\langle F, \nu\rangle \le \max\left(4 \max_{M_0 \cap B_R(0)} H,\ \frac{8n}{R}\right) \langle F, \nu\rangle := C_1(R)\langle F, \nu\rangle \tag{4.10}$$

and therefore

$$0 < \frac{\delta_i}{C_1(R)} \le \langle F, \nu\rangle \le R \tag{4.11}$$

holds everywhere on $\Sigma^i(t) \cap B_{R/2}(0)$ when $R_i > 2R$. Hence $\Sigma^i(t)$ is uniformly starshaped in $B_{R/2}(0)$ and we may use the local curvature bound in ([14], Theorem 3.6 and Remark 3.7) to conclude that

$$|A|^2 \le C_2 \max\left(\max_{M_0 \cap B_R(0)} |A|^2, R^{-1} \max_{M_0 \cap B_R(0)} H + R^{-2}\right). \tag{4.12}$$

holds everywhere on $\Sigma^i(t) \cap B_{R/4}(0)$ when $R_i > 2R$, where C_2 depends on n and $(R \max_{M_0 \cap B_R(0)} H)$. Thus the solutions satisfy uniform curvature estimates independent of i on any compact set. Higher regularity then follows from known theory, see in [20]. To obtain a subsolution we choose for each $T < \infty$ an $0 < \alpha_0 = \alpha_0(T) < \infty, \kappa(T) > -\infty$

such that the conical solutions

$$\zeta(x,t) = \alpha(t)\,|x| + \kappa(T), \quad \alpha(0) = \alpha_0$$

from Sect. 3 provide a lower barrier for all $\Sigma^i(t)$ on $t \in [0, T)$. Here α_0 is chosen so that T is the lifetime of the cones $\zeta(\cdot, t)$. It follows that we can pass to the limit to obtain a solution M_t of the inverse mean curvature flow which is defined for all $t \in (0, +\infty)$ and is C^∞ smooth. Note that M_t is again an entire graph: For each $\rho > 0$ the initial hypersurface M_0 is δ-starshaped also with respect to $\bar{x}_\rho = \rho\omega$ since $H\langle F - \bar{x}_\rho, \nu\rangle = H\langle F, \nu\rangle - \rho H\langle \omega, \nu\rangle \geq \delta > 0$ as $\langle \omega, \nu\rangle \leq 0$. If $R_i > \rho$ this will also be true for $\Sigma_i(0)$ and hence, by the maximum principle, on all $\Sigma_i(t)$. Thus $\langle F - \rho\omega, \nu\rangle > 0$ for all $\rho > 0$ everywhere on all M_t. Dividing by ρ and letting $\rho \to \infty$ on compact subsets yields $\langle -\omega, \nu\rangle \geq 0$ and hence $\langle -\omega, \nu\rangle > 0$ by the strong maximum principle as desired.

If the initial surface M_0 is convex, then each $\Sigma_i(0)$ is uniformly convex by Lemma 4.2. Then in view of the result of Urbas [22] the surfaces $\Sigma^i(t)$, $t \in [0, \infty)$ are also uniformly convex proving that all limit surfaces M_t are convex in this case. This completes the proof of the longtime existence of solutions with superlinear, δ-starshaped initial data, as stated in Theorem 4.1. □

We will give next a short time existence result for convex initial data M_0 which lies between two cones as in condition (1.5).

Theorem 4.3 (Short time existence of asymptotically conical solutions) *Let M_0 be an entire convex graph $x_{n+1} = \bar{u}_0(x)$ over $\mathbb{R}^n$ which satisfies condition* (1.5)*. Assume in addition that the mean curvature H of M_0 satisfies the global bounds*

$$0 < c_0 \leq H\,\langle F, \omega\rangle \leq C_0. \tag{4.13}$$

Then, there exists $\tau > 0$ and a unique C^∞ smooth solution M_t of (1.1) *for $t \in (0, \tau]$ which is an entire convex graph $x_{n+1} = \bar{u}(x,t)$ over $\mathbb{R}^n$ and satisfies condition* (1.7)*. Moreover, on M_t we have*

$$c_\tau \leq H\,\langle F, \omega\rangle \leq C, \quad \textit{for all} \quad t \in (0, \tau] \tag{4.14}$$

for a constant $c_\tau > 0$ depending on τ and $C := \max(C_0, 2n)$.

Remark 4.3 We note that on the graph M_0 of any convex function $\bar{u}_0 \in C^2_{\text{loc}}(\mathbb{R}^n)$ which satisfies condition (1.5) one has

$$0 \leq -\langle F, \nu\rangle \leq C_0 \tag{4.15}$$

for some constant C_0 which can be taken without loss of generality to be equal to C_0 in (4.13).

Proof For $\epsilon \in (0, 1)$, consider the approximations M_0^ϵ of the initial surface M_0 defined as entire graphs $x_{n+1} = \bar{u}_{0,\epsilon}(x)$, with

$$\bar{u}_{0,\epsilon}(x) = \bar{u}_0(x) + \epsilon\,(|x|^2 + 1), \quad x \in \mathbb{R}^n. \tag{4.16}$$

Then, each $\bar{u}_{0,\epsilon}$ satisfies the conditions of Theorem 4.1 (with $\bar{u}_{\epsilon,0} \in C^2_{\text{loc}}$ instead of $\bar{u}_{0,\epsilon} \in C^2$) and in addition it is strictly convex. By Theorem 4.1 and Remark 4.2 there exists a solution M_t^ϵ to (1.1) on $t \in (0, +\infty)$ with initial data M_0^ϵ. In addition M_t^ϵ are smooth entire convex graphs given by $x_{n+1} = \bar{u}_\epsilon(x,t)$, $x \in \mathbb{R}^n$. The functions $\bar{u}_\epsilon$ satisfy equation (1.4). Since $\bar{u}_{0,\epsilon}$

satisfies $\alpha_0\,|x| \le \bar u_{0,\epsilon}(x) \le u_{0,1}(x)$ for all $\epsilon \in (0,1)$, the comparison principle implies that for any $0 < \epsilon_1 < \epsilon_2 < 1$ we have

$$\alpha(t)\,|x| \le \bar u_{\epsilon_1}(x,t) \le \bar u_{\epsilon_2}(x,t) \le \bar u_1(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,T)$$

where T denotes the extinction time of $\alpha(t)$. In particular, the monotone limit $\bar u := \lim_{\epsilon \to 0} \bar u_\epsilon$ exists and satisfies

$$\alpha(t)\,|x| \le \bar u(x,t) \le \bar u_1(x,t). \tag{4.17}$$

We will show next that $\bar u$ is a solution of (1.4) with initial data $\bar u_0$.

Claim 4.1 *There exists $\tau > 0$ for which the limit $\bar u$ is a smooth convex solution of (1.4) on $\mathbb{R}^n \times (0,\tau)$ with initial data $\bar u_0$.*

Proof of Claim 4.1 Consider the approximations $\bar u_\epsilon$ and denote by H_ϵ the mean curvature of M_t^ϵ. Set $v_\epsilon := \langle \hat F_\epsilon, \nu\rangle\, H_\epsilon$, where $\langle \hat F_\epsilon, \nu\rangle$ denotes the quantity $\langle \hat F_\epsilon, \nu\rangle := -\langle F_\epsilon, \omega\rangle\,\langle \omega, \nu_\epsilon\rangle$ on M_t^ϵ. Each v_ϵ satisfies the equation (2.8) and since each M_t^ϵ is convex the last term on the righthand side of (2.8) is nonnegative. Since $\langle \omega, \nu_\epsilon\rangle \le 1$, we conclude that each v_ϵ satisfies

$$\frac{\partial}{\partial t} v_\epsilon - \nabla_i \left(\frac{1}{H^2} \nabla_i v_\epsilon \right) \ge -2. \tag{4.18}$$

Moreover, our initial conditions on $\bar u_0$ guarantee that $v_\epsilon \ge c_0 > 0$ for a uniform in ϵ constant c_0. The differential inequality (4.18) implies that

$$v_\epsilon := \langle \hat F_\epsilon, \nu\rangle\, H_\epsilon \ge c_0/2 > 0, \quad \text{on} \quad M_t^\epsilon, \quad t \in [0,\tau] \tag{4.19}$$

if we choose $\tau := c_0/4$. On the other hand, our initial assumption that $H\,\langle F, \omega\rangle \le C_0$ on M_0 implies that $H\,|F| \le C_1$ on M_0, which in turn gives a uniform in ϵ bound $H_\epsilon \le C_2/(1+|x|)$ on M_0^ϵ, for a uniform in ϵ constant C_2. Proposition 2.11, implies the bound

$$H_\epsilon \le C\,(1+|x|)^{-1}, \quad \text{on} \quad M_t^\epsilon, \quad t \in [0,\tau]. \tag{4.20}$$

Combining the two estimates yields

$$0 < c_R \le H_\epsilon(\cdot,t) \le C, \quad \text{on} \quad |x| \le R, \quad t \in [0,\tau], \tag{4.21}$$

for uniform in ϵ and t constants c_R, C. These bounds guarantee that the equation (1.4) is uniformly parabolic in ϵ on compact sets and by standard regularity arguments the limit $\bar u$ will be a smooth convex solution of (1.4) with initial data $\bar u_0$. By passing to the limit in (4.19) and (4.20) we conclude that

$$\langle \hat F, \nu\rangle\, H \ge c > 0 \quad \text{and} \quad H \le C\,(1+|x|)^{-1}, \quad \text{on} \quad M_t, \quad t \in [0,\tau] \tag{4.22}$$

for $c := c_0/2$. □

It remains to show that the solution $\bar u$ satisfies the upper bound $\bar u(x,t) \le \alpha(t)\,|x| + \kappa$ in (1.7). To this end, we will first show that $\bar u(\cdot,t)$ has linear growth at infinity which will allow us to apply the comparison principle Proposition 2.10.

Claim 4.2 *The limit $\bar u$ satisfies the linear bound*

$$\bar u(x,t) \le \theta\,|x| + \kappa_1$$

for some constants $\theta > 0$ and $\kappa_1 > 0$.

Proof of Claim 4.2 First notice that for any pair (θ, κ) with $\theta > 0, \kappa \in \mathbb{R}$ an elementary calculation shows that the subgraph of the conical surface $\zeta(x) = \theta r + \kappa$, $r = |x|$, is equal to the complement of a natural family of spheres lying above it:

$$\{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \mid x_{n+1} < \theta|x| + \kappa\} = \bigcap_{\rho>0, \tilde{\kappa}\geq\kappa} \left(\mathbb{R}^{n+1} \setminus B_\rho(0, \rho\sqrt{1+\theta^2} + \tilde{\kappa})\right) \tag{4.23}$$

Since $\bar{u}_0(x) \leq \alpha_0|x| + \kappa$, for each $\rho_0 > 0$ and $\tilde{\epsilon} > 0$ we may now choose $\epsilon_0 = \epsilon_0(\rho_0, \tilde{\epsilon})) > 0$ such that for all $0 < \epsilon < \epsilon_0$, $0 < \rho < \rho_0$ the balls $B_\rho(0, \rho\sqrt{1+\alpha_0^2} + \tilde{\kappa} + \tilde{\epsilon})$, $\tilde{\kappa} \geq \kappa$, are contained in the epigraph

$$\{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \mid x_{n+1} > \bar{u}_{0,\epsilon}(x)\}$$

of the approximating functions $\bar{u}_{0,\epsilon}$ given by (4.16). Using the barrier principle for IMCF applied to the resulting graphs of $\bar{u}_{t,\epsilon}$ and the balls expanding by IMCF, $B_{\rho(t)}(0, \rho\sqrt{1+\alpha_0^2} + \tilde{\kappa} + \tilde{\epsilon})$, $\rho(t) = \rho\exp(t/n)$ we conclude from the monotone convergence of the $\bar{u}_{t,\epsilon}$ in the limit $\tilde{\epsilon} \to 0$ that the balls $B_{\rho(t)}(0, \rho\sqrt{1+\alpha_0^2} + \tilde{\kappa})$, $\tilde{\kappa} \geq \kappa$ are contained in the epigraph of $\bar{u}_t = \lim \bar{u}_{t,\epsilon}$ for each $\rho > 0$. In other words,

$$\{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \mid x_{n+1} = \bar{u}(x,t)\} \subset \bigcap_{\rho>0, \tilde{\kappa}\geq\kappa} \left(\mathbb{R}^{n+1} \setminus B_{\rho(t)}(0, \rho\sqrt{1+\alpha_0^2} + \tilde{\kappa})\right).$$

Now note that

$$\rho\sqrt{\alpha_0^2 + 1} = \rho(t)\exp(-t/n)\sqrt{\alpha_0^2 + 1} = \rho(t)\sqrt{\theta(t)^2 + 1},$$

where

$$\theta(t) = \sqrt{\exp\left(\frac{-2t}{n}\right)(1+\alpha_0^2) - 1}, \quad t \in [0, n\log\sqrt{1+\alpha_0^2}) \cap [0, T_{\max}(\bar{u})).$$

Since $\cap_{\rho(t)} = \cap_{\rho>0}$ we get

$$\{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \mid x_{n+1} = \bar{u}(x,t)\} \subset \bigcap_{\rho(t), \tilde{\kappa}\geq\kappa} \left(\mathbb{R}^{n+1} \setminus B_{\rho(t)}(0, \rho(t)\sqrt{1+\theta^2(t)} + \tilde{\kappa})\right)$$

which implies the claim

$$\bar{u}(x,t) \leq \theta(t)|x| + \kappa$$

in view of (4.23) as required. Notice that $\theta(t) > \alpha(t)$ for $t > 0$ such that this estimate cannot yet yield the optimal upper bound. □

It remains to show that $\bar{u}(x,t) \leq \alpha(t)\,|x| + \kappa$. This simply follows from the next claim, by comparing $\bar{u}$ with the conical solution $\zeta_2(x,t) := \alpha(t)\,|x| + \kappa$.

Claim 4.3 *Assume that $\bar{u}_1, \bar{u}_2$ are two smooth and convex entire graph solutions of equation (1.4) on $\mathbb{R}^n \times (0, \tau]$ for some $\tau > 0$ which satisfy the bounds*

$$\alpha\,|x| \leq \bar{u}_i(x,t) \leq \theta\,|x| + \kappa, \quad i = 1, 2, \quad on \quad \mathbb{R}^n \times [0, \tau] \tag{4.24}$$

for some constants $0 < \alpha \leq \theta$ and $\kappa > 0$. Assume in addition that $\bar{u}_i$, $i = 1, 2$ both satisfy conditions (4.22). If $\bar{u}_1(\cdot, 0) \leq \bar{u}_2(\cdot, 0)$, then $\bar{u}_1(\cdot, t) \leq \bar{u}_2(\cdot, t)$ on $\mathbb{R}^n \times (0, \tau]$.

Proof of Claim 4.3 To simplify the notation for any function $\bar{u}$ on $\mathbb{R}^n \times [0, \tau]$, we set

$$\mathcal{F}(D^2\bar{u}, D\bar{u}) := -\sqrt{1+|D\bar{u}|^2}\left(\operatorname{div}\left(\frac{D\bar{u}}{\sqrt{1+|D\bar{u}|^2}}\right)\right)^{-1} = -\frac{\sqrt{1+|D\bar{u}|^2}}{H}. \quad (4.25)$$

Since both $\bar{u}_1$ and $\bar{u}_2$ satisfy

$$\frac{\partial}{\partial t}\bar{u} = \mathcal{F}(D^2\bar{u}, D\bar{u}) \quad (4.26)$$

setting $\bar{u}_s := s\,u_1 + (1-s)u_2$, we have

$$\begin{aligned}\frac{\partial}{\partial t}(\bar{u}_1 - \bar{u}_2) &= \mathcal{F}(D^2\bar{u}_1, D\bar{u}_1) - \mathcal{F}(D^2\bar{u}_2, D\bar{u}_2)\\ &= \int_0^1 \frac{d}{ds}\mathcal{F}(D^2\bar{u}_s, D\bar{u}_s)\,ds\\ &= \int_0^1 \frac{\partial \mathcal{F}}{\partial \sigma_{ij}}\,D_{ij}(\bar{u}_1 - \bar{u}_2) + \frac{\partial \mathcal{F}}{\partial p_i}\,D_i(\bar{u}_1 - \bar{u}_2)\,ds\\ &= a_{ij}\,D_{ij}(\bar{u}_1 - \bar{u}_2) + b_i\,D_i(\bar{u}_1 - \bar{u}_2)\end{aligned}$$

where

$$a_{ij} := \int_0^1 \frac{\partial \mathcal{F}}{\partial \sigma_{ij}}(D^2\bar{u}_s, D\bar{u}_s)\,ds, \quad b_i := \int_0^1 \frac{\partial \mathcal{F}}{\partial p_i}(D^2\bar{u}_s, D\bar{u}_s)\,ds.$$

The uniqueness assertion of our theorem will directly follow from Proposition 2.10 if we show that the coefficients a_{ij} and b_i satisfy conditions (2.10) and (2.11). To this end, we observe using (4.25) that

$$\frac{\partial \mathcal{F}}{\partial \sigma_{ij}} = \frac{\sqrt{1+|D\bar{u}|^2}}{H^2}\,\frac{\partial H}{\partial \sigma_{ij}}.$$

Since

$$H = \frac{1}{\sqrt{1+|D\bar{u}|^2}}\left(\delta_{ij} - \frac{D_i\bar{u}D_j\bar{u}}{1+|D\bar{u}|^2}\right)D_{ij}\bar{u}$$

we conclude that

$$\frac{\partial \mathcal{F}}{\partial \sigma_{ij}} = \frac{1}{H^2}\left(\delta_{ij} - \frac{D_i\bar{u}D_j\bar{u}}{1+|D\bar{u}|^2}\right).$$

Moreover, a direct calculation shows that

$$\left|\frac{\partial \mathcal{F}}{\partial p_i}\right| \le C\,|x|^2\,\frac{|D^2\bar{u}|}{(1+|D\bar{u}|^2)^{1/2}}.$$

Observe next that (4.22), (1.7) and the convexity of $\bar{u}$ imply the uniform bound $H \ge c/(1+|x|)$ on M_t, $t \in [0,\tau]$ for some $c > 0$ and we also have the uniform bound from above $H \le C/(1+|x|)$. It is easy to conclude then that a_{ij} satisfies

$$\lambda\,\xi^2(1+|x|^2) \le a_{ij}(x,t)\,\xi_i\xi_j \le \Lambda\,\xi^2\,(1+|x|^2)$$

for some positive constants λ, Λ. Also, convexity implies the bound $|D^2\bar{u}|/(1+|D\bar{u}|^2)^{1/2} \le C\,H$ which in turn gives

$$|b_i(x,t)| \le C\,|x|^2\,\frac{|D^2\bar{u}|}{(1+|D\bar{u}|^2)^{1/2}} \le C\,|x|^2\,H \le C\,(1+|x|).$$

We can then apply Proposition 2.10 to conclude that $\bar{u}_1 \le \bar{u}_2$, as claimed. □

Let us now conclude the proof of the Theorem 4.3. We have shown above that the limit $\bar{u}$ is a smooth convex solution of (1.4) which satisfies conditions (1.7) and (4.22), which in turn imply (4.14). The uniqueness assertion of the theorem readily follows by Claim 4.3 since (1.7) and (4.14) also imply (4.22). □

We will next compute the behavior at infinity of $v := \langle \hat{F}, \nu\rangle\, H$, where $\langle \hat{F}, \nu\rangle := -\langle F, \omega\rangle\, \langle \omega, \nu\rangle$. This will be crucial for the proof of Theorem 6.2 which is the main a'priori estimate in this work.

Proposition 4.4 *Under the assumptions of Theorem 4.3, the function $v := \langle \hat{F}, \nu\rangle\, H$ on M_t satisfies the asymptotic behavior*

$$\lim_{|F(z,t)|\to+\infty} v(F(z,t),t) = \gamma(t), \quad \textit{for all} \quad t \in (0, \tau] \tag{4.27}$$

with $\gamma(t)$ given by (3.7).

Proof We will use the graph representation $x_{n+1} = \bar{u}(x, t)$, $(x, t) \in \mathbb{R}^n \times [0, T)$, of the solution M_t of (1.1) for $t \in (0, \tau]$, as given by Theorem 4.3 and we will show that

$$\lim_{|x|\to+\infty} v(x, \bar{u}(x,t), t) = \gamma(t), \quad \text{uniformly on} \quad [\tau_0, \tau] \tag{4.28}$$

for all $\tau_0 \in (0, \tau/2)$, which readily yields (4.27). The function $\bar{u}$ satisfies the equation (1.4) and conditions (1.7) and (4.22). We may then consider H and $\langle \hat{F}, \nu\rangle$ as functions of $(x, t) \in \mathbb{R}^n \times [0, \tau]$. Throughout the proof c, C will denote positive constants which may change from line to line but always remain uniform in t, for $t \in [0, \tau]$.

Since $\langle \hat{F}, \nu\rangle = \bar{u}/(1 + |D\bar{u}|^2)^{1/2}$, by (1.7) we have $\langle \hat{F}, \nu\rangle \le C\,|x|$, on $[0, \tau]$. Combining this with (4.22) yields

$$c\,(1 + |x|)^{-1} \le H(x, t) \le C\,(1 + |x|)^{-1}, \quad \text{on} \quad \mathbb{R}^n \times [0, \tau]. \tag{4.29}$$

In addition, (1.7) and the convexity of $\bar{u}$ imply the gradient estimate

$$\sup_{\mathbb{R}^n\times[0,\tau]} |D\bar{u}(x, t)| \le C. \tag{4.30}$$

The function $\bar{u}$ satisfies (4.26), where the fully nonlinear operator $\mathcal{F}$ is given by (4.25). For this proposition we will use an a priori estimate for fully-nonlinear parabolic equations which was proven by G. Tian and X-J. Wang in [22] (Theorem 1.1 in [22]) to show that the mean curvature H of our surface remains sufficiently close to that of the cone $\zeta_1(x, t) = \alpha(t)\,|x|$, for $|x|$ sufficiently large, because of condition (1.7). To this end, let $\tilde{u}$ be the function defined by

$$\bar{u}(x, t) = \alpha(t)\,|x|\,\big(1 + \tilde{u}(x, t)\big). \tag{4.31}$$

We notice that the ellipticity of the operator $\mathcal{F}$ depends on $H(x, t) \sim |x|^{-1}$, for $|x|$ large. Hence, equation (4.26) becomes singular as $|x| \to \infty$. To avoid this issue it is more convenient to work in cylindrical coordinates for $|x| \ge \rho$, with ρ large. Let $\tilde{u}$ be the function defined by (4.31) and express $\tilde{u}$ in polar coordinates $\tilde{u}(r, \theta_1, \ldots, \theta_{n-1}, t)$, $r = |x|$. We introduce cylindrical $(s, \theta_1, \ldots, \theta_{n-1})$ with $s := \log r$ and define the function $\hat{u}(s, \psi, t)$ in terms of $\tilde{u}$, setting

$$\tilde{u}(r, \psi, t) = \hat{u}(s, \psi, t), \quad s := \log r, \quad \psi := (\theta_1, \ldots, \theta_{n-1}) \in S^{n-1}.$$

It follows by a direct calculation that $\hat{u}$ satisfies an equation of the form

$$\hat{u}_t = \hat{\mathcal{F}}(D^2\hat{u}, D\hat{u}, \hat{u})$$

where $D^2\hat{u} := D^2_{s,\psi}\hat{u}$ and $D\hat{u} := (\hat{u}_s, D_\psi \hat{u})$ denote first and second derivatives with respect to the cylindrical variable (s, ψ). The nonlinearity $\hat{\mathcal{F}}$ also depends on $\hat{\gamma}$.

Observe first that (1.7) implies the bound

$$0 \le \hat{u}(s,\psi,t) \le \frac{\kappa}{\alpha(t)}\, e^{-s} \le C\, e^{-s}, \quad \text{on} \quad \mathbb{R} \times S^{n-1} \times [0,\tau] \tag{4.32}$$

with $C = C(\tau)$. Also, a direct calculation shows that

$$\langle F, \nu \rangle = \frac{\alpha(t)\, e^s\, \hat{u}_s}{\sqrt{1 + \alpha^2(t)(1+\hat{u}+\hat{u}_s)^2 + |D_\psi \hat{u}|^2)}} \tag{4.33}$$

where by (4.30), $\sqrt{1 + \alpha^2(t)(1+\hat{u}+\hat{u}_s)^2 + |D_\psi \hat{u}|^2)} = 1 + |D\bar{u}|^2 \le C$. On the other hand, the condition $|\langle F, \nu\rangle| \le C_0$ on M_0 (see in (4.15)) and the maximum principle on the evolution of $\langle F, \nu \rangle$ given in Lemma 2.1 implies that $|\langle F, \nu\rangle| \le C$ on M_t for $t \in [0, \tau]$. Hence, (4.33) implies the bound

$$0 \le \hat{u}_s(\cdot, t) \le C\, e^{-s}, \quad \text{on} \quad \mathbb{R} \times S^{n-1} \times [0,\tau]. \tag{4.34}$$

For any $s_0 \ge 0$, and $\tau_0 \in (0, \tau)$, we set

$$Q^2_{s_0,\tau_0} := [s_0 - 2, s_0 + 2] \times S^{n-1} \times [\tau_0/2, \tau],$$
$$Q^1_{s_0,\tau_0} := [s_0 - 1, s_0 + 1] \times S^{n-1} \times [\tau_0, \tau]$$

so that $Q^1_{s_0,\tau_0} \subset Q^2_{s_0,\tau_0}$. It is not difficult to verify (using (4.32) and (4.34)) that the nonlinearity $\hat{\mathcal{F}}$ satisfies the assumptions of Theorem 1.1 in [22] on $Q^2_{s_0,\tau_0}$, for any $s_0 \ge 0$ with bounds that are independent from s_0 (as long as $s_0 \ge 0$). It follows that from Theorem 1.1 in [22] that for any $s_0 \ge 0$

$$\|D^2_{s,\psi}\hat{u}\|_{C^{\alpha,\alpha/2}(Q^1_{s_0,\tau_0})} \le C_{\tau_0} \quad \|\hat{u}\|_{L^\infty(Q^2_{s_0,\tau_0})}$$

for an exponent $\alpha > 0$. Here C_{τ_0} depends on the initial data, τ and τ_0, but is independent of s_0. This also implies the bound

$$\|e^{s_0}\hat{u}\|_{C^{\alpha,\alpha/2}(Q^1_{s_0,\tau_0})} \le C_{\tau_0} \quad \|e^{s_0}\hat{u}\|_{L^\infty(Q^2_{s_0,\tau_0})}. \tag{4.35}$$

Combining (4.35) with the bounds (4.32) and (4.34) gives $\|e^{s_0}\hat{u}\|_{C^{2+\alpha}_{\text{cyl}}(Q^1_{s_0,\tau_0})} \le C_{\tau_0} < \infty$. Since the constant C_{τ_1} is independent of s_0 (as long as $s_0 \ge 0$) we finally obtain the bound

$$\|e^{s}\hat{u}\|_{C^{2+\alpha}_{\text{cyl}}(\mathcal{C}\times[\tau_0,\tau])} \le C_{\tau_0} < \infty \tag{4.36}$$

where $\mathcal{C}$ denotes the half cylinder given by $\mathcal{C} := [0, +\infty) \times S^{n-1}$. This estimate shows that $\hat{u}$ is uniformly small in $C^{2+\alpha}_{\text{cyl}}$ norm as $s \to +\infty$. Expressing the mean curvature $v := \langle \hat{F}, \nu\rangle\, H$ in cylindrical coordinates we readily deduce that (4.28) holds, which also implies (4.27). □

5 L^p bounds on $1/H$

We will assume in this section that M_t is a solution of (1.1) on $[0,\tau]$ as given by Theorem 4.3 which satisfies condition (1.7) and that $\tau < T - 3\delta$, for some $\delta > 0$ and small, where $T = T(\alpha_0)$ denotes the extinction time of $\alpha(t)$ given in terms of α_0 by (3.8). Recall that $v := \langle \hat{F}, \nu\rangle\, H$, where $\langle \hat{F}, \nu\rangle = -\langle F, \omega\rangle\,\langle \omega, \nu\rangle$. Our goal is to establish a'priori bounds on suitably weighted L^p norms of $v^{-1}(\cdot, t)$ on M_t, for any $p \geq 1$, that depend on δ but are independent of τ. In the next section we will use these L^p bounds and a Moser iteration argument to bound the L^∞ norm of v^{-1} on M_t. This L^∞ bound constitutes the main a priori estimate on which the proof of the long time existence of the flow is based upon. We begin with the following straightforward observation which will be frequently used in the sequel.

Lemma 5.1 *Assume that M_t is an entire convex graph over $\mathbb{R}^n$ satisfying (1.7) with $\alpha(t) \leq \alpha_0$. Then, $\langle \hat{F}, \nu\rangle := -\langle F, \omega\rangle\,\langle \omega, \nu\rangle$, satisfies*

$$\langle \hat{F}, \nu\rangle \geq \lambda(n, \alpha_0)\,\sqrt{\gamma(t)}\,|F|. \tag{5.1}$$

Proof We begin by noticing that the lower bound in condition (1.7) implies the bound

$$\langle F, \omega\rangle \geq \frac{\alpha(t)}{\sqrt{1+\alpha^2(t)}}\,|F|.$$

In addition, it follows from the convexity of M_t and (1.7) that $-\langle \omega, \nu\rangle \geq \dfrac{1}{\sqrt{1+\alpha(t)^2}}$. Thus,

$$\langle \hat{F}, \nu\rangle := -\langle F, \omega\rangle\,\langle \omega, \nu\rangle \geq \frac{\alpha(t)}{1+\alpha(t)^2}\,|F| \geq \lambda(\alpha_0, n)\,\sqrt{\gamma(t)}\,|F|.$$

The last inequality follows from the definition of $\gamma(t) := (n-1)\alpha(t)^2/(1+\alpha(t)^2)$ and $\alpha(t) \leq \alpha_0$. ☐

We recall that $v^{-1} := (\langle \hat{F}, \nu\rangle\, H)^{-1}$ satisfies equation (2.9) and by (4.27) $\lim_{|F|\to\infty} v(z,t) = \gamma(t)$, where $\gamma(t)$ satisfies the ODE (3.6) with initial condition $\gamma(0) := \gamma_0 := (n-1)\alpha_0^2/(1+\alpha_0^2)$.

Let $\hat{\gamma}(t)$ denote the solution of the ODE (3.6) with initial condition $\hat{\gamma}(0) := \hat{\gamma}_0 := (n-1)\hat{\alpha}_0^2/(1+\hat{\alpha}_0^2)$ for some number $\hat{\alpha}_0$ that satisfies $0 < \hat{\alpha}_0 < \alpha_0$. Then, $\hat{\gamma}(t) < \gamma(t)$. Denote by $\hat{T} = \hat{T}(\hat{\alpha}_0)$ the vanishing time of $\hat{\gamma}$ that clearly satisfies $\hat{T} < T = T(\alpha_0)$. For a given number $\delta > 0$ (small) we will choose from now on $\hat{\alpha}_0$ such that the vanishing time $\hat{T}$ of $\hat{\gamma}$ satisfies

$$T - 2\delta \leq \hat{T} \leq T - \delta.$$

For that choice of $\hat{\gamma}$ we will have $\hat{\gamma}(t) < \gamma(t)$, for all $t < \hat{T}$. Hence, if we set

$$w(\cdot, t) := \hat{\gamma}(t)\, v(\cdot, t)^{-1} = \hat{\gamma}(t)\,(\langle \hat{F}, \nu\rangle\, H)^{-1} \tag{5.2}$$

then by (4.27) we have

$$\lim_{|F(z,t)|\to\infty} w(z,t) = \hat{\gamma}(t)\,\gamma(t)^{-1} < 1, \quad t \in [0,\tau], \quad \tau < \hat{T}. \tag{5.3}$$

We will next compute the evolution of w from the evolution of v^{-1}, shown in (2.9), and the ODE for $\hat{\gamma}$, shown in (3.6). Indeed, if we multiply (2.9) by $\hat{\gamma}(t)$ and use (3.6), we obtain

$$\frac{\partial}{\partial t} w - \nabla_i\left(\frac{1}{H^2}\nabla_i w\right) \leq -\frac{2}{H^2 w}|\nabla w|^2 + 2\,\langle \omega, \nu\rangle^2 \hat{\gamma}^{-1} w^2 + c_1 \hat{\gamma}^{-1}\, w \tag{5.4}$$

with

$$c_1 := \frac{2\hat{\gamma}(t)}{n-1} - 2 < 0, \quad \text{since} \quad \hat{\gamma} < \gamma(t) := \frac{(n-1)\,\alpha^2(t)}{1+\alpha^2(t)} \le (n-1).$$

Next, we set

$$\hat{w} := (w-1)_+.$$

Because of (5.3), for each given $t \in [0, \tau]$, $\tau < \hat{T} < T$, the function $\hat{w}(\cdot, t)$ satisfies

$$\hat{w}(\cdot, t) \equiv 0, \quad \text{for} \quad |F| \ge R(t) \tag{5.5}$$

for some $R(t) < \infty$. Notice that the main difficulty in our proof comes from the fact that we don't know that $R(t)$ is uniform in t.

Lemma 5.2 (Energy inequality) *Under the assumptions of Theorem 4.3, for any* $p \ge 0$, $q := (p+3)/2$ *the function* $\hat{w} := (w-1)_+$ *with* $w := \hat{\gamma}\,(\langle \hat{F}, \nu\rangle\, H)^{-1}$ *satisfies*

$$\begin{aligned} &\frac{d}{dt}\int_{M_t} \hat{w}^{p+1}\, d\mu + 2\lambda^2 \hat{\gamma}^{-2}\gamma \int_{M_t} |F|^2 |\nabla \hat{w}^q|^2\, d\mu \le \\ &\le C(p)\,\hat{\gamma}^{-1} \left(\int_{M_t} \hat{w}^{p+2}\, d\mu + \int_{M_t} \hat{w}^{p+1}\, d\mu + \int_{M_t} c_0(z,t)\,\hat{w}^p\, d\mu \right) \end{aligned} \tag{5.6}$$

with $\lambda, C(p)$ *positive constants that depend on the initial data (and* $C(p)$ *also on linearly* p*) and*

$$c_0(z,t) := 2\left(\langle \omega, \nu\rangle^2 + \frac{\hat{\gamma}}{n-1} - 1 \right)_+. \tag{5.7}$$

Proof If we first set $\bar{w} := w - 1$, we see from (5.4) that

$$\begin{aligned} \frac{\partial}{\partial t}\bar{w} - \nabla_i\Big(\frac{1}{H^2}\nabla_i \bar{w}\Big) &\le -\frac{2}{H^2 w}|\nabla \bar{w}|^2 + 2\hat{\gamma}^{-1}\langle \omega, \nu\rangle^2 (\bar{w}+1)^2 + \hat{\gamma}^{-1} c_1 (\bar{w}+1) \\ &\le -\frac{2}{H^2 w}|\nabla \bar{w}|^2 + 2\hat{\gamma}^{-1}\langle \omega, \nu\rangle^2 \bar{w}^2 + \hat{\gamma}^{-1}(4\langle \omega, \nu\rangle^2 + c_1)\bar{w} \\ &\quad + \hat{\gamma}^{-1}(2\langle \omega, \nu\rangle^2 + c_1). \end{aligned}$$

We next observe that since $\langle \omega, \nu\rangle^2 \le 1$ and $c_1 \le 0$, we have $4\langle \omega, \nu\rangle^2 + c_1 \le 4$, thus

$$\frac{\partial}{\partial t}\bar{w} - \nabla_i\Big(\frac{1}{H^2}\nabla_i \bar{w}\Big) \le -\frac{2}{H^2 w}|\nabla \bar{w}|^2 + 2\hat{\gamma}^{-1}\bar{w}^2 + 4\hat{\gamma}^{-1}\bar{w}_+ + \hat{\gamma}^{-1} c_0(z,t)$$

with $c_0(z,t) := (2\langle \omega, \nu\rangle^2 + c_1)_+$, $c_1 = 2\hat{\gamma}/(n-1) - 2$, hence given by (5.7). Let $\hat{w} := (w-1)_+ = \bar{w}_+$. If we multiply the last inequality by $\hat{w}^p = \bar{w}_+^p$, for some number $p \ge 0$, and integrate by parts (recalling that by (5.5) $\hat{w}(\cdot, t)$ has compact support in M_t), we obtain

$$\begin{aligned} \frac{1}{p+1}\frac{d}{dt}\int_{M_t} \hat{w}^{p+1}\, d\mu \le &- p\int_{M_t} \frac{1}{H^2}\hat{w}^{p-1}|\nabla \hat{w}|^2\, d\mu - 2\int_{M_t} \frac{1}{H^2 w}\hat{w}^p |\nabla \hat{w}|^2\, d\mu \\ &+ 2\hat{\gamma}^{-1}\int_{M_t} \hat{w}^{p+2}\, d\mu + (4\hat{\gamma}^{-1}+1)\int_{M_t} \hat{w}^{p+1}\, d\mu \\ &+ \hat{\gamma}^{-1}\int_{M_t} c_0(z,t)\,\hat{w}^p\, d\mu. \end{aligned} \tag{5.8}$$

Here we also used that $\partial(d\mu)/\partial t = d\mu$. Also, for $p = 0$ we use the inequality

$$\int_{M_t} \nabla_i\big(\frac{1}{H^2}\nabla_i \bar{w}\big)\chi_{\{\bar{w}>0\}}\, d\mu = \int_{M_t\cap\partial\{\bar{w}>0\}} \frac{1}{H^2}\frac{\partial \bar{w}}{\partial \nu}\, d\sigma \le 0.$$

We next remark that from the definition of $w := (\langle \hat{F}, \nu\rangle H)^{-1}\hat{\gamma}$, we may express $H^{-1} = \hat{\gamma}^{-1} w \langle \hat{F}, \nu\rangle$. Also, $w\,\chi_{\{w>1\}} \ge (w-1)_+ = \hat{w}$. Hence, we may combine the two gradient terms on the right hand side of (5.8) to conclude

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_{M_t} \hat{w}^{p+1}\, d\mu \le &-(p+2)\,\hat{\gamma}^{-2}\int_{M_t} \langle \hat{F}, \nu\rangle^2 \hat{w}^{p+1}|\nabla \hat{w}|^2\, d\mu\\ &+ 2\hat{\gamma}^{-1}\int_{M_t} \hat{w}^{p+2}\, d\mu + (4\hat{\gamma}^{-1}+1)\int_{M_t} \hat{w}^{p+1}\, d\mu\\ &+ \hat{\gamma}^{-1}\int_{M_t} c_0(z,t)\,\hat{w}^p\, d\mu.\end{aligned}$$

Writing

$$\hat{w}^{p+1}|\nabla \hat{w}|^2 = \frac{4}{(p+3)^2}\,|\nabla w^{\frac{p+3}{2}}|^2$$

and using (5.1) we obtain

$$\begin{aligned}\frac{d}{dt}\int_{M_t} \hat{w}^{p+1}\, d\mu \le &-2\,\lambda^2\,\hat{\gamma}^{-2}\gamma\int_{M_t} |F|^2|\nabla \hat{w}^{\frac{p+3}{2}}|^2\, d\mu + c_2(p)\,\hat{\gamma}^{-1}\int_{M_t} \hat{w}^{p+2}\, d\mu\\ &+ c_1(p)\,\hat{\gamma}^{-1}\int_{M_t} \hat{w}^{p+1}\, d\mu + \hat{\gamma}^{-1}(p+1)\int_{M_t} c_0(z,t)\,\hat{w}^p\, d\mu\end{aligned}$$

for some new positive constants $c_i(p)$ depending (linearly) on p and the initial data. This readily gives (5.6) by setting $q := (p+3)/2$. □

We will next prove the following variant of Hardy's inequality adapted to our situation (see in [2] and [19] for standard Hardy inequalities on complete non-compact manifolds).

Proposition 5.3 (Hardy inequality) *Let M_t be a solution of (1.1) as in Theorem 4.3. Then, there exists a constant $C_n > 0$ depending only on dimension n such that any function g that is compactly supported on M_t, we have*

$$\int_{M_t} g^2\, d\mu \le C(n)\left(\int_{M_t} |\nabla g|^2|F|^2\, d\mu + \int_{M_t} g^2|H|\,|F|\, d\mu\right). \tag{5.9}$$

Proof To simplify the notation, set $\rho(F) := |F|$ and recall that from our assumptions on M_t we have $\rho > 0$ everywhere. We begin by computing $\Delta\rho$. We have

$$\Delta\rho = \nabla_i\nabla_i(\langle F, F\rangle^{1/2}) = \nabla_i(\langle F, F\rangle^{-1/2}\langle \mathbf{e_i}, F\rangle) = \frac{n}{|F|} - \frac{|F^T|^2}{|F|^3} + H\,\frac{\langle F, \nu\rangle}{|F|}$$

from which we conclude the lower bound

$$\Delta\rho \ge \frac{n-1}{\rho} - H. \tag{5.10}$$

Let $g := \rho^\gamma\psi$ for some $\gamma < 0$ to be chosen momentarily. We then have

$$|\nabla g|^2 = |\nabla(\rho^\gamma\psi)|^2 = |\gamma\rho^{\gamma-1}\psi\nabla\rho + \rho^\gamma\nabla\psi|^2 \ge 2\gamma\rho^{2\gamma-1}\psi\nabla\rho\cdot\nabla\psi.$$

We next observe that it is convenient to choose $\gamma = -1/2$ which gives

$$|\nabla g|^2 \geq -\psi \rho^{-2} \nabla \rho \cdot \nabla \psi$$

or equivalently (using that $\psi^2 = g^2 \rho$)

$$|\nabla g|^2 \rho^2 \geq -\frac{1}{2}\, \nabla \rho \cdot \nabla \psi^2 = -\frac{1}{2}\, \nabla \rho \cdot \nabla (g^2 \rho).$$

After integrating by parts we obtain

$$\int_{M_t} |\nabla g|^2 \rho^2 \, d\mu \geq \frac{1}{2} \int_{M_t} g^2 \rho \, \Delta \rho \, d\mu.$$

Combining this with inequality (5.10) yields

$$\int_{M_t} |\nabla g|^2 \rho^2 \, d\mu \geq \frac{n-1}{2} \int_{M_t} g^2 d\mu - \frac{1}{2} \int_{M_t} g^2 \rho \, H d\mu \tag{5.11}$$

from which (5.9) readily follows. □

We will now combine (5.6) with the above Hardy inequality to prove the following L^{p+1} bound on $\hat{w}$ in terms of its initial data.

Theorem 5.4 (L^{p+1} estimate on $\hat{w}$) *Assume that M_t is a solution to* (1.1) *as in Theorem* 4.3 *defined for $t \in (0, \tau]$, and assume that $\tau < T - 3\delta$ with T given by* (3.8) *and $\delta > 0$. Then, for any $p \geq 0$ there exists a constant $C = C(p)$ depending on p, T, δ and also on the constants κ, α_0 such that*

$$\sup_{t \in [0,\tau]} \int_{M_t} \hat{w}^{p+1}(\cdot, t) \, d\mu \leq C(p, \delta, T) \left(1 + \int_{M_0} \hat{w}^{p+1} \, d\mu \right). \tag{5.12}$$

Proof We recall that $\hat{\gamma}(t)$ is a solution of the ODE (3.6) with initial value $0 < \hat{\gamma}(0) < \gamma(0)$ so that that its vanishing time $\hat{T}$ satisfies $T - 2\delta < \hat{T} < T - \delta$, for the given small number $\delta > 0$. For any number $p > 0$, set $q := (p+3)/2$. Applying (5.11) for $g = \hat{w}^q$ gives

$$\int_{M_t} |\nabla \hat{w}^q|^2 |F|^2 \, d\mu \geq \frac{n-1}{2} \int_{M_t} \hat{w}^{2q} d\mu - \frac{1}{2} \int_{M_t} \hat{w}^{2q} \, |F| H d\mu. \tag{5.13}$$

We will next estimate $|F| H \chi_{\{\hat{w}>0\}}$ in terms of $\hat{w}$. Recall that by definition $w(\cdot, t) := \hat{\gamma}(t) \, (\langle \hat{F}, \nu \rangle \, H)^{-1}$ and that from (5.1) we have $|F| \leq \lambda^{-1} \gamma^{-1/2} \langle \hat{F}, \nu \rangle$. Thus,

$$|F| \, H \leq \lambda^{-1} \gamma^{-1/2} \, \langle \hat{F}, \nu \rangle H = \lambda^{-1} \, \gamma^{-1/2} \, \hat{\gamma} \, w^{-1}.$$

Since $w \, \chi_{\{w>1\}} \leq (w-1) \, \chi_{\{w>1\}} = (w-1)_+ = \hat{w}$, we have

$$|F| \, H \, \chi_{\{w>1\}} \leq \lambda^{-1} \, \gamma^{-1/2} \, \hat{\gamma} \, \hat{w}^{-1}.$$

Thus (5.13) yields

$$\int_{M_t} |\nabla \hat{w}^q|^2 |F|^2 \, d\mu \geq \frac{n-1}{2} \int_{M_t} \hat{w}^{2q} d\mu - \frac{1}{2} \lambda^{-1} \, \gamma^{-1/2} \, \hat{\gamma} \int_{M_t} \hat{w}^{2q-1} d\mu. \tag{5.14}$$

Recall that $q = (p+3)/2$, so that $2q - 1 = p + 2$. If we now combine this last estimate with (5.6) and also use that $\hat{\gamma}^{-1} \gamma > 1$ and $n - 1 \geq 1$, we obtain the differential inequality

$$\begin{aligned} \frac{d}{dt} \int_{M_t} \hat{w}^{p+1} \, d\mu \leq & - \lambda^2 \, \hat{\gamma}^{-1} \int_{M_t} \hat{w}^{p+3} d\mu + \\ & + C(p) \, \hat{\gamma}^{-1} \left(\int_{M_t} \hat{w}^{p+2} \, d\mu + \int_{M_t} \hat{w}^{p+1} \, d\mu + \int_{M_t} c_0(z, t) \, \hat{w}^p d\mu \right) \end{aligned}$$

for constant $C(p)$ depending on p and also on κ, α_0 and with $c_0(z,t)$ given by (5.7). We may apply the interpolation inequality

$$\int_{M_t} |g|^{p+2}\, d\mu = \int_{M_t} |g|^{\frac{p+1}{2}}\, |g|^{\frac{p+3}{2}}\, d\mu \le \frac{\lambda^2}{2C(p)} \int_{M_t} |g|^{p+3}\, d\mu + \frac{C(p)}{2\lambda^2} \int_{M_t} |g|^{p+1}\, d\mu$$

to $g := \hat{w}$ to conclude that

$$\frac{d}{dt} \int_{M_t} \hat{w}^{p+1}\, d\mu \le C(p)\, \hat{\gamma}^{-1} \left(\int_{M_t} \hat{w}^{p+1}\, d\mu + \int_{M_t} c_0(z,t)\, \hat{w}^p d\mu \right) \tag{5.15}$$

for a new constant $C(p)$ that depends on p and also on our initial data α_0, κ and dimension n.

Because M_t is non-compact, in order to estimate the last term in (5.15) in terms of $\int_{M_t} \hat{w}^{p+1}\, d\mu$ we will need to look more carefully into the coefficient $c_0(z,t)$. We claim the following.

Claim 5.1 *Assume that $\hat{\gamma}(t)$ is chosen so that its vanishing time $\hat{T}$ satisfies $T-2\delta < \hat{T} \le T-\delta$ for the given small number $\delta > 0$. Then, there exists a number $R_\delta \ge 1$ (depending on δ) such that*

$$c_0(\cdot,t) \equiv 0 \quad on \quad M_t \cap \{|F| \ge R_\delta\}, \quad 0 \le t < \hat{T}. \tag{5.16}$$

Proof of claim 5.1 Recall that $c_0(\cdot,t) := 2\left(\langle \omega, \nu\rangle^2 + \hat{\gamma}(t)/(n-1) - 1\right)_+$ and that $\hat{\gamma}(t) < \gamma(t)$ for all $t < \hat{T}$. Since by definition $\gamma(t) = (n-1)\,\alpha(t)^2/(1+\alpha(t)^2)$ we may also express $\hat{\gamma}(t) = (n-1)\,\hat{\alpha}(t)^2/(1+\hat{\alpha}(t)^2)$ for some function of time function $\hat{\alpha}(t)$. It follows from the condition $T - 2\delta < \hat{T} \le T - \delta$ that

$$0 < \mu_1(\delta) \le \alpha(t) - \hat{\alpha}(t) \le \mu_2(\delta), \quad \forall\, t < \hat{T}$$

for some positive constants $\mu_1(\delta)$, $\mu_2(\delta)$ that tend to zero as $\delta \to 0$. Consider the cones defined by the graphs $x_{n+1} = \alpha(t)\,|x|$ and $x_{n+1} = \hat{\alpha}(t)\,|x| + \kappa$ over $x \in \mathbb{R}^n$. These cones intersect at $|x| = r(t) := \kappa/(\alpha(t) - \hat{\alpha}(t))$. Let $R(t) := \sqrt{1+\alpha^2(t)}\, r(t)$. It follows from (1.7) and a simple geometric consideration that uses the convexity of M_t that

$$c_0(\cdot,t) \equiv 0 \quad \text{on} \quad M_t \cap \{|F| \ge R(t)\}.$$

Since,

$$R(t) := \kappa\, \frac{\sqrt{1+\alpha^2(t)}}{\alpha(t) - \hat{\alpha}(t)} \le \kappa\, \frac{\sqrt{1+\alpha_0^2}}{\mu_1(\delta)} := R_\delta, \quad 0 \le t < \hat{T}$$

the claim follows. □

Using the above claim and the bound $c_0(z,t) \le 2$, we may now estimate the term $\int_{M_t} c_0(z,t)\, \hat{w}^p d\mu$ in (5.15) as

$$\begin{aligned} \int_{M_t} c_0(z,t)\, \hat{w}^p d\mu &\le 2 \int_{M_t \cap \{c_0 > 0\}} \hat{w}^p\, d\mu \le C(R_\delta, p) \left(\int_{M_t} \hat{w}^{p+1}\, d\mu \right)^{p/(p+1)} \\ &\le C(R_\delta, p) \left(\int_{M_t} \hat{w}^{p+1}\, d\mu + 1 \right). \end{aligned}$$

Combining the last estimate with (5.15), we obtain

$$\frac{d}{dt}\int_{M_t} \hat{w}^{p+1}\, d\mu \le C(p,\delta)\,\hat{\gamma}(t)^{-1}\left(1+\int_{M_t} \hat{w}^{p+1}\, d\mu\right). \tag{5.17}$$

Since we have assumed that $\hat{\gamma}(t)$ vanishes at $\hat{T}$ and $T-2\delta < \hat{T} < T-\delta$, it follows that $\hat{\gamma}(t)^{-1} \le C(\delta)$ for all $t < T-3\delta$. We conclude from (5.17) that

$$\frac{d}{dt}\int_{M_t} \hat{w}^{p+1}\, d\mu \le C(p,\delta)\left(1+\int_{M_t} \hat{w}^{p+1}\, d\mu\right)$$

for another constant $C(p,\delta)$. After integrating this inequality in time t we conclude that if $\tau < T-3\delta$, (5.12) holds. □

6 L^∞ estimates on $1/H$

We will assume throughout this section that M_t is an entire graph convex solution of (1.1) on $[0,\tau]$ as in Theorem 4.3 and that $\tau < T-3\delta$, for some $\delta > 0$, where T is the number given by (3.8). We will establish a local L^∞ bound on $(\langle \hat{F},\nu\rangle\, H)^{-1}$ which holds on M_t for all $t \in [0,\tau]$ and depends only on the initial data, on T and on δ. This bound constitutes the main step in the proof of the long time existence result Theorem 1.1. It states as follows.

Theorem 6.1 (L^∞ bound on w in terms of its spatial averages) *Assume that M_t is a solution to (1.1) as in Theorem 4.3 defined for $t \in (0,\tau]$, and assume that $\tau < T-3\delta$ with T given by (3.8) and $\delta > 0$. There exist absolute constants $\mu > 0$ and $\sigma > 0$ and a constant C that depends on α_0, κ, on δ, and the initial bound $\sup_{M_0}\langle F,\omega\rangle\, H$, for which $w := \hat{\gamma}(t)\,(\langle \hat{F},\nu\rangle\, H)^{-1}$ satisfies the bound*

$$\sup_{t\in(t_0,\tau]} \|w\|_{L^\infty(M_t)} \le C\, t_0^{-\mu}\left(1+\sup_{t\in(t_0/4,\tau]}\sup_{R\ge 1} R^{-n}\int_{M_t\cap\{|F|\le R\}} w(\cdot,t)\, d\mu\right)^{\sigma}. \tag{6.1}$$

for any $t_0 \in (0,\tau/2]$.

For the proof of this theorem we will use a parabolic variant of Moser's iteration on the differential inequality (5.4) that is satisfied by $w := \hat{\gamma}(t)\,(\langle \hat{F},\nu\rangle\, H)^{-1}$. Such technique was first introduced in the nonlinear parabolic context by Dahlberg and Kenig in [5]. In fact we will closely follow the proof in [5] (see also in the proof of Lemma 1.2.6 in [7]). For the inverse mean curvature flow in the compact setting, a similar bound was shown in [18] via a variant of the Stampacchia iteration method.

The estimate (6.1) will be shown in two steps Propositions 6.2 and 6.5 below. Let us begin by introducing some notation. For any given number $t_0 \in (0,\tau]$ we set

$$S_{t_0} := \{(P,t) \in \mathbb{R}^{n+1}\times(0,t_0] : \quad P \in M_t, \quad t \in (0,t_0]\} = \cup_{t\in(0,t_0]} M_t \times \{t\}.$$

Also, for any given numbers $\rho_0 > 1$, $t_0 \in (0,\tau]$ and $r \in (0,1)$ we consider the cylinders in $\mathbb{R}^{n+1}\times(0,+\infty)$ given by

$$Q^r_{\rho_0,t_0} := \{(\mathbf{x},t) \in \mathbb{R}^{n+1}\times(0,+\infty) : \quad \rho_0(1-r) < |\mathbf{x}| < \rho_0(1+r), \quad (1-r)\,t_0 < t \le t_0\}.$$

In particular, we set

$$Q_{\rho_0,t_0} := Q^{1/4}_{\rho_0,t_0}, \quad Q^*_{\rho_0,t_0} := Q^{1/2}_{\rho_0,t_0}, \quad Q^{**}_{\rho_0,t_0} := Q^{3/4}_{\rho_0,t_0}.$$

Notice that since in equation (1.4) one cannot scale the time t, it is not necessary to use the standard parabolic scaling in the above cylinders, one can just use the same scale in $\mathbf{x}$ and t.

Proposition 6.2 *Assume that M_t is a solution to (1.1) as in Theorem 4.3 defined for $t \in (0, \tau]$, and assume that $\tau < T - 3\delta$ with T given by (3.8) and $\delta > 0$. There exist absolute constants $\mu > 0$ and $\sigma > 0$ and a constant C that depends on α_0, κ, on δ, and the initial bound $\sup_{M_0} \langle F, \omega \rangle H$, for which $w := \hat{\gamma}(t) (\langle \hat{F}, \nu \rangle H)^{-1}$ satisfies the bound*

$$\|w\|_{L^\infty(Q_{\rho_0,t_0} \cap S_{t_0})} \leq C\, t_0{}^{-\mu} \left(1 + \sup_{t \in (t_0/4, t_0]} \rho_0^{-n} \int_{M_t \cap Q^{**}_{\rho_0,t_0}} w(\cdot, t)\, d\mu \right)^{\sigma} \tag{6.2}$$

which holds for any $\rho_0 > 2$ such that $Q_{\rho_0,t_0} \cap S_{t_0}$ is not empty.

Remark 6.1 For the remaining of this section we will call *uniform constants* the constants that may depend on the number $\delta > 0$, the constants α_0, κ, but that are *independent* of ρ_0 and t_0.

Since w satisfies the differential inequality (5.4), if we set $\bar{w} := \max(w, 1)$ it follows that $\bar{w}$ satisfies the same differential inequality and since $\bar{w} \leq \bar{w}^2$ we have

$$\frac{\partial}{\partial t} \bar{w} - \nabla_i \left(\frac{1}{H^2} \nabla_i \bar{w} \right) \leq -\frac{2}{H^2 \bar{w}} |\nabla \bar{w}|^2 + c_2\, \hat{\gamma}^{-1} \bar{w}^2. \tag{6.3}$$

for some new constant $c_2 > 0$.

Remark 6.2 In the following we shall not distinguish between the image $F(z, t)$ of a point $z \in M$ and its coordinate vector in $\mathbb{R}^{n+1}$.

For the given numbers $\rho_0 > 1$ and $t_0 \in (0, \tau]$ and any numbers $1/4 < r < \bar{r} < 1/2$, we consider a radial cutoff function $\psi = \psi(\rho, t)$, $\rho = |\mathbf{x}|$, $\mathbf{x} \in \mathbb{R}^{n+1}$ with $\psi \in C_c^\infty(Q^{\bar{r}}_{\rho_0,t_0})$ satisfying

$$\psi \equiv 1 \quad \text{on} \quad Q^r_{\rho_0,t_0}, \quad 0 \leq \psi \leq 1, \quad \rho_0\, |\psi_\rho| + t_0\, |\psi_t| \leq C\, (\bar{r} - r)^{-1}. \tag{6.4}$$

We extend ψ to be equal to zero outside $Q^{\bar{r}}_{\rho_0,t_0}$ and define the function η on S_{t_0} by

$$\eta(F, t) := \psi(|F|, t), \quad F \in M_t. \tag{6.5}$$

Lemma 6.3 *Under the assumptions of Theorem 6.2, for any $p \geq 1$ and $\theta := (p + 2)/2$, we have*

$$\begin{aligned} &\sup_{t \in (0, t_0]} \int_{M_t} (\eta^2 \bar{w}^p)(\cdot, t)\, d\mu + \int_0^{t_0} \int_{M_t} \rho_0^2\, |\nabla(\eta \bar{w}^\theta)|^2 d\mu\, dt \\ &\leq C\, t_0^{-1}\, (\bar{r} - r)^{-2} \int_0^{t_0} \int_{M_t \cap \{\eta > 0\}} \bar{w}^{2\theta} d\mu\, dt \end{aligned} \tag{6.6}$$

where $\eta \in C_c(S_{t_0})$ is the cutoff function defined by (6.5)

Proof We begin by observing that the cutoff function defined by (6.5) satisfies

$$|\nabla \eta| \leq C\, \rho_0^{-1}\, (\bar{r} - r)^{-1} \quad \text{and} \quad |\partial_t \eta| \leq C\, (\bar{r} - r)^{-1} \left(\rho_0^{-2}\, H^{-1} |\langle F, \nu \rangle| + t_0^{-1} \right) \tag{6.7}$$

where we have denoted by $\nabla \eta$ the gradient of η on M. The first inequality follows from (6.4) and the calculation

$$|\nabla_i \eta| = \frac{1}{2}\, |\psi_\rho|\, |F|^{-1}\, |\nabla_i \langle F, F \rangle| \leq C\, \rho_0^{-1}\, (\bar{r} - r)^{-1}$$

while the second inequality follows from (6.4) and the calculation

$$|\partial_t \eta| = |\psi_\rho|\,|F|^{-1}\,|\langle F, F_t\rangle| + |\psi_t| \le C\,(\bar r - r)^{-1}\left(\rho_0^{-2}\, H^{-1}|\langle F, \nu\rangle| + t_0^{-1}\right).$$

Using equation (6.3) and that $\partial(d\mu)/\partial t = d\mu$, $\bar w \ge 1$ and $H^{-1} = \hat\gamma^{-1}\langle \hat F, \nu\rangle\, w$, we have

$$\begin{aligned}
\frac{d}{dt}\int \bar w^p \eta^2\, d\mu &= p\int \bar w^{p-1}\,\eta^2\,\bar w_t\, d\mu + 2\int \bar w^p \eta\,\eta_t\, d\mu + \int \bar w^p\,\eta^2\, d\mu\\
&\le -p(p-1)\int \frac{1}{H^2}\,\bar w^{p-2}\eta^2\,|\nabla \bar w|^2\, d\mu - 2p\int \frac{1}{H^2}\bar w^{p-2}\eta^2|\nabla \bar w|^2\, d\mu\\
&\quad - 2p\int \frac{1}{H^2}\,\bar w^{p-1}\eta\nabla_i \bar w\nabla_i\eta\, d\mu + 2\int \bar w^p\eta\,|\eta_t|\, d\mu\\
&\quad + c_2\, p\,\hat\gamma^{-1}\int \bar w^{p+1}\,\eta^2\, d\mu + \int \bar w^p\,\eta^2\, d\mu\\
&\le -p(p+1)\hat\gamma^{-2}\int \langle \hat F, \nu\rangle^2\,\bar w^p\eta^2\,|\nabla\bar w|^2\, d\mu\\
&\quad - 2p\,\hat\gamma^{-2}\int \langle \hat F, \nu\rangle^2\,\bar w^{p+1}\eta\nabla_i\bar w\nabla_i\eta\, d\mu\\
&\quad + 2\int \bar w^p\eta\,|\eta_t|\, d\mu + \bar c_2\int \bar w^{p+1}\,\eta^2\, d\mu.
\end{aligned}$$

with $\bar c_2 := c_2\, p\,\hat\gamma^{-1} + 1$. Let $\theta = (p+2)/2$. Writing

$$\bar w^p\,|\nabla \bar w|^2 = \theta^{-2}\,|\nabla \bar w^\theta|^2 \quad\text{and}\quad \bar w^{p+1}\nabla\bar w = \theta^{-1}\bar w^\theta\,\nabla\bar w^\theta$$

we obtain

$$\begin{aligned}
\frac{d}{dt}\int \bar w^p\eta^2\, d\mu &\le -p(p+1)\theta^{-2}\hat\gamma^{-2}\int \langle \hat F, \nu\rangle^2\eta^2|\nabla\bar w^\theta|^2\, d\mu\\
&\quad - 2p\,\theta^{-1}\hat\gamma^{-2}\int \langle \hat F, \nu\rangle^2\,\eta\,\bar w^\theta|\nabla\eta|\,|\nabla\bar w^\theta|\, d\mu\\
&\quad + 2\int \bar w^p\eta\,|\eta_t|\, d\mu + \bar c_2\int \bar w^{p+1}\,\eta^2\, d\mu.
\end{aligned}$$

We estimate

$$\begin{aligned}
\int \langle \hat F, \nu\rangle^2\bar w^\theta\eta|\nabla\eta|\,|\nabla\bar w^\theta|\, d\mu &\le \frac{(p+1)}{4\theta}\int \langle \hat F, \nu\rangle^2\eta^2|\nabla\bar w^\theta|^2\, d\mu\\
&\quad + \frac{\theta}{(p+1)}\int \bar w^{2\theta}\langle \hat F, \nu\rangle^2|\nabla\eta|^2\, d\mu
\end{aligned}$$

to conclude

$$\begin{aligned}
&\frac{d}{dt}\int \bar w^p\eta^2\, d\mu + \frac{1}{2}\,p\,(p+1)\theta^{-2}\hat\gamma^{-2}\int \langle \hat F, \nu\rangle^2\eta^2\,|\nabla\bar w^\theta|^2\, d\mu\\
&\le C\left(\hat\gamma^{-2}\frac{p}{p+1}\int \langle \hat F, \nu\rangle^2\bar w^{2\theta}|\nabla\eta|^2\, d\mu + \hat\gamma^{-1}\int \bar w^{p+1}\eta^2\, d\mu + \int \bar w^p\eta\,|\eta_t|\, d\mu\right)
\end{aligned} \tag{6.8}$$

for a uniform constant C that is in particular independent of p. Also, by (6.7) we have

$$\int \bar w^p\eta\,|\eta_t|\, d\mu \le C\,(\bar r - r)^{-1}\left(t_0^{-1}\int \bar w^p\eta\, d\mu + \rho_0^{-2}\int \bar w^p\eta\; H^{-1}|\langle F, \nu\rangle|\, d\mu\right).$$

Using that

$$H^{-1} = \hat{\gamma}^{-1} \langle \hat{F}, \nu \rangle \, w \leq \hat{\gamma}^{-1} \langle \hat{F}, \nu \rangle \, \bar{w}$$

and

$$\langle \hat{F}, \nu \rangle \, |\langle F, \nu \rangle| \leq C \, |F|^2 \leq C \, \rho_0^2$$

on the support of η and also that $\bar{w} \geq 1$, we obtain the bound

$$\begin{aligned} \int \bar{w}^p \eta \, |\eta_t| \, d\mu &\leq C \, (\bar{r} - r)^{-1} \left(t_0^{-1} \int \bar{w}^p \, \eta \, d\mu + \int \hat{\gamma}^{-1} \, \bar{w}^{p+1} \, \eta \, d\mu \right) \\ &\leq C \, t_0^{-1} (\bar{r} - r)^{-1} \int \hat{\gamma}^{-1} \, \bar{w}^{p+1} \, \eta \, d\mu. \end{aligned} \tag{6.9}$$

Since $p \geq 1$, we have

$$\frac{4}{9} \leq p(p+1)\theta^{-2} = \frac{4p(p+1)}{(p+2)^2} \leq 4.$$

Integrating (6.8) in time on $(0, t]$ for all $t \in (0, t_0]$ and using (6.7) and (6.9) yields

$$\begin{aligned} &\sup_{t \in (0, t_0]} \int_{M_t} \bar{w}^p \, \eta^2 \, d\mu \, dt + \hat{\gamma}^{-2} \int_0^{t_0} \int_{M_t} \langle \hat{F}, \nu \rangle^2 \eta^2 |\nabla \bar{w}^\theta|^2 \, d\mu \, dt \\ &\leq C (\bar{r} - r)^{-2} \left(\rho_0^{-2} \int_0^{t_0} \int_{M_t \cap \{\eta > 0\}} \hat{\gamma}^{-2} \langle \hat{F}, \nu \rangle^2 \bar{w}^{2\theta} d\mu \, dt \right. \\ &\left. + t_0^{-1} \int_0^{t_0} \int_{M_t \cap \{\eta > 0\}} \hat{\gamma}^{-1} \bar{w}^{p+1} d\mu \, dt \right). \end{aligned}$$

Using the bounds $\bar{w} \geq 1$, (5.1) and $c \, \rho_0^2 \leq \langle \hat{F}, \nu \rangle^2 \leq |F|^2 \leq C \rho_0^2$ and $\hat{\gamma}^{-1} \leq C_\delta$, for $t_0 < T - 3\delta$, we obtain

$$\begin{aligned} &\sup_{t \in (0, t_0]} \int_{M_t} (\eta^2 \bar{w}^p)(\cdot, t) \, d\mu + \int_0^{t_0} \int_{M_t} \rho_0^2 \eta^2 \, |\nabla \bar{w}^\theta|^2 d\mu \, dt \\ &\leq C \, t_0^{-1} \, (\bar{r} - r)^{-2} \int_0^{t_0} \int_{M_t \cap \{\eta > 0\}} \bar{w}^{2\theta} d\mu \, dt. \end{aligned}$$

Finally, using the estimate

$$\begin{aligned} \int_0^{t_0} \int_{M_t} \rho_0^2 \, \eta^2 |\nabla \bar{w}^\theta|^2 d\mu \, dt &\geq \frac{1}{2} \int_0^{t_0} \int_{M_t} \rho_0^2 \, |\nabla (\eta \bar{w}^\theta)|^2 d\mu \, dt \\ &\quad - 4 \int_0^{t_0} \int_{M_t} \rho_0^2 \, |\nabla \eta|^2 \bar{w}^{2\theta} d\mu \, dt \\ &\geq \frac{1}{2} \int_0^{t_0} \int_{M_t} \rho_0^2 \, |\nabla (\eta \bar{w}^\theta)|^2 d\mu \, dt \\ &\quad - C \, (\bar{r} - r)^{-2} \int_0^{t_0} \int_{M_t \cap \{\eta > 0\}} \bar{w}^{2\theta} d\mu \, dt \end{aligned}$$

we conclude (6.6). □

We will prove next a variant of the following Sobolev inequality which holds on any complete manifold N^n, with $n \geq 3$

$$\left(\int_{N^n} |f|^{\frac{2n}{n-2}} \, d\mu \right)^{\frac{n-2}{n}} \leq C(n) \int_{N^n} |\nabla f|^2 + H^2 f^2 \, d\mu \tag{6.10}$$

and for any $f \in C_c^1(N^n)$. When $n = 2$ we will use instead the inequality

$$\left(\int_{N^2} |f|^4 \, d\mu \right)^{1/2} \leq C \, |N^2 \cap \operatorname{supp} f|^{1/2} \int_{N^2} |\nabla f|^2 + H^2 f^2 \, d\mu \tag{6.11}$$

which holds for any $f \in C_c^1(N^2)$.

Lemma 6.4 *We set $q^* := q/(q-1)$ with $q = n/2$ if $n \geq 3$ and $q = 2$ if $n = 2$. Then, for any $k \in (0, q^*)$ and $h \in C_c^{1,0}(S_{t_0})$ we have*

$$\begin{aligned} \int_0^{t_0} \int_{M_t} h^{2k} \, d\mu \, dt \leq & C \left\{ \int_0^{t_0} |M_t \cap \operatorname{supp} h|^\lambda \int_{M_t} |\nabla h|^2 + H^2 h^2 \, d\mu \, dt \right. \\ & \left. \cdot \sup_{t \in (0, t_0]} \left(\int_{M_t} h^{2(k-1)q}(\cdot, t) \, d\mu \right)^{1/q} \right\}. \end{aligned} \tag{6.12}$$

with $\lambda = 0$ if $n \geq 3$ and $\lambda = 1/2$ if $n = 2$.

Proof Since $h(\cdot, t) \in C_c^1(M_t)$, it follows from (6.10) that for $n \geq 3$ and any $t \in (0, t_0]$ we have

$$\left(\int_{M_t} |h|^{q^*} \, d\mu \right)^{2/q^*} \leq C(n) \quad \int_{M_t} |\nabla h|^2 + H^2 h^2 \, d\mu.$$

Hence, for any $t \in (0, t_0]$ we have

$$\begin{aligned} \int_{M_t} h^{2k} \, d\mu &= \int_{M_t} h^2 \, h^{2(k-1)} \, d\mu \\ &\leq \left(\int_{M_t} |h|^{2q^*} \, d\mu \right)^{1/q^*} \left(\int_{M_t} |h|^{2(k-1)q} \, d\mu \right)^{1/q} \\ &\leq C \left(\int_{M_t} |\nabla h|^2 + H^2 h^2 \, d\mu \right) \sup_{t \in (-0, t_0]} \left(\int_{M_t} h^{2(k-1)q}(\cdot, t) \, d\mu \right)^{1/q}. \end{aligned}$$

Inequality (6.12) with $\lambda = 0$ now follows by integrating in t. When $n = 2$ one uses the same calculation as above with the only difference that now $q^* = 2$ and by (6.11) we have

$$\left(\int_{M_t} |h|^{2q^*} \, d\mu \right)^{1/q^*} \leq C \, |M_t \cap \operatorname{supp} h|^{1/2} \int_{M_t} |\nabla f|^2 + H^2 f^2 \, d\mu$$

leading to (6.12) with $\lambda = 1/2$. □

We will next combine (6.6) and (6.12) to conclude the proof of Proposition 6.2 via a Moser iteration argument.

Proof of Proposition 6.2 For the given numbers $\rho_0 > 1$ and $t_0 \in (0, \tau]$ and any numbers $1/4 < r < \bar{r} < 1/2$, we let $\eta \in C_c(S_{t_0} \cap Q^{**}_{\rho_0, r_0})$ be the cutoff function given by (6.4)–(6.5). Clearly,

$$\frac{\rho_0}{2} \leq |F| \leq 2 \, \rho_0, \quad \text{on} \quad S_{t_0} \cap Q^*_{\rho_0, r_0}$$

which includes the support of η.

We first apply (6.12) to $h := \eta\, \bar{w}^\theta \in C^{1,0}(S_{t_0})$ to obtain

$$\rho_0^{-n} \int_0^{t_0} \int_{M_t} (\eta \bar{w}^\theta)^{2k}\, d\mu \le C \left\{ \rho_0^{-n} \int_0^{t_0} \int_{M_t} \rho_0^2\, |\nabla (\eta \bar{w}^\theta)|^2 + \rho_0^2\, H^2 (\eta \bar{w}^\theta)^2\, d\mu\, dt \right. \\ \left. \cdot \sup_{t \in (0, t_0]} \left(\rho_0^{-n} \int_{M_t} (\eta \bar{w}^\theta)^{2(k-1)q} (\cdot, t)\, d\mu \right)^{1/q} \right\}. \tag{6.13}$$

Notice that we have multiplied by ρ_0^{-n} to make the inequality scaling invariant in space. For $n \ge 3$ inequality (6.13) simply follows from (6.12), since $\lambda = 0$ and $q = n/2$. When $n = 2$ we apply (6.12) with $\lambda = 1/2$ and $q = 2$ and use the fact that $|M_t \cap \operatorname{supp} \eta| \le C\, \rho_0^2$.

From Proposition 2.12 we have

$$\rho_0^2\, H^2 \le C\, |F|^2 H^2 \le C \tag{6.14}$$

since $|F| \ge \rho_0/2$ on the support of η. We next choose $k = k(p) > 1$ such that

$$2\theta(k-1)q = p.$$

Since $\theta = (p+2)/2$ this means that $(p+2)(k-1)q = p$, hence $(k-1)q = p/(p+2) < 1$ or $k < (q+1)/q < q/(q-1) := q^*$. In addition $k > 1 + p/(p+2)q \ge 1 + 1/(3q)$, since $p \ge 1$. Summarizing, for future reference we have

$$1 + \frac{1}{3q} < k = k(p) < q^* \tag{6.15}$$

with q, q^* as in Lemma 6.4. Thus, from (6.13) and (6.14) we obtain

$$\rho_0^{-n} \int_0^{t_0} \int_{M_t} \eta^2 \bar{w}^{2\theta k}\, d\mu \le C \left\{ \rho_0^{-n} \int_0^{t_0} \int_{M_t} \rho_0^2\, |\nabla (\eta \bar{w}^\theta)|^2 + \eta^2 \bar{w}^{2\theta}\, d\mu\, dt \right. \\ \left. \cdot \sup_{t \in (0, t_0]} \left(\rho_0^{-n} \int_{M_t} \eta^{2(k-1)q}\, \bar{w}^p (\cdot, t)\, d\mu \right)^{1/q} \right\}. \tag{6.16}$$

To simplify the notation, set

$$S^r_{\rho_0, t_0} := Q^r_{\rho_0, t_0} \cap S_{t_0} \quad \text{and} \quad S^{\bar{r}}_{\rho_0, t_0} := Q^{\bar{r}}_{\rho_0, t_0} \cap S_{t_0}$$

and recall that from its definition $\eta \equiv 1$ on $Q^r_{\rho_0, t_0}$ and $\eta \equiv 0$ outside $Q^{\bar{r}}_{\rho_0, t_0}$. Also, set

$$B := (\bar{r} - r)^{-2}\, t_0^{-1}$$

Combining (6.6) and (6.16) yields

$$\rho_0^{-n} \iint_{S^r_{\rho_0, t_0}} \bar{w}^{2\theta k}\, d\mu dt \le C\, B^{1+1/q}\, t_0^{-(1+1/q)} \left(\rho_0^{-n} \iint_{S^{\bar{r}}_{\rho_0, t_0}} \bar{w}^{2\theta}\, d\mu dt \right)^{1+1/q}. \tag{6.17}$$

We will now iterate this inequality to obtain the desired L^∞ bound on $\bar{w}$. To this end, we define $p_0, p_1, \ldots$ and $\theta_0, \theta_1, \ldots$ by letting $p_0 = 1$ and setting

$$\theta_\nu = \frac{p_\nu + 2}{2}, \quad \theta_{\nu+1} = k_\nu\, \theta_\nu, \quad k_\nu = k_\nu(p_\nu) = 1 + \frac{p_\nu}{(p_\nu + 2)\, q}. \tag{6.18}$$

We also define

$$\rho_\nu := \frac{(1+\nu)}{2(1+2\nu)} \quad \text{and} \quad Q_\nu := Q^{r_\nu}_{\rho_0, t_0}, \quad S_\nu := S^r_{\rho_0, t_0} = Q^{\rho_\nu}_{\rho_0, t_0} \cap S_{t_0}.$$

Observe that under this notation $Q_0 = Q^*_{\rho_0,t_0}$ while $\lim_{\nu\to\infty} Q_\nu = Q^*_{\rho_0,t_0}$. Also, set

$$M_\nu := \left(\rho_0^{-n} \iint_{S_\nu} v^{2\theta_\nu}\, d\mu\, dt\right)^{1/2\theta_\nu}.$$

It then follows from (6.17) that

$$M_{\nu+1}^{2\theta_{\nu+1}} \le C\, B_\nu^{1+1/q}\, M_\nu^{2\theta_\nu(1+1/q)} \tag{6.19}$$

with

$$B_\nu = (r_\nu - r_{\nu+1})^{-2}\, t_0^{-1} \le C\, \nu^4\, t_0^{-1}.$$

Since $q > 1$, it follows from (6.19) that

$$M_{\nu+1} \le (C\, \nu^8\, t_0{}^{-2})^{1/2\theta_{\nu+1}}\ M_\nu^{\lambda_\nu} \tag{6.20}$$

with $\lambda_\nu = (1+1/q)/k(p_\nu)$. Since $\lim_{\nu\to\infty} p_\nu = +\infty$ we have $\lim_{\nu\to\infty} k(p_\nu) = 1+1/q$. It follows that

$$E^\nu < \theta_\nu < (E^*)^\nu \quad \text{and} \quad E^\nu < p_\nu < (E^*)^\nu$$

for some numbers $1 < E < E^* < \infty$. Also, $1 < \bar{\lambda}_\nu < 1 + C\, E^{-\nu}$. We conclude from the bounds above that

$$\lim_{\nu\to\infty} M_\nu \le C\, t_0^{-\mu_0} M_0^{\sigma_0}$$

for some absolute constants μ_0 and σ_0. Thus,

$$\|\bar{w}\|_{L^\infty(Q_{\rho_0,t_0}\cap S_{t_0})} \le C\, t_0^{-\mu_1} \left(\rho_0^{-n} \iint_{M_t\cap Q^*_{\rho_0,t_0}} \bar{w}^3\, d\mu\, dt\right)^{\sigma_1} \tag{6.21}$$

with $2\theta_0 = p_0 + 2 = 3$ and for some new positive absolute constants μ_1 and σ_1. The constant C is independent of ρ_0 and t_0.

To finish the proof of the proposition it will be sufficient to estimate the integral on the right hand side of (6.21) in terms of

$$I := \sup_{t\in(t_0/4,t_0]} \left(\rho_0^{-n} \int_{M_t\cap Q^{**}_{\rho_0,t_0}} \bar{w}(\cdot,t)\, d\mu\right).$$

To this end, we set again $B := (\bar{r} - r)^{-2}\, t_0^{-1}$ and combine (6.13) with (6.6) and the bound (6.14), to obtain for $\theta_0 := 3/2$ the bound

$$\left(\rho_0^{-n} \iint_{S^r_{\rho_0,t_0}} \bar{w}^{2\theta_0 k}\, d\mu dt\right) \le C\, B \left(\rho_0^{-n} \iint_{S^{\bar{r}}_{\rho_0,t_0}} \bar{w}^{2\theta_0}\, d\mu\, dt\right) \cdot \sup_{t\in((1-\bar{r})\,t_0,t_0])} \left(\rho_0^{-n} \int_{M_t\cap Q^{\bar{r}}_{r_0,t_0}} \bar{w}^{2\theta_0(k-1)q}(x,t)\, d\mu\right)^{1/q}. \tag{6.22}$$

If we choose $k > 1$ so that $2\theta_0(k-1)q = 1$, the above bound yields

$$\rho_0^{-n} \iint_{S^r_{\rho_0,t_0}} \bar{w}^{2\theta_0 k}\, d\mu\, dt \le C\, B\, I^{1/q} \left(\rho_0^{-n} \iint_{S^{\bar{r}}_{\rho_0,t_0}} \bar{w}^{2\theta_0}\, d\mu\, dt\right). \tag{6.23}$$

Setting,

$$m(r,k) := \iint_{S^r_{\rho_0,t_0}} \bar{w}^{2\theta_0 k}\, d\mu\, dt$$

follows from (6.22) that for any $1/4 < r < \bar{r} < 3/4$, we have

$$m(r,k) \le C\,(\bar{r}-r)^{-2}\bar{t_0}^{-1}\, I^{1/q}\, m(\bar{r},1). \tag{6.24}$$

Using Hölder's inequality we have

$$m(\bar{r},1) \le m(\bar{r},k)^{\lambda/k}\, m(\bar{r},s)^{(1-\lambda)/s}$$

for any $s \in (0,1)$ with $\lambda = \frac{(1-s)k}{k-s}$. For $\gamma > 1$ and $r \in [2/3, 1]$, (6.24) shows that

$$\begin{aligned}\log m(3r^\gamma/4,k) &\le \log C + \log {t_0}^{-1} + \frac{1}{q}\log I + \log(3\,(r-r^\gamma)/4)^{-2}\\ &\quad + \frac{\lambda}{k}\log m(3r/4,k) + \frac{1-\lambda}{q}\log m(3/4,s)\end{aligned}$$

since $m(3r/4, q) \le m(3/4, q)$. Integrating in r with respect to dr/r on $[2/3, 1]$ we find after a change of variable that

$$\begin{aligned}&\gamma^{-1}\int_{2/3}^1 \log m(3r/4,k)\,\frac{dr}{r}\\ &\quad\le C_1\log I + C_2\log {t_0}^{-1} + C_2\log m(3/4,s) + C_3 + \frac{\lambda}{k}\int_{2/3}^1 m(3r/4,k)\,\frac{dr}{r}.\end{aligned} \tag{6.25}$$

Now choose s so that $2\theta_0 s = 1$ (recall that we have set $\theta_0 = 3$) and γ so close to 1 so that $\gamma^{-1} > \lambda/k$. If $m(1/2, k) \le 1$, then since $k > 1$ we conclude that $m(1/2, 1) \le C$ and the bound $\|\bar{w}\|_{L^\infty(Q_{\rho_0,t_0}\cap S_{t_0})} \le C$ follows from (6.21). Otherwise, $\log m(3r4, k) > 0$ for $r \in [2/3, 1]$ and from (6.25) we obtain

$$\left(\gamma^{-1}-\frac{\lambda}{k}\right)\int_{2/3}^1 \log m(3r/4,k)\,\frac{dr}{r} \le C_1\log I + C_2\log {t_0}^{-1} + C_2\log m(3/4,s) + C_3$$

which yields

$$m(1/2,k) \le C\, t_0^{-\mu_2}\, I^{\sigma_4}\, m(3/4,s)$$

or equivalently

$$\rho_0^{-n}\iint_{Q^*_{\rho_0,t_0}\cap S_{t_0}} \bar{w}^{2\theta_0 k}\, d\mu\, dt \le C\,\bar{t_0}^{-\mu_2} I^{\sigma_2}\left(\rho_0^{-n}\iint_{Q^{**}_{\rho_0,t_0}\cap S_{t_0}} \bar{w}\, d\mu\, dt\right)^{\sigma_3} \tag{6.26}$$

Since, $\iint_{Q^{**}_{\rho_0,t_0}\cap S_{t_0}} \bar{w}\, d\mu\, dt \le C\, I$, combining (6.26) with (6.21) yields the bound

$$\|\bar{w}\|_{L^\infty(Q_{\rho_0,t_0}\cap S_{t_0})} \le C\,\bar{t_0}^{-\mu} I^{\sigma} \tag{6.27}$$

for some new absolute constants $\sigma > 0$ and $\mu > 0$. The constant C is independent of r_0 and $\bar{t_0}$. Recalling that $\bar{w} = \max(w, 1)$ we conclude (6.2). □

Proposition 6.2 provides an L^∞ bound on $w(\cdot, t)$ on $M_t \cap \{|F| \geq 2\}$, $0 < t \leq \tau$. The next result gives an L^∞ bound on $w(\cdot, t)$ on $M_t \cap \{|F| \leq 1\}$. For any given $t_0 \in (0, \tau]$ and $r \in (0, 1)$ we consider the parabolic cylinders in $\mathbb{R}^{n+1} \times (0, +\infty)$ given by

$$Q_{t_0} := B_2(0) \times (t_0/2, t_0] \quad \text{and} \quad Q^{**}_{t_0} := B_4(0) \times (t_0/4, t_0]$$

where $B_r(0) := \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}| < r\}$ denotes the ball in $\mathbb{R}^{n+1}$ centered at the origin of radius r. We have the following estimate.

Proposition 6.5 *Assume that M_t is a solution to (1.1) as in Theorem 4.3 defined for $t \in (0, \tau]$ and assume that $\tau < T - 3\delta$ with T given by (3.8) and $\delta > 0$. There exist absolute constants $\mu > 0$ and $\sigma > 0$ and a constant C that depends on α_0, κ, on δ, and the initial bound $\sup_{M_0} \langle F, \omega \rangle H$, for which $w := \hat{\gamma}(t)\,(\langle \hat{F}, \nu \rangle H)^{-1}$ satisfies the bound*

$$\|w\|_{L^\infty(Q_{t_0} \cap S_{t_0})} \leq C\, t_0^{-\mu} \left(1 + \sup_{t \in (t_0/4, t_0]} \int_{M_t \cap Q^{**}_{t_0}} w(\cdot, t)\, d\mu \right)^{\sigma} \tag{6.28}$$

which holds for any $t_0 > 0$ such that $Q_{t_0} \cap S_{t_0}$ is not empty.

Proof The proof is the very similar as the proof of Proposition 6.2. It is actually simpler as it doesn't need to be scaled with respect to ρ_0. □

Proof of Theorem 6.1 Readily follows by combining the two estimates in Propositions 6.2 and 6.5. □

We will next combine Theorems 5.4 and 6.1 to obtain the following L^∞ bound on w in terms of the initial data.

Theorem 6.6 (L^∞ bound on w in terms of the initial data) *Assume that M_t is a solution to (1.1) as in Theorem 4.3 defined for $t \in (0, \tau)$, and assume that $\tau < T - 3\delta$ with $T = T(\alpha_0)$ given by (3.8) and $\delta > 0$. Then, of any $t_0 \in (0, \tau/2]$ there exists a constant $C_\delta\big(t_0, \alpha_0, \kappa, \sup_{M_0} w, \inf_{M_0} w\big)$ such that*

$$\sup_{t \in (t_0, \tau)} \|w(\cdot, t)\|_{L^\infty(M_t)} \leq C_\delta\big(t_0, \alpha_0, \kappa, \sup_{M_0} w, \inf_{M_0} w\big). \tag{6.29}$$

Proof We recall the definition of $\hat{w} := (w-1)_+$, with $w := \hat{\gamma}(t)\, v^{-1} = \hat{\gamma}(t)\,(\langle \hat{F}, \nu \rangle H)^{-1}$ and $\hat{\gamma}$ as defined at the beginning of this section. Since $\langle \hat{F}, \nu \rangle = -\langle F, \omega \rangle \langle \omega, \nu \rangle$ and $-\langle \omega, \nu \rangle := (\sqrt{1 + |D\bar{u}|^2})^{-1}$ satisfies $(\sqrt{1+\alpha_0^2})^{-1} \leq \langle \omega, \nu \rangle \leq 1$, it follows that the assumed initial bound (1.8) and the definition of w imply the bound

$$\bar{c}_0 \leq w(\cdot, 0) \leq \bar{C}_0 \tag{6.30}$$

for some positive constants $\bar{c}_0, \bar{C}_0$ depending on the constants c_0, C_0 in (1.8) and $\alpha_0, \hat{\gamma}_0 := \hat{\gamma}(0)$.

For any $t_0 \in (0, \tau/2]$ we have that (6.1) holds. Hence, it is sufficient to bound the righthand side of (6.1) in terms of the initial data and t_0. Since $w \leq \hat{w} + 1$ and $\hat{w}$ is compactly supported for each $t \in (t_0/4, \tau]$ (the latter follows from (5.3) and the fact that $\hat{\gamma}(t) < \gamma(t)$), we have

$$\sup_{t \in (t_0/4, \tau]} \sup_{R \geq 1} R^{-n} \int_{M_t \cap \{|F| \leq R\}} w(\cdot, t)\, d\mu \leq 1 + \sup_{t \in (t_0/4, \tau]} \int_{M_t} \hat{w}(\cdot, t)\, d\mu. \tag{6.31}$$

We next want to apply the L^{p+1} bound (5.12) for $p = 0$, to bound $\sup_{t \in (t_0/4, \tau]} \int_{M_t} \hat{w}(\cdot, t)\, d\mu$ in terms of the initial data and t_0. Notice that we cannot use (5.12) on the interval $[0, \tau]$, as we have not assumed that (5.3) holds at $t = 0$ which would imply that $\hat{w}(\cdot, 0)$ is compactly supported. It holds only for $t > 0$ as a consequence of parabolic regularity (see Proposition 4.4). Thus, we first apply (5.12) on $(t_0/4, \tau]$ to obtain

$$\sup_{t\in(t_0/4,\tau]} \int_{M_t} \hat{w}(\cdot, t)\, d\mu \le C(\delta, T)\left(1 + \int_{M_{t_0/4}} \hat{w}(\cdot, t_0/4)\, d\mu\right). \tag{6.32}$$

To conclude our proof we will bound $\int_{M_{t_0/4}} \hat{w}(\cdot, t_0/4)\, d\mu$ in terms of $\sup_{M_0} w$ and the size of the support of $\hat{w}(\cdot, t_0/4)$. Let us first bound $\sup_{M_{t_0/4}} w$ in terms of $\sup_{M_0} w$. We will do that for $t_0/4 \le \tau_0$, for a $\tau_0 > 0$ depending only on the initial data. This is sufficient since t_0 in (6.29) can be chosen small. To this end, we will use the maximum principle on w to equation (5.4). Indeed, setting $m(t) := \sup_{M_t} w$, a straightforward application of the maximum principle on equation (5.4), using also the facts that $\langle \omega, \nu\rangle \le 1$, $c_1 < 0$ and $\gamma^{-1}(t) \le \gamma^{-1}(\tau_0)$ on $[0, \tau_0]$, gives that

$$\frac{dm(t)}{dt} \le 2\langle \omega, \nu\rangle^2\, \hat{\gamma}^{-1}(t)\, m(t)^2 + c_1\, \hat{\gamma}^{-1}(t)\, m(t) \le 2\hat{\gamma}^{-1}(\tau_0)\, m(t)^2$$

yielding

$$\sup_{t\in[0,\tau_0]} m(t) \le \frac{m(0)\, \hat{\gamma}(\tau_0)}{\hat{\gamma}(\tau_0) - 2m(0)\, \tau_0}.$$

If τ_0 is sufficiently small such that $\hat{\gamma}_0/2 \le \hat{\gamma}(\tau_0) \le \hat{\gamma}_0$, we conclude that

$$\sup_{t\in[0,\tau_0]} m(t) \le \frac{2m(0)\, \hat{\gamma}_0}{\hat{\gamma}_0 - 4m(0)\, \tau_0}.$$

By decreasing τ_0 is necessary we may assume that $\hat{\gamma}_0 - 4m(0)\, \tau_0 \ge \hat{\gamma}_0/2$. We conclude using also (6.30) that for such a τ_0 we have

$$\sup_{t\in[0,\tau_0]} w(\cdot, t) \le 2\, m(0) \le 2\bar{C}_0. \tag{6.33}$$

Since we may assume without loss of generality that $t_0/4 \le \tau_0$, the last bound and $\hat{w} \le w$ imply that $\sup_{M_{t_0/4}} \hat{w} \le 2\bar{C}_0$. On the other hand, by (5.3) and the fact that $\hat{\gamma}(t) < \gamma(t)$, we have that $\hat{w} := (w - 1)_+$ is compactly supported for all $t \in (0, \tau)$ and in particular for $t := t_0/4$. This means that its support is contained in a ball in $\mathbb{R}^{n+1}$ of radius $R_0 := R_0(t_0)$. Hence,

$$\int_{M_{t_0/4}} \hat{w}(\cdot, t_0/4)\, d\mu \le C\big(R_0, \bar{C}_0\big). \tag{6.34}$$

Finally, by combining (6.1) with (6.31), (6.32) and (6.34) we conclude that (6.29) holds. □

7 Long time existence Theorem 1.1

In this final section we will give the proof of our long time existence Theorem 1.1, which says that our solution M_t of the inverse mean curvature flow will exist up to time T, where T denotes the critical time where the cone at infinity becomes flat and is given by (3.8).

Proof of Theorem 1.1 Our short time existence Theorem 4.3 implies the existence of a maximal time $\tau_{\max} > 0$ for which a convex solution M_t of (1.1) exists on $[0, \tau_{\max})$ and the following hold:

i. M_t, $t \in [0, \tau_{\max})$ is an entire convex graph $x_{n+1} = \bar{u}(x, t)$ over $\mathbb{R}^n$ which satisfies condition (1.7);
ii. $\bar{u}$ is C^∞ smooth on $\mathbb{R}^n \times (0, \tau_{\max})$;
iii. $c_{\tau_1} < H\,\langle F, \omega\rangle \le C_0$, on $t \in [0, \tau_1]$, for all $0 < \tau_1 < \tau_{\max}$.

It $\tau_{\max} = T$, we are done, otherwise $\tau_{\max} < T - \delta$, for some $\delta > 0$. We claim that

$$\inf_{t\in[0,\tau_{\max})} \inf_{M_t} H\,\langle \hat{F}, \nu\rangle \ge c_\delta > 0. \tag{7.1}$$

To this end, we will combine (6.29) with (6.33). We have seen in the proof of Theorem 6.6 that there exists $\tau_0 > 0$ depending only on the initial data such that (6.33) holds. Assume without loss of generality that $\tau_0 < T/2$. Since $w := \hat{\gamma}(t)(H\,\langle \hat{F}, \nu\rangle)^{-1}$, it follows from (6.33) that

$$\inf_{t\in[0,\tau_0]} \inf_{M_t} H\,\langle \hat{F}, \nu\rangle \ge c_1(\tau_0) > 0.$$

Now we apply (6.29) for $t_0 := \tau_0$ and $\tau := \tau_{\max}$ to obtain the bound

$$\sup_{t\in(\tau_0,\tau_{\max})} \|w(\cdot, t)\|_{L^\infty(M_t)} \le C_\delta\big(\tau_0\big).$$

This can be done since conditions i.–iii. above imply that Theorem 6.6 holds on $(0, \tau_{\max})$. It follows that

$$\inf_{t\in(\tau_0,\tau_{\max})} \inf_{M_t} H\,\langle \hat{F}, \nu\rangle \ge c_2(\delta, \tau_0) > 0.$$

Combining the last two bounds yields that (7.1) holds and since $\langle \hat{F}, \nu\rangle = -\langle F, \omega\rangle\,\langle \omega, \nu\rangle \le \langle F, \omega\rangle$ we also have

$$\inf_{t\in(0,\tau_{\max})} \inf_{M_t} H\,\langle F, \omega\rangle \ge c_\delta > 0. \tag{7.2}$$

In addition, by Proposition 2.12 we have

$$\sup_{M_t} H\,\langle F, \omega\rangle \le \sup_{M_t} H\,\langle F, \omega\rangle \le C_0. \tag{7.3}$$

On the other hand, $\bar{u}_t \le 0$ and (1.7) imply that the pointwise limit $\bar{u}(x, \tau_{\max}) := \lim_{t\to\tau_{\max}} \bar{u}(x, t)$ exists for all $x \in \mathbb{R}^n$ and it defines a convex graph. Moreover, it satisfies (1.7) at $t = \tau_{\max}$. Now the lower and upper bounds (7.1), (7.3) and (1.7) for $t \in [0, \tau_{\max}]$, imply that the fully-nonlinear equation (1.4) satisfied by $\bar{u}$ is strictly parabolic on compact subsets of $\mathbb{R}^n \times [0, \tau_{\max}]$. It follows by standard local regularity results on fully-nonlinear equations that $\bar{u}(\cdot, \tau_{\max})$ is C^∞ smooth. Moreover, the above bounds show that $c_\delta \le H\,\langle F, \omega\rangle \le C_0$ on $M_{\tau_{\max}}$. Also, since $|\langle F, \nu\rangle| \le C_0$ on M_0 (see in (4.15)) its evolution equation given in Lemma 2.1 and convexity imply the bound $|\langle F, \nu\rangle| \le C(T)$ on $M_{\tau_{\max}}$. We conclude from the above discussion that at time $t = \tau_{\max}$ the entire graph $M_{\tau_{\max}}$ given by $x_{n+1} = \bar{u}(\cdot, \tau_{\max})$ satisfies all the assumptions of our short time existence result Theorem 4.3, hence the flow can be extended beyond $\tau_{\max}$ contradicting its maximality. This shows that $\tau_{\max} = T$, showing that our solution exists for all $t \in (0, T)$.

Let us now observe that as $t \to T$, the solution converges to a horizontal plane of height $h \in [0, \kappa]$. First, the pointwise limit $\bar{u}(x, T) := \lim_{t\to T} \bar{u}(x, t)$ exists, since $\bar{u}_t(x, t) \ge 0$ for all $t < T$. Second, $\bar{u}(\cdot, T)$ is convex and lies between the two horizontal planes $x_{n+1} = 0$ and $x_{n+1} = \kappa$. The latter simply follows from (1.7) and the fact that $\alpha(T) = 0$. In addition, our a'priori local bound from above on the mean curvature H shown in Proposition 2.11, which holds uniformly up to $t = T$, implies that $\bar{u}(\cdot, T) \in C^{1,1}_{loc}(\mathbb{R}^n)$. It follows that $\bar{u}(\cdot, T)$ must be a horizontal plane of height $h \in [0, \kappa]$, and that the convergence $\lim_{t \to T} \bar{u}(\cdot, t) \equiv h$ is in $C^{1,\alpha}$, on any compact subset of $\mathbb{R}^n$ and for all $\alpha < 1$. □

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References

1. Alessandroni, R., Sinestrari, C.: Evolution of convex entire graphs by curvature flows. Geom. Flows **1**, 111–125 (2015)
2. Carron, G.: Inégalités de Hardy sur les variétés riemanniennes non-compactes. J. Math. Pures Appl. **76**, 883–891 (1997)
3. Choi, K., Daskalopoulos, P.: The Q^k flow on complete non compact graphs, arXiv: 1603.03453
4. Choi, K., Daskalopoulos, P., Kim, L., Lee, L.: The evolution of complete non-compact graphs by powers of Gauss curvature, arXiv: 1603.04286
5. Dahlberg, B.E.J., Kenig, C.E.: Nonnegative solutions to the porous medium equation. Commun. Part. Differ. Equ. **9**, 409–437 (1984)
6. Daskalopoulos, P., Huisken, G., King, J.R.: Self-similar entire convex solutions to the inverse mean curvature flow, private communication
7. Daskalopoulos, P., Kenig, C.: Degenerate Diffusions: Initial Value Problems and Local Regularity Theory, EMS Tracts in Mathematics, 1. European Mathematical Society (EMS), Zürich (2007)
8. Ecker, K., Huisken, G.: Mean curvature evolution of entire graphs. Ann. Math. **130**, 453–471 (1989)
9. Ecker, K., Huisken, G.: Interior estimates for hypersurfaces moving by mean curvature. Invent. Math. **105**, 547–569 (1991)
10. Franzen, M.: Existence of convex entire graphs evolving by powers of the mean curvature, preprint (2011) arXiv:1112.4359
11. Halldorsson, H.P.: Self-similar solutions to the curve shortening flow. Trans. Am. Math. Soc. **364**(10), 5285–5309 (2012)
12. Gerhardt, C.: Flow of nonconvex hypersurfaces into spheres. J. Differ. Geom. **32**, 299–314 (1990)
13. Heidusch, M.E.: Zur Regularität des inversen mittleren Krümmungsflusses, Dissertation Tübingen Univ. (2001)
14. Holland, J.: Interior estimates for hypersurfaces evolving by their k-th Weingarten curvature and some applications. Indiana Univ. Math. J. **63**, 1281–1310 (2014)
15. Huisken, G., Ilmanen, T.: The Riemannian Penrose inequality. Int. Math. Res. Not. **20**, 1045–1058 (1997)
16. Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the Riemannian Penrose inequality. J. Differ. Geom. **59**, 353–438 (2001)
17. Huisken, G., Ilmanen, T.: Higher regularity of the inverse mean curvature flow. J. Differ. Geom. **80**, 433–451 (2008)
18. Kombe, I., Özaydin, M.: Improved Hardy and Rellich inequalities on Riemannian manifolds. Trans. Am. Math. Soc. **361**, 6191–6203 (2009)
19. Krylov, N.V.: Nonlinear Elliptic and Parabolic Equations of the Second Order. Reidel, Dordrecht (1987)
20. Smoczyk, K.: Remarks on the inverse mean curvature flow. Asian J. Math. **4**, 331–335 (2000)
21. Tian, G., Wang, X.-J.: A priori estimates for fully nonlinear parabolic equations. Int. Math. Res. Not **2013**, 3857–3877 (2013)
22. Urbas, J.: On the expansion of strashaped hypersurfaces by symmetric functions of their principal curvatures. Math. Z. **205**, 355–372 (1990)